SIAM J. MATH. ANAL.
Vol. 40, No. 6, pp. 2478–2495

ASYMPTOTIC STABILITY OF PERIODIC SOLUTIONS FOR NONSMOOTH DIFFERENTIAL EQUATIONS WITH APPLICATION TO THE NONSMOOTH VAN DER POL OSCILLATOR[*]

ADRIANA BUICĂ[†], JAUME LLIBRE[‡], AND OLEG MAKARENKOV[§]

Abstract. In this paper we study the existence, uniqueness, and asymptotic stability of the periodic solutions of the Lipschitz system $\dot{x} = \varepsilon g(t, x, \varepsilon)$, where $\varepsilon > 0$ is small. Our results extend the classical second Bogoliubov theorem for the existence of stable periodic solutions to nonsmooth differential systems. As an application we prove the existence of asymptotically stable 2π-periodic solutions of the nonsmooth van der Pol oscillator $\ddot{u} + \varepsilon (|u| - 1)\, \dot{u} + (1 + a\varepsilon)u = \varepsilon\lambda \sin t$. Moreover, we construct the so-called resonance curves that describe the dependence of the amplitude of these solutions as a function of the parameters a and λ. Finally we compare such curves with the resonance curves of the classical van der Pol oscillator $\ddot{u} + \varepsilon \left(u^2 - 1\right) \dot{u} + (1 + a\varepsilon)u = \varepsilon\lambda \sin t$.

Key words. periodic solution, asymptotic stability, averaging theory, nonsmooth differential system, nonsmooth van der Pol oscillator

AMS subject classifications. 34C29, 34C25, 47H11

DOI. 10.1137/070701091

1. Introduction.

In this paper we study the existence, uniqueness, and asymptotic stability of the T-periodic solutions of the system

$$\dot{x} = \varepsilon g(t, x, \varepsilon), \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, and the function $g \in C^0(\mathbb{R} \times \mathbb{R}^k \times [0, 1], \mathbb{R}^k)$ is T-periodic in the first variable and locally Lipschitz with respect to the second. For this class of differential systems, the study of the T-periodic solutions can be made using the averaging function

$$g_0(v) = \int_0^T g(\tau, v, 0)d\tau \tag{1.2}$$

and looking for the periodic solutions that starts near some $v_0 \in g_0^{-1}(0)$.

In the case that g is of class C^1, we recall the stable periodic case of the second Bogoliubov's theorem [6, Chap. 1, section 5, Theorem II] which states *If* $\det (g_0)'(v_0) \neq 0$ *and* $\varepsilon > 0$ *is sufficiently small, then system* (1.1) *has a unique* T*-periodic solution in a neighborhood of* v_0*. Moreover, if all the eigenvalues of the Jacobian matrix*

[*]Received by the editors August 24, 2007; accepted for publication (in revised form) November 8, 2008; published electronically February 25, 2009.
http://www.siam.org/journals/sima/40-6/70109.html

[†]Department of Applied Mathematics, Babeş-Bolyai University, Cluj-Napoca, Romania (abuica@math.ubbcluj.ro).

[‡]Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain (jllibre@mat.uab.cat). This author was partially supported by MEC/FEDER grant MTM2008-03437 and by CICYT grant 2005SGR 00550.

[§]Research Institute of Mathematics, Voronezh State University, Voronezh, Russia (omakarenkov@math.vsu.ru). This author was partially supported by grant BF6M10 of the Russian Federation Ministry of Education and U.S. CRDF (BRHE), by RFBR grant 06-01-72552, by the President of the Russian Federation Young Researcher grant MK-1620.2008.1, and by Marie Curie IIF GA-2008-221331.

$(g_0)'(v_0)$ *have negative real part, then this periodic solution is asymptotically stable.* This theorem has a long history and includes results by Fatou [15], Mandelstam and Papaleksi [31], and Krylov and Bogoliubov [25, section 2].

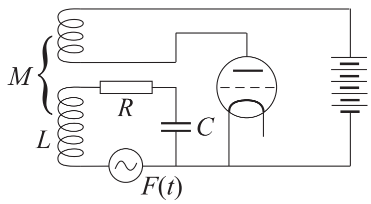

FIG. 1.1. *Circuit scheme for the classical triode oscillator (see* [3, *Chap.* VIII, section 2, *Figure* 348]*,* [30, *Chap.* I, section 5, *Figure* 1]*, and* [36, section 3.1.7, *Figures* 3–5]*).*

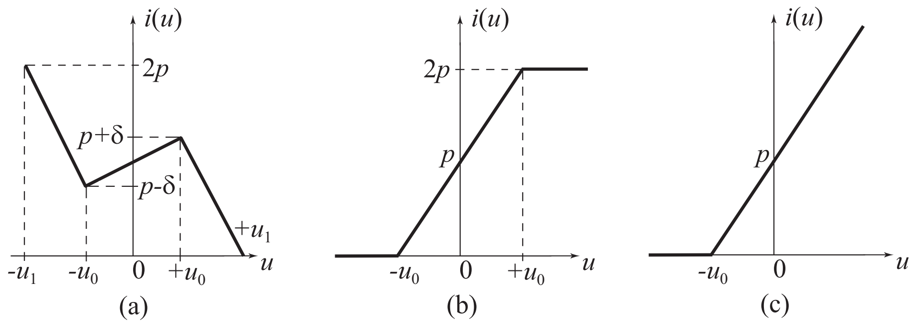

FIG. 1.2. *Characteristics of the triode of the circuit of Figure* 1.1. (a) *Triode in a harsh regime (see* [3, *Chap.* IV, section 7, *Figure* 212b]*,* [30, *Chap.* I, section 5, *comments for eqs.* 5.3–5.4]*);* (b) *triode with saturation (see* [3, *Chap.* VIII, section 3, *Figure* 364]*);* (c) *triode without saturation (see* [3, *Chap.* IX, section 7, *Figure* 482]*).*

The Bogoliubov result provided a theoretical justification for some resonance phenomena which appear in many real physical systems. One of the most significant examples is the classical triode oscillator whose scheme is drawn in Figure 1.1 and whose current u is described by the second order differential equation

$$\ddot{u} + \frac{1}{LC}\left(RC - Mi'(u)\right)\dot{u} + \omega^2 u = \frac{1}{LC}F(t), \tag{1.3}$$

where $R = \varepsilon R_0$, $M = \varepsilon M_0$, $\omega^2 = 1 + \varepsilon b$, $F(t) = \varepsilon\lambda \sin t$, $\varepsilon > 0$, is assumed to be small and the triode characteristic $i(u)$ is drawn in Figure 1.2(a). The analysis of the diagram of bifurcation of the periodic solutions in this system is performed in almost every book on nonlinear oscillations (see Andronov, Witt, and Khaikin [3, Chap. VIII, section 2], Malkin [30, Chap. I, section 5], and Nayfeh and Mook [36, section 3.1.7]) but with the smooth approximation $i(u) = i_{(a)}(u) = S_0 + S_1 u - \frac{1}{3}S_3 u^3$ (leading to the classical van der Pol equation). Therefore it is natural to look for a technique that permits one to avoid this smooth approximation and allows one to work with the original shape of the triode characteristic drawn in Figure 1.2(a).

Though the unforced equation (1.3) (i.e., for $F = 0$) with i described by Figure 1.2(b) and Figure 1.2(c) is well studied (see [3, Chap. VIII, section 3 and Chap. IX, section 7]), the question about resonances in these equations when $F \neq 0$ (e.g., $F(t) = \varepsilon\lambda \sin t$) is still partially open. In this direction Levinson [29] uncovered a family of solutions of (1.3) of remarkable singular structure and Levi [28] completed the study of the limit behavior of all solutions. The present paper complements these results by describing the location of asymptotically stable periodic solutions of (1.3).

Studying (1.3) with the triode characteristic given by Figure 1.2(a), 1.2(b), or 1.2(c) we finally note that there exists a change of variables (see, for example, how Levinson changed equation 2.0 in [29]) that allows one to rewrite (1.3) into the form (1.1) with some function g that is not C^1 but is Lipschitz with respect to the second variable. Therefore the goal of this work is to generalize the results on the existence of a stable periodic solution of the second Bogoliubov theorem to the case that the function g of (1.1) is only Lipschitz.

