Quasilinearization for the forced Düffing equation

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Abstract

In this paper we present the quasilinearization method for the periodic problem related to the forced Düffing equation. We obtain two monotone sequences of approximate solutions, with quadratic order of convergence. We work in the presence of lower and upper solutions. The approximate problems are linear.

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1 Introduction

In this paper we apply the quasilinearization method to the periodic problem for the forced Düffing equation

$$\begin{cases} x'' + kx' + f(t,x) = 0 \\ x(0) = x(T), \quad x'(0) = x'(T) \end{cases} \tag{1.1}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $k \in \mathbb{R}$. Existence of a lower and an upper solution is assumed. We say that α_0 is a lower solution of the problem (1.1) if $\alpha_0 \in C^2[0,T]$ and

$$\begin{cases} \alpha_0'' + k\alpha_0' + f(t,\alpha_0) \geq 0 \\ \alpha_0(0) = \alpha_0(T), \quad \alpha_0'(0) = \alpha_0'(T) \end{cases} \tag{1.2}$$

Whenever the reversed inequality holds for some function $\beta_0 \in C^2[0,T]$, we say that β_0 is an upper solution.
We consider the following iterative schemes

$$\begin{cases} \alpha''_{n+1} + k\alpha'_{n+1} + f(t,\alpha_n) + \frac{\partial f}{\partial x}(t,\alpha_n)(\alpha_{n+1}-\alpha_n) = 0, \\ \alpha_{n+1}(0) = \alpha_{n+1}(T), \quad \alpha'_{n+1}(0) = \alpha'_{n+1}(T) \end{cases} \tag{1.3}$$

$$\begin{cases} \beta''_{n+1} + k\beta'_{n+1} + f(t,\beta_n) + \frac{\partial f}{\partial x}(t,\alpha_n)(\beta_{n+1}-\beta_n) = 0, \\ \beta_{n+1}(0) = \beta_{n+1}(T), \quad \beta'_{n+1}(0) = \beta'_{n+1}(T). \end{cases} \tag{1.4}$$

The sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ obtained as solutions of the linear problems (1.3) and (1.4) are monotone and converge quadratically to the solution of (1.1). In addition, we require, roughly speaking, that the nonlinear function f is decreasing and convex.
We say that a sequence $(\alpha_n)_{n\geq 0}$ converges quadratically to x^* in $C[0,T]$ (with the supremum norm), whenever there exist $c > 0$ and $n_0 \in \mathbb{N}$ such that

$$||x^* - \alpha_{n+1}|| \leq c||x^* - \alpha_n||^2, \quad \text{for all } n \geq n_0.$$

The type of problems which is the object of our work is extensively studied in the literature. Let us remind only some references which are related to the technique used in our paper. The method of lower and upper solutions for (1.1) is presented by Wang-Cabada-Nieto in [11], together with a monotone iterative method . C. Wang [10] studied the case of reversedly lower and upper solutions.

The quasilinearization method is a tool for obtaining approximate solutions to nonlinear equations with rapide convergence. It was applied to a variety of problems (see the monograph [8] by Lakshmikantham-Vatsala and the references therein), and even some very efficient abstract schemes were given in [2, 3, 4]. Some boundary value problems were studied with the quasilinearization method in [5, 6, 8, 9]. Our approach is closely to [6] and some examples in [8], since we prefer to assume convexity for the nonlinear part and obtain the approximations as solutions of corresponding linear problems, rather than do not impose convexity but consider nonlinear approximate problems (like in [5, 9]). Anyway, our results can be easily extended to the case of nonlinearities of DC-type (i.e. $f = f_1 - f_2$, where f_1 and f_2 are convex), as in [8].

2 Preliminaries

The aim of this section is to establish some comparison and existence results for the linear problem of the form (1.1), which will be needed later on.

Lemma 2.1 *Let $g, l : [0,T] \to \mathbb{R}$ be two continuous functions with $l(t) < 0$ for every $t \in [0,T]$. Let $x \in C^2[0,T]$ be such that*

$$\begin{cases} -(x'' + kx' + l(t)x) = g(t) \\ x(0) = x(T), \quad x'(0) = x'(T). \end{cases} \tag{2.5}$$

If $g(t) \geq 0$ for all $t \in [0,T]$ then $x(t) \geq 0$ for all $t \in [0,T]$.

