ELLIPTIC AND PARABOLIC DIFFERENTIAL INEQUALITIES

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1 Introduction

The maximum principles or the Gronwall-type inequalities are powerful instruments in the theory of differential equations (uniqueness or a-priori boundedness of the solutions can be obtained, for example).
The aim of this work is to establish this kind of results for semilinear elliptic and parabolic problems. Usually, these are called comparison results (see [4, 13]).
In section 2 we consider the following semilinear elliptic inequality.

$$(1) \qquad \begin{cases} -\Delta u + \omega u \leq f(x,u), \ a.a. \ x \in \Omega \\ u \in H_0^1(\Omega), \ \Delta u \in L^2(\Omega). \end{cases}$$

Throughout this paper, Ω will be a bounded domain in R^n.
In Theorem 1 below, f is monotone increasing and Lipschitz with respect to the last variable, with a constant which satisfies $a - \omega < \lambda_1$, (where $\omega \geq 0$ and λ_1 is the first eigenvalue of the Laplace operator). A more general comparison result is Corollary 1, where f satisfies only condition (9) below,

$$f(x,u_1) - f(x,u_2) \leq g(x, u_1 - u_2), \text{ a.a. } x \in \Omega, u_1 \geq u_2,$$

with g fulfilling the hypothesis of Theorem 1 (in particular, f can be monotone decreasing).

In section 3 we consider parabolic problems of the following forms.

$$(2) \qquad \begin{cases} \frac{\partial u}{\partial t} - \Delta u \leq f(x,t,u), \ a.a. \ x \in \Omega, \ \text{ for all } t > 0 \\ u(x,0) \leq u_0(x), \ \ a.a. \ x \in \Omega \\ u(\cdot,t) \in H_0^1(\Omega), \ \Delta u(\cdot,t) \in L^2(\Omega) \ \text{ for all } t > 0 \\ \frac{\partial u}{\partial t} \in L^1(0,T;L^2(\Omega)). \end{cases}$$

$$(3) \qquad \begin{cases} \frac{\partial u}{\partial t} - \Delta u \leq f(x,t,u), \ a.a. \ x \in R^n, \ \ \text{ for all } t > 0 \\ u(x,0) \leq u_0(x), \ \ a.a. \ x \in R^n \\ u(\cdot,t) \in H^2(R^n) \ \ \text{ for all } t > 0 \\ \frac{\partial u}{\partial t} \in L^1(0,T;L^2(R^n)). \end{cases}$$

The main result for these problems is Theorem 3, where the function f satisfies the Caratheodory conditions, is monotone increasing and Lipschitz with respect to the last variable (without any restriction for the Lipschitz constant).
Mainly, we shall use the abstract Gronwall lemma of Rus ([8]).

Lemma 1 *Let X be an ordered metric space and $A : X \to X$ an order preserving and a Picard operator (i.e. A has a unique fixed point, u^*, which is the limit of the sequence $(A^n u)_{n \geq 1}$ for every $u \in X$).*
If $u \leq Au$ then $u \leq u^*$.
If $Au \leq u$ then $u^* \leq u$.

Let us mention that Lemma 1 generalizes Proposition 7.15 from [15], where is considered the case of a linear mapping A and, also, Lemma 1 from [16] where the mapping A is linearly bounded. This lemma is a powerful tool for obtaining Gronwall-type inequalities.
The existence results from this paper (i.e. the first part of Theorem 1, and Theorem 2) are known. They are obtained by using the Banach contraction mapping theorem. For completness, we shall present a sketch of the proof. For the elliptic equation we have used [6] and [7]. In order to study existence in [6, 11, 10], or stability in [5] for certain nonlinear evolution problems, some fixed point theorems and the theory of semigroups were used. Using the same ideas, we have stated and proved Theorem 2.
Similar comparison results for elliptic equations contain [13] and [4]. In Corollary 1 we do not assume continuity and differentiability for the function

f, as in Theorem 10.1, page 263 [4], where the theorem of the mean is used. One of the conditions for the function f, used in the comparison result from [13], Theorem 1, is

$$\limsup_{|u|\to\infty} \frac{f(x,u)}{u} \le a \le \lambda_1.$$

Let us mention the idea from [13] of using these results in order to obtain that a sub-solution is less or equal than a super-solution of a certain elliptic equation. The same fact can be obtained for parabolic problems using our results. This is an essential claim in the method of sub- and super- solutions ([15]), an efficient tool for obtaining existence results for nonlinear equations. This method was used, for example, in the periodic parabolic problems in [3].