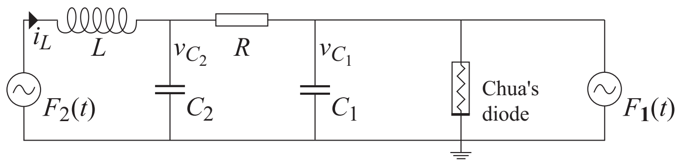

FIG. 1.3. *Forced Chua's circuit (see* [5, 11, 20, 35, 40]*).*

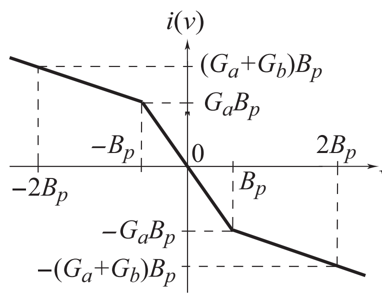

FIG. 1.4. *Nonlinear characteristic of the Chua's diode of the circuit drawn at Figure* 1.3 *given by* $i(v) = G_b v + (1/2)(G_a - G_b)\,(|v + B_p| - |v - B_p|)$, *where* $G_a, G_b, B_p \in \mathbb{R}$ *are some constants depending on the properties of the Chua's diode (see* [10]*).*

Another motivation of this paper comes from the forced Chua's circuit (see Figure 1.3) studied in a large number of papers in the modern electrical engineering. This circuit is described by the three-dimensional system

$$\begin{aligned} C_1 \frac{dv_{C_1}}{dt} &= \frac{v_{C_2} - v_{C_1}}{R} - i(v_{C_1}) + F_1(t), \\ C_2 \frac{dv_{C_2}}{dt} &= \frac{v_{C_1} - v_{C_2}}{R} + i_L, \\ L \frac{di_L}{dt} &= -v_{C_2} + F_2(t, v_{C_2}), \end{aligned} \tag{1.4}$$

where $i(v)$ (the characteristic of the Chua's diode) is a piecewise linear function, as it is represented in Figure 1.4. The recent literature provides insight into the numerical simulations of (1.4) (see [40, 20], where $F_1 \neq 0$ and $F_2 \neq 0$, [5, 35], where $F_1 = 0$ and F_2 is periodic, or [11] where both F_1 and F_2 are periodic). Generalization of the Bogoliubov result for (1.1) with Lipschitz right-hand part will allow for the first time the theoretical detection of asymptotically stable periodic solutions of (1.4) in the case that C_1 is large enough. Eventually this theoretical analysis may provide new interesting parameters of the forced Chua's circuit for doing additional numerical experiments.

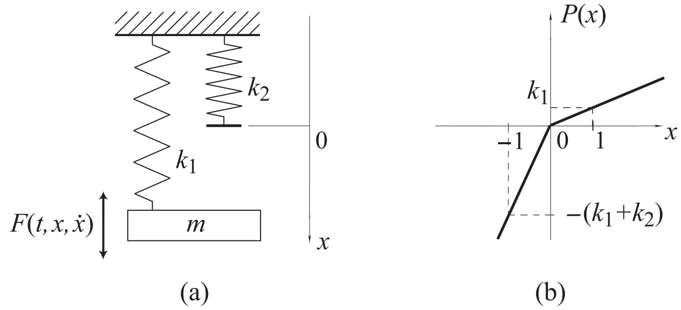

FIG. 1.5. *A prototypic device presented in* (a) *where a driven mass is attached to a immovable beam via a spring with piecewise linear stiffness like in* (b)*; see, e.g.,* [7], [24, *Chap.* I, *p.* 16 *and Chap.* IV, *p.* 100]*, and* [38].

On the other hand, part of the interest in generalizing the Bogoliubov result comes from mechanics, where differential systems with piecewise linear stiffness describe various oscillating processes. One of these systems is exhibited by the device drawn in Figure 1.5(a), where a forced mass is attached to a spring whose stiffness changes from k_1 to $k_1 + k_2$ when the mass coordinate crosses 0 in the negative direction. This device is governed by the second order differential equation

$$m\ddot{x} + P(x) = F(t, x, \dot{x}), \tag{1.5}$$

where the piecewise linear stiffness P is drawn in Figure 1.5(b). Depending on the particular configuration of the device of Figure 1.5(a), different expressions for F in (1.5) must be considered. Thus we have that $F(t, x, \dot{x}) = -f(x)\dot{x} + M \cos \omega t$ with piecewise constant f for a shock-absorber and jigging conveyor (see [24, Chap. I, p. 16 and Chap. IV, p. 100]), where the original Bogoliubov result is employed without justification). The function F takes the simpler form $F(t, x, \dot{x}) = -c\dot{x} + MQ(t)$ for an impact resonator, and $F(t, x, \dot{x}) = -c\dot{x} + M \sin \omega t$ for a cracked-body model (see [38, 7], where only numerical experiments are performed). In each of these situations (1.5) can be rewritten in the form (1.1) with g Lipschitz, provided that the constant k_2 and the amplitude of the force F are sufficiently small. Therefore the extension of the Bogoliubov result to the nonsmooth case that we shall do will allow one to justify the resonances that appeared in all these results. We note that the recent report by Los Alamos National Laboratory [13] describes the increasing interest in a specific form of the model of Figure 1.5(a) called the cracked-body model and, particularly, in the suspension bridge models. Consequently the results of this paper can be applied to such models.

A first model of a one-dimensional suspended bridge is drawn in Figure 1.6(a). It is represented (see [16, 27]) by the beam bending under its own weight and being supported by cables whose restoring force due to elasticity is proportional to u^+ (see Figure 1.6(b)), where $u = u(t, x)$ is the displacement of a point at a distance x from one end of the bridge at time t and u is measured in the downward direction. Looking for u of the form $u(t, x) = z(t) \sin(\pi x/L)$ and considering $F(x, t) = h(t) \sin(\pi x/L)$,

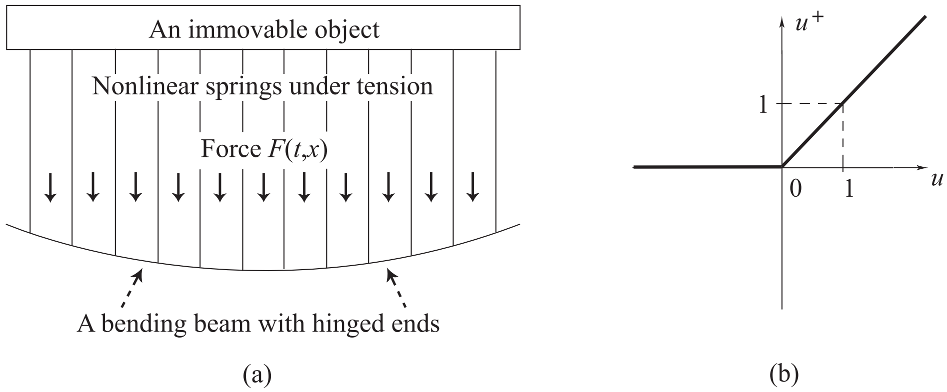

FIG. 1.6. (a) *The first idealization of the suspension bridge: the beam bending under its own weight is supported by the nonlinear cables (see* [27, *Figure* 2]*);* (b) *characteristic of stiffness of nonlinear springs.*

we arrive (see [16]) at the following particular case of differential equation (1.5):

$$m\ddot{z} + \delta\dot{z} + c(\pi/L)^4 z + dz^+ = mg + h(t), \tag{1.6}$$

where the constant $m > 0$ is the mass per unit of length, $\delta > 0$ is a small viscous damping coefficient, $c > 0$ measures the flexibility or stiffness of the bridge, $L > 0$ is the length of the bridge, $d > 0$ represents the stiffness of nonlinear springs, and h is a continuous T-periodic force modelling wind, marching troops, or cattle (see [19] for details). Considering $c > 0$ and $d > 0$ fixed and assuming that either $c > 0$ and $h(t)$ are sufficiently small, or that $c > 0$ is fixed and $h(t)$ is sufficiently large, or that $c > 0$ is sufficiently small and $h(t)$ fixed, Glover, Lazer, and McKenna [16], Lazer and McKenna [27], and Fabry [14] proved various theorems on the location of asymptotically stable T-periodic solutions in (1.6). The question *What happens with these solutions when $d > 0$, $\delta > 0$, and $h(t)$ are all sufficiently small?* was open and can be solved using the generalization of the Bogoliubov result that we provide. Lazer and McKenna proved in [26] that the Poincaré map for (1.6) is differentiable, but we note that this is not sufficient for applying the original Bogoliubov result.

We end the list of possible applications noting that system (1.4) describing the Chua's circuit (Figure 1.3) appeared recently for studying the so-called negative slope mechanical systems (see Awrejcewicz [4, section 8.2.2]). So our results can also be applied to these mechanical systems.

These applications require generalizations of the second Bogoliubov theorem for Lipschitz right-hand parts. To the best of our knowledge Mitropol'skii was the first to consider such a kind of generalization. Assuming that g is Lipschitz, $g_0 \in C^3(\mathbb{R}^k, \mathbb{R}^k)$, and all the eigenvalues of the matrix $(g_0)'(v_0)$ have negative real part, Mitropol'skii [34] developed the Bogoliubov result proving the existence and uniqueness of a T-periodic solution of system (1.1) in a neighborhood of v_0. There was great progress weakening the assumptions of the existence result (see Samoilenko [39] and Mawhin [32]), but this progress did not take place in the case of the uniqueness. Moreover, the asymptotic stability of the T-periodic solution remained unstudied in the case of Lipschitz systems for a long time. It has been done recently by Buică and Daniilidis in [8] for Lipschitz systems (1.1) assuming that the function $v \mapsto g(t, v, 0)$ is differentiable at v_0 for almost any $t \in [0, T]$ and that the eigenvectors of the matrix $(g_0)'(v_0)$ are orthogonal.

In section 2, assuming that g is piecewise differentiable in the second variable, we prove in Theorem 2.5 that the stable periodic solution of the Bogoliubov theorem persists when g is not necessary C^1. Theorem 2.5 follows from this more general Theorem 2.1 whose hypotheses do not use any differentiability—neither of g, nor of g_0. Assuming only continuity for g, we show in Theorem 2.9 the existence of a nonasymptotically stable T-periodic solution of system (1.1) if the Brouwer topological degree of $-g_0$ is negative. In section 3 we illustrate our results constructing the resonance curves of the nonsmooth van der Pol oscillator, also studied in [18], and compare these curves with the resonance curves of the classical van der Pol oscillator, which were constructed by Andronov and Witt [1, 2].[1]

2. Main results. Throughout the paper $\Omega \subset \mathbb{R}^k$ will be an open set. For any $\delta > 0$ we denote $B_\delta(v_0) = \left\{v \in \mathbb{R}^k \, : \, \|v - v_0\| \le \delta\right\}$. We have the following main result on the existence, uniqueness, and asymptotic stability of T-periodic solutions for system (1.1).