Proof. First we prove by contradiction that $x(0) \geq 0$. Let us assume that $x(0) < 0$. We distinguish three cases: $x'(0) = 0$; $x'(0) < 0$ and $x'(0) > 0$. Every case lead to

(S) *there exists $t_1 \in (0,T)$ such that $x(t_1) < 0$ and $x'(t_1) = 0$.*

Then t_1 is a local minimum for x, which also implies that $x''(t_1) > 0$. When we replace these in the following relation

$$-\left[x''(t_1) + kx'(t_1) + l(t_1)x(t_1)\right] = g(t_1)$$

we get a contradiction.

Let us prove now the above statement (S).

Case 1. Whenever $x'(0) = 0$, if we replace in the differential equation of x, we obtain $x''(0) \leq -l(0)x(0) < 0$. Then x' is strictly decreasing in some neighborhood of 0, V. But $x'(0) = 0$. Thus $x'(t) < 0$ for all $t \in V$. Hence x is strictly decreasing in V. Relation $x(0) = x(T)$ assures that (S) is valid.

Case 2. Whenever $x'(0) < 0$ we have that $x'(t) < 0$ in some neighborhood of 0. The rest is like in Case 1.

Case 3. Whenever $x'(0) > 0$ we have that, also, $x'(T) > 0$. Then x is strictly increasing in some neighborhood of T. Relation $x(0) = x(T)$ guarantees (S).

Hence we know that $x(0) = x(T) \geq 0$. It is easy to see that the existence of some $t^* \in (0,T)$ with $x(t^*) < 0$ assures that (S) hold. But this lead to a contradiction, as we have already proved. Then $x(t) \geq 0$ for all $t \in [0,T]$. □

Lemma 2.2 *Let $l : [0,T] \to \mathbb{R}$ be a continuous function with $l(t) < 0$ for all $t \in [0,T]$. Then the problem (2.5) has a unique solution for every $g \in C[0,T]$.*

Proof. We apply Theorem 3.1, page 214 from [7] and deduce that is is sufficient if we prove that the only solution of the corresponding homogeneous equation with $x(0) = x(T)$ and $x'(0) = x'(T)$ is the null solution. It is easy to see that this is valid on the base of Lemma 2.1. □

Throughout this paper let us consider

$$D = \left\{x \in C^2[0,T] : \quad x(0) = x(T),\ x'(0) = x'(T)\right\}.$$

Lemma 2.3 *Let $l : [0,T] \to \mathbb{R}$ be a continuous function with $l(t) < 0$ for all $t \in [0,T]$. The linear operator $L : D \to C[0,T]$, $Lx = -\left(x'' + kx' + l(t)x\right)$ is bijective and its inverse is positive and completely continuous between $C[0,T]$ to itself.*

Proof. The bijectivity of L is assured by Lemma 2.2. It is easy to see that L is continuous from D endowed with C^2 norm

$$||x||_{C^2} = ||x|| + ||x'|| + ||x''||,$$

to $C[0,T]$ with the supremum norm, denoted here $||\cdot||$. Then L^{-1} exists and is continuous between $C[0,T]$ and D. Of course, is continuous between $C[0,T]$ to itself. Complete continuity of L^{-1} is assured because, in addition, D is compactly imbedded in $C[0,T]$. The positivity of L^{-1}, i.e. $y \geq 0$ implies $L^{-1}y \geq 0$, follows by Lemma 2.1. □

3 Main results

Throughout this section let us denote

$$\Omega = \{(t,u) \in [0,T] \times \mathbb{R} : \quad \alpha_0(t) \leq u \leq \beta_0(t)\}$$

and consider the order interval in the space $C[0,T]$,

$$[\alpha_0, \beta_0] = \{x \in C[0,T], \quad \alpha_0(t) \leq x(t) \leq \beta_0(t) \text{ for all } t \in [0,T]\},$$

where $\alpha_0, \beta_0 \in C[0,T]$ with $\alpha_0(t) \leq \beta_0(t)$ for all $t \in [0,T]$. The following Lemma is a unicity result for the nonlinear problem (1.1).