In the end, let us notice that our results remain valid if the Laplace operator, $-\Delta$ is replaced by a general elliptic differential operator,

$$Lu = -\sum_{j,k=1}^{n} D_j(a_{jk}D_k u) + \sum_{j=1}^{n} a_j D_j u + a_0 u,$$

with $a_{jk} = a_{kj} \in C^1(\Omega)$, $a_j, a_0 \in C(\Omega)$ and $a_0 \le 0$ (see [1, 4]).

2 Main results in the elliptic case

In this section we investigate problems $(1)_=$ and (1). We denote by $(1)_=$, $(1)_\ge$ the coresponding problems obtained by replacing $\le$ with $=$ and $\ge$, respectively, in (1).

Let us list first the following hypothesis.
(H1) $|f(x,u_1) - f(x,u_2)| \le a\,|u_1 - u_2|$, a.a. $x \in \Omega$ and for all $u_1, u_2 \in R$.
(H2) $f(\cdot,u)$ is measurable for all $u \in R$, and $f(\cdot,0) \in L^2(\Omega)$.
(H3) $f(x,u_1) \le f(x,u_2)$, a.a. $x \in \Omega$ and for all $u_1, u_2 \in R$ with $u_1 \le u_2$.

We consider the usual order relation on $L^2(\Omega)$, i.e., $v_1 \le v_2$ if and only if $v_1(x) \le v_2(x)$ a.a. $x \in \Omega$.
We shall denote by λ_1 the first eigenvalue of $(-\Delta) : H_0^1(\Omega) \to H^{-1}(\Omega)$ and we shall use the weak maximum principle (see [6]).

Lemma 2 *Let $v \in L^2(\Omega)$ such that $v \ge 0$. If $\omega \ge 0$ then $(-\Delta + \omega)^{-1} v \ge 0$.*

The following Theorem 1 and Corollary 1 are the main results of this section.

Theorem 1 *Let $f : \Omega \times R \to R$ be a function such that (H1), (H2) are satisfied and ω be a nonnegative real number. Let us suppose that the following relation holds,*

$$a < \lambda_1 + \omega. \tag{4}$$

Then the Dirichlet problem $(1)_=$ has a unique weak solution $u^* \in H_0^1(\Omega)$. If, in addition, (H3) is fulfilled, then every solution $\underline{u}$ of (1) satisfies

$$\underline{u}(x) \le u^*(x)$$

for almost all $x \in \Omega$.

Proof. Let us consider the following mapping,

$$A : L^2(\Omega) \to L^2(\Omega) \qquad A(v) = f(\cdot, (-\Delta + \omega)^{-1}v). \tag{5}$$

As it is shown in [7], we have that the hypothesis for f assure that A is well defined and is a contraction mapping with the Lipschitz constant

$$\frac{a}{\lambda_1 + \omega} < 1. \tag{6}$$

Then, by the contraction mapping theorem, A is a Picard mapping. We denote by $v^* \in L^2(\Omega)$ the unique fixed point of A.
Let $v_1, v_2 \in L^2(\Omega)$ be such that $v_1 \le v_2$. Using Lemma 2 and (H3), it is easy to see that $Av_1 \le Av_2$. Thus A is order preserving.
By Lemma 1, every $v \in L^2(\Omega)$ with $v \le Av$ is such that

$$v \le v^*. \tag{7}$$

Also, let us note that, via the equality

$$v = -\Delta u + \omega u,$$

the problem $(1)_=$ is equivalent with

$$v = A(v)$$

and the problem (1) consists in finding $\underline{v}$ such that

$$v \le A(v).$$

Now, from (7) and Lemma 2, the conclusion follows.