THEOREM 2.1. *Let $g \in C^0(\mathbb{R} \times \Omega \times [0,1], \mathbb{R}^k)$ and $v_0 \in \Omega$. Assume the following four conditions.*

(i) *For some $L > 0$ we have that $\|g(t, v_1, \varepsilon) - g(t, v_2, \varepsilon)\| \le L\,\|v_1 - v_2\|$ for any $t \in [0,T],\ v_1, v_2 \in \Omega,\ \varepsilon \in [0,1]$.*

(ii) *For any $\gamma > 0$ there exists $\delta > 0$ such that*

$$\left\| \int_0^T g(\tau, v_1 + u(\tau), \varepsilon) d\tau - \int_0^T g(\tau, v_2 + u(\tau), \varepsilon) d\tau \right.$$
$$\left. - \int_0^T g(\tau, v_1, 0) d\tau + \int_0^T g(\tau, v_2, 0) d\tau \right\| \le \gamma \|v_1 - v_2\|$$

for any $u \in C^0([0,T], \mathbb{R}^k)$, $\|u\| \le \delta$, $v_1, v_2 \in B_\delta(v_0)$, and $\varepsilon \in [0,\delta]$.

(iii) *Let g_0 be the averaged function given by (1.2) and consider that $g_0(v_0) = 0$.*

(iv) *There exist $q \in [0,1)$, $\alpha, \delta_0 > 0$, and a norm $\|\cdot\|_0$ on $\mathbb{R}^k$ such that $\|v_1 + \alpha g_0(v_1) - v_2 - \alpha g_0(v_2)\|_0 \le q \|v_1 - v_2\|_0$ for any $v_1, v_2 \in B_{\delta_0}(v_0)$.*

Then there exists $\delta_1 > 0$ such that for every $\varepsilon \in (0, \delta_1]$, system (1.1) has exactly one T-periodic solution x_ε with $x_\varepsilon(0) \in B_{\delta_1}(v_0)$. Moreover, the solution x_ε is asymptotically stable and $x_\varepsilon(0) \to v_0$ as $\varepsilon \to 0$.

When the solution $x(\cdot, v, \varepsilon)$ of system (1.1) with the initial condition $x(0, v, \varepsilon) = v$ is well defined on $[0,T]$ for any $v \in B_{\delta_0}(v_0)$, the map $v \mapsto x(T, v, \varepsilon)$ is also well defined and is called the *Poincaré map* at time T of system (1.1). In order to prove the existence, uniqueness, and stability of the T-periodic solutions of system (1.1) stated in Theorem 2.1, it is sufficient to study the same properties for the fixed points of this Poincaré map.

Before proving Theorem 2.1 we state and prove two lemmas. In order to state the first lemma, we need to introduce the function

$$g_\varepsilon(v) = \int_0^T g(\tau, x(\tau, v, \varepsilon), \varepsilon) d\tau$$

[1] At the final stage of publishing of this paper, we have been informed by Prof. Michael Guevara that the resonance curves mentioned appeared for the first time in [Balth. van der Pol, *Tijdschr. Ned Rad Gen.*, (1924) (in Dutch)], an English translation appeared in [*Phil. Mag.*, vol. 3, 1927, p. 65]. We thank Prof. Guevara for calling our attention to those papers and some other historical background on resonance curves.

and to note that by writing the equivalent integral equation of system (1.1) we have

$$x(T, v, \varepsilon) = v + \varepsilon g_\varepsilon(v).$$

LEMMA 2.2. *Let $g \in C^0(\mathbb{R}\times\Omega\times[0,1],\mathbb{R}^k)$ and $\delta_0 > 0$ be such that $B_{\delta_0}(v_0) \subset \Omega$. If* (i) *is satisfied, then there exist $\delta \in [0, \delta_0]$ and $L_1 > 0$ such that the map $(v, \varepsilon) \mapsto g_\varepsilon(v)$ is well defined and continuous on $B_{\delta_0}(v_0) \times [0, \delta]$ and*

$$\|g_\varepsilon(v_1) - g_\varepsilon(v_2)\| \le L_1\|v_1 - v_2\| \quad \textit{for any } \varepsilon \in [0,\delta],\ v_1, v_2 \in B_{\delta_0}(v_0).$$

If both (i) *and* (ii) *are satisfied, then for any $\gamma > 0$ there exists $\delta \in [0, \delta_0]$ such that*

$$\|g_\varepsilon(v_1) - g_0(v_1) - g_\varepsilon(v_2) + g_0(v_2)\| \le \gamma\|v_1 - v_2\|$$

for any $v_1, v_2 \in B_\delta(v_0)$ and $\varepsilon \in [0, \delta]$.

Proof. Using the continuity of the solution of a differential system with respect to the initial data and the parameter (see [37, Chap. 4, section 23, statements G and D]), we obtain the existence of $\varepsilon_0 > 0$ such that $x(t, v, \varepsilon) \in \Omega$ for any $t \in [0, T]$, $v \in B_{\delta_0}(v_0)$, and $\varepsilon \in [0, \varepsilon_0]$. Using the Grönwall–Bellman lemma (see [17, Lemma 6.2] or [12, Chap. II, section 11]) from the representation $x(t, v, \varepsilon) = v + \varepsilon \int_0^t g(\tau, x(\tau, v, \varepsilon), \varepsilon)d\tau$ and the property (i), we obtain $\|x(t, v_1, \varepsilon) - x(t, v_2, \varepsilon)\| \le e^{\varepsilon LT}\|v_1 - v_2\|$ for all $t \in [0, T]$, v_1, $v_2 \in B_{\delta_0}(v_0)$, and $\varepsilon \in [0, \varepsilon_0]$. Therefore $y(t, v, \varepsilon) = \int_0^t g(\tau, x(\tau, v, \varepsilon), \varepsilon)d\tau$ satisfies the property

$$\|y(t, v_1, \varepsilon) - y(t, v_2, \varepsilon)\| \le L_1\|v_1 - v_2\| \tag{2.1}$$

for all $t \in [0, T]$, v_1, $v_2 \in B_{\delta_0}(v_0)$, $\varepsilon \in [0, \varepsilon_0]$, and $L_1 = LTe^{\varepsilon_0 LT}$. Since $g_\varepsilon(v) = y(T, v, \varepsilon)$ the first part of the lemma has been proven.

Taking into account that $x(t, v, \varepsilon) = v + \varepsilon y(t, v, \varepsilon)$, we have

$$y(T, v_1, \varepsilon) - y(T, v_1, 0) - y(T, v_2, \varepsilon) + y(T, v_2, 0) = I_1(v_1, v_2, \varepsilon) + I_2(v_1, v_2, \varepsilon), \tag{2.2}$$

where

$$I_1(v_1, v_2, \varepsilon) = \int_0^T [g(\tau, v_2 + \varepsilon y(\tau, v_1, \varepsilon), \varepsilon) - g(\tau, v_2 + \varepsilon y(\tau, v_2, \varepsilon), \varepsilon)]d\tau,$$

$$I_2(v_1, v_2, \varepsilon) = \int_0^T [(g(\tau, v_1 + \varepsilon y(\tau, v_1, \varepsilon), \varepsilon) - g(\tau, v_2 + \varepsilon y(\tau, v_1, \varepsilon), \varepsilon))]d\tau$$
$$- \int_0^T (g(\tau, v_1, 0) - g(\tau, v_2, 0))d\tau.$$

Since $(t, v, \varepsilon) \mapsto y(t, v, \varepsilon)$ is bounded on $[0, T] \times B_{\delta_0}(v_0) \times [0, \varepsilon_0]$, we have that $\varepsilon y(t, v, \varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly with respect to $t \in [0, T]$ and $v \in B_{\delta_0}(v_0)$. Decreasing $\varepsilon_0 > 0$, if necessary, we get that $v_2 + \varepsilon y(t, v_1, \varepsilon) \in \Omega$ for any $t \in [0, T]$, $v_1, v_2 \in B_{\delta_0}(v_0)$, $\varepsilon \in [0, \varepsilon_0]$. By assumption (i) and relation (2.1) we obtain that $\|I_1(v_1, v_2, \varepsilon)\| \le T \cdot \varepsilon LL_1\|v_1 - v_2\|$ for all $\varepsilon \in [0, \varepsilon_0]$, $v_1, v_2 \in B_{\delta_0}(v_0)$.

