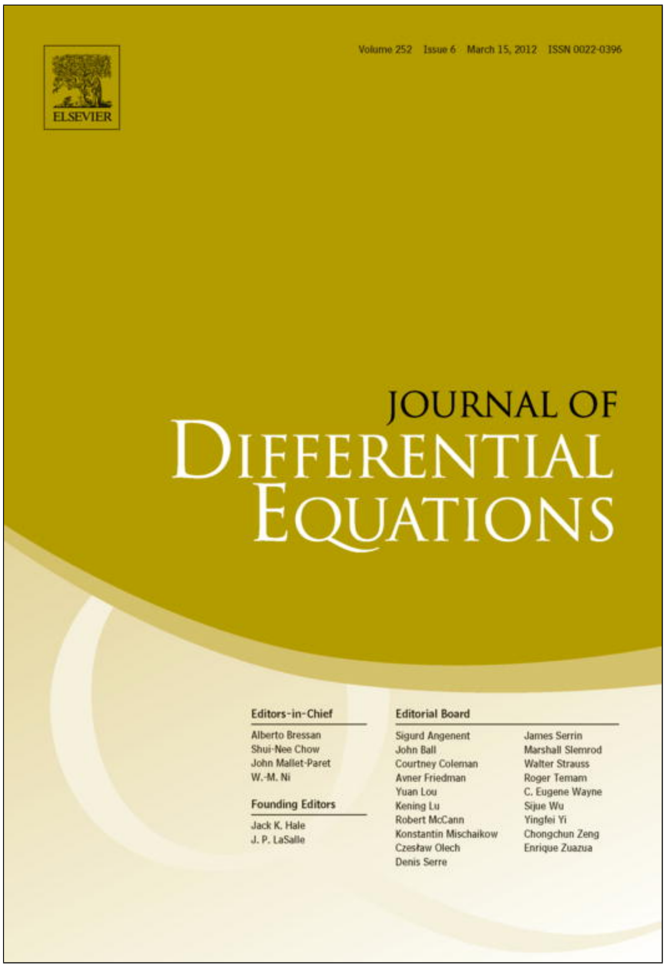
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Bifurcations from nondegenerate families of periodic solutions in Lipschitz systems

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ABSTRACT

The paper addresses the problem of bifurcation of periodic solutions from a normally nondegenerate family of periodic solutions of ordinary differential equations under perturbations. The approach to solve this problem can be described as transforming (by a Lyapunov–Schmidt reduction) the initial system into one which is in the standard form of averaging, and subsequently applying the averaging principle. This approach encounters a fundamental problem when the perturbation is only Lipschitz (nonsmooth) as we do not longer have smooth Lyapunov–Schmidt projectors. The situation of Lipschitz perturbations has been addressed in the literature lately and the results obtained conclude the existence of the bifurcated branch of periodic solutions. Motivated by recent challenges in control theory, we are interested in the uniqueness problem. We achieve this in the case when the Lipschitz constant of the perturbation obeys a suitable estimate.

1. Introduction

In [21] Malkin developed a perturbation theory to study the existence, uniqueness and stability of T-periodic solutions in the n-dimensional T-periodic systems of the form

$$\dot{x} = f(t,x) + \varepsilon g(t,x,\varepsilon), \tag{1}$$

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where both functions f and g are *sufficiently smooth* and $\varepsilon > 0$ is *small*. It is assumed in [21] that the unperturbed system (namely (1) with $\varepsilon = 0$) has a family of T-periodic solutions, denoted $x(\cdot, \xi(h))$, whose initial conditions are given by a smooth function $\xi : \mathbb{R}^k \to \mathbb{R}^n$. In these settings the adjoint linearized differential system

$$\dot{u} = -\big(D_x f\big(t, x\big(t, \xi(h)\big)\big)\big)^* u \tag{2}$$

has k linearly independent T-periodic solutions $u_1(\cdot, h), \ldots, u_k(\cdot, h)$, thus the *geometric multiplicity* of the Floquet multiplier $+1$ of (2) is at least k for each $h \in \mathbb{R}^k$. Assuming, in addition, that the *algebraic multiplicity*[1] of $+1$ is exactly k (thus the geometric one is also k), Malkin proved [21] that if the *bifurcation function*

