ASYMPTOTIC STABILITY OF PERIODIC SOLUTIONS FOR NONSMOOTH DIFFERENTIAL EQUATIONS WITH APPLICATION TO THE NONSMOOTH VAN DER POL OSCILLATOR

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Abstract. In this paper we study the existence, uniqueness, and asymptotic stability of the periodic solutions of the Lipschitz system \( \dot{x} = \varepsilon g(t, x, \varepsilon) \), where \( \varepsilon > 0 \) is small. Our results extend the classical second Bogoliubov theorem for the existence of stable periodic solutions to nonsmooth differential systems. As an application we prove the existence of asymptotically stable 2\( \pi \)-periodic solutions of the nonsmooth van der Pol oscillator \( \ddot{u} + \varepsilon (|u| - 1) \dot{u} + (1 + a\varepsilon)u = \varepsilon \lambda \sin t. \) Moreover, we construct the so-called resonance curves that describe the dependence of the amplitude of these solutions as a function of the parameters \( a \) and \( \lambda \). Finally we compare such curves with the resonance curves of the classical van der Pol oscillator \( \ddot{u} + \varepsilon (u^2 - 1) \dot{u} + (1 + a\varepsilon)u = \varepsilon \lambda \sin t. \)

Key words. periodic solution, asymptotic stability, averaging theory, nonsmooth differential system, nonsmooth van der Pol oscillator

AMS subject classifications. 34C29, 34C25, 47H11

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1. Introduction. In this paper we study the existence, uniqueness, and asymptotic stability of the \( T \)-periodic solutions of the system

\begin{equation}
\dot{x} = \varepsilon g(t, x, \varepsilon),
\end{equation}

where \( \varepsilon > 0 \) is a small parameter, and the function \( g \in C^0(\mathbb{R} \times \mathbb{R}^k \times [0, 1], \mathbb{R}^k) \) is \( T \)-periodic in the first variable and locally Lipschitz with respect to the second. For this class of differential systems, the study of the \( T \)-periodic solutions can be made using the averaging function

\begin{equation}
g_0(v) = \int_0^T g(\tau, v, 0) d\tau
\end{equation}

and looking for the periodic solutions that starts near some \( v_0 \in g_0^{-1}(0) \).

In the case that \( g \) is of class \( C^1 \), we recall the stable periodic case of the second Bogoliubov’s theorem [6, Chap. 1, section 5, Theorem II] which states If \( \det (g_0)'(v_0) \neq 0 \) and \( \varepsilon > 0 \) is sufficiently small, then system (1.1) has a unique \( T \)-periodic solution in a neighborhood of \( v_0 \). Moreover, if all the eigenvalues of the Jacobian matrix...
If \((g_0)'(v_0)\) have negative real part, then this periodic solution is asymptotically stable. This theorem has a long history and includes results by Fatou [15], Mandelstam and Papaleksi [31], and Krylov and Bogoliubov [25, section 2].

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\[ \frac{d}{dt} (v_0) + M'(v_0) \frac{d}{dt} v_0 + \frac{1}{2} \omega^2 v_0 = \frac{1}{LC} F(t), \]

where \(R = \varepsilon R_0\), \(M = \varepsilon M_0\), \(\omega^2 = 1 + \varepsilon \omega^2\), \(F(t) = \varepsilon \lambda \sin t\), \(\varepsilon > 0\), is assumed to be small and the triode characteristic \(i(u)\) is drawn in Figure 1.2(a). The analysis of the diagram of bifurcation of the periodic solutions in this system is performed in almost every book on nonlinear oscillations (see Andronov, Witt, and Khaikin [3, Chap. VIII, section 2], Malkin [30, Chap. I, section 5], and Nayfeh and Mook [36, section 3.1.7]) but with the smooth approximation \(i(u) = i_a(u) = S_0 + S_1 u - \frac{1}{3} S_3 u^3\) (leading to the classical van der Pol equation). Therefore it is natural to look for a technique that permits one to avoid this smooth approximation and allows one to work with the original shape of the triode characteristic drawn in Figure 1.2(a).

Though the unforced equation (1.3) (i.e., for \(F = 0\)) with \(i\) described by Figure 1.2(b) and Figure 1.2(c) is well studied (see [3, Chap. VIII, section 3 and Chap. IX, section 7]), the question about resonances in these equations when \(F \neq 0\) (e.g., \(F(t) = \varepsilon \lambda \sin t\)) is still partially open. In this direction Levinson [29] uncovered a
family of solutions of (1.3) of remarkable singular structure and Levi [28] completed
the study of the limit behavior of all solutions. The present paper complements these
results by describing the location of asymptotically stable periodic solutions of (1.3).

Studying (1.3) with the triode characteristic given by Figure 1.2(a), 1.2(b), or 1.2(c)
we finally note that there exists a change of variables (see, for example, how Levinson
changed equation 2.0 in [29]) that allows one to rewrite (1.3) into the form (1.1) with
some function $g$ that is not $C^1$ but is Lipschitz with respect to the second variable.
Therefore the goal of this work is to generalize the results on the existence of a stable
periodic solution of the second Bogoliubov theorem to the case that the function $g$ of
(1.1) is only Lipschitz.

Another motivation of this paper comes from the forced Chua’s circuit (see Figure 1.3) studied in a large number of papers in the modern electrical engineering. This
circuit is described by the three-dimensional system

\[
\begin{align*}
C_1 \frac{dv C_1}{dt} &= \frac{v C_2 - v C_1}{R} - i(v C_1) + F_1(t), \\
C_2 \frac{dv C_2}{dt} &= \frac{v C_1 - v C_2}{R} + i L, \\
L \frac{di L}{dt} &= -v C_2 + F_2(t, v C_2),
\end{align*}
\]

where $i(v)$ (the characteristic of the Chua’s diode) is a piecewise linear function, as it
is represented in Figure 1.4. The recent literature provides insight into the numerical
simulations of (1.4) (see [40, 20], where $F_1 \neq 0$ and $F_2 \neq 0$, [5, 35], where $F_1 = 0$
and $F_2$ is periodic, or [11] where both $F_1$ and $F_2$ are periodic). Generalization of
the Bogoliubov result for (1.1) with Lipschitz right-hand part will allow for the first

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig13}
\caption{Forced Chua’s circuit (see [5, 11, 20, 35, 40]).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig14}
\caption{Nonlinear characteristic of the Chua’s diode of the circuit drawn at Figure 1.3 given
by $i(v) = G_a v + (1/2)(G_a - G_b) (|v + B_p| - |v - B_p|)$, where $G_a, G_b, B_p \in \mathbb{R}$
are some constants depending on the properties of the Chua’s diode (see [10]).}
\end{figure}
time the theoretical detection of asymptotically stable periodic solutions of (1.4) in the case that $C_1$ is large enough. Eventually this theoretical analysis may provide new interesting parameters of the forced Chua’s circuit for doing additional numerical experiments.