We fix $\gamma > 0$ and take $\delta > 0$ given by (ii). Without loss of generality we can consider that $\delta \le \min\{\delta_0, \varepsilon_0, \gamma/(2TLL_1)\}$. Therefore assumption (ii) implies that $\|I_2(v_1, v_2, \varepsilon)\| \le (\gamma/2)\|v_1 - v_2\|$ for any $\varepsilon \in [0, \delta]$, $v_1, v_2 \in B_\delta(v_0)$. Substituting the obtained estimations for I_1 and I_2 into (2.2) we have $\|y(T, v_1, \varepsilon) - y(T, v_1, 0) - y(T, v_2, \varepsilon)+$

$y(T, v_2, 0)\| \leq (\varepsilon TLL_1 + \gamma/2)\|v_1 - v_2\| \leq \gamma\|v_1 - v_2\|$ for any $\varepsilon \in [0, \delta]$, $v_1, v_2 \in B_\delta(v_0)$. Hence the proof is complete. □

LEMMA 2.3. *Let $g_0 : \Omega \to \mathbb{R}^k$, satisfying assumption* (iv) *for some $q \in (0, 1)$, $\alpha, \delta_0 > 0$, and a norm $\|\cdot\|_0$ on $\mathbb{R}^k$. Then $\|v_1 + \varepsilon g_0(v_1) - v_2 - \varepsilon g_0(v_2)\|_0 \leq (1 - \varepsilon(1 - q)/\alpha)\,\|v_1 - v_2\|_0$ for any $v_1, v_2 \in B_{\delta_0}(v_0)$ and any $\varepsilon \in [0, \alpha]$.*

Proof. Indeed, the equality $v + \varepsilon g_0(v) = (1 - \varepsilon/\alpha)v + \varepsilon/\alpha\,(v + \alpha g_0(v))$ implies that the Lipschitz constant of the function $I + \varepsilon g_0$ with respect to the norm $\|\cdot\|_0$ is $(1 - \varepsilon/\alpha) + \varepsilon/\alpha\, q = 1 - \varepsilon(1 - q)/\alpha$. □

Proof of Theorem 2.1. By Lemma 2.2 we have that there exists $\delta_1 \in [0, \delta_0]$ such that

$$\|g_\varepsilon(v_1) - g_0(v_1) - g_\varepsilon(v_2) + g_0(v_2)\|_0 \leq ((1 - q)/(2\alpha))\|v_1 - v_2\|_0 \tag{2.3}$$

for any $\varepsilon \in [0, \delta_1]$, $v_1, v_2 \in B_{\delta_1}(v_0)$. First we prove that there exists $\varepsilon_1 \in [0, \delta_1]$ such that for every $\varepsilon \in [0, \varepsilon_1]$ there exists $v_\varepsilon \in B_{\delta_1}(v_0)$ such that $x(\cdot, v_\varepsilon, \varepsilon)$ is a T-periodic solution of (1.1) by showing that there exists v_ε such that $x(T, v_\varepsilon, \varepsilon) = v_\varepsilon$. Using (iii) and (iv) we have

$$\|v + \alpha g_0(v) - v_0\|_0 \leq q\|v - v_0\|_0 \quad \text{for any } v \in B_{\delta_1}(v_0).$$

Therefore we have that the map $I + \alpha g_0$ maps $B_{\delta_1}(v_0)$ into itself. From Lemma 2.2 we have that there exists $\varepsilon_0 > 0$ such that the map $(v, \varepsilon) \mapsto g_\varepsilon(v)$ is well defined and continuous on $B_{\delta_1}(v_0) \times [0, \varepsilon_0]$. We deduce that there exists $\varepsilon_1 > 0$ sufficiently small such that, for every $\varepsilon \in [0, \varepsilon_1]$, the map $I + \alpha g_\varepsilon$ maps $B_{\delta_1}(v_0)$ into itself as well. Therefore, by the Brouwer theorem (see, for example, [23, Theorem 3.1]) we have that $B_{\delta_1}(v_0)$ contains at least one fixed point of the map $I + \alpha g_\varepsilon$ for any $\varepsilon \in [0, \varepsilon_1]$. Denote this fixed point by v_ε. Then we have $g_\varepsilon(v_\varepsilon) = 0$ and $x(T, v_\varepsilon, \varepsilon) = v_\varepsilon$ for any $\varepsilon \in [0, \varepsilon_1]$.

Now we prove that $x(\cdot, v_\varepsilon, \varepsilon)$ is the only T-periodic solution of (1.1) starting near v_0 and that, moreover, it is asymptotically stable. Knowing that $x(T, v, \varepsilon) = v + \varepsilon g_\varepsilon(v)$ we write the following identity:

$$x(T, v, \varepsilon) = v + \varepsilon g_0(v) + \varepsilon\,(g_\varepsilon(v) - g_0(v))\,. \tag{2.4}$$

Using Lemma 2.3 we have from (2.3) and (2.4) that

$$\begin{aligned}\|x(T, v_1, \varepsilon) - x(T, v_2, \varepsilon)\|_0 \leq{}& (1 - \varepsilon(1 - q)/\alpha + \varepsilon(1 - q)/(2\alpha))\|v_1 - v_2\|_0\\ ={}& (1 - \varepsilon(1 - q)/(2\alpha))\|v_1 - v_2\|_0\end{aligned}$$

for all $v_1, v_2 \in B_{\delta_1}(v_0)$ and $\varepsilon \in [0, \delta_1]$. We proved before that there exists $\varepsilon_1 > 0$ such that for every $\varepsilon \in [0, \varepsilon_1]$ there exists $v_\varepsilon \in B_{\delta_1}(v_0)$ with $x(\cdot, v_\varepsilon, \varepsilon)$ a T-periodic solution of (1.1). Since $\varepsilon(1 - q)/(2\alpha) > 0$ and $\varepsilon_1 \leq \delta_1$, the last inequality implies that for each $\varepsilon \in [0, \delta_1]$, the T-periodic solution $x(\cdot, v_\varepsilon, \varepsilon)$ is the only T-periodic solution of (1.1) in $B_{\delta_1}(v_0)$ and, moreover (see [23, Lemma 9.2]), it is asymptotically stable. □

Remark 2.4. We note that a similar result close to Theorem 2.1 is contained in [8, Theorem 3.5]. But instead of the assumption (iv) with a fixed $\alpha > 0$, it is assumed in [8] with any $\alpha > 0$ sufficiently small. Anyway, notice that Lemma 2.3 implies that it is the same to assume (iv) for only one $\alpha > 0$ or for all $\alpha > 0$ sufficiently small. The advantage of our Theorem 2.1 is that it does not require differentiability of $g(t, \cdot, \varepsilon)$ at any point, while [8] needs it at v_0. See also Remark 2.8.

In general it is not easy to check assumptions (ii) and (iv) in the applications of Theorem 2.1. Thus we also give the following theorem based on Theorem 2.1 which assumes certain type of piecewise differentiability instead of (ii) and deals with properties of the matrix $(g_0)'(v_0)$ instead of the Lipschitz constant of g_0.

For any set $M \subset [0,T]$ measurable in the sense of Lebesgue we denote by $\mathrm{mes}(M)$ the Lebesgue measure of M (see [21, Chap. V, section 3]).

THEOREM 2.5. *Let $g \in C^0(\mathbb{R}\times\Omega\times[0,1],\mathbb{R}^k)$ satisfying* (i). *Let g_0 be the averaged function given by* (1.2) *and consider $v_0 \in \Omega$ such that $g_0(v_0) = 0$. Assume that*

(v) *given any $\widetilde{\gamma} > 0$ there exist $\widetilde{\delta} > 0$ and $M \subset [0,T]$ with $\mathrm{mes}(M) < \widetilde{\gamma}$ such that for every $v \in B_{\widetilde{\delta}}(v_0)$, $t \in [0,T]\setminus M$, and $\varepsilon \in [0,\widetilde{\delta}]$ we have that $g(t,\cdot,\varepsilon)$ is differentiable at v and $\|g'_v(t,v,\varepsilon) - g'_v(t,v_0,0)\| \le \widetilde{\gamma}$.*

Finally assume that

(vi) *g_0 is continuously differentiable in a neighborhood of v_0 and the real parts of all the eigenvalues of $(g_0)'(v_0)$ are negative.*

Then there exists $\delta_1 > 0$ such that for every $\varepsilon \in (0,\delta_1]$, system (1.1) *has exactly one T-periodic solution x_ε with $x_\varepsilon(0) \in B_{\delta_1}(v_0)$. Moreover, the solution x_ε is asymptotically stable and $x_\varepsilon(0) \to v_0$ as $\varepsilon \to 0$.*

For proving Theorem 2.5 we need two preliminary lemmas.

LEMMA 2.6. *Let $g \in C^0(\mathbb{R}\times\Omega\times[0,1],\mathbb{R}^k)$ satisfying* (i). *If* (v) *holds, then* (ii) *is satisfied.*

Proof. Let $\gamma > 0$ be an arbitrary number. We show that (ii) holds with $\delta = \widetilde{\delta}/2$, where $\widetilde{\delta}$ is given by (v) applied with $\widetilde{\gamma} = \min\{\gamma/(4L), \gamma/(4T)\}$. We consider also $M \subset [0,T]$ given by (v) applied with the same value of $\widetilde{\gamma}$.