$$M(h) = \int_0^T \begin{pmatrix} \langle u_1(\tau, h), g(\tau, x(\tau, \xi(h)), 0)\rangle \\ \cdots \\ \langle u_k(\tau, h), g(\tau, x(\tau, \xi(h)), 0)\rangle \end{pmatrix} d\tau$$

has *a simple zero* $h_0 \in \mathbb{R}^k$, then for any $\varepsilon > 0$ sufficiently small, system (1) has *a unique T-periodic solution* x_ε such that $x_\varepsilon(0) \to \xi(h_0)$ as $\varepsilon \to 0$. Here simple zero means that $M(h_0) = 0$ and the Jacobian determinant of M at h_0 is nonzero. As usual $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$. Moreover, Malkin related *the asymptotic stability* of the solution x_ε with the eigenvalues of the Jacobian matrix $DM(h_0)$. The same result has been proved independently in Loud [19]. Since then, this result has been refined and developed in various directions, and the problem itself has been treated from different perspectives, some of them leading to other expressions of the bifurcation function [6,9,14,15,25]. In particular, Rhouma and Chicone [25] studied the situation where only the *geometric multiplicity of the multiplier* $+1$ *of* (2) *is k*. The mentioned property has been termed *normal nondegeneracy* of the manifold $\xi(\mathbb{R}^k)$.

Perturbation theory for Eq. (1) found many applications in engineering and still helps solving important problems in synchronization (see e.g. Várkonyi and Holmes [23], Makarenkov, Nistri and Papini [20]).

In this paper (Theorem 7 below is our main result) we prove bifurcation of isolated branches from *normally nondegenerate manifold* $\xi(\mathbb{R}^k)$ assuming only Lipschitz continuity for the perturbation g, and continuity of the Lipschitz constant of the map $z \mapsto g(t, z+\zeta, \varepsilon) - g(t, z, 0)$ with respect to its entries. This condition, denoted below by (A9), has its roots in Glover, Lazer and McKenna [10] and has recently proved its effectiveness in semi-linear perturbation problems [5]. We are aware that less regular perturbations g have been already considered in the literature, where a topological degree method is employed to prove the existence of bifurcation (see e.g. Fečkan [9] and Kamenskii, Makarenkov and Nistri [14]). However, our paper seems to be the first contribution that takes advantage of the Lipschitz continuity of g to achieve uniqueness of the bifurcating branch of periodic solutions.

Our interest in Lipschitz differential equations is motivated by applications in control and optimization. In optimization, Lipschitz ingredients come from variational inequalities constraining a differential equation (see e.g. [26]). The solutions of boundary-value problems (that includes T-periodic problems) for this class of equations are investigated in [27], but the role of bifurcation problems for this kind of solutions hasn't yet been discussed in the literature. In control, Lipschitz right-hand sides often appear in the analysis of the so-called dithered T-periodic systems. An averaging method (see [22] and [12]) allows to analyze the dynamics of such an uncertain system over a suitable averaged system, which appears to be deterministic. One application of the perturbation result that we propose could be a further simplifying of the averaged equations in the case where the undithered system possesses families of T-periodic solutions (i.e. resonate with the dither). An example of such a situation is when the undithered system is autonomous and possesses a cycle of period T.

[1] λ^* is a Floquet multiplier of the T-periodic linear system (2) if it is an eigenvalue of $U(T, h)$, where $U(t, h)$ is a fundamental matrix solution of this system. The Floquet multiplier λ^* has geometric multiplicity k when the dimension of the kernel of $U(T, h) - \lambda^* I_{n\times n}$ is k, while its algebraic multiplicity is counted as a root of the algebraic equation $\det(U(T, h) - \lambda I_{n\times n}) = 0$.

Stability of solutions of differential equations with Lipschitz right-hand sides is quite well understood and goes back to Lasota and Strauss [13] and can also be achieved based on the Clarke Implicit Function Theorem [7]. This paper represents the next natural step in the development of the perturbation theory for this class of differential equations. This theory has been pioneered by Glover, Lazer and McKenna in [10] and recently put in a more general context by Buică, Llibre and Makarenkov [5] (where the unperturbed systems were assumed Hamiltonian or linear).

