Contributions to coincidence degree theory of homogeneous operators

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Abstract. *The aim of this paper is to establish some formulas for the coincidence degree of nonlinear homogeneous operators in Banach spaces. As applications, we shall consider the periodic problem for general functional differential equations. We succeed to abstract and even to extend similar results of A. Capietto, J. Mawhin and F. Zanolin for the periodic problem for ordinary differential equations and retarted functional differential equations.*
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1 Introduction

A. Capietto, J. Mawhin and F. Zanolin have established some formulas for the coincidence degree of some nonlinear homogeneous differential operators [1], and retarted functional differential operators [2] in spaces of periodic functions. The aim of this paper is to prove similar formulas for the coincidence degree of nonlinear homogeneous abstract operators in Banach spaces. As applications, we shall consider the periodic problem for general functional differential equations (FDE). We succeed to abstract and even to extend the mentioned results from [1,2].
More exactly, the abstract equations and FDE have the forms

$$Lx = N(x) + N_p(x), \tag{1}$$

$$x'(t) = F(x)(t) + F_p(x)(t). \tag{2}$$

$L : X \to Y$ is a Fredholm linear continuous operator of index 0 and $N, N_p : X \to Y$ are completely continuous between Banach spaces X and Y.
F , $F_p : C_\omega \to L_\omega$ are continuous operators, taking bounded sets into bounded sets. We denote by C_ω the space of ω- periodic continuous functions $x : R \to R^m$ and by L_ω the space of ω-periodic integrable functions $x : R \to R^m$, with the usual norms.
For the nonlinear operators involved, N, N_p, F, F_p we shall assume to be positively homogeneous.
Our results are formulated in terms of computation of coincidence degrees (see [3]). Existence results are easy consequences.

2 Preliminaries

In this section we present with an adapted proof two theorems, which are essentially from [3] (Theorem III.3, page 24 and Theorem IV.2, page 31). They will be used in the next section.
Let us denote $X_1 = KerL$, $Y_1 = \mathrm{co}ImL$ and consider the linear isomorphism $J : X_1 \to Y_1$ and the continuous projectors $P : X \to X_1$ and $Q : Y \to Y_1$. Let us remember that the operator $M : X \to X$, $M = (L + JP)^{-1}(N + JP)$ is completely continuous (see also [6]) and the coincidence degree of (L, N) is defined as the Leray-Schauder degree of M (on some open and bounded subset of X).
Ω will denote an open and bounded subset of X.

Theorem 1 *Assume that $Lx \neq N(x)$ for each $x \in \partial\Omega$.*
Then there exists $\mu > 0$ such that $\mu \leq inf_{x\in\partial\Omega}||Lx - N(x)||$ and for every N_p with $sup_{x\in\partial\Omega}||N_p(x)|| < \mu$,

$$d((L, N + N_p), \Omega) = d((L, N), \Omega).$$

Proof. By using formula $L - N = (L + JP)(I - M)$ and the facts that $(L + JP) : X \to Y$ is a homeomorfism and $(I - M) : \overline{\Omega} \to X$ is proper, we deduce that $(L - N) : \overline{\Omega} \to Y$ is proper, too. Thus, $(L - N)$ is a closed operator, which implies that $(L - N)(\partial\Omega)$ is a closed subset of Y. From our hypothesis, $0 \notin (L - N)(\partial\Omega)$. Then we obtain

$$inf_{x\in\partial\Omega}||Lx - N(x)|| \ = \ dist(0, (L - N)(\partial\Omega) \geq \mu > 0.$$

Finally, let us consider the homotopy $H(x,\lambda) = N(x) + \lambda N_p(x)$, which is admissible because for all $x \in \partial\Omega$ and $\lambda \in [0,1]$ the following inequalities are valid.

$$\begin{aligned} ||Lx - H(x,\lambda)|| &= ||Lx - N(x) - \lambda N_p(x)|| \\ &\geq ||Lx - N(x)|| \; - \; \lambda ||N_p(x)|| \\ &> \mu - \lambda\mu = (1-\lambda)\mu \geq 0. \end{aligned}$$

Using the invariance of the coincidence degree under an admissible homotopy, we obtain the conclusion.

Theorem 2 *Assume that $QN(x) \neq 0$ for all $x \in \partial\Omega \cap X_1$. Then there exists $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1]$ the equation $Lx = \lambda\varepsilon N(x)$ has no solutions on $\partial\Omega$ for every $\lambda \in (0,1]$ and*

$$d((L, \varepsilon N), \Omega) = d_B(J^{-1}QN, \Omega \cap X_1, 0).$$

Proof. It is readily checked that $Lx \neq QN(x)$ for each $x \in \partial\Omega$. Then, by Theorem 1, there exists $\mu > 0$ such that $\mu \leq \inf_{x\in\partial\Omega} ||Lx - QN(x)||$.
Let us consider now for $\lambda \in (0,1]$, the operator $N_p = \lambda(I-Q)(\varepsilon N)$ and notice that the solutions of equation