Let $u \in C^0([0,T],\mathbb{R}^k)$, $\|u\| \le \delta$, and $F(v) = \int_0^T g(\tau, v+u(\tau),\varepsilon)d\tau - \int_0^T g(\tau,v,0)d\tau$. Let $v_1, v_2 \in B_\delta(v_0)$ and $\varepsilon \in [0,\delta]$. We have $F(v) = F_1(v) + F_2(v)$, where $F_1(v) = \int_M (g(\tau,v+u(\tau),\varepsilon) - g(\tau,v,0))d\tau$ and $F_2(v) = \int_{[0,T]\setminus M}(g(\tau,v+u(\tau),\varepsilon) - g(\tau,v,0))d\tau$. By (i) we have that $\|F_1(v_1) - F_1(v_2)\| \le 2L\cdot\mathrm{mes}(M)\|v_1 - v_2\| < 2L\widetilde{\gamma}\|v_1 - v_2\| \le (\gamma/2)\|v_1 - v_2\|$. On the other hand, using (v), we will prove that a similar relation holds for F_2. In order to do this, we denote $h(\tau,v) = g(\tau,v+u(\tau),\varepsilon) - g(\tau,v,0)$. Notice that for each $\tau \in [0,T]\setminus M$ we can write $h'_v(\tau,v) = (g'_v(\tau,v+u(\tau),\varepsilon) - g'_v(\tau,v_0,0)) - (g'_v(\tau,v,0) - g'_v(\tau,v_0,0))$. As a direct consequence of (v) we deduce that $\|h'_v(\tau,v)\| \le 2\widetilde{\gamma}$ for all $v \in B_\delta(v_0)$ and $\tau \in [0,T]\setminus M$. Now applying the mean value theorem for the function $h(\tau,\cdot)$, we have $\|h(\tau,v_1) - h(\tau,v_2)\| \le 2\widetilde{\gamma}\|v_1 - v_2\|$ for all $\tau \in [0,T]\setminus M$ and all $v_1, v_2 \in B_\delta(v_0)$. Then $\|F_2(v_1) - F_2(v_2)\| \le \int_{[0,T]\setminus M}\|h(\tau,v_1) - h(\tau,v_2)\|d\tau \le 2T\widetilde{\gamma}\|v_1 - v_2\| \le (\gamma/2)\|v_1 - v_2\|$. Therefore we have proved that $\|F(v_1) - F(v_2)\| \le \gamma\|v_1 - v_2\|$, which coincides with (ii). ∎

LEMMA 2.7. *Let $g_0 : \Omega \to \mathbb{R}^k$ satisfying assumption* (vi) *for some $v_0 \in \Omega$. Then there exist $q \in [0,1)$, α, $\delta_0 > 0$ and a norm $\|\cdot\|_0$ on $\mathbb{R}^k$ such that* (iv) *is satisfied.*

Proof. If λ is an eigenvalue of $\alpha(g_0)'(v_0)$, then $\lambda + 1$ is an eigenvalue of $I + (\alpha g_0)'(v_0)$. Since the eigenvalues of $\alpha(g_0)'(v_0)$ tend to 0 as $\alpha \to 0$ and have negative real parts, then there exists $\alpha \in [0,1)$ such that the absolute values of all the eigenvalues of $I + \alpha(g_0)'(v_0)$ are less than one. Therefore (see [22, p. 90, Lemma 2.2]) there exist $\widetilde{q} \in [0,1)$ and a norm $\|\cdot\|_0$ on $\mathbb{R}^k$ such that $\sup_{\|\xi\|_0 \le 1}\|\xi + \alpha(g_0)'(v_0)\xi\|_0 \le \widetilde{q}$.

By continuous differentiability of g_0 in a neighborhood of v_0 we have that $\|g_0(v_1) - g_0(v_2) - (g_0)'(v_0)(v_1 - v_2)\| / \|v_1 - v_2\| \le \|g_0(v_1) - g_0(v_2) - (g_0)'(v_2)(v_1 - v_2)\| + \|(g_0)'(v_2)(v_1 - v_2) - (g_0)'(v_0)(v_1 - v_2)\| / \|v_1 - v_2\| \to 0$ as $\max\{\|v_1 - v_0\|, \|v_2 - v_0\|\} \to 0$. Therefore taking into account that all norms on $\mathbb{R}^k$ are equivalent, there exists $\delta_0 > 0$ such that $\|g_0(v_1) - g_0(v_2) - (g_0)'(v_0)(v_1 - v_2)\|_0 \le (1 - \widetilde{q})/(2\alpha)\,\|v_1 - v_2\|_0$ for all

$v_1, v_2 \in B_{\delta_0}(v_0)$. Then

$$\begin{aligned}
&\|v_1+\alpha g_0(v_1) - v_2 - \alpha g_0(v_2)\|_0 \\
&\quad \le \alpha\|g_0(v_1) - g_0(v_2) - (g_0)'(v_0)(v_1 - v_2)\|_0 + \|v_1 - v_2 + \alpha(g_0)'(v_0)(v_1 - v_2)\|_0 \\
&\quad \le (1 + \widetilde{q})/2\,\|v_1 - v_2\|_0
\end{aligned}$$

for all $v_1, v_2 \in B_{\delta_0}(v_0)$. ◻

Proof of Theorem 2.5. Lemmas 2.6 and 2.7 imply that assumptions (ii) and (iv) of Theorem 2.1 are satisfied. Therefore the conclusion of the theorem follows by applying Theorem 2.1. ◻

It was observed by Mitropol'skii in [34] that in spite of the fact that $g(t, \cdot, \varepsilon)$ in (1.1) is only Lipschitz, sometimes the function g_0 turns out to be differentiable in applications. In particular we will see in section 3 that this is the case for the nonsmooth van der Pol oscillator.

Clearly if $g \in C^1(\mathbb{R} \times \mathbb{R}^k \times [0, 1], \mathbb{R}^k)$, then (i) and (v) hold in any open bounded set $\Omega \subset \mathbb{R}^k$. Therefore Theorem 2.5 is a generalization of the stable periodic case of the second Bogoliubov theorem formulated in the introduction.

Remark 2.8. Theorem 2.5 does not require the eigenvectors of $(g_0)'(v_0)$ to be orthogonal as in [8, Theorem 3.6]. Moreover, assumption (H_2) of [8] is more restrictive than (v).

For completeness we also give the following theorem on the existence of nonasymptotically stable T-periodic solutions for (1.1). In the theorem below, $d(F, V)$ denotes the Brouwer topological degree of the vector field $F \in C^0(\mathbb{R}^k, \mathbb{R}^k)$ on the open and bounded set $V \subset \mathbb{R}^k$ (see [23, Chap. 2, section 5.2]).

THEOREM 2.9. *Let $g \in C^0(\mathbb{R} \times \mathbb{R}^k \times [0, 1], \mathbb{R}^k)$. Assume that there exists an open bounded set $V \subset \mathbb{R}^k$ such that $g_0(v) \neq 0$ for any $v \in \partial V$ and*

(vii) $d(-g_0, V) < 0$.

Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ system (1.1) *has at least one nonasymptotically stable T-periodic solutions x_ε with $x_\varepsilon(0) \in V$.*

Proof. Since $g_0(v) \neq 0$ for any $v \in \partial V$, then from Mawhin's theorem [32] (or [33, section 5]) we have that there exists $\varepsilon_0 > 0$ such that

$$d(-g_0, V) = d(I - x(T, \cdot, \varepsilon), V) \quad \text{for any } \varepsilon \in (0, \varepsilon_0]. \tag{2.5}$$

By [23, Theorem 9.6] for any asymptotically stable T-periodic solution x_ε of (1.1) we have that $d(I - x(T, \cdot, \varepsilon), B_\delta(x_\varepsilon(0))) = 1$ for $\delta > 0$ sufficiently small. Therefore if all the possible T-periodic solutions of (1.1) with $\varepsilon \in (0, \varepsilon_0]$ had been asymptotically stable, then the degree $d(I - x(T, \cdot, \varepsilon), V)$ would have been nonnegative, contradicting (vii) and (2.5). ◻

Remark 2.10. Assumptions (iii) and (iv) imply that $d(-g_0, V) = 1$ (see [23, Theorem 5.16]).

Finally thinking in terms of the application to the nonsmooth van der Pol oscillator, we formulate the following theorem which combines Mawhin's theorem (see [32] or [33, Theorem 3]) and Theorems 2.5 and 2.9. In this theorem $([g_0]_i)'_{(j)}$ stays for the derivative of the ith component of the function g_0 with respect to the jth variable.

THEOREM 2.11. *Let $g \in C^0(\mathbb{R} \times \Omega \times [0, 1], \mathbb{R}^2)$. Let $v_0 \in \Omega$ be such a point that $g_0(v_0) = 0$ and g_0 is continuously differentiable in a neighborhood of v_0.*

(a) *If $\det (g_0)'(v_0) \neq 0$, then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ system* (1.1) *has at least one T-periodic solution x_ε satisfying $x_\varepsilon(0) \to v_0$ as $\varepsilon \to 0$.*

(b) *If* (*i*) *and* (*v*) *hold and*

$$\det (g_0)'(v_0) > 0 \quad \textit{and} \quad ([g_0]_1)'_{(1)}(v_0) + ([g_0]_2)'_{(2)}(v_0) < 0, \tag{2.6}$$

then there exists $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0]$ *system* (1.1) *has exactly one* T*-periodic solution* x_ε *such that* $x_\varepsilon(0) \to v_0$ *as* $\varepsilon \to 0$. *Moreover, the solution* x_ε *is asymptotically stable.*

(c) *If* $\det (g_0)'(v_0) < 0$, *then there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0]$ *system* (1.1) *has at least one nonasymptotically stable* T*-periodic solution* x_ε *such that* $x_\varepsilon(0) \to v_0$ *as* $\varepsilon \to 0$.

Proof. Statement (a) is added for the completeness of the formulation of Theorem 2.11 and it follows from Mawhin's theorem (see [32] or [33, Theorem 3]).

On the other hand, it is a simple calculation to show that (2.6) implies that all the eigenvalues of $(g_0)'(v_0)$ have negative real part. Therefore assumption (vi) of Theorem 2.5 is also satisfied and statement (b) follows from this theorem.

