REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 35, N° 1, 2006, pp. 11–15

NOTE ON THE ABSTRACT GENERALIZED QUASILINEARIZATION METHOD

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Abstract. The abstract generalized quasilinearization method established in [2] for ordered Banach spaces with regular or normal cone and continuous mappings, is revisited for strongly minihedral cones and (o)-continuous operators.

MSC 2000. 47J25, 47H07.
Keywords. Quasilinearization, ordered Banach space, cone, nonlinear operator equation, (o)-continuous operator, approximation.

1. INTRODUCTION

In the paper [2], for the first time, an abstract theory of the generalized quasilinearization method for semilinear equations of coincidence type was developed in ordered Banach spaces in order to cover lots of specific results given for particular classes of continuous problems (see [1], [3] and [6]). The cone in [2] was assumed to be regular or normal, the mappings were all continuous (topologically) and the approximations of the solutions were understood with respect to the norm. Thus the theory in [2] does not apply to discontinuous problems. The aim of this note is to show that the results in [2] can be easily adapted to the case of a strongly minihedral cone and of (o)-continuous (continuous with respect to the order) mappings. Here the approximations will be first understood with respect to the order. For example, an increasing sequence (x_n) will approximate x if x is the lowest upper bound of (x_n).

Recall that the positive cone K of an ordered Banach space $(X, |.|_X)$ is said to be *strongly minihedral* if every subset of X which is bounded from above (i.e. has an upper bound with respect to the order) has a supremum (lowest upper bound) (see [5, p. 219]).

Let $(X, \leq)$ be an ordered set. If (a_n) is an increasing sequence (i.e. $a_n \leq a_{n+1}$) of elements of X, we shall write $a_n \uparrow a$ when $a = \sup\{a_n\}$. The notation $b_n \downarrow b$ has a similar meaning. One says that a sequence (x_n) of elements of X is (o)-*convergent* to an element x and that x is the (o)-*limit* of this sequence in X, if there exist two sequences (a_n) and (b_n) in X such that $a_n \leq x_n \leq b_n$ for all n, $a_n \uparrow x$ and $b_n \downarrow x$. We denote $x_n \to^o x$ (see [4, p. 2]). Let X and Y be

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ordered sets and $F : X \to Y$ be a mapping. We say that F is (o)-*continuous* if $x_n \to^o x$ implies $Fx_n \to^o Fx$ ([4, p. 154]).

2. MONOTONE ITERATIVE TECHNIQUE FOR (o)-CONTINUOUS OPERATORS

Consider the coincidence operator equation

$$Lu = N(u), \quad u \in D. \tag{1}$$

First we shall complement Theorem 2.1 [2] by hypothesis (iii) (c) in the following theorem:

THEOREM 1. *Let X be an ordered Banach space, Z be an ordered topological linear space, D a linear subspace of X and $\alpha_0, \beta_0 \in D$. Let $L : D \to Z$ be a linear operator and $N : X \to Z$ be a mapping. Assume that the following conditions are satisfied:*

(i) $\alpha_0 \le \beta_0$, $L\alpha_0 \le N(\alpha_0)$ *and* $L\beta_0 \ge N(\beta_0)$;

(ii) *for every $u, v \in X$ with $\alpha_0 \le u \le v \le \beta_0$, there is a linear operator $P(u,v) : X \to Z$ such that $L - P(u,v) : D \to Z$ is bijective with positive inverse,*

$$N(u) \le N(v) - P(u,v)(v-u)$$

and

$$-P(u,v)z \le -P(\alpha,\beta)z$$

for all $\alpha, \beta, u, v, z \in X$ with $\alpha_0 \le \alpha \le u \le v \le \beta \le \beta_0$ and $z \ge 0$;

(iii) (c) *the positive cone of X is strongly minihedral and the operators*

$$\begin{cases} (L - P(\alpha_0,\beta_0))^{-1} N, \ \ (L - P(\alpha_0,\beta_0))^{-1} P(\alpha_0,\beta_0), \\ (L - P(\alpha_0,\beta_0))^{-1} P(u,u), \ \ u \in X, \ \alpha_0 \le u \le \beta_0 \end{cases} \tag{2}$$

are (o)*-continuous on the order interval* $[\alpha_0, \beta_0]$.

Then the sequences $(\alpha_n), (\beta_n)$ given by the iterative schemes

$$L\alpha_{n+1} = N(\alpha_n) + P(\alpha_n,\beta_n)(\alpha_{n+1} - \alpha_n),$$
$$L\beta_{n+1} = N(\beta_n) + P(\alpha_n,\beta_n)(\beta_{n+1} - \beta_n)$$

$(n \in \mathbf{N})$ are well and uniquely defined in D. In addition, $\alpha_n \uparrow u_*$ and $\beta_n \downarrow u^*$, where u_*, u^* are the minimal and, respectively, the maximal solution in $[\alpha_0, \beta_0]$ of (1).