$$Lx = QN(x) + N_p(x)$$

are the same as for equation

$$Lx = \lambda\varepsilon N(x).$$

The facts that N is completely continuous and $(I-Q)$ is a linear bounded operator imply that the set $N_p(\partial\Omega)$ is bounded and contained in a ball of radius $R(\varepsilon)$ such that $R(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then, there exists $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1]$, $\sup_{x\in\partial\Omega} ||N_p(x)|| < \mu$. By Theorem 1, this fact implies the first part of conclusion.
Now, by applying the well-known continuation theorem ([3], Theorem IV.1, page 27), we obtain that

$$d((L, \varepsilon N), \Omega) = d_B(J^{-1}QN, \Omega \cap X_1, 0).$$

3 Main results

In this section we shall give perturbations results in the coincidence degree theory, using the good properties of homogeneous operators.
We denote by $B_r(0)$ the ball of center 0 and radius $r > 0$ from the Banach space X.

Theorem 3 *Let $N, N_p : X \to Y$ be two positively homogeneous operators of order α and β, respectively.*
Assume that $QN(y) \neq 0$ for all $y \in X_1$ with $|y| = 1$. Then there exists $r_0 > 0$ such that, if $r \leq r_0$ and $1 < \alpha$ or $r \geq r_0$ and $\alpha < 1$ equation $Lx = N(x)$ has no solutions x with $|x| = r$ and

(i) $d((L, N),\ B_r(0)) = d_B(J^{-1}QN,\ B_r(0) \cap X_1, 0).$

Similarly, there exists $r_1 > 0$ such that, if $r \leq r_1$ and ($1 < \alpha \leq \beta$ or $1 = \alpha < \beta$), or $r \geq r_1$ and ($\beta \leq \alpha < 1$ or $\beta < \alpha = 1$), (1) has no solutions x with $|x| = r$ and

(ii) $d((L, N + N_p),\ B_r(0)) \ = \ d((L, N),\ B_r(0)).$

Proof. It can be easily proved that, in our hypotheses, the Brouwer degree of $J^{-1}QN$ is well-defined on $B_r(0) \cap X_1$ and is constant with respect to all $r > 0$.
From Theorem 2 we have that there exists $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1]$ and for all $\lambda \in (0, 1]$,

$$Ly \neq \lambda\varepsilon N(y) \text{ for every } y \in B_1(0). \tag{3}$$

Now let us consider $x \in B_r(0)$ such that, for some $\lambda \in (0, 1]$

$$Lx = \lambda N(x).$$

Then $y = \frac{1}{r}x$ satisfies $|y| = 1$ and $Ly = \lambda r^{\alpha-1}N(y)$. If we consider $r \leq r_0 = \varepsilon_1^{\frac{1}{\alpha-1}}$ when $1 < \alpha$, or $r \geq r_0$ when $\alpha < 1$, this last relation contradicts (3). Thus, for all $\lambda \in (0, 1]$,

$$Lx \neq \lambda N(x), \text{ for all } x \in B_r(0)$$

and (see [3]) the conclusion (i) follows.
In order to prove (ii) we shall consider two cases.
a) The case $\alpha \neq 1$.
Repeating the arguments from the proof of Theorem 2 we obtain that there exists $\varepsilon_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_2]$, $\gamma \geq 1$ and $\lambda \in (0, 1]$ we have

$$Lz \neq \varepsilon(N(z) + \lambda\varepsilon^{\gamma-1}N_p(z)), \text{ for every } z \in B_1. \tag{4}$$

If we consider $x \in B_r(0)$ such that for some $\lambda \in [0, 1]$,

$$Lx = N(x) + \lambda N_p(x),$$

then $z = \frac{1}{r}x$ satisfies $|z| = 1$ and

$$Lz = r^{\alpha-1}N(z) + \lambda r^{\beta-1}N_p(z),$$

or, equivalently,

$$Lz = r^{\alpha-1}(N(z) + \lambda(r^{\alpha-1})^{\frac{\beta-1}{\alpha-1}-1}N_p(z)).$$

If we take $r_1 = \varepsilon_2^{\frac{1}{\alpha-1}}$ and $r \leq r_1$ when $1 < \alpha \leq \beta$ or $r \geq r_1$ when $\beta \leq \alpha < 1$, then this contradicts (4). For such an r and for all $\lambda \in [0, 1]$ we have

$$Lx \neq N(x) + \lambda N_p(x), \text{ for all } x \in B_r(0).$$

Using the homotopy invariance of the coincidence degree we obtain the conclusion.
b) The case $\alpha = 1$.
Now the hypotheses assure that the only solution of $Lx = N(x)$ is $x = 0$. From Theorem 1 we obtain that there exists $\mu_1 > 0$ such that for all $\mu \in (0, \mu_1]$ and every $\lambda \in [0, 1]$,

$$Lz \neq N(z) + \lambda\mu N_p(z), \text{ for all } z \in B_1.$$

The conclusion follows like in the previous case.