Lemma 3.1 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous and $\alpha_0, \beta_0 \in D$, be a lower and, respectively, an upper solution of (1.1), such that*

$$\alpha_0(t) \leq \beta_0(t) \textit{ for all } t \in [0,T].$$

Assume that $f(t,\cdot)$ is C^1 on $\mathbb{R}$ and $\frac{\partial f}{\partial x}(t,u) < 0$ for all $(t,u) \in \Omega$. Then (1.1) has at most one solution in $[\alpha_0, \beta_0]$.

Proof. Whenever x and y are two solutions of (1.1) in $[\alpha_0, \beta_0]$, we have that $z = x - y$ satisfies the following relations

$$-(z'' + kz') = f(t, x(t)) - f(t, y(t)) = l(t)z,$$

where

$$l(t) = \begin{cases} \frac{f(t,x(t)) - f(t,y(t))}{x(t)-y(t)}, & x(t) \neq y(t) \\ \frac{\partial f}{\partial x}(t, x(t)), & x(t) = y(t). \end{cases} \tag{3.6}$$

It easy to see that $l(t) < 0$ for all $t \in [0, T]$ and that $z \in D$. We apply Lemma 2.2 and obtain that $z = 0$, i.e. $x = y$. $\square$

The next theorem is our main result.

Theorem 3.1 *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be continuous and $\alpha_0, \beta_0 \in D$, be a lower and, respectively, an upper solution of (1.1), such that*

$$\alpha_0(t) \leq \beta_0(t) \text{ for all } t \in [0, T].$$

Assume that $f(t, \cdot)$ is C^2 on $\mathbb{R}$ and convex on $[\alpha_0(t), \beta_0(t)]$ for all $t \in [0, T]$, and that $\frac{\partial f}{\partial x}(t, u) < 0$ for all $(t, u) \in \Omega$. Then the sequences (α_n) and (β_n) given by the iterative schemes (1.3) and (1.4) are well and uniquely defined in D, and converge monotonically and quadratically in $C[0, T]$ to the unique solution of (1.1) in $[\alpha_0, \beta_0]$.

Proof. The fact that α_n and β_n are well and uniquely defined in D is assured by Lemma 2.2.
The differentiability and convexity of $f(t, \cdot)$ on $[\alpha_0(t), \beta_0(t)]$ imply the following relations

$$\frac{\partial f}{\partial x}(t, u)(v - u) \leq f(t, v) - f(t, u) \leq \frac{\partial f}{\partial x}(t, v)(v - u), \tag{3.7}$$

for all $\alpha_0(t) \leq u \leq v \leq \beta_0(t)$.
We shall prove by induction that the following proposition is valid for all $n \geq 0$.

$$(P_n) \begin{cases} \alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n \\ \alpha_{n+1} \text{ is a lower solution of (1.1)} \\ \beta_{n+1} \text{ is an upper solution of (1.1)} \end{cases} \tag{3.8}$$

Let us verify first for $n = 0$. In order to avoid some complicated formulas, let us denote $L_0 x = -\left(x'' + kx' + \frac{\partial f}{\partial x}(t, \alpha_0)x\right)$. Using this notation, we can write (1.3) for $n = 0$ in the form

$$L_0 \alpha_1 = f(t, \alpha_0) - \frac{\partial f}{\partial x}(t, \alpha_0)\alpha_0.$$

Then, using also the fact that α_0 is a lower solution, we obtain

$$L_0(\alpha_1 - \alpha_0) = L_0\alpha_1 + \alpha_0'' + k\alpha_0' + \frac{\partial f}{\partial x}(t, \alpha_0)\alpha_0 = \alpha_0'' + k\alpha_0' + f(t, \alpha_0) \geq 0.$$

By Lemma 2.1, it follows that

$$\alpha_0 \leq \alpha_1.$$

Analogously one can prove that $\beta_1 \leq \beta_0$.
Using one of the inequalities (3.7) we have

$$L_0(\beta_1 - \alpha_1) = f(t, \beta_0) - \frac{\partial f}{\partial x}(t, \alpha_0)\beta_0 - f(t, \alpha_0) + \frac{\partial f}{\partial x}(t, \alpha_0)\alpha_0 \geq 0.$$