On the other hand, part of the interest in generalizing the Bogoliubov result comes from mechanics, where differential systems with piecewise linear stiffness describe various oscillating processes. One of these systems is exhibited by the device drawn in Figure 1.5(a), where a forced mass is attached to a spring whose stiffness changes from $k_1$ to $k_1 + k_2$ when the mass coordinate crosses 0 in the negative direction. This device is governed by the second order differential equation

\[ m \ddot{x} + P(x) = F(t, x, \dot{x}), \]

where the piecewise linear stiffness $P$ is drawn in Figure 1.5(b). Depending on the particular configuration of the device of Figure 1.5(a), different expressions for $F$ in (1.5) must be considered. Thus we have that $F(t, x, \dot{x}) = -f(x)\dot{x} + M\cos\omega t$ with piecewise constant $f$ for a shock-absorber and jigging conveyor (see [24, Chap. I, p. 16 and Chap. IV, p. 100]), where the original Bogoliubov result is employed without justification). The function $F$ takes the simpler form $F(t, x, \dot{x}) = -c\dot{x} + MQ(t)$ for an impact resonator, and $F(t, x, \dot{x}) = -c\dot{x} + M\sin\omega t$ for a cracked-body model (see [38, 7], where only numerical experiments are performed). In each of these situations (1.5) can be rewritten in the form (1.1) with $g$ Lipschitz, provided that the constant $k_2$ and the amplitude of the force $F$ are sufficiently small. Therefore the extension of the Bogoliubov result to the nonsmooth case that we shall do will allow one to justify the resonances that appeared in all these results. We note that the recent report by Los Alamos National Laboratory [13] describes the increasing interest in a specific form of the model of Figure 1.5(a) called the cracked-body model and, particularly, in the suspension bridge models. Consequently the results of this paper can be applied to such models.

A first model of a one-dimensional suspended bridge is drawn in Figure 1.6(a). It is represented (see [16, 27]) by the beam bending under its own weight and being supported by cables whose restoring force due to elasticity is proportional to $u^+$ (see Figure 1.6(b)), where $u = u(t, x)$ is the displacement of a point at a distance $x$ from one end of the bridge at time $t$ and $u$ is measured in the downward direction. Looking for $u$ of the form $u(t, x) = z(t)\sin(\pi x/L)$ and considering $F(x, t) = h(t)\sin(\pi x/L)$,
An immovable object

Nonlinear springs under tension

Force $F(t,x)$

A bending beam with hinged ends

Fig. 1.6. (a) The first idealization of the suspension bridge: the beam bending under its own weight is supported by the nonlinear cables (see [27, Figure 2]); (b) characteristic of stiffness of nonlinear springs.

we arrive (see [16]) at the following particular case of differential equation (1.5):

$$m\ddot{z} + \delta \dot{z} + c(\pi/L)^4 z + dz^+ = mg + h(t),$$

where the constant $m > 0$ is the mass per unit of length, $\delta > 0$ is a small viscous damping coefficient, $c > 0$ measures the flexibility or stiffness of the bridge, $L > 0$ is the length of the bridge, $d > 0$ represents the stiffness of nonlinear springs, and $h$ is a continuous $T$-periodic force modelling wind, marching troops, or cattle (see [19] for details). Considering $c > 0$ and $d > 0$ fixed and assuming that either $c > 0$ and $h(t)$ are sufficiently small, or that $c > 0$ is fixed and $h(t)$ is sufficiently large, or that $c > 0$ is sufficiently small and $h(t)$ fixed, Glover, Lazer, and McKenna [16], Lazer and McKenna [27], and Fabry [14] proved various theorems on the location of asymptotically stable $T$-periodic solutions in (1.6). The question What happens with these solutions when $d > 0$, $\delta > 0$, and $h(t)$ are all sufficiently small? was open and can be solved using the generalization of the Bogoliubov result that we provide. Lazer and McKenna proved in [26] that the Poincaré map for (1.6) is differentiable, but we note that this is not sufficient for applying the original Bogoliubov result.

We end the list of possible applications noting that system (1.4) describing the Chua’s circuit (Figure 1.3) appeared recently for studying the so-called negative slope mechanical systems (see Awrejcewicz [4, section 8.2.2]). So our results can also be applied to these mechanical systems.

These applications require generalizations of the second Bogoliubov theorem for Lipschitz right-hand parts. To the best of our knowledge Mitropol’skii was the first to consider such a kind of generalization. Assuming that $g$ is Lipschitz, $g_0 \in C^3(\mathbb{R}^k, \mathbb{R}^k)$, and all the eigenvalues of the matrix $(g_0)^{(v_0)}$ have negative real part, Mitropol’skii [34] developed the Bogoliubov result proving the existence and uniqueness of a $T$-periodic solution of system (1.1) in a neighborhood of $v_0$. There was great progress weakening the assumptions of the existence result (see Samoilenko [39] and Mawhin [32]), but this progress did not take place in the case of the uniqueness. Moreover, the asymptotic stability of the $T$-periodic solution remained unstudied in the case of Lipschitz systems for a long time. It has been done recently by Buică and Daniilidis in [8] for Lipschitz systems (1.1) assuming that the function $v \mapsto g(t,v,0)$ is differentiable at $v_0$ for almost any $t \in [0,T]$ and that the eigenvectors of the matrix $(g_0)^{(v_0)}$ are orthogonal.
In section 2, assuming that $g$ is piecewise differentiable in the second variable, we prove in Theorem 2.5 that the stable periodic solution of the Bogoliubov theorem persists when $g$ is not necessary $C^1$. Theorem 2.5 follows from this more general Theorem 2.1 whose hypotheses do not use any differentiability—neither of $g$, nor of $g_0$. Assuming only continuity for $g$, we show in Theorem 2.9 the existence of a nonasymptotically stable $T$-periodic solution of system (1.1) if the Brouwer topological degree of $-g_0$ is negative. In section 3 we illustrate our results constructing the resonance curves of the nonsmooth van der Pol oscillator, which were studied in [18], and compare these with the resonance curves of the classical van der Pol oscillator, which were constructed by Andronov and Witt [1, 2].

2. Main results. Throughout the paper $\Omega \subset \mathbb{R}^k$ will be an open set. For any $\delta > 0$ we denote $B_\delta(v_0) = \{ v \in \mathbb{R}^k : \|v - v_0\| \leq \delta \}$. We have the following main result on the existence, uniqueness, and asymptotic stability of $T$-periodic solutions for system (1.1).

**Theorem 2.1.** Let $g \in C^0(\mathbb{R} \times \Omega \times [0, 1], \mathbb{R}^k)$ and $v_0 \in \Omega$. Assume the following four conditions.

(i) For some $L > 0$ we have that $\|g(t, v_1, \varepsilon) - g(t, v_2, \varepsilon)\| \leq L \|v_1 - v_2\|$ for any $t \in [0, T], v_1, v_2 \in \Omega, \varepsilon \in [0, 1]$.

(ii) For any $\gamma > 0$ there exists $\delta > 0$ such that

$$\left\| \int_0^T g(\tau, v_1 + u(\tau), \varepsilon)d\tau - \int_0^T g(\tau, v_2 + u(\tau), \varepsilon)d\tau - \int_0^T g(\tau, v_1, 0)d\tau + \int_0^T g(\tau, v_2, 0)d\tau \right\| \leq \gamma \|v_1 - v_2\|$$

for any $u \in C^0([0, T], \mathbb{R}^k), \|u\| \leq \delta, v_1, v_2 \in B_\delta(v_0)$, and $\varepsilon \in [0, \delta]$.

(iii) Let $g_0$ be the averaged function given by (1.2) and consider that $g_0(v_0) = 0$.