Corollary 1 *Let $g : \Omega \times R \to R$ be a continuous function which satisfies (H1) with $a < \lambda_1 + \omega$, (H3) and*

$$g(x, 0) = 0, \ \ a.a. \ x \in \Omega. \tag{8}$$

Let $f : \Omega \times R \to R$ be a function such that

$$f(x, u_1) - f(x, u_2) \leq g(x, u_1 - u_2), \ \ a.a. \ x \in \Omega, \ \textit{for all } u_1 \geq u_2. \tag{9}$$

The following statements are valid.
(s1) If there exist $\underline{u} \in C(\overline{\Omega}) \cup C^2(\Omega)$ a solution of (1) and $\overline{u} \in C(\overline{\Omega}) \cup C^2(\Omega)$ a solution of $(1)_{\geq}$, then

$$\underline{u}(x) \leq \overline{u}(x), \ \ a.a. \ x \in \Omega.$$

(s2) The Dirichlet problem $(1)_=$ has at most one solution in $C(\overline{\Omega}) \cup C^2(\Omega)$.

Proof. (s1) Let us denote by $\tilde{w} = (\underline{u} - \overline{u})^+$ (where $u^+ = \max\{0, u\}$).
For every $x_1 \in \Omega$ such that

$$\underline{u}(x_1) > \overline{u}(x_1),$$

from the continuity of $(\underline{u} - \overline{u})$ it follows that there exists a neighborhood of x_1, $V_1 \subset \Omega$ such that $\tilde{w}(x) = \underline{u}(x) - \overline{u}(x) > 0$ for all $x \in V_1$. The following relations hold for all $x \in V_1$ (using, also, condition (9)).

$$\begin{aligned} -\Delta \tilde{w} + \omega \tilde{w}(x) &= (-\Delta \underline{u} + \omega \underline{u}(x)) - (-\Delta \overline{u} + \omega \overline{u}(x)) \leq \\ &\leq f(x, \underline{u}(x)) - f(x, \overline{u}(x)) \leq g(x, \tilde{w}(x)). \end{aligned} \tag{10}$$

For every $x_2 \in \Omega$ such that

$$\underline{u}(x_2) < \overline{u}(x_2),$$

there exists a neighborhood of x_2, $V_2 \subset \Omega$ such that $\tilde{w}(x) = 0$ for all $x \in V_2$. The following relations hold for all $x \in V_2$.

$$-\Delta \tilde{w} + \omega \tilde{w}(x) = 0 = g(x, \tilde{w}). \tag{11}$$

For every $x_3 \in \Omega$ such that

$$\underline{u}(x_3) = \overline{u}(x_3),$$

either there exists a neighborhood of x_3, $V_3 \subset \Omega$ such that $\underline{u}(x) = \overline{u}(x)$ for all $x \in V_3$, or there exists a sequence $x_n \to x_3$ such that $\underline{u}(x_n) < (>)\overline{u}(x_n)$. By the continuity of $(\underline{u} - \overline{u})$ and $\Delta(\underline{u} - \overline{u})$, and relations (10) and (11), we can deduce that

$$-\Delta\tilde{w}(x_3) + \omega\tilde{w}(x_3) \leq g(x_3, \tilde{w}(x_3)).$$

Then $\tilde{w}$ is a solution of the following problem

$$\left\{ \begin{array}{c} -\Delta w + \omega w \leq g(x, w),\ a.a.\ x \in \Omega \\ w \in H_0^1(\Omega)\,,\ \Delta w \in L^2(\Omega). \end{array} \right. \tag{12}$$

Let us notice that $w^* = 0$ is a solution of $(12)_=$ (by (8)). In the case of this problem the hypothesis of Theorem 1 are fulfilled. Then, $\tilde{w}(x) \leq 0$ for all $x \in \Omega$ and, by the definition of $\tilde{w}$, $\tilde{w} = 0$. Hence, the conclusion holds.
(s2) Let us denote by u_1 and u_2 two solutions of $(1)_=$. We apply twice (s1) and obtain that $u_1 \leq u_2$ and, also, $u_2 \leq u_1$. Then $u_1 = u_2$.