In order to prove our main result we extend the *Lyapunov–Schmidt reduction method* (see [6]) to the case of nonsmooth Lipschitz functions and derive suitable estimates for the dependence of the Lipschitz constant of the implicit function on state variables and parameter ε. In addition, we need to discover new Lipschitz analogues of the smooth dependence of the solution of system (1) on the parameter and the initial condition.

The paper is organized as follows. In the next section we summarize our notations. In Section 3 we generalize the Lyapunov–Schmidt reduction method for nonsmooth functions. In Section 4 we prove Theorem 3 and the main result of the paper, Theorem 7.

2. Notations

The following notations will be used throughout this paper.

Let $n, m, k \in \mathbb{N}$, $k \leqslant n$, $i \in \mathbb{N} \cup \{0\}$.

We denote the projection onto the first k coordinates by $\pi : \mathbb{R}^n \to \mathbb{R}^k$, and the one onto the last $n-k$ coordinates by $\pi^\perp : \mathbb{R}^n \to \mathbb{R}^{n-k}$.

We denote by $I_{n\times n}$ the identity $n \times n$ matrix, while $0_{n\times m}$ denotes the null $n \times m$ matrix. For an $n \times n$ matrix A we denote by A^* the adjoint of A, that in the case the matrix is real reduces to the transpose.

We consider a norm in $\mathbb{R}^n$ denoted by $\|\cdot\|$. Let Ψ be an $n \times n$ real matrix. Then $\|\Psi\|$ denotes the operator norm, i.e. $\|\Psi\| = \sup_{\|\xi\|=1} \|\Psi\xi\|$.

Let $\xi \in \mathbb{R}^n$ and $\mathcal{Z} \subset \mathbb{R}^n$ be compact, then we denote by $\rho(\xi, \mathcal{Z}) = \min_{\zeta\in\mathcal{Z}} \|\xi - \zeta\|$ the distance between ξ and $\mathcal{Z}$. For $\delta > 0$ and $z \in \mathbb{R}^n$ the ball in $\mathbb{R}^n$ centered in z of radius δ will be denoted by $B_\delta(z)$.

For a subset $\mathcal{U} \subset \mathbb{R}^n$ we denote by $\mathrm{int}(\mathcal{U})$, $\overline{\mathcal{U}}$ and $\overline{\mathrm{co}}\,\mathcal{U}$ its interior, closure and closure of the convex hull, respectively.

We denote by $C^i(\mathbb{R}^n, \mathbb{R}^m)$ the set of all continuous and i times continuously differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}^m$.

Let $\mathcal{F} \in C^0(\mathbb{R}^n, \mathbb{R}^n)$ be a function that does not have zeros on the boundary of some open bounded set $\mathcal{U} \subset \mathbb{R}^n$. Then $d(\mathcal{F}, \mathcal{U})$ denotes the Brouwer topological degree of $\mathcal{F}$ on $\mathcal{U}$ (see [3] or [18, Ch. 1, § 3]).

For $\mathcal{F} \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, $D\mathcal{F}$ denotes the Jacobian matrix of $\mathcal{F}$. If $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ and $\alpha \in \mathbb{R}^k, \beta \in \mathbb{R}^{n-k}$, then $D_\alpha \mathcal{F}(\cdot, \beta)$ denotes the Jacobian matrix of $\mathcal{F}(\cdot, \beta)$. For $\mathcal{F} \in C^2(\mathbb{R}^n, \mathbb{R})$, $H\mathcal{F}$ denotes the Hessian matrix of $\mathcal{F}$, i.e. the Jacobian matrix of the gradient of $\mathcal{F}$.

Let $\delta > 0$ be sufficiently small. With $o(\delta)$ we denote a function of variable δ such that $o(\delta)/\delta \to 0$ as $\delta \to 0$, while $O(\delta)$ denotes a function of δ such that $O(\delta)/\delta$ is bounded as $\delta \to 0$. Besides these classical notations, we introduce now $\tilde{o}(\delta)$ for a function of variable δ such that $\tilde{o}(\delta) \to 0$ as $\delta \to 0$. Here the functions o, O or $\tilde{o}$ may depend also on other variables, but the above properties hold uniformly when these variables lie in a fixed bounded region.