(iv) There exist $g \in [0, 1], \alpha, \delta_0 > 0$, and a norm $\|\cdot\|_0$ on $\mathbb{R}^k$ such that $\|v_1 + \alpha g_0(v_1)\|_0 \leq \|g\|_0 \|v_1 - v_2\|_0$ for any $v_1, v_2 \in B_{\delta_0}(v_0)$.

Then there exists $\delta_1 > 0$ such that for every $\varepsilon \in (0, \delta_1], system (1.1)$ has exactly one $T$-periodic solution $x_\varepsilon$ with $x_\varepsilon(0) \in B_{\delta_1}(v_0)$. Moreover, the solution $x_\varepsilon$ is asymptotically stable and $x_\varepsilon(0) \rightarrow v_0$ as $\varepsilon \rightarrow 0$.

When the solution $x(\cdot, v, \varepsilon)$ of system (1.1) with the initial condition $x(0, v, \varepsilon) = v$ is well defined on $[0, T]$ for any $v \in B_{\delta_0}(v_0)$, the map $v \mapsto x(T, v, \varepsilon)$ is also well defined and is called the Poincaré map at time $T$ of system (1.1). In order to prove the existence, uniqueness, and stability of the $T$-periodic solutions of system (1.1) stated in Theorem 2.1, it is sufficient to study the same properties for the fixed points of this Poincaré map.

Before proving Theorem 2.1 we state and prove two lemmas. In order to state the first lemma, we need to introduce the function

$$g_\varepsilon(v) = \int_0^T g(\tau, x(\tau, v, \varepsilon), \varepsilon)d\tau$$

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1At the final stage of publishing of this paper, we have been informed by Prof. Michael Guevara that the resonance curves mentioned appeared for the first time in [Balth. van der Pol, *Tijdschr. Ned Rad Gen.*, (1924) (in Dutch)], an English translation appeared in [Phil. Mag., vol. 3, 1927, p. 65]. We thank Prof. Guevara for calling our attention to those papers and some other historical background on resonance curves.
and to note that by writing the equivalent integral equation of system (1.1) we have

\[ x(T, v, \varepsilon) = v + \varepsilon g_\varepsilon(v). \]

**Lemma 2.2.** Let \( g \in C^0(\mathbb{R} \times \Omega \times [0, 1], \mathbb{R}^k) \) and \( \delta_0 > 0 \) be such that \( B_{\delta_0}(v_0) \subset \Omega \). If (i) is satisfied, then there exist \( \delta \in [0, \delta_0] \) and \( L_1 > 0 \) such that the map \((v, \varepsilon) \mapsto g_\varepsilon(v)\) is well defined and continuous on \( B_{\delta_0}(v_0) \times [0, \delta] \) and

\[ ||g_\varepsilon(v_1) - g_\varepsilon(v_2)|| \leq L_1 ||v_1 - v_2|| \quad \text{for any } \varepsilon \in [0, \delta], \, v_1, v_2 \in B_{\delta_0}(v_0). \]

If both (i) and (ii) are satisfied, then for any \( \gamma > 0 \) there exists \( \delta \in [0, \delta_0] \) such that

\[ ||g_\varepsilon(v_1) - g_\varepsilon(v_2)|| \leq \gamma ||v_1 - v_2|| \]

for any \( v_1, v_2 \in B_{\delta_0}(v_0) \) and \( \varepsilon \in [0, \delta] \).

**Proof.** Using the continuity of the solution of a differential system with respect to the initial data and the parameter (see [37, Chap. 4, section 23, statements G and D]), we obtain the existence of \( \varepsilon_0 > 0 \) such that \( x(t, v, \varepsilon) \in \Omega \) for any \( t \in [0, T], \, v \in B_{\delta_0}(v_0) \), and \( \varepsilon \in [0, \varepsilon_0] \). Using the Grönwall–Bellman lemma (see [17, Lemma 6.2] or [12, Chap. II, section 11]) from the representation \( x(t, v, \varepsilon) = v + \varepsilon \int_0^t g(\tau, x(\tau, v, \varepsilon), \varepsilon)d\tau \) and the property (i), we obtain

\[ ||x(t, v_1, \varepsilon) - x(t, v_2, \varepsilon)|| \leq e^{L_1 \varepsilon} ||v_1 - v_2|| \quad \text{for all } t \in [0, T], \, v_1, v_2 \in B_{\delta_0}(v_0), \quad \text{and } \varepsilon \in [0, \varepsilon_0]. \]

Therefore \( y(t, v, \varepsilon) = \int_0^t g(\tau, x(\tau, v, \varepsilon), \varepsilon)d\tau \) satisfies the property

\[ ||y(t, v_1, \varepsilon) - y(t, v_2, \varepsilon)|| \leq L_1 ||v_1 - v_2|| \]

for all \( t \in [0, T], \, v_1, \, v_2 \in B_{\delta_0}(v_0), \, \varepsilon \in [0, \varepsilon_0] \), and \( L_1 = LT e^{\varepsilon_0 LT}. \) Since \( g_\varepsilon(v) = y(T, v, \varepsilon) \) the first part of the lemma has been proven.

Taking into account that \( x(t, v, \varepsilon) = v + \varepsilon y(t, v, \varepsilon) \), we have

\[ y(T, v_1, \varepsilon) - y(T, v_1, 0) - y(T, v_2, \varepsilon) + y(T, v_2, 0) = I_1(v_1, v_2, \varepsilon) + I_2(v_1, v_2, \varepsilon), \]

where

\[ I_1(v_1, v_2, \varepsilon) = \int_0^T [g(\tau, v_2 + \varepsilon y(\tau, v_1, \varepsilon), \varepsilon) - g(\tau, v_2 + \varepsilon y(\tau, v_2, \varepsilon), \varepsilon)]d\tau, \]

\[ I_2(v_1, v_2, \varepsilon) = \int_0^T [(g(\tau, v_1 + \varepsilon y(\tau, v_1, \varepsilon), \varepsilon) - g(\tau, v_2 + \varepsilon y(\tau, v_1, \varepsilon), \varepsilon))]d\tau \]

\[ - \int_0^T (g(\tau, v_1, 0) - g(\tau, v_2, 0))d\tau. \]

Since \((t, v, \varepsilon) \mapsto y(t, v, \varepsilon)\) is bounded on \([0, T] \times B_{\delta_0}(v_0) \times [0, \varepsilon_0] \), we have that \( \varepsilon y(t, v, \varepsilon) \to 0 \) as \( \varepsilon \to 0 \) uniformly with respect to \( t \in [0, T] \) and \( v \in B_{\delta_0}(v_0) \). Decreasing \( \varepsilon_0 > 0 \), if necessary, we get that \( v_2 + \varepsilon y(t, v_1, \varepsilon) \in \Omega \) for any \( t \in [0, T], \, v_1, v_2 \in B_{\delta_0}(v_0), \, \varepsilon \in [0, \varepsilon_0] \). By assumption (i) and relation (2.1) we obtain that

\[ ||I_1(v_1, v_2, \varepsilon)|| \leq T \cdot \varepsilon L_1 ||v_1 - v_2|| \quad \text{for all } \varepsilon \in [0, \varepsilon_0], \, v_1, v_2 \in B_{\delta_0}(v_0). \]