Statement (c) follows from Theorem 2.9. Indeed, since $\det (g_0)'(v_0) < 0$, it implies (see [23, Theorem 5.9]) that $d(g_0, B_\rho(v_0))$ is defined for any $\rho > 0$ sufficiently small and that $d(g_0, B_\rho(v_0)) = \det(g_0)'(v_0) < 0$. ◻

3. Application to the nonsmooth van der Pol oscillator. In [18] Hogan first demonstrated the existence of a limit cycle for the nonsmooth van der Pol equation $\ddot{u}+\varepsilon(|u|-1)\dot{u}+u = 0$. This equation governs the circuit drawn at Figure 1.1 with the triode characteristic $i(u) = S_0 + S_1 u - S_2 u|u|$ whose derivative $i'(u) = S_1 - 2S_2|u|$ is nondifferentiable (see Nayfeh and Mook [36, section 3.3.4], where the same stiffness characteristic appears in mechanics). In this paper we extend this study by considering the van der Pol problem on the location of stable and unstable periodic solutions of the perturbed equation

$$\ddot{u} + \varepsilon\,(|u| - 1)\,\dot{u} + (1 + a\varepsilon)u = \varepsilon\lambda \sin t, \tag{3.1}$$

where a is a detuning parameter and $\varepsilon\lambda \sin t$ is an external force. We assume that $\varepsilon > 0$ is sufficiently small, and we consider that the parameters a and λ vary in $\mathbb{R}$.

We finally note that standard change of variables (see example in section 3) brings (1.3) into the form (1.1), with g sufficiently smooth to satisfy the hypotheses of the second Bogoliubov theorem. But we remind the reader that our aim is to apply directly Theorem 2.5, in the same way that Andronov and Witt applied Bogoliubov theorem to the classical van der Pol oscillator

$$\ddot{u} + \varepsilon\left(u^2 - 1\right)\dot{u} + (1 + a\varepsilon)u = \varepsilon\lambda \sin t, \tag{3.2}$$

which can be found in [1, Figure 4] or in [30, Chap. I, section 16, Figure 15].

A function u is a solution of (3.1) if and only if $(z_1, z_2) = (u, \dot{u})$ is a solution of the system

$$\begin{aligned} \dot{z}_1 &= \dot{z}_2, \\ \dot{z}_2 &= -z_1 + \varepsilon[-a z_1 - (|z_1| - 1)z_2 + \lambda \sin t]. \end{aligned} \tag{3.3}$$

After the change of variables

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

system (3.3) takes the form

$$
\begin{aligned}
\dot{x}_1 &= \varepsilon\sin(-t)\left[-a(x_1\cos t + x_2\sin t)\right.\\
&\quad \left.-(|x_1\cos t + x_2\sin t| - 1)(-x_1\sin t + x_2\cos t) + \lambda\sin t\right],\\
\dot{x}_2 &= \varepsilon\cos(-t)\left[-a(x_1\cos t + x_2\sin t)\right.\\
&\quad \left.-(|x_1\cos t + x_2\sin t| - 1)(-x_1\sin t + x_2\cos t) + \lambda\sin t\right].
\end{aligned} \tag{3.4}
$$

The corresponding averaged function g_0, calculated using (1.2), is given by

$$
\begin{aligned}
[g_0]_1(M,N) &= \pi a N - \pi\lambda + \pi M - \tfrac{4}{3}M\sqrt{M^2+N^2},\\
[g_0]_2(M,N) &= -\pi a M + \pi N - \tfrac{4}{3}N\sqrt{M^2+N^2},
\end{aligned} \tag{3.5}
$$

and it is continuously differentiable in $\mathbb{R}^2\backslash\{0\}$.

In short, by statement (a) of Theorem 2.11, the zeros $(M,N)\in\mathbb{R}^2$ of this function with the property that $\det(g_0)'(M,N)\neq 0$ determine the 2π-periodic solutions of (3.3) emanating from the solution of the unperturbed system

$$
\begin{aligned}
u_1(t) &= M\cos t + N\sin t,\\
u_2(t) &= -M\sin t + N\cos t.
\end{aligned} \tag{3.6}
$$

We have the following expression for the determinant:

$$
\det(g_0)'(M,N) = \pi^2(1+a^2) + \frac{32}{9}(M^2+N^2) - 4\pi\sqrt{M^2+N^2}. \tag{3.7}
$$

Following Andronov and Witt [1] we are concerned with the dependence of the amplitude of the solution (3.6) with respect to a and λ. Thus we decompose this solution as follows:

$$
u_1(t) = A\sin(t+\phi), \quad u_2(t) = A\cos(t+\phi), \tag{3.8}
$$

where (M,N) is related to (A,ϕ) by

$$
M = A\sin\phi, \quad N = A\cos\phi. \tag{3.9}
$$

Substituting (3.9) into (3.5) and (3.7) we obtain

$$
\begin{aligned}
[g_0((A\sin\phi, A\cos\phi))]_1 &= -(4/3)\cdot A|A|\sin\phi + \pi a A\cos\phi + \pi A\sin\phi - \pi\lambda,\\
[g_0((A\sin\phi, A\cos\phi))]_2 &= -(4/3)\cdot A|A|\cos\phi - \pi a A\sin\phi + \pi A\cos\phi,
\end{aligned} \tag{3.10}
$$

and, respectively,

$$
\det(g_0)'((A\sin\phi, A\cos\phi)) = \pi^2(1+a^2) + \frac{32}{9}A^2 - 2\pi|A|. \tag{3.11}
$$

Looking for the zeros (A,ϕ) of (3.10), we find the implicit formula

$$
A^2\left(a^2 + \left(1 - \frac{4}{3\pi}|A|\right)^2\right) = \lambda^2 \tag{3.12}
$$

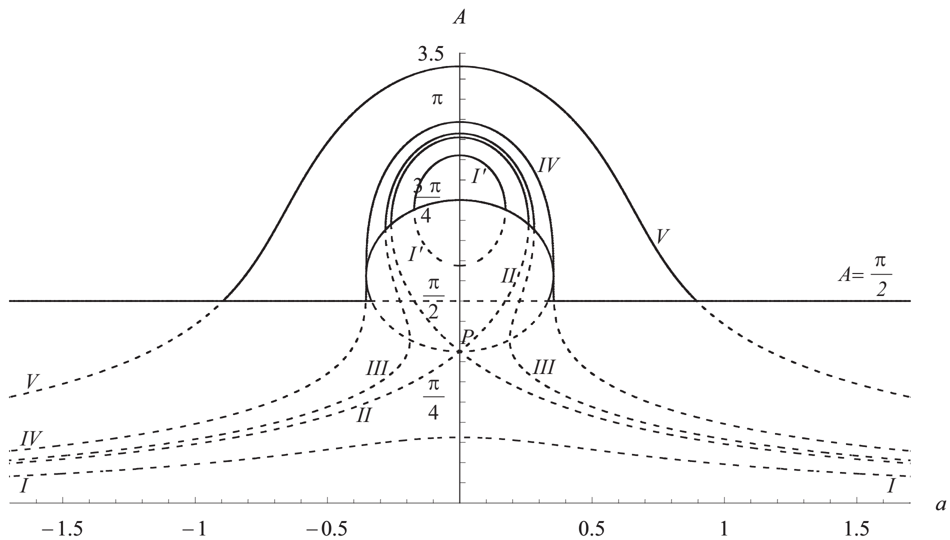

FIG. 3.1. *Dependence of the amplitude of stable (solid curves) and unstable (dash curves)* 2π*-periodic solutions of the nonsmooth periodically perturbed van der Pol equation* (3.1) *on the detuning parameter* a *obtained over formulas* (3.12), (3.16), *and* (3.17) *for different values of* λ. *The curve* I *is plotted with* $\lambda = 0.4$, II *with* $\lambda = 3\pi/16$, III *with some* $\lambda = \sqrt{0.4} \in (3\pi/16, 9\sqrt{3}\pi/64)$, IV *with* $\lambda = 9\sqrt{3}\pi/64$, *and* V *with* $\lambda = 1.5$. *Point* P *is* $2/\sqrt{3}$.

for determining A. Observe that the number of positive zeros of (3.12) coincide with the number of zeros of the equation $A^2\big(a^2 + \big(1 - \frac{4}{3\pi}A\big)^2\big) = \lambda^2$. To estimate this number we define

$$f(A) = A^2\left(a^2 + \left(1 - \frac{4}{3\pi}A\right)^2\right) - \lambda^2,$$

and we have

$$f'(A) = 2A\left(a^2 + \left(1 - \frac{4}{3\pi}A\right)^2\right) - \frac{8}{3\pi}A^2\left(1 - \frac{4}{3\pi}A\right).$$

Since f' has one or two zeros, then (3.12) has one, two, or three positive solutions A for any fixed a and λ. In order to understand the different situations that can appear, we follow Andronov and Witt, who suggested in [1] (see also [2]) to construct the so-called *resonance curves*, namely, the curves A in function of a, for λ fixed. The equation of this curve is given by formula (3.12). Some curves (3.12) corresponding to different values of λ are drawn in Figure 3.1. The way for describing these resonance curves (3.12) is borrowed from [30, Ch. 1, section 5], where the classical van der Pol equation is considered.