Proof. As in [2], the sequences (α_n), (β_n) are well and uniquely defined in D,

$$\alpha_0 \le \alpha_1 \le ... \le \alpha_n \le ... \le \beta_n \le ...\beta_1 \le \beta_0$$

and for each $u \in D$ with $\alpha_0 \le u \le \beta_0$ and $Lu = N(u)$, one has

$$\alpha_n \le u \le \beta_n \ \text{ for all } n \in \mathbf{N}.$$

Since the positive cone is strongly minihedral, there are u_*, u^* with $\alpha_n \uparrow u_*$ and $\beta_n \downarrow u^*$. Clearly $\alpha_n \le u_* \le u \le u^* \le \beta_n$. It remains to prove that

$u_*, u^* \in D$ and $Lu_* = N(u_*)$, $Lu^* = N(u^*)$. Now, as in the proof of Theorem 2.1 [2], we find that

$$(3)\qquad (L - P(\alpha_0, \beta_0))^{-1} [N(\alpha_n) - P(\alpha_0, \beta_0)\alpha_n] \le \alpha_{n+1}$$
$$\le \quad (L - P(\alpha_0, \beta_0))^{-1} [N(\alpha_n) + P(u_*, u_*)(\alpha_{n+1} - \alpha_n) - P(\alpha_0, \beta_0)\alpha_{n+1}].$$

From $0 \le \alpha_{n+1} - \alpha_n \le u_* - \alpha_n$ and $u_* - \alpha_n \downarrow 0$, we have that $\alpha_{n+1} - \alpha_n \to^o 0$. Next, using (3) and the (o)-continuity of operators (2), we obtain

$$u_* = (L - P(\alpha_0, \beta_0))^{-1} [N(u_*) - P(\alpha_0, \beta_0) u_*].$$

Hence $u_* \in D$ and $Lu_* = N(u_*)$. The same argument holds for u^*. □

Remark 2. In particular, if $P(u, v) = 0$ for every u, v, Theorem 1 reduces to the monotone iterative method for the equation $Lu = N(u)$ with an increasing mapping N. In this case condition (iii) (c) requires that $L^{-1}N$ is (o)-continuous. If in addition $Z = D$ and L is the identity map of D, we have the following proposition which gives more information than Theorem 19.1 (a) in [5]. □

Corollary 3. *Let X be a Banach space, partially ordered by the cone $K \subset X$. Let $[\alpha_0, \beta_0]$ be an order interval and*

$$N : [\alpha_0, \beta_0] \to [\alpha_0, \beta_0]$$

increasing. If K is strongly minihedral and N is (o)-continuous, then N has a minimal u_* and a maximal u^*fixed point in $[\alpha_0, \beta_0]$ and

$$N^n(\alpha_0) \uparrow u_* \quad \text{and} \quad N^n(\beta_0) \downarrow u^*.$$

3. QUADRATIC MONOTONE APPROXIMATION

Our next result complements Theorem 2.2 in [2] and gives conditions so that (1) has a unique solution u^* in $[\alpha_0, \beta_0]$ and that the sequences $(p_n), (q_n)$ given by

$$p_n = u^* - \alpha_n \quad \text{and} \quad q_n = \beta_n - u^*,$$

satisfy

$$(4)\qquad |p_{n+1}|_X,\ |q_{n+1}|_X \le a\left(|p_n|_X^2 + |q_n|_X^2\right)$$

for all $n \in \mathbf{N}$ and some constant $a > 0$. Under additional conditions, we prove that (α_n), (β_n) converge in norm to u^*. In this case (4) shows that the convergence is quadratic.

Theorem 4. *Assume all the assumptions of Theorem 1 hold. If*

(iv) *for every $u, v \in D$ with $\alpha_0 \le u \le v \le \beta_0$, there exists a mapping $R(v, u) : D \to Z$ such that*

$$N(u) \ge N(v) - R(v, u)(v - u);$$

(v) *$L - R(v, u)$ is inverse positive, i.e. $(L - R(v, u))z \ge 0$ implies $z \ge 0$,* *then* (1) *has a unique solution* u^* *in* $[\alpha_0, \beta_0]$.

Moreover, if the norm $|.|_X$ *is monotone and the following conditions are satisfied:*

(vi) $\Gamma := (L - P(u,u))^{-1} : Z \to X$ *is continuous for every* $u \in D$, $\alpha_0 \leq u \leq \beta_0$;