We fix \( \gamma > 0 \) and take \( \delta > 0 \) given by (ii). Without loss of generality we can consider that \( \delta \leq \min\{\delta_0, \varepsilon_0, \gamma/(2T L_1)\} \). Therefore assumption (ii) implies that

\[ ||I_2(v_1, v_2, \varepsilon)|| \leq (\gamma/2)||v_1 - v_2|| \quad \text{for any } \varepsilon \in [0, \delta], \, v_1, v_2 \in B_5(v_0). \]

Substituting the obtained estimations for \( I_1 \) and \( I_2 \) into (2.2) we have

\[ ||y(T, v_1, \varepsilon) - y(T, v_1, 0) - y(T, v_2, \varepsilon)|| \leq \gamma ||v_1 - v_2||. \]
y(T, v_2, 0) \leq \varepsilon(\gamma/2)\|v_1 - v_2\| \leq \gamma\|v_1 - v_2\| \text{ for any } \varepsilon \in [0, \delta], v_1, v_2 \in B_\delta(v_0).$

Hence the proof is complete. \[\square\]

**Lemma 2.3.** Let \( g_0 : \Omega \to \mathbb{R}^k \), satisfying assumption (iv) for some \( q \in (0, 1) \), \( \alpha, \delta_0 > 0 \), and a norm \( \| \cdot \|_0 \) on \( \mathbb{R}^k \). Then

\[ \|v_1 + \varepsilon g_0(v_1) - v_2 - \varepsilon g_0(v_2)\|_0 \leq (1 - \varepsilon(1-q)/\alpha)\|v_1 - v_2\|_0 \text{ for any } v_1, v_2 \in B_{\delta_0}(v_0) \text{ and any } \varepsilon \in [0, \alpha]. \]

**Proof.** Indeed, the equality \( v + \varepsilon g_0(v) = (1 - \varepsilon/\alpha)v + \varepsilon/\alpha (v + \alpha g_0(v)) \) implies that the Lipschitz constant of the function \( I + \varepsilon g_0 \) with respect to the norm \( \| \cdot \|_0 \) is

\[ (1 - \varepsilon/\alpha) + \varepsilon/\alpha q = 1 - \varepsilon(1-q)/\alpha. \]

**Proof of Theorem 2.1.** By Lemma 2.2 we have that there exists \( \delta_1 \in [0, \delta_0] \) such that

\[ \|g_\varepsilon(v_1) - g_0(v_1) - g_\varepsilon(v_2) + g_0(v_2)\|_0 \leq ((1 - q)/(2\alpha))\|v_1 - v_2\|_0 \]

for any \( \varepsilon \in [0, \delta_1] \), \( v_1, v_2 \in B_{\delta_1}(v_0) \). First we prove that there exists \( \varepsilon_1 \in [0, \delta_1] \) such that for every \( \varepsilon \in [0, \varepsilon_1] \) there exists \( v_\varepsilon \in B_{\delta_1}(v_0) \) such that \( x(\cdot, v_\varepsilon, \varepsilon) \) is a T-periodic solution of (1.1) by showing that there exists \( v_\varepsilon \) such that \( x(T, v_\varepsilon, \varepsilon) = v_\varepsilon \). Using (iii) and (iv) we have

\[ \|v + \varepsilon g_0(v) - v_0\|_0 \leq q\|v - v_0\|_0 \text{ for any } v \in B_{\delta_1}(v_0). \]

Therefore we have that the map \( I + \alpha g_\varepsilon \) maps \( B_{\delta_1}(v_0) \) into itself. From Lemma 2.2 we have that there exists \( \varepsilon_0 > 0 \) such that the map \( (v, \varepsilon) \mapsto g_\varepsilon(v) \) is well defined and continuous on \( B_{\delta_1}(v_0) \times [0, \varepsilon_0] \). We deduce that there exists \( \varepsilon_1 > 0 \) sufficiently small such that, for every \( \varepsilon \in [0, \varepsilon_1] \), the map \( I + \alpha g_\varepsilon \) maps \( B_{\delta_1}(v_0) \) into itself as well. Therefore, by the Brouwer theorem (see, for example, [23, Theorem 3.1]) we have that \( B_{\delta_1}(v_0) \) contains at least one fixed point of the map \( I + \alpha g_\varepsilon \) for any \( \varepsilon \in [0, \varepsilon_1] \). Denote this fixed point by \( v_\varepsilon \). Then we have \( g_\varepsilon(v_\varepsilon) = 0 \) and \( x(T, v_\varepsilon, \varepsilon) = v_\varepsilon \) for any \( \varepsilon \in [0, \varepsilon_1] \).

Now we prove that \( x(\cdot, v_\varepsilon, \varepsilon) \) is the only T-periodic solution of (1.1) starting near \( v_0 \) and that, moreover, it is asymptotically stable. Knowing that \( x(T, v, \varepsilon) = v + \varepsilon g_\varepsilon(v) \) we write the following identity:

\[ x(T, v, \varepsilon) = v + \varepsilon g_0(v) + \varepsilon (g_\varepsilon(v) - g_0(v)). \]

Using Lemma 2.3 we have from (2.3) and (2.4) that

\[ \|x(T, v_1, \varepsilon) - x(T, v_2, \varepsilon)\|_0 \leq (1 - \varepsilon(1-q)/\alpha + \varepsilon(1-q)/(2\alpha))\|v_1 - v_2\|_0 \]

for all \( v_1, v_2 \in B_{\delta_1}(v_0) \) and \( \varepsilon \in [0, \delta_1] \). We proved before that there exists \( \varepsilon_1 > 0 \) such that for every \( \varepsilon \in [0, \varepsilon_1] \) there exists \( v_\varepsilon \in B_{\delta_1}(v_0) \) with \( x(\cdot, v_\varepsilon, \varepsilon) \) a T-periodic solution of (1.1). Since \( (1-q)/(2\alpha) > 0 \) and \( \varepsilon_1 \leq \delta_1 \), the last inequality implies that for each \( \varepsilon \in [0, \delta_1] \), the T-periodic solution \( x(\cdot, v_\varepsilon, \varepsilon) \) is the only T-periodic solution of (1.1) in \( B_{\delta_1}(v_0) \) and, moreover (see [23, Lemma 9.2]), it is asymptotically stable. \[\square\]

**Remark 2.4.** We note that a similar result close to Theorem 2.1 is contained in [8, Theorem 3.5]. But instead of the assumption (iv) with a fixed \( \alpha > 0 \), it is assumed in [8] with any \( \alpha > 0 \) sufficiently small. Anyway, notice that Lemma 2.3 implies that it is the same to assume (iv) for only one \( \alpha > 0 \) or for all \( \alpha > 0 \) sufficiently small. The advantage of our Theorem 2.1 is that it does not require differentiability of \( g(t, \cdot, \varepsilon) \) at any point, while [8] needs it at \( v_0 \). See also Remark 2.8.
In general it is not easy to check assumptions (ii) and (iv) in the applications of Theorem 2.1. Thus we also give the following theorem based on Theorem 2.1 which assumes certain type of piecewise differentiability instead of (ii) and deals with properties of the matrix \((g_0)'(v_0)\) instead of the Lipschitz constant of \(g_0\).

For any set \(M \subset [0, T]\) measurable in the sense of Lebesgue we denote by \(\text{mes}(M)\) the Lebesgue measure of \(M\) (see [21, Chap. V, section 3]).