When $\lambda = 0$ the curve (3.12) is formed by the axis $A = 0$ and the isolated point $(0, 3\pi/4)$. When $\lambda > 0$ but sufficiently small, the resonance curve consists of two branches: instead of $A = 0$ we have the curve of the type $I - I$, and instead of the point $(0, 3\pi/4)$ we obtain an oval $I' - I'$ surrounding this point. When $\lambda > 0$ increases, the oval $I' - I'$ and the branch $I - I$ tend to each other, and, for a certain λ, there exists only one branch $II - II$ with a double point P. The value of this λ can be obtained assuming that (3.12) has for $a = 0$ a double root and, therefore, (3.11) should be zero. Solving jointly (3.12) and (3.11) with $a = 0$ we obtain $\lambda = 3\pi/16$ and $P = 2\pi/8$. If $\lambda > 3\pi/16$, then we have curves of the type III which take form V when $\lambda > 0$ crosses the value $\lambda = 9\sqrt{3}\pi/64$. From here, if $\lambda < 3\pi/16$, then (3.12) has three real roots when $|a|$ is sufficiently small, and only one root when $|a|$ is greater than a certain number which depends on λ. When $3\pi/16 < \lambda < 9\sqrt{3}\pi/64$ (3.12) has one, three, and again one solution according to whether $a < a_1$, $a_1 < a < a_2$, and $a > a_2$, respectively, where a_1, a_2 depend on λ. The amplitude curves of type V provide exactly one solution of (3.12) for any value of a. The value $\lambda = 9\sqrt{3}\pi/64$, which separates the curves where (3.12) has three solutions from the curves where (3.12) has one solution, is obtained from the property that (3.12) with this λ has a double root for some a and thus this value of a vanishes (3.11). Therefore $\lambda = 9\sqrt{3}\pi/64$ is the point separating the interval $(0, \lambda)$ where the system formed by (3.12) and

$$\pi^2(1 + a^2) + \frac{32}{9}A^2 - 2\pi|A| = 0 \tag{3.13}$$

has at least one solution from the interval (λ, ∞) where (3.12)–(3.13) has no solutions.

In short, we have studied the amplitudes of the 2π-periodic solutions of system (3.3) depending on a and λ, when a physical system described by (3.3) possesses 2π-periodic oscillations and when some of them are asymptotically stable. To find the answer we have used statement (b) of Theorem 2.11. Assumption (i) is obviously satisfied with $\Omega = \mathbb{R}^2$. The next statement shows that the right-hand side of system (3.4) satisfies (v).

PROPOSITION 3.1. *Let $v_0 \in \mathbb{R}^2$, $v_0 \neq 0$. Then the right-hand side of* (3.4) *satisfies* (v) *for any $a, \lambda \in \mathbb{R}$.*

The proof of Proposition 3.1 is given in section 4.

Thus we have to study the signs of (3.11) and $([g_0]_1)'_M(A\sin\phi, A\cos\phi) + ([g_0]_2)'_N(A\sin\phi, A\cos\phi)$. We have

$$([g_0]_1)'_M(M,N) + ([g_0]_2)'_N(M,N) = 2\left(\pi - 2\sqrt{M^2+N^2}\right),$$

and therefore the conditions for the asymptotic stability of the 2π-periodic solutions of (3.3) near (3.6) are

$$\pi^2(1 + a^2) + \frac{32}{9}(M^2+N^2) - 4\pi\sqrt{M^2+N^2} > 0 \tag{3.14}$$

and

$$2\left(\pi - 2\sqrt{M^2+N^2}\right) < 0. \tag{3.15}$$

Substituting (3.9) into the inequalities (3.14) and (3.15), we obtain the following equivalent inequalities in terms of the amplitude A:

$$\pi^2(1 + a^2) + \frac{32}{9}A^2 - 2\pi|A| > 0 \tag{3.16}$$

and

$$2\pi - 4|A| < 0. \tag{3.17}$$

Conditions (3.16) and (3.17) mean that the asymptotically stable 2π-periodic solutions of (3.3) correspond to those parts of resonance curves under consideration which are outside the ellipse (3.13) and above the line $A = \pi/2$. All the results are collected in Figure 3.1, where it is easy to see that for any detuning parameter a and any amplitude $\lambda > 0$, (3.1) possesses at least one asymptotically stable 2π-periodic solution with amplitude close to A obtained from (3.12). Among all the asymptotically stable 2π-periodic solutions of (3.1), there exists exactly one whose fixed neighborhood does not contain any nonasymptotically stable 2π-periodic solution of (3.1) for sufficiently small $\varepsilon > 0$. The amplitude of this asymptotically stable 2π-periodic solution is obtained from (3.16)–(3.17).

To compare the changes due to nonsmoothness in the behavior of the resonance curves, we give in Figure 3.2 the resonance curves of the classical van der Pol oscillator (3.2).

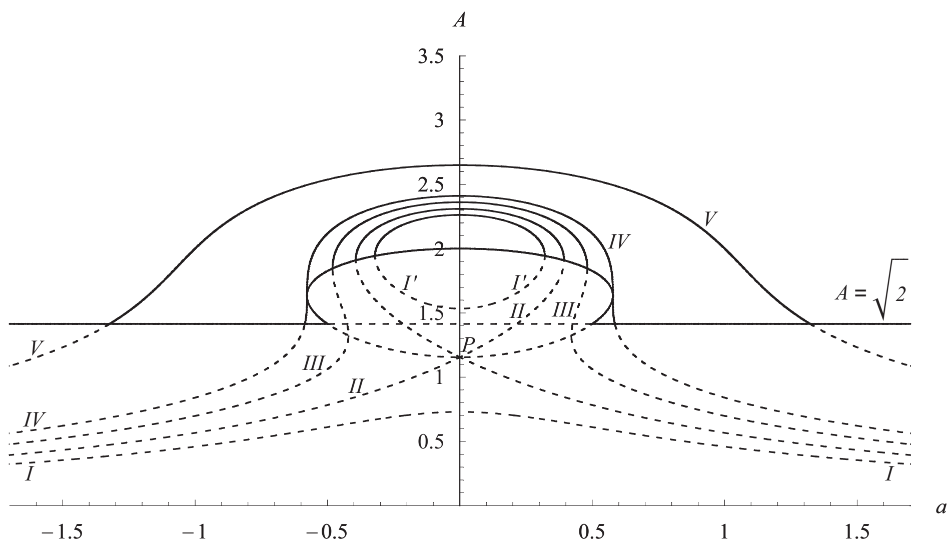

FIG. 3.2. *Dependence of the amplitude of stable (solid curves) and unstable (dash curves)* 2π*-periodic solutions of the classical periodically perturbed van der Pol equation* (3.2) *on the detuning parameter a for different values of λ. Following Andronov and Witt (see* [1, *Figure* 4]*), curve I is plotted with* $\lambda = \sqrt{0.4}$*, II with* $\lambda = 4\sqrt{3}/9$*, III with some* $4\sqrt{3}/9 < \lambda < \sqrt{32/27}$*, IV with* $\lambda = \sqrt{32/27}$*, and V with* $\lambda = 2$*. Point P is* $2/\sqrt{3}$*.*

The formulas of Figure 3.1 can be compared with the formulas for Figure 3.2. In fact, the corresponding expressions (3.12)–(3.13) and (3.14)–(3.15) are (see the formulas (5.21)–(5.22) and (16.6)–(16.7) of [30])

$$A^2\left(a^2 + \left(1 - \frac{A^2}{4}\right)^2\right) = \lambda^2,$$

$$1 - a^2 - A^2 + \frac{3}{16}A^4 = 0,$$

and

$$1 + a^2 - (M^2 + N^2) + \frac{3}{16}(M^2 + N^2)^2 > 0,$$

$$2 - (M^2 + N^2) < 0,$$

respectively, when we consider the classical van der Pol equation (3.2).

It can be checked that the eigenvectors of the matrix $(g_0)'((A\sin\phi, A\cos\phi))$ are orthogonal only for $A = 0$, so Theorem 3.6 from [8] cannot be applied. At the same time assumption (H2) from [8] is not satisfied for our problem (see Remark 2.8).

4. Appendix.

Proof of Proposition 3.1. As before, $[v]_i$ is the ith component of the vector $v \in \mathbb{R}^2$. Let $g(t,v) = |[v]_1\cos t + [v]_2 \sin t|$, and notice that it is enough to prove that $g : [0,2\pi]\times\mathbb{R}^2 \to \mathbb{R}$ satisfies (v). In the case that $[v_0]_2 \neq 0$, denote $\theta(v) = \arctan(-[v]_1/[v]_2)$ if $[v_0]_2 = 0$, denote $\theta(v) = \arctan(-[v]_1/[v]_2)$ if $[v_0]_1[v]_2 < 0$, $\theta(v_0) = \pi/2$, and, respectively, $\theta(v) = \arctan(-[v]_1/[v]_2) + \pi$ if $[v_0]_1[v]_2 > 0$. In any case notice that the function $v \mapsto \theta(v)$ is continuous in every sufficiently small neighborhood of v_0. Fix $\widetilde{\gamma} > 0$. Let M be the union of two intervals centered in $\theta(v_0)$ (respectively, $\theta(v_0) + 2\pi$ if $\theta(v_0) < 0$) and in $\theta(v_0)+\pi$, each of length $\widetilde{\gamma}/2$. Denote them M_1 and M_2. Take $\widetilde{\delta} > 0$ such that $\theta(v) \in M_1$ for all $v \in B_{\widetilde{\delta}}(v_0)$. Of course, also $\theta(v) + \pi \in M_2$ for all $\|v - v_0\| \leq \widetilde{\delta}$. This implies that for fixed $t \in [0,2\pi]\setminus M$, $[v]_1\cos t + [v]_2 \sin t$ has constant sign for all $v \in B_{\widetilde{\delta}}(v_0)$, which further gives that $g(t,\cdot)$ is differentiable and $g'_v(t,v) = g'_v(t,v_0)$ for all $v \in B_{\widetilde{\delta}}(v_0)$. Hence (v) is fulfilled. ∎

Acknowledgments. The authors are grateful to the anonymous referee who motivated us to include a list of several applications of Theorem 2.5 in the introduction—that definitely improved the paper. The authors also thank Aris Daniilidis for helpful discussions and Rafael Ortega who called our attention to the change of variables used in the Levinson paper [29]. Part of this work was done during a visit of the first and the third author to the Centre de Recerca Matemàtica, Barcelona (CRM). They express their gratitude to the CRM for providing very nice working conditions.