Thus, by Lemma 2.1,

$$\alpha_1 \leq \beta_1.$$

Let us prove now that α_1 is a lower solution of (1.1). We have

$$\alpha_1'' + k\alpha_1' + f(t, \alpha_1) = f(t, \alpha_1) - f(t, \alpha_0) - \frac{\partial f}{\partial x}(t, \alpha_0)(\alpha_1 - \alpha_0) \geq 0,$$

where we have used (1.3) and (3.7) for $\alpha_0 \leq \alpha_1$.
Analogously, β_1 is an upper solution for (1.1).
The proof of the fact that, if (P_n) is valid then (P_{n+1}) is true, can be done in the same manner as above. In order to avoid the repetion, let us skip it.
At this moment we have that for every $n \geq 0$, $\alpha_{n+1} \in D$ is a solution of the linear differential equation (1.3) and that

$$\alpha_0(t) \leq \alpha_1(t) \leq ... \leq \alpha_n(t) \leq ... \leq \beta_0(t) \text{ for all } t \in [0, T].$$

We shall prove that the sequence (α_n) converges uniformly on $[0, T]$ and its limit is a solution of (1.1).
For each $t \in [0, T]$, let us denote by $x^*(t)$ the limit of the numerical sequence $(\alpha_n(t))$ and $\sigma_n(t) = L\alpha_{n+1}(t)$, where L is the linear operator between D and $C[0, T]$ given by $Lx = -(x'' + kx' - x)$. Using (1.3) we get that

$$\sigma_n(t) = f(t, \alpha_n) + \alpha_{n+1}(t) + \frac{\partial f}{\partial x}(t, \alpha_n(t)). \tag{3.9}$$

Because the functions f and $\frac{\partial f}{\partial x}$ are continuous and the sequence (α_n) is bounded in $C[0, T]$, we have that (σ_n) is bounded in $C[0, T]$. Also, we can write

$$\alpha_{n+1} = L^{-1}\sigma_n. \tag{3.10}$$

By Lemma 2.3, L^{-1} is completely continuous. Hence the sequence (α_n) is compact in $C[0,T]$. It is also monotone. Then it is uniformly convergent to x^*. When we pass to the limit for $n \to \infty$ in (3.10) and (3.9) we get that $x^* = L^{-1}[f(t,x^*) + x^*]$. Thus $x^* \in D$ and $Lx^* = f(t,x^*) + x^*$, which is equivalent to the fact that x^* is a solution of the problem (1.1).
Analogously, the sequence (β_n) converges uniformly on $[0,T]$, and its limit is a solution of 1.1. By Lemma 3.1, the solution is unique in $[\alpha_0, \beta_0]$.
In order to justify that the order of convergence of the sequence (α_n) to x^* is 2, we denote

$$p_n = x^* - \alpha_n$$

and consider the linear operator $L_* x = -\left[x'' + kx' + \frac{\partial f}{\partial x}(t,x^*)x\right]$. Let us remember that, by convexity of f, $\frac{\partial f}{\partial x}(t,x^*) \geq \frac{\partial f}{\partial x}(t,\alpha_n)$, since $x^* \geq \alpha_n$. The following inequalities hold.

$$\begin{aligned}
L_* p_{n+1} &\leq -\left[p''_{n+1} + kp'_{n+1} + \frac{\partial f}{\partial x}(t,\alpha_n)p_{n+1}\right] \\
&= -(x'' + kx') - \frac{\partial f}{\partial x}(t,\alpha_n)x^* + \left[\alpha''_{n+1} + k\alpha'_{n+1} + \frac{\partial f}{\partial x}(t,\alpha_n)\alpha_{n+1}\right] \\
&= f(t,x^*) - \frac{\partial f}{\partial x}(t,\alpha_n)p_n - f(t,\alpha_n) \\
&\leq \left[\frac{\partial f}{\partial x}(t,x^*) - \frac{\partial f}{\partial x}(t,\alpha_n)\right]p_n \\
&\leq a \cdot p_n^2.
\end{aligned}$$

We have used relation (3.7) for $\alpha_n \leq x^*$. The last inequality is true because the function $\frac{\partial f}{\partial x}(t,\cdot)$ is monotone increasing and Lipschitz on the compact interval $[\alpha_0(t), \beta_0(t)]$ for each $t \in [0,T]$. Using the positivity of L_*^{-1}, assured by Lemma 2.3, we obtain

$$0 \leq p_{n+1} \leq aL_*^{-1}\left(p_n^2\right),$$

and than, continuity of L_*^{-1} gives that there exists $c > 0$ with

$$||p_{n+1}|| \leq c||p_n||^2.$$

In the same manner one can prove the quadratic convergence of (β_n). □

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