**Theorem 2.5.** Let \(g \in C^0(\mathbb{R} \times \Omega \times [0, 1], \mathbb{R}^k)\) satisfying (i). Let \(g_0\) be the averaged function given by (1.2) and consider \(v_0 \in \Omega\) such that \(g_0(v_0) = 0\). Assume that

\[
(v) \quad \text{given any } \tilde{\gamma} > 0 \text{ there exist } \delta > 0 \text{ and } M \subset [0, T] \text{ with } \text{mes}(M) < \tilde{\gamma} \text{ such that for every } v \in B_{\delta}(v_0), t \in [0, T] \setminus M, \text{ and } \varepsilon \in [0, \delta] \text{ we have that } g(t, \cdot, \varepsilon) \text{ is differentiable at } v \text{ and } \|g'_e(t, v, \varepsilon) - g'_e(t, v_0, 0)\| \leq \tilde{\gamma}.
\]

Finally assume that

\[
(vi) \quad g_0 \text{ is continuously differentiable in a neighborhood of } v_0 \text{ and the real parts of all the eigenvalues of } (g_0)'(v_0) \text{ are negative.}
\]

Then there exists \(\delta_1 > 0\) such that for every \(\varepsilon \in (0, \delta_1]\), system (1.1) has exactly one \(T\)-periodic solution \(x_\varepsilon\) with \(x_\varepsilon(0) \in B_{\delta_1}(v_0)\). Moreover, the solution \(x_\varepsilon\) is asymptotically stable and \(x_\varepsilon(0) \rightarrow v_0\) as \(\varepsilon \rightarrow 0\).

For proving Theorem 2.5 we need two preliminary lemmas.

**Lemma 2.6.** Let \(g \in C^0(\mathbb{R} \times \Omega \times [0, 1], \mathbb{R}^k)\) satisfying (i). If (v) holds, then (ii) is satisfied.

**Proof.** Let \(\gamma > 0\) be an arbitrary number. We show that (ii) holds with \(\delta = \delta/2\), where \(\delta\) is given by (v) applied with \(\tilde{\gamma} = \min \{\gamma/(4L), \gamma/(4T)\}\). We consider also \(M \subset [0, T]\) given by (v) applied with the same value of \(\tilde{\gamma}\).

Let \(u \in C^0([0, T], \mathbb{R}^k), \|u\| \leq \delta, \text{ and } F(v) = \int_0^T g(\tau, v + u(\tau), \varepsilon) \text{d}\tau - \int_0^T g(\tau, v, 0) \text{d}\tau\). Let \(v_1, v_2 \in B_\delta(v_0)\) and \(\varepsilon \in [0, \delta]\). We have \(F(v) = F_1(v) + F_2(v)\), where \(F_1(v) = \int_M (g(\tau, v + u(\tau), \varepsilon) - g(\tau, v, 0)) \text{d}\tau\) and \(F_2(v) = \int_{[0, T] \setminus M} (g(\tau, v + u(\tau), \varepsilon) - g(\tau, v, 0)) \text{d}\tau\). By (i) we have that \(\|F(v_1) - F(v_2)\| \leq 2L \cdot \text{mes}(M)\|v_1 - v_2\| < 2L\tilde{\gamma}\|v_1 - v_2\| \leq (\gamma/2)\|v_1 - v_2\|.\)

On the other hand, using (v), we will prove that a similar relation holds for \(F_2\). In order to do this, we denote \(h(\tau, v) = g(\tau, v + u(\tau), \varepsilon) - g(\tau, v, 0)\). Notice that for each \(\tau \in [0, T] \setminus M\) we can write \(h'_e(\tau, v) = (g'_e(\tau, v + u(\tau), \varepsilon) - g'_e(\tau, v, 0)) - (g'_e(\tau, v, 0) - g'_e(\tau, v_0, 0)).\) As a direct consequence of (v) we deduce that \(\|h'_e(\tau, v, \varepsilon)\| \leq \tilde{\gamma}\) for all \(v \in B_\delta(v_0)\) and \(\tau \in [0, T] \setminus M\). Now applying the mean value theorem for the function \(h(\tau, \cdot)\), we have \(\|h(\tau, v_1) - h(\tau, v_2)\| \leq 2\tilde{\gamma}\|v_1 - v_2\|\) for all \(\tau \in [0, T] \setminus M\) and all \(v_1, v_2 \in B_\delta(v_0)\). Then \(\|F_2(v_1) - F_2(v_2)\| \leq \int_{[0, T] \setminus M} \|h(\tau, v)\| \text{d}\tau \leq 2T\tilde{\gamma}\|v_1 - v_2\| \leq (\gamma/2)\|v_1 - v_2\|.\)

Therefore we have proved that \(\|F(v_1) - F(v_2)\| \leq \gamma\|v_1 - v_2\|\), which coincides with (ii).

**Lemma 2.7.** Let \(g_0 : \Omega \rightarrow \mathbb{R}^k\) satisfying assumption (vi) for some \(v_0 \in \Omega\). Then there exist \(q \in [0, 1], \alpha, \delta_0 > 0\) and a norm \(\|\cdot\|_0\) on \(\mathbb{R}^k\) such that (iv) is satisfied.

**Proof.** If \(\lambda\) is an eigenvalue of \((\alpha g_0)'(v_0)\), then \(\lambda + 1\) is an eigenvalue of \(I + (\alpha g_0)'(v_0)\). Since the eigenvalues of \((\alpha g_0)'(v_0)\) tend to 0 as \(\alpha \rightarrow 0\) and have negative real parts, then there exists \(\alpha \in [0, 1]\) such that the absolute values of all the eigenvalues of \(I + (\alpha g_0)'(v_0)\) are less than one. Therefore (see [22, p. 90, Lemma 2.2]) there exist \(\tilde{q} \in [0, 1]\) and a norm \(\|\cdot\|_0\) on \(\mathbb{R}^k\) such that \(\sup_{0 \leq \xi \leq 1} \|\xi + \alpha (g_0)'(v_0)\|_0 \leq \tilde{q}\).

By continuous differentiability of \(g_0\) in a neighborhood of \(v_0\) we have that \(\|g_0(v_1) - g_0(v_2)\|/\|v_1 - v_2\| \leq \|g_0(v_1) - g_0(v_2)\| + \|g_0'(v_1)(v_1 - v_2) - g_0'(v_2)(v_1 - v_2)\|/\|v_1 - v_2\| \rightarrow 0\) as \(\max\{\|v_1 - v_0\|, \|v_2 - v_0\|\} \rightarrow 0\). Therefore taking into account that all norms on \(\mathbb{R}^k\) are equivalent, there exists \(\delta_0 > 0\) such that \(\|g_0(v_1) - g_0(v_2) - (g_0)'(v_0)(v_1 - v_2)\|_0 \leq (1 - \tilde{q})/(2\alpha)\|v_1 - v_2\|_0\) for all
\[v_1, v_2 \in B_{0}(v_0)\]  

Then \[
\|v_1 + \alpha g_0(v_1) - v_2 - \alpha g_0(v_2)\|_0 \\
\leq \alpha \|g_0(v_1) - g_0(v_2) - (g_0)'(v_0)(v_1 - v_2)\|_0 + \|v_1 - v_2 + (g_0)'(v_0)(v_1 - v_2)\|_0 \\
\leq (1 + \bar{q})/2 \|v_1 - v_2\|_0
\]

for all \(v_1, v_2 \in B_{0}(v_0)\). \(\square\)

**Proof of Theorem 2.5.** Lemmas 2.6 and 2.7 imply that assumptions (ii) and (iv) of Theorem 2.1 are satisfied. Therefore the conclusion of the theorem follows by applying Theorem 2.1. \(\square\)

It was observed by Mitropol’skii in [34] that in spite of the fact that \(g(t, \cdot, \varepsilon)\) in (2.5) is only Lipschitz, sometimes the function \(g_0\) turns out to be differentiable in applications. In particular we will see in section 3 that this is the case for the nonsmooth van der Pol oscillator.