Theorem 4 *Assume that the following conditions are fulfilled:*
(i) the operator N is positively 1-homogeneous;
(ii) the equation $Lx = N(x)$ has only the solution $x = 0$;
(iii) there exist $\varepsilon, b, r_0 > 0$ such that $||N_p(x)|| \leq \varepsilon||x|| + b$ for all $x \in X$ with $||x|| \geq r_0$.
Then there exist ε sufficiently small and r sufficiently large such that

$$d((L, N + N_p), B_r(0)) = d((L, N), B_r(0)).$$

Proof. The operator $M : X \to X$, $M = (L + JP)^{-1}(N + JP)$ is completely continuous, positively 1-homogeneous and has a unique fixed point $x = 0$. Then, (by Theorem 1, [7]) we obtain that there exists $c > 0$ such that

$$||x - M(x)|| \geq c||x||, \text{ for all } x \in X.$$

Further,

$$\begin{aligned} c||x|| &\leq ||x - M(x)|| = ||(L + JP)^{-1}(L + JP)(x - M(x))|| = \\ &= ||(L + JP)^{-1}(Lx - N(x))|| \leq ||(L + JP)^{-1}|| \cdot ||Lx - N(x)||. \end{aligned}$$

It follows that there exists $d > 0$ such that $d||x|| \leq ||Lx - N(x)||$.
Finally, we choose $\varepsilon < d$ and $r \geq \frac{b}{d-\varepsilon}$, $r \geq r_0$ and apply Theorem 1.

4 Applications to FDE

The aim of this section is to obtain some consequences of Theorems 1-4 above in the case of ω-periodic problems for FDE.
In order to skip the detailed proof of these results, let us make the following considerations.
We denote (see also [4]) $X = Y = C_\omega$, $Lx = x(t) - x(0)$, $N(x) = \int_0^t F(x)(s)ds$, $N_p(x) = \int_0^t F_p(x)(s)ds$.
Then $X_1 \simeq R^m$, $Y_1 \simeq R^m$ and $P(x) = x(0)$, $Q(x) = x(\omega)$, $J(a) = \omega a$. The function $J^{-1}QN$ is $f : \Omega \cap R^m \to R^m$ given by the formula

$$f(a) = \frac{1}{\omega}\int_0^\omega F(a)(s)ds.$$

Let us consider also the ω-periodic problem for the equation

$$x'(t) = F(x)(t). \tag{5}$$

The main results of this section are the following.

1. Small perturbations

Theorem 5 *There exists $\varepsilon > 0$ sufficiently small such that, providing $||F_p(x)|| < \varepsilon$ for all $x \in \partial\Omega$,*

$$d((L, N + N_p), \Omega) = d((L, N), \Omega).$$

2. Small right hand side

Theorem 6 *If $f(a) \neq 0$, for all $a \in \partial\Omega \cap R^m$ then there exists $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1]$*

$$d((L, \varepsilon N), \Omega) = d_B(f, \Omega \cap R^m, 0).$$

3. Homogeneous right hand side

Theorem 7 *Let F, F_p be positively homogeneous operators of order α and β, respectively. Assume that $f(a) \neq 0$ for all $a \in R^m$ with $|a| = 1$. Then there exists $r_0 > 0$ such that if $r \leq r_0$ and ($1 < \alpha \leq \beta$ or $1 = \alpha < \beta$) or $r \geq r_0$ and ($\beta \leq \alpha < 1$ or $\beta < \alpha = 1$)*

$$d((L, N + N_p), B_r(0)) = d((L, N), B_r(0)) = d_B(f, B_r(0) \cap R^m, 0).$$

4. Asymptotically homogeneous perturbations

Theorem 8 *Suppose that the following conditions are satisfied:*
(i) F is positively 1-homogeneous;
(ii) the ω-periodic problem for the equation $x' = F(x)$ has only the solution $x = 0$;
(iii) there exist $\varepsilon, b, r_0 > 0$ such that $||F_p(x)|| \leq \varepsilon||x|| + b$ for all $x \in C_\omega$ with $||x|| \geq r_0$.
Then there exist ε sufficiently small and r sufficiently large such that

$$d((L, N + N_p), B_r(0)) = d((L, N), B_r(0)).$$

We consider now the retarted FDE

$$x' = g(t, x_t) + g_p(t, x_t),$$

where $g, g_p : R \times \mathcal{C}_r \to R^m$ are continuous mappings, taking bounded sets into bounded sets, ω-periodic in t, and positively homogeneous of order α and β, respectively, in the second variable. $\mathcal{C}_r = C([-r, 0], R^m)$ and, for each t, x_t is the element of $\mathcal{C}_r$ defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$.
If we denote by $F(x)(t) = g(t, x_t)$ and $F_p(x)(t) = g_p(t, x_t)$ we could apply Theorem 5-8. The corollary of Theorem 7 obtained in this way generalize Theorem 2 [2], where g_p does not depend on x_t.
In the case of the periodic problem for ordinary differential equations, corollaries of Theorem 7 and Theorem 8 are little extansions of Example 2 [1], Corollary 8 [1] and Theorem 2 [7], respectively (where the nonperturbed equation is autonomous).

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