(vi) *there exists two constants* $c_1, c_2 > 0$ *such that*

$$|(R(w,\alpha) - P(\alpha,\beta))\, z|_Z \leq c_1 |w - \alpha|_X |z|_X + c_2 |\alpha - \beta|_X |z|_X$$

for all $\alpha, \beta, w, z \in D$, $\alpha_0 \leq \alpha \leq w \leq \beta \leq \beta_0$, $z \geq 0$,

then $(\alpha_n), (\beta_n)$ *satisfy* (4) *for some* $a > 0$. *If in addition*

$$|\Gamma|\, |\alpha_0 - \beta_0|_X (c_1 + c_2) < 1, \tag{5}$$

then $(\alpha_n), (\beta_n)$ *converge in norm to* u^* *quadratically.*

Proof. Inequality (4) can be proved as in [2]. To show that $|p_n|_X \to 0$, notice first that the monotonicity of the norm guarantees that the sequences $(|p_n|_X)$ and $(|\alpha_n - \beta_n|_X)$ are decreasing, so convergent to some $x, z \in \mathbf{R}_+$, respectively. Clearly, $x, z \leq |\alpha_0 - \beta_0|_X$. Then, using the inequality

$$|p_{n+1}|_X \leq |\Gamma| \left(c_1 |p_n|_X^2 + c_2 |p_n|_X |\alpha_n - \beta_n|_X \right)$$

established in [2] (here $|\Gamma|$ is the norm of the operator Γ), we obtain

$$x \leq |\Gamma| \left(c_1 x^2 + c_2 x z \right) \leq |\Gamma|\, x \,(c_1 + c_2)\, |\alpha_0 - \beta_0|_X .$$

This together with (5) guarantees that $x = 0$. Hence (α_n) converges in norm to u^*. A similar argument can be used to show that (β_n) converges in norm to u^*. □

Remark 5. If we know a priori that (1) has a unique solution u^* in $[\alpha_0, \beta_0]$, then we do not need assumption (iii) (c) in order to guarantee the quadratic convergence to u^* of the sequences (α_n) and (β_n). □

By means of the next condition (d) (3), our last result complements Theorem 2.3 in [2].

Theorem 6. *Let* X *be an ordered Banach space,* Z *be another Banach space,* D *a linear subspace of* X *and* $\alpha_0, \beta_0 \in D$. *Let* $L : D \to Z$ *be a linear operator and* $N : X \to Z$ *be a mapping. Assume that the following conditions are satisfied:*

(a) $\alpha_0 \leq \beta_0$, $L\alpha_0 \leq N(\alpha_0)$ *and* $L\beta_0 \geq N(\beta_0)$;

(b) $N = N_1 - N_2$, *where* $N_1, N_2 : X \to Z$ *are Gâteaux differentiable mappings which are convex on* $[\alpha_0, \beta_0]$, *and for every* $u, v, z \in X$ *with* $\alpha_0 \leq u \leq v \leq \beta_0$ *and* $z \geq 0$,

$$N_i'(u)\, z \leq N_i'(v)\, z, \quad i = 1, 2;$$

(c) $L - N_1'(u) + N_2'(v) : D \to Z$ *is bijective and has positive inverse for every* $u, v \in [\alpha_0, \beta_0]$ *with* $u \leq v$ *or* $v \leq u$;

(d) (3) *the positive cone of X is strongly minihedral and the operators*

$$\begin{cases} \left(L - N_1'(\alpha_0) + N_2'(\beta_0)\right)^{-1} N, \\ \left(L - N_1'(\alpha_0) + N_2'(\beta_0)\right)^{-1} \left(N_1'(\alpha_0) - N_2'(\beta_0)\right), \\ \left(L - N_1'(\alpha_0) + N_2'(\beta_0)\right)^{-1} N'(u), \quad u \in X,\ \alpha_0 \le u \le \beta_0 \end{cases}$$

are (o)*-continuous.*

Then (1) *has a unique solution u^* in $[\alpha_0, \beta_0]$ and the sequences $(\alpha_n), (\beta_n)$ given by the iterative schemes*

$$L\alpha_{n+1} = N(\alpha_n) + \left(N_1'(\alpha_n) - N_2'(\beta_n)\right)(\alpha_{n+1} - \alpha_n),$$
$$L\beta_{n+1} = N(\beta_n) + \left(N_1'(\alpha_n) - N_2'(\beta_n)\right)(\beta_{n+1} - \beta_n)$$

$(n \in \mathbf{N})$ *are well and uniquely defined in D and*

$$\alpha_n \uparrow u^*, \quad \beta_n \downarrow u^*.$$

Moreover, if the norm $|.|_X$ is monotone, $\Gamma := \left(L - N'(u)\right)^{-1} : Z \to X$ is continuous for every $u \in D$, $\alpha_0 \le u \le \beta_0$ and N_1', N_2' are Lipschitz on $[\alpha_0, \beta_0]$, then $(\alpha_n), (\beta_n)$ satisfy (4) *for some $a > 0$. If in addition either $|\Gamma|$, $|\alpha_0 - \beta_0|_X$, or the Lipschitz constants of N_1' and N_2' are small enough, then (α_n), (β_n) converge in norm to u^* quadratically.*

Proof. Apply Theorem 1 and Theorem 4 with

$$P(u, v) = N_1'(u) - N_2'(v), \quad R(v, u) = N_1'(v) - N_2'(u).$$

□

Notice that Remark 5 is also true in this case.

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Received by the editors: March 9, 2006.