Remark 1. Let $\omega = 0$, $\tilde{a} \in R$ and $f(x, u) = \tilde{a}u$. The comparison result (equivalently, the maximum principle for the Laplace operator), is valid if and only if $\tilde{a} < \lambda_1$ (see [6]).
Let us make here a more detailed discussion concerning the relation between this result and ours, given also a simple example on the real line.
The case $\tilde{a} \leq -\lambda_1$ is not covered by Theorem 1, but it is covered by Corollary 1 (when (9) is fulfilled with $g = 0$).
If $\tilde{a}$ is one of the eigenvalues of $(-\Delta)$, we can not talk about comparison, since problem $(1)_=$ does not have either a minimal or a maximal solution.
If $\tilde{a}$ is strictly between two eigenvalues of $(-\Delta)$, then the solution is unique, but the comparison result does not hold. This is the case of the following example. For other interesting considerations of this type we recommend [9].
Let us consider the problem:

$$\left\{ \begin{array}{c} -u'' \leq 2u\ \ ,\ x \in (0, \pi) \\ u(0) = u(\pi) = 0 \end{array} \right. \tag{13}$$

In this case $\lambda_1 = 1$, $\lambda_2 = 4$ and the unique solution of $(9)_=$ is $u^* = 0$. We have that

$$u(x) = \frac{1}{2} - \frac{1}{2cos\pi\sqrt{2}/2}cos(x\sqrt{2} - \pi\sqrt{2}/2)$$

is a solution of our problem. But, $u(\pi/2) > 0$.

Remark 2. For the equation $-\Delta u = g(x,u)$, if we add ωu, the condition for $g(x,u) + \omega u$ to be nondecreasing in u (i.e., the hypothesis (H3)) becomes (see, also, [1])

$$g(x,u_1) - g(x,u_2) \leq -\omega(u_1 - u_2), \text{ for all } u_1, u_2 \in R \text{ with } u_1 \leq u_2.$$

Example 1. Let $B_1(0)$ be the unit ball in R^n with $n = 2$ or $n = 3$. If u is a solution of

$$\left\{ \begin{array}{c} -\Delta u \leq (n - |x|^2)(u + \frac{1}{\sqrt{e}}) \ \text{, a.e. } x \in B_1(0) \\ u \in H_0^1(B_1(0)) \cap H^2(B_1(0)) \end{array} \right. \tag{14}$$

then $\quad u(x) \leq e^{-\frac{1}{2}|x|^2} - \frac{1}{\sqrt{e}}$.

This follows from Theorem 1. Let us notice that here $\omega = 0$.
Let $f : B_1(0) \times R \to R, \quad f(x,u) = (n - |x|^2)(u + \frac{1}{\sqrt{e}})$.
This is a continuous function on the closure of its domain, so, (H2) is fulfilled. It is easy to see that (H3) is valid. We also have,

$$|f(x,u_1) - f(x,u_2)| \leq n\,|u_1 - u_2|.$$

Thus, (H1) is satisfied with $a = n$, and relation (4) becomes

$$\lambda_1 > n,$$

which is true in this case (see [12]).
The unique solution of

$$\left\{ \begin{array}{c} -\Delta u = (n - |x|^2)(u + \frac{1}{\sqrt{e}}) \quad , x \in B_1(0) \\ u = 0 \quad , x \in \partial B_1(0) \end{array} \right.$$

is

$$u(x) = e^{-\frac{1}{2}|x|^2} - \frac{1}{\sqrt{e}}.$$

So, we can apply Theorem 1 and obtain the estimation.

3 Main results in the parabolic case

In this section we study problems (2), (3) and $(2)_=$, $(3)_=$ (i.e. the corresponding Cauchy-Dirichlet problem of (2) and Cauchy problem of (3) for the heat equation), using Lemma 1. To this purpose we interpret the evolution problems as differential equations in Hilbert spaces and take advantage of the theory of semigroups of nonexpansive operators and the Bielecki norms technique. For inequalities, we also need the maximum principle for parabolic differential operators.