We say that the function $Q : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is *locally uniformly Lipschitz with respect to its first variable* if for each compact $K \subset \mathbb{R}^n \times \mathbb{R}^m$ there exists $L > 0$ such that $\|Q(z_1, \lambda) - Q(z_2, \lambda)\| \leqslant L\|z_1 - z_2\|$ for all $(z_1, \lambda), (z_2, \lambda) \in K$.

For any Lebesgue measurable set $M \subset [0, T]$ we denote by $mes(M)$ the Lebesgue measure of M.

3. Lyapunov–Schmidt reduction method for nonsmooth Lipschitz functions

If the continuously differentiable function $P : \mathbb{R}^n \to \mathbb{R}^n$ vanishes on some set $\mathcal{Z} \subset \mathbb{R}^n$, then sufficient conditions for the existence of zeros near $\mathcal{Z}$ of the perturbed function

$$F(z, \varepsilon) = P(z) + \varepsilon Q(z, \varepsilon), \quad z \in \mathbb{R}^n,\ \varepsilon > 0 \text{ small enough} \tag{3}$$

can be expressed in terms of the restrictions to $\mathcal{Z}$ of the functions $z \mapsto DP(z)$ and $z \mapsto Q(z, 0)$. Roughly speaking, this is what is known in the literature as the Lyapunov–Schmidt reduction method, as it is presented for instance in [6,4] or [18, §24.8]. In these references it is assumed that Q is a continuously differentiable function. We show in this section that this last assumption can be relaxed to just Lipschitz continuity, where the Lipschitz constant of Q must obey a suitable estimate.

Theorem 1. *Let $P \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, let $Q \in C^0(\mathbb{R}^n \times [0, 1], \mathbb{R}^n)$ be locally uniformly Lipschitz with respect to its first variable, and let $F : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ be given by* (3)*. Assume that P satisfies the following hypotheses.*

(A1) *There exist an invertible $n \times n$ matrix S, an open ball $V \subset \mathbb{R}^k$ with $k \leqslant n$, and a function $\beta_0 \in C^1(\overline{V}, \mathbb{R}^{n-k})$ such that P vanishes on the set $\mathcal{Z} = \bigcup_{\alpha \in \overline{V}} \{S\binom{\alpha}{\beta_0(\alpha)}\}$.*
(A2) *For any $z \in \mathcal{Z}$ the matrix $DP(z)S$ has in its upper right corner the null $k \times (n-k)$ matrix and in the lower right corner the $(n-k) \times (n-k)$ matrix $\Delta(z)$ with $\det(\Delta(z)) \neq 0$.*

For any $\alpha \in \overline{V}$ we define

$$\widehat{Q}(\alpha) = \pi Q\left(S\begin{pmatrix}\alpha \\ \beta_0(\alpha)\end{pmatrix}, 0\right). \tag{4}$$

Then the following statements hold.

(C1) *For any sequences $(z_m)_{m \geqslant 1}$ from $\mathbb{R}^n$ and $(\varepsilon_m)_{m \geqslant 1}$ from $[0, 1]$ such that $z_m \to z_0 \in \mathcal{Z}$, $\varepsilon_m \to 0$ as $m \to \infty$ and $F(z_m, \varepsilon_m) = 0$ for any $m \geqslant 1$, we have $\widehat{Q}(\pi S^{-1} z_0) = 0$.*
(C2) *If $\widehat{Q} : \overline{V} \to \mathbb{R}^k$ is such that $\widehat{Q}(\alpha) \neq 0$ for all $\alpha \in \partial V$ and $d(\widehat{Q}, V) \neq 0$, then there exists $\varepsilon_1 > 0$ sufficiently small such that for each $\varepsilon \in (0, \varepsilon_1]$ there exists at least one $z_\varepsilon \in \mathbb{R}^n$ with $F(z_\varepsilon, \varepsilon) = 0$ and $\rho(z_\varepsilon, \mathcal{Z