Finally we thank Martin Golubitsky and Andre Vanderbauwhede, who invited us to present the paper at their minisimposia "Recent developments in bifurcation theory" of Equadiff 2007 (see [9]), giving a significant impact to its recognition.

REFERENCES

[1] A. Andronov and A. Witt, *On mathematical theory of entrainment*, Z. Prikl. Phis., 6 (1930), pp. 3–17 (in Russian).

[2] A. Andronow and A. Witt, *Zur Theorie des Mitnehmens von van der Pol*, Arch. Elektrotechnik, 24 (1930), pp. 99–110 (in German).

[3] A. A. Andronov, A. A. Witt, and S. E. Khaikin, *Theory of Oscillators*, (translated from the Russian by F. Immirzi; translation edited and abridged by W. Fishwick) Pergamon Press, Oxford, 1966.

[4] J. Awrejcewicz and C. H. Lamarque, *Bifurcation and Chaos in Nonsmooth Mechanical Systems*, World Scientific, River Edge, NJ, 2003.

[5] M. S. Baptista, T. P. Silva, J. C. Sartorelli, and I. L. Caldas, *Phase synchronization in the perturbed Chua circuit*, Phys. Rev. E (3), 74 (2006), no. 056707.

[6] N. N. Bogoliubov, *On Some Statistical Methods in Mathematical Physics*, Akademiya Nauk Ukrainskoi SSR, Kiev, 1945 (in Russian).

[7] A. P. Bovsunovskii, *Comparative analysis of nonlinear resonances of a mechanical system with unsymmetrical piecewise characteristic of restoring force*, Strength Matherials, 39 (2007), pp. 159–169.

[8] A. Buică and A. Daniilidis, *Stability of periodic solutions for Lipschitz systems obtained via the averaging method*, Proc. Amer. Math. Soc., 135 (2007), pp. 3317–3327.

[9] A. Buică, J. Llibre, and O. Makarenkov, *Lipschitz constant of integral with respect to a parameter versus derivative of integral with respect to a parameter in the theory of nonsmooth bifurcations*, Book of abstracts, Equadiff 2007, Vienna University of Technology, 2007, pp. 88–89.

[10] L. O. Chua, *Global unfolding of Chua's circuit*, IEICE Trans. Fundamentals, E76-A, 5 (1993), pp. 704–734.
[11] D. Cofagna and G. Grassi, *Chaotic beats in a modified Chua's circuit: Dynamic behaviour and circuit design*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 17 (2007), pp. 209–226.
[12] B. P. Demidovich, *Lectures on the Mathematical Theory of Stability*, Izdat. Nauka, Moscow, 1967 (in Russian).
[13] S. W. S. W. Doebling, C. R. Farrar, M. B. Prime, and D. W. Shevitz, *Damage Identification and Health Monitoring of Structural and Mechanical Systems from Changes in Their Vibration Characteristics: A Literature Review*, LA-13070-MS, UC-900, Los Alamos National Laboratory, 1996.
[14] C. Fabry, *Large-amplitude oscillations of a nonlinear asymmetric oscillator with damping*, Nonlinear Anal. Ser. A: Theory Methods, 44 (2001), pp. 613–626.
[15] P. Fatou, *Sur le mouvement d'un système soumis à des forces à courte période*, Bull. Soc. Math. France, 56 (1928), pp. 98–139 (in French).
[16] J. Glover, A. C. Lazer, and P. J. McKenna, *Existence and stability of large scale nonlinear oscillations in suspension bridges*, Z. Angew. Math. Phys., 40 (1989), pp. 172–200.
[17] J. K. Hale, *Ordinary Differential Equations*, Robert E. Krieger Publishing Co., Inc., Huntington, NY, 1980.
[18] S. J. Hogan, *Relaxation oscillations in a system with a piecewise smooth drag coefficient*, J. Sound Vibration, 263 (2003), pp. 467–471.
[19] A. A. Jakkula, *A history of suspension bridges in bibliographical form*, Bull. Agricultural and Mechanical College of Texas 4th ser., vol. 12 Federal Works Agency, Washington, D.C., 1941.
[20] G. A. Johnston and E. R. Hunt, *Derivative control of the steady state in Chua's circuit driven in the chaotic region*, IEEE Trans. Circuits Systems I Fund. Theory Appl., 40 (2000), pp. 833–835.
[21] A. N. Kolmogorov and S. V. Fomin, *Introductory Real Analysis* (translated and edited by R. A. Silverman, Selected Russian Publications in the Mathematical Sciences, Prentice-Hall, Englewood Cliffs, NJ, 1970.
[22] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations* (translated from the Russian by R. E. Flaherty; edited by Leo F. Boron), P. Noordhoff Ltd., Groningen, The Netherlands, 1964.
[23] M. A. Krasnosel'skii, *The Operator of Translation Along the Trajectories of Differential Equations*, Translations of Mathematical Monographs 19 (translated from the Russian by Scripta Technica), American Mathematical Society, Providence, RI, 1968.
[24] B. I. Kryukov, *Dynamics of Resonance-Type Vibration Machines*, Naukova Dnka, Kiev, 1967 (in Russian).
[25] N. M. Krylov and N. N. Bogoliubov, *Introduction to Non-linear Mechanics*, Akademiya Nauk Ukrainskoi SSR, 1937 (in Russian); English translation: Annals of Mathematics Studies 11, Princeton University Press, Princeton, NJ, 1943.
[26] A. C. Lazer and P. J. McKenna, *Existence, uniqueness, and stability of oscillations in differential equations with asymmetric nonlinearities*, Trans. Amer. Math. Soc., 315 (1989), pp. 721–739.
[27] A. C. Lazer and P. J. McKenna, *Large-amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis*, SIAM Rev., 32 (1990), pp. 537–578.
[28] M. Levi, *Qualitative analysis of the periodically forced relaxation oscillations*, Mem. Amer. Math. Soc., 32 (1981), no. 244.
[29] N. Levinson, *A second order differential equation with singular solutions*, Ann. Math. (2), 50 (1949), pp. 127–153.
[30] I. G. Malkin, *Some problems of the theory of nonlinear oscillations*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956 (in Russian); English translation: AEC tr 3766 (book 1), US Atomic Energy Commission, 1959.
[31] L. I. Mandelstam and N. D. Papaleksi, *On justification of one approximation method for solving differential equations*, Zh. Eksper. Teoret. Fiz., IV (1934), no. 117.
[32] J. Mawhin, *Le Problème des Solutions Périodiques en Mécanique non Linéaire*, Thèse de doctorat en sciences, Université de Liège, 1969; published in *Degré topologique et solutions périodiques des systèmes différentiels non linéaires*, Bull. Soc. Roy. Sci. Liège, 38 (1969), pp. 308–398.
[33] J. Mawhin, *Periodic solutions in the golden sixties: The birth of a continuation theorem*, in Ten Mathematical Essays on Approximation in Analysis and Topology, Elsevier B. V., Amsterdam, 2005, pp. 199–214.

[34] Yu. A. Mitropol'skii, *On periodic solutions of systems of nonlinear differential equations with non-differentiable right-hand sides*, Ukrain. Mat. Ž., 11 (1959), pp. 366–379 (in Russian).

[35] K. Murali and M. Lakshmanan, *Chaotic dynamics of the driven Chua's circuit*, IEEE Trans. Circuits Systems I Fund. Theory Appl., 50 (2003), pp. 1503–1508.

[36] A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations*, Wiley-Interscience, New York, 1979.

[37] L. S. Pontrjagin, *Ordinary differential equations* (translated from the Russian by L. Kacinskas and W. B. Counts), Adiwes International Series in Mathematics, Addison–Wesley Publishing, Reading, MA, 1962.

[38] R. Puers, J. Bienstman, and J. Vandewalle, *The autonomous impact resonator: A new operation principle for a silicon resonant strain gauge*, in Int. Conf. on Solid-State Sensors and Actuators, Digest of Technical Papers, Chicago, 1997, pp. 1105–1108.

[39] A. M. Samoilenko, *On periodic solutions of differential equations with nondifferentiable right–hand sides*, Ukrain. Mat. Ž., 15 (1963), pp. 328–332 (in Russian).

[40] E. Sánchez, M. A. Matías, and V. Pérez-Muñuzuri, *Chaotic synchronization in small assemblies of driven Chua's circuits*, IEEE Trans. Circuits Systems I Fund. Theory Appl., 47 (2000), pp. 644–654.