3.1 The initial value problem in a Hilbert space

We consider the problem

$$\left\{ \begin{array}{c} \frac{du}{dt} + Bu = f(t, u(t)), \quad t \in [0, T] \\ u(0) = u_0 \end{array} \right. \tag{15}$$

where H is a Hilbert space, $B : D(B) \subset H \to H$ is a linear m-accretiv operator, $u_0 \in H$ and $f : [0, T] \times H \to H$ such that

$$f(\cdot, u) \in L^1(0, T\,; H) \;\text{ for all } u \in H \tag{16}$$

$$f(t, \cdot) \in C(H; H) \quad \text{for all } t \in [0, T] \tag{17}$$

We denote by $\{S(t)\}_{t \geq 0}$ the semigroup of nonexpansive operators generated by B.
The function $u \in C([0, T]; H)$ given by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds$$

is said to be a *mild solution* (see [2, 14]) for the IVP

$$\left\{ \begin{array}{c} \frac{du}{dt} + Bu = f(t), \quad t \in [0, T] \\ u(0) = u_0. \end{array} \right. \tag{18}$$

Theorem 2 *If there exists $a \in L^1(0, T)$ such that*

$$|f(t, u) - f(t, \overline{u})| \leq a(t)|u - \overline{u}| \quad a.a.\ t \in (0, T), \text{ for all } u, \overline{u} \in H, \tag{19}$$

then (15) has a unique mild solution $u^* \in C([0, T]; H)$ which can be obtained by succesive approximations starting from every $u \in C([0, T]; H)$.

Proof. Let us consider the operator

$$A : C([0,T]; H) \to C([0,T]; H) \tag{20}$$

$$Au(t) = S(t)u_0 + \int_0^t S(t-s)f(s,u(s))ds,$$

and the following Bielecki norm

$$||u|| = \max_{t\in[0,T]} |u(t)|e^{-2\int_0^t a(s)ds}.$$

Let $u, \overline{u} \in C([0,T]; H)$ and $t \in [0,T]$. From the estimations

$$\begin{aligned} |Au(t) - A\overline{u}(t)| &= |\int_0^t S(t-s)[f(s,u(s)) - f(s,\overline{u}(s))]|_H \le \\ &\le \int_0^t a(s)|u(s) - \overline{u}(s)|ds \le \frac{1}{2}e^{2\int_0^t a(s)ds}||u - \overline{u}|| \end{aligned}$$

we have

$$||Au - A\overline{u}|| \le \frac{1}{2}||u - \overline{u}||.$$

Thus, the mapping A is a contraction on $C([0,T]; H)$, which yields that it has a unique fixed point, u^*, that can be obtained by succesive approximations. Of course, u^* is the unique solution of (15).

3.2 Parabolic inequalities

In the study of problem (2) we shall use the results from the previous section for the Hilbert space $L^2(\Omega)$ and the linear m-accretive operator (see [2])

$$B : D(B) \subset L^2(\Omega) \to L^2(\Omega), \quad Bu = -\Delta u,$$

$$D(B) = \{u \in H_0^1(\Omega) \ ; \Delta u \in L^2(\Omega)\}.$$

For problem (3) we shall consider the Hilbert space $L^2(R^n)$ and the linear m-accretive operator

$$B : D(B) \subset L^2(R^n) \to L^2(R^n), \quad Bu = -\Delta u, \quad D(B) = H^2(R^n).$$

We consider the usual order relations on $C([0,T]; L^2(\Omega))$ and $L^1(0,T; L^2(\Omega))$.

Let us consider the operator

$$\mathcal{L} : L^2(\Omega) \times L^1(0,T;L^2(\Omega)) \to C([0,T];L^2(\Omega))$$

$$\mathcal{L}(u_0,f)(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds,$$

and, in the coresponding form, when we replace Ω by R^n.
Here we give the weak maximum principle in the following form.