Clearly if \(g \in C^1(\mathbb{R} \times \mathbb{R}^k \times [0, 1], \mathbb{R}^k)\), then (i) and (v) hold in any open bounded set \(\Omega \subset \mathbb{R}^k\). Therefore Theorem 2.5 is a generalization of the stable periodic case of the second Bogoliubov theorem formulated in the introduction.

**Remark 2.8.** Theorem 2.5 does not require the eigenvectors of \((g_0)'(v_0)\) to be orthogonal as in [8, Theorem 3.6]. Moreover, assumption \((H_2)\) of [8] is more restrictive than (v).

For completeness we also give the following theorem on the existence of nonasymptotically stable \(T\)-periodic solutions for (1.1). In the theorem below, \(d(F, V)\) denotes the Brouwer topological degree of the vector field \(F \in C^0(\mathbb{R}^k, \mathbb{R}^k)\) on the open and bounded set \(V \subset \mathbb{R}^k\) (see [23, Chap. 2, section 5.2]).

**Theorem 2.9.** Let \(g \in C^0(\mathbb{R} \times \mathbb{R}^k \times [0, 1], \mathbb{R}^k)\). Assume that there exists an open bounded set \(V \subset \mathbb{R}^k\) such that \(g_0(v) \neq 0\) for any \(v \in \partial V\) and

\[(vii)\] \(d(-g_0, V) < 0\).

Then there exists \(\varepsilon_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0]\) system (1.1) has at least one nonasymptotically stable \(T\)-periodic solutions \(x_\varepsilon\) with \(x_\varepsilon(0) \in V\).

**Proof.** Since \(g_0(v) \neq 0\) for any \(v \in \partial V\), then from Mawhin’s theorem [32] (or [33, section 5]) we have that there exists \(\varepsilon_0 > 0\) such that

\[(2.5)\] \(d(-g_0, V) = d(I - x(T, \cdot, \varepsilon), V)\) for any \(\varepsilon \in (0, \varepsilon_0]\).

By [23, Theorem 9.6] for any asymptotically stable \(T\)-periodic solution \(x_\varepsilon\) of (1.1) we have that \(d(I - x(T, \cdot, \varepsilon), B_\delta(x_\varepsilon(0))) = 1\) for \(\delta > 0\) sufficiently small. Therefore if all the possible \(T\)-periodic solutions of (1.1) with \(\varepsilon \in (0, \varepsilon_0]\) had been asymptotically stable, then the degree \(d(I - x(T, \cdot, \varepsilon), V)\) would have been nonnegative, contradicting (vii) and (2.5). \(\square\)

**Remark 2.10.** Assumptions (iii) and (iv) imply that \(d(-g_0, V) = 1\) (see [23, Theorem 5.16]).

Finally thinking in terms of the application to the nonsmooth van der Pol oscillator, we formulate the following theorem which combines Mawhin’s theorem (see [32] or [33, Theorem 3]) and Theorems 2.5 and 2.9. In this theorem \((g_0)'(v_0)\) stays for the derivative of the \(i\)th component of the function \(g_0\) with respect to the \(i\)th variable.

**Theorem 2.11.** Let \(g \in C^0(\mathbb{R} \times \mathbb{R}^2 \times [0, 1], \mathbb{R}^2)\). Let \(v_0 \in \Omega\) be such a point that \(g_0(v_0) = 0\) and \(g_0\) is continuously differentiable in a neighborhood of \(v_0\).

(a) If \(\det (g_0)'(v_0) \neq 0\), then there exists \(\varepsilon_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0]\) system (1.1) has at least one \(T\)-periodic solution \(x_\varepsilon\) satisfying \(x_\varepsilon(0) \to v_0\) as \(\varepsilon \to 0\).
(b) If (i) and (v) hold and
\( \det (g_0)'(v_0) > 0 \) and \( ([g_0]_1)'(v_0) + ([g_0]_2)'(v_0) < 0, \)
then there exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) system (1.1) has exactly one \( T \)-periodic solution \( x_\varepsilon \) such that \( x_\varepsilon(0) \to v_0 \) as \( \varepsilon \to 0 \). Moreover, the solution \( x_\varepsilon \) is asymptotically stable.

(c) If \( \det (g_0)'(v_0) < 0 \), then there exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) system (1.1) has at least one nonasymptotically stable \( T \)-periodic solution \( x_\varepsilon \) such that \( x_\varepsilon(0) \to v_0 \) as \( \varepsilon \to 0 \).

**Proof.** Statement (a) is added for the completeness of the formulation of Theorem 2.11 and it follows from Mawhin’s theorem (see [32] or [33, Theorem 3]).

On the other hand, it is a simple calculation to show that (2.6) implies that all the eigenvalues of \( (g_0)'(v_0) \) have negative real part. Therefore assumption (vi) of Theorem 2.5 is also satisfied and statement (b) follows from this theorem.

Statement (c) follows from Theorem 2.9. Indeed, since \( \det (g_0)'(v_0) < 0 \), it implies (see [23, Theorem 5.9]) that \( \det (g_0, B_\rho(v_0)) \) is defined for any \( \rho > 0 \) sufficiently small and that \( \det (g_0, B_\rho(v_0)) = \det (g_0)'(v_0) < 0. \)

### 3. Application to the nonsmooth van der Pol oscillator.

In [18] Hogan first demonstrated the existence of a limit cycle for the nonsmooth van der Pol equation \( \ddot{\theta} + \varepsilon(|\theta| - 1)\dot{\theta} + (1 + \alpha \varepsilon)\theta = \varepsilon \lambda \sin t, \)
where \( \alpha \) is a detuning parameter and \( \varepsilon \lambda \sin t \) is an external force. We assume that \( \varepsilon > 0 \) is sufficiently small, and we consider that the parameters \( \alpha \) and \( \lambda \) vary in \( \mathbb{R} \).

We finally note that standard change of variables (see example in section 3) brings (1.3) into the form (1.1), with \( g \) sufficiently smooth to satisfy the hypotheses of the second Bogoliubov theorem. But we remind the reader that our aim is to apply directly Theorem 2.5, in the same way that Andronov and Witt applied Bogoliubov theorem to the classical van der Pol oscillator

\[
\ddot{\theta} + \varepsilon (u^2 - 1) \dot{\theta} + (1 + \alpha \varepsilon)\theta = \varepsilon \lambda \sin t,
\]
which can be found in [1, Figure 4] or in [30, Chap. I, section 16, Figure 15].