Lemma 3 *For both, Ω a bounded domain, and R^n, $u_0^1 \leq u_0^2$ and $f^1 \leq f^2$ imply that $\mathcal{L}(u_0^1, f^1) \leq \mathcal{L}(u_0^2, f^2)$.*

Proof. It is sufficient to prove that $u_0 \leq 0$ and $f \leq 0$ imply $\mathcal{L}(u_0, f) \leq 0$.
(a) The case of a bounded domain.
This follows by the positivity of the semigroup $S(t)$ (see [1]) (i.e., $S(t)v \geq 0$ for all $v \in L^2(\Omega)$).
Also, let us mention that, for $u_0 \in H_0^1(\Omega)$ with $\Delta u_0 \in L^2(\Omega)$ and $f \in C^1([0,T];L^2(\Omega))$, this is exactly the maximum principle for the heat equation, which is found in [6].
(b) The case of R^n.
This time we have that the operator $S(t) : L^2(R^n) \to L^2(R^n)$ is given by the formula

$$S(t)v(x) = \int_{R^n} E(x-y,t)v(y)dy$$

for a.a. $x \in R^n$, for all $v \in L^2(R^n)$ and $t \geq 0$. E is the fundamental solution of the heat operator

$$E(x,t) = \frac{1}{(2\sqrt{\pi t})^n} e^{-\frac{||x||^2}{4t}}.$$

From this formulas we obtain the conclusion.

Remark. In what follows is considered only the case of a bounded domain. But the results remain valid in the case of R^n (i.e. problem (3)).

We make the following assumptions.
(H4) there exists $a \in L^1(0,T)$ such that

$$|f(x,t,z) - f(x,t,\overline{z})| \leq a(t)\,|z - \overline{z}|\,,$$

for a.a. $x \in \Omega$, $t \in (0,T)$ and for all $z, \overline{z} \in R$;
(H5) $f(\cdot,\cdot,z)$ is measurable for all $z \in R$, and $f(\cdot,\cdot,0) \in L^2(\Omega \times (0,T))$;
(H6) for all $z, \overline{z} \in R$ with $z \leq \overline{z}$, we have that $f(x,t,z) \leq f(x,t,\overline{z})$ a.a. $x \in \Omega$, $t \in (0,T)$.

Theorem 3 *Let (H4) and (H5) be satisfied. Then* $(2)_=$ *has a unique solution* $u^* \in C([0,T]; L^2(\Omega))$. *If, in addition, (H6) is fulfilled, then every solution* $\underline{u}$ *of (2) satisfies*

$$\underline{u}(x,t) \leq u^*(x,t)$$

for almost all $x \in \Omega$ *and for all* $t \in [0,T]$.

Proof. Let us consider the mapping A given by (20) in the conditions of this section. The hypothesis assure that we can apply Theorem 2. Thus, A is a Picard operator and u^* is its unique fixed point, which is the unique solution of $(2)_=$.
Let $\underline{u}$ be a solution of (2) and $f_1(x,t) = \frac{\partial \underline{u}}{\partial t} - \Delta \underline{u}$, $u_1(x) = \underline{u}(x,0)$ and $f_2(x,t) = f(x,t,\underline{u}(x,t))$. We have that $f_1 \leq f_2$, $u_1 \leq u_0$ and, also $\underline{u} = \mathcal{L}(u_1, f_1)$, $A\underline{u} = \mathcal{L}(u_0, f_2)$. Then, by Lemma 3,

$$\underline{u} \leq A\underline{u}.$$

Using (H6) and Lemma 3, A is order preserving.
Applying Lemma 1,

$$\underline{u} \leq u^*.$$

All these considerations assure that the conclusion holds.

Example 2. If $u \in C([0,\infty); H^2(0,\pi))$ is such that:

$$(21) \qquad \begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \leq \frac{t+2}{t+1} u, \; a.a. \; x \in (0,\pi) \quad \text{for all } t > 0 \\ u(0,t) = u(\pi,t) = 0 \quad \text{for all } t > 0 \\ u(x,0) \leq \sin x \;\; \text{a.a. } x \in (0,\pi) \end{cases}$$

then

$$u(x,t) \leq (t+1)\sin x, \quad \text{a.a. } x \in (0,\pi), \text{ for all } t \geq 0.$$

Let us consider an arbitrary constant $T > 0$ and $f : [0,\pi] \times [0,T] \times R \to R$, $f(x,t,z) = \frac{t+2}{t+1} z$. This function is continuous and monotone increasing with respect to the last variable. It is easy to see that it satisfies the hypothesis (H4), (H5) and (H6). Thus we can apply Theorem 3 and, noticing that $u^*(x,t) = (t+1)\sin x$ is the unique solution of $(21)_=$, we deduce the conclusion.

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