A function \( u \) is a solution of (3.1) if and only if \( (z_1, z_2) = (u, \dot{u}) \) is a solution of the system
\[
\begin{align*}
\dot{z}_1 &= \dot{z}_2, \\
\dot{z}_2 &= -z_1 + \varepsilon [-a z_2 - (|z_1| - 1)z_2 + \lambda \sin t].
\end{align*}
\]

After the change of variables
\[
\begin{pmatrix}
z_1(t) \\
z_2(t)
\end{pmatrix} =
\begin{pmatrix}
\cos t & \sin t \\
-\sin t & \cos t
\end{pmatrix}
\begin{pmatrix}
x_1(t) \\
x_2(t)
\end{pmatrix},
\]
system (3.3) takes the form
\begin{align}
\dot{x}_1 &= \varepsilon \sin(-t) \left[ -a(x_1 \cos t + x_2 \sin t) 
- (|x_1 \cos t + x_2 \sin t| - 1)(-x_1 \sin t + x_2 \cos t) + \lambda \sin t \right], \\
\dot{x}_2 &= \varepsilon \cos(-t) \left[ -a(x_1 \cos t + x_2 \sin t) 
- (|x_1 \cos t + x_2 \sin t| - 1)(-x_1 \sin t + x_2 \cos t) + \lambda \sin t \right].
\end{align}

The corresponding averaged function \( g_0 \), calculated using (1.2), is given by
\begin{equation}
\begin{aligned}
[g_0]_1(M,N) &= \pi a N - \pi \lambda - \pi M - \frac{2}{3} M \sqrt{M^2 + N^2}, \\
[g_0]_2(M,N) &= -\pi a M - \pi N - \frac{2}{3} N \sqrt{M^2 + N^2},
\end{aligned}
\end{equation}
and it is continuously differentiable in \( \mathbb{R}^2 \setminus \{0\} \).

In short, by statement (a) of Theorem 2.11, the zeros \( (M,N) \in \mathbb{R}^2 \) of this function with the property that \( \det (g_0)'(M,N) \neq 0 \) determine the \( 2\pi \)-periodic solutions of (3.3) emanating from the solution of the unperturbed system
\begin{equation}
\begin{aligned}
u_1(t) &= M \cos t + N \sin t, \\
u_2(t) &= -M \sin t + N \cos t.
\end{aligned}
\end{equation}

We have the following expression for the determinant:
\begin{equation}
\begin{aligned}
\det (g_0)'(M,N) &= \pi^2 (1 + a^2) + \frac{32}{9} (M^2 + N^2) - 4\pi \sqrt{M^2 + N^2}.
\end{aligned}
\end{equation}

Following Andronov and Witt [1] we are concerned with the dependence of the amplitude of the solution (3.6) with respect to \( a \) and \( \lambda \). Thus we decompose this solution as follows:
\begin{equation}
\begin{aligned}
u_1(t) &= A \sin(t + \phi), \\
u_2(t) &= A \cos(t + \phi),
\end{aligned}
\end{equation}
where \( (M,N) \) is related to \( (A, \phi) \) by
\begin{equation}
M = A \sin \phi, \quad N = A \cos \phi.
\end{equation}

Substituting (3.9) into (3.5) and (3.7) we obtain
\begin{equation}
\begin{aligned}
[g_0((A \sin \phi, A \cos \phi))]_1 &= -\left( \frac{4}{3} \right) \cdot A |A| \sin \phi + \pi a A \sin \phi + \pi A \sin \phi - \pi \lambda, \\
[g_0((A \sin \phi, A \cos \phi))]_2 &= -\left( \frac{4}{3} \right) \cdot A |A| \cos \phi - \pi a A \sin \phi + \pi A \cos \phi,
\end{aligned}
\end{equation}
and, respectively,
\begin{equation}
\begin{aligned}
\det (g_0)'((A \sin \phi, A \cos \phi)) &= \pi^2 (1 + a^2) + \frac{32}{9} A^2 - 2\pi |A|. 
\end{aligned}
\end{equation}

Looking for the zeros \( (A, \phi) \) of (3.10), we find the implicit formula
\begin{equation}
\begin{aligned}
A^2 \left( a^2 + \left( 1 - \frac{4}{3\pi} |A| \right)^2 \right) = \lambda^2.
\end{aligned}
\end{equation}
Fig. 3.1. Dependence of the amplitude of stable (solid curves) and unstable (dash curves) 2π-periodic solutions of the nonsmooth periodically perturbed van der Pol equation (3.1) on the detuning parameter $a$ obtained over formulas (3.12), (3.16), and (3.17) for different values of $\lambda$. The curve I is plotted with $\lambda = 0.4$, II with $\lambda = 3\pi/16$, III with some $\lambda = \sqrt{0.4} \in (3\pi/16, 9\sqrt{3}\pi/64)$, IV with $\lambda = 9\sqrt{3}\pi/64$, and V with $\lambda = 1.5$. Point $P$ is $2/\sqrt{3}$.

For determining $A$. Observe that the number of positive zeros of (3.12) coincide with the number of zeros of the equation $A^2 \left( a^2 + \left(1 - \frac{4}{3\pi}A \right)^2 \right) = \lambda^2$. To estimate this number we define

$$f(A) = A^2 \left( a^2 + \left(1 - \frac{4}{3\pi}A \right)^2 \right) - \lambda^2,$$

and we have

$$f'(A) = 2A \left( a^2 + \left(1 - \frac{4}{3\pi}A \right)^2 \right) - \frac{8}{3\pi} A^2 \left(1 - \frac{4}{3\pi}A \right).$$

Since $f'$ has one or two zeros, then (3.12) has one, two, or three positive solutions $A$ for any fixed $a$ and $\lambda$. In order to understand the different situations that can appear, we follow Andronov and Witt, who suggested in [1] (see also [2]) to construct the so-called resonance curves, namely, the curves $A$ in function of $a$, for $\lambda$ fixed. The equation of this curve is given by formula (3.12). Some curves (3.12) corresponding to different values of $\lambda$ are drawn in Figure 3.1. The way for describing these resonance curves (3.12) is borrowed from [30, Ch. 1, section 5], where the classical van der Pol equation is considered.

When $\lambda = 0$ the curve (3.12) is formed by the axis $A = 0$ and the isolated point $(0, 3\pi/4)$. When $\lambda > 0$ but sufficiently small, the resonance curve consists of two branches: instead of $A = 0$ we have the curve of the type I - I, and instead of the point $(0, 3\pi/4)$ we obtain an oval $I' - I'$ surrounding this point. When $\lambda > 0$ increases, the oval $I' - I'$ and the branch I - I tend to each other, and, for a certain $\lambda$, there exists only one branch $II - II$ with a double point $P$. The value of this $\lambda$ can be obtained assuming that (3.12) has for $a = 0$ a double root and, therefore, (3.11)
should be zero. Solving jointly (3.12) and (3.11) with \(a = 0\) we obtain \(\lambda = 3\pi/16\) and \(P = 2\pi/8\). If \(\lambda > 3\pi/16\), then we have curves of the type III which take form \(V\) when \(\lambda > 0\) crosses the value \(\lambda = 9\sqrt{3}\pi/64\). From here, if \(\lambda > 3\pi/16\), then we have curves of the type III which take form \(V\) when \(\lambda > 0\) crosses the value \(\lambda = 9\sqrt{3}\pi/64\).

If \(\lambda > 3\pi/16\), then we have curves of the type III which take form \(V\) when \(\lambda > 0\) crosses the value \(\lambda = 9\sqrt{3}\pi/64\). From here, if \(\lambda < 3\pi/16\), then (3.12) has three real roots when \(|a|\) is sufficiently small, and only one root when \(|a|\) is greater than a certain number which depends on \(\lambda\). The amplitude curves of type \(V\) provide exactly one solution of (3.12) for any value of \(a\). The value \(\lambda = 9\sqrt{3}\pi/64\), which separates the curves where (3.12) has three solutions from the curves where (3.12) has one solution, is obtained from the property that (3.12) with this \(\lambda\) has a double root for some \(a\) and thus this value of \(a\) vanishes (3.11). Therefore \(\lambda = 9\sqrt{3}\pi/64\) is the point separating the interval \((0, \lambda\)) where the system formed by (3.12) and (3.13) has at least one solution from the interval \((\lambda, \infty)\) where (3.12)–(3.13) has no solutions.

In short, we have studied the amplitudes of the \(2\pi\)-periodic solutions of system (3.3) depending on \(a\) and \(\lambda\), when a physical system described by (3.3) possesses \(2\pi\)-periodic oscillations and when some of them are asymptotically stable. To find the answer we have used statement (b) of Theorem 2.11. Assumption (i) is obviously satisfied with \(\Omega = \mathbb{R}^2\). The next statement shows that the right-hand side of system (3.4) satisfies (v).

**Proposition 3.1.** Let \(v_0 \in \mathbb{R}^2\), \(v_0 \neq 0\). Then the right-hand side of (3.4) satisfies (v) for any \(a, \lambda \in \mathbb{R}\).

The proof of Proposition 3.1 is given in section 4.

Thus we have to study the signs of (3.11) and \(((g_0)_1)'_M(A\sin \phi, A\cos \phi) + ((g_0)_2)'_N(A\sin \phi, A\cos \phi)\). We have

\[
((g_0)_1)'_M(M, N) + ((g_0)_2)'_N(M, N) = 2 \left(\pi - 2\sqrt{M^2 + N^2}\right),
\]

and therefore the conditions for the asymptotic stability of the \(2\pi\)-periodic solutions of (3.3) near (3.6) are

\[
\pi^2(1 + a^2) + \frac{32}{9}A^2 - 2\pi|A| > 0
\]

and

\[
2 \left(\pi - 2\sqrt{M^2 + N^2}\right) < 0.
\]

Substituting (3.9) into the inequalities (3.14) and (3.15), we obtain the following equivalent inequalities in terms of the amplitude \(A\):

\[
\pi^2(1 + a^2) + \frac{32}{9}A^2 - 2\pi|A| > 0
\]

and

\[
2\pi - 4|A| < 0.
\]

Conditions (3.16) and (3.17) mean that the asymptotically stable \(2\pi\)-periodic solutions of (3.3) correspond to those parts of resonance curves under consideration which are
outside the ellipse (3.13) and above the line $A = \pi/2$. All the results are collected in Figure 3.1, where it is easy to see that for any detuning parameter $a$ and any amplitude $\lambda > 0$, (3.1) possesses at least one asymptotically stable $2\pi$-periodic solution with amplitude close to $A$ obtained from (3.12). Among all the asymptotically stable $2\pi$-periodic solutions of (3.1), there exists exactly one whose fixed neighborhood does not contain any nonasymptotically stable $2\pi$-periodic solution of (3.1) for sufficiently small $\varepsilon > 0$. The amplitude of this asymptotically stable $2\pi$-periodic solution is obtained from (3.16)–(3.17).

To compare the changes due to nonsmoothness in the behavior of the resonance curves, we give in Figure 3.2 the resonance curves of the classical van der Pol oscillator (3.2).

\[ A^2 \left( a^2 + \left( 1 - \frac{A^2}{4} \right)^2 \right) = \lambda^2, \]
\[ 1 - a^2 - A^2 + \frac{3}{16} A^4 = 0, \]
and
\[ 1 + a^2 - (M^2 + N^2) + \frac{3}{16} (M^2 + N^2)^2 > 0, \]
\[ 2 - (M^2 + N^2) < 0, \]

Fig. 3.2. Dependence of the amplitude of stable (solid curves) and unstable (dash curves) $2\pi$-periodic solutions of the classical periodically perturbed van der Pol equation (3.2) on the detuning parameter $a$ for different values of $\lambda$. Following Andronov and Witt (see [1, Figure 4]), curve I is plotted with $\lambda = \sqrt{0.4}$, II with $\lambda = 4\sqrt{3}/9$, III with some $4\sqrt{3}/9 < \lambda < \sqrt{32/27}$, IV with $\lambda = \sqrt{32/27}$, and V with $\lambda = 2$. Point P is $2/\sqrt{3}$.

The formulas of Figure 3.1 can be compared with the formulas for Figure 3.2. In fact, the corresponding expressions (3.12)–(3.13) and (3.14)–(3.15) are (see the formulas (5.21)–(5.22) and (16.6)–(16.7) of [30])

\[ A^2 \left( a^2 + \left( 1 - \frac{A^2}{4} \right)^2 \right) = \lambda^2, \]
\[ 1 - a^2 - A^2 + \frac{3}{16} A^4 = 0, \]
respectively, when we consider the classical van der Pol equation (3.2).

It can be checked that the eigenvectors of the matrix \((g_0)'((A \sin \phi, A \cos \phi))\) are orthogonal only for \(A = 0\), so Theorem 3.6 from [8] cannot be applied. At the same time assumption (H2) from [8] is not satisfied for our problem (see Remark 2.8).

4. Appendix.

Proof of Proposition 3.1. As before, \([v]_i\) is the \(i\)th component of the vector \(v \in \mathbb{R}^2\). Let \(g(t, v) = \|(v)_1 \cos t + (v)_2 \sin t\|\), and notice that it is enough to prove that \(g : [0, 2\pi] \times \mathbb{R}^2 \to \mathbb{R}\) satisfies (v). In the case that \(|v_0|_2 \neq 0\), denote \(\theta(v) = \arctan(-[v]_1/[v]_2)\) if \(|v_0|_2 = 0\), denote \(\theta(v) = \arctan([-v]_1/[v]_2)\) if \(|v_0|_1|v|_2 < 0\), \(\theta(v_0) = \pi/2\), and, respectively, \(\theta(v) = \arctan([-v]_1/[v]_2) + \pi\) if \(|v_0|_1|v|_2 > 0\). In any case notice that the function \(v \mapsto \theta(v)\) is continuous in every sufficiently small neighborhood of \(v_0\). Fix \(\tilde{\delta} > 0\). Let \(M\) be the union of two intervals centered in \(\theta(v_0)\) (respectively, \(\theta(v_0) + 2\pi\) if \(\theta(v_0) < 0\)) and in \(\theta(v_0) + \pi\), each of length \(\tilde{\delta}/2\). Denote then \(M_1\) and \(M_2\). Take \(\tilde{\delta} > 0\) such that \(\theta(v) \in M_1\) for all \(v \in B_{\tilde{\delta}}(v_0)\). Of course, also \(\theta(v) + \pi \in M_2\) for all \(\|v - v_0\| \leq \tilde{\delta}\). This implies that for fixed \(t \in [0, 2\pi] \setminus M\), \([v]_1 \cos t + [v]_2 \sin t\) has constant sign for all \(v \in B_{\tilde{\delta}}(v_0)\), which further gives that \(g(t, \cdot)\) is differentiable and \(g'_t(t, v) = g'_t(t, v_0)\) for all \(v \in B_{\tilde{\delta}}(v_0)\). Hence (v) is fulfilled. \(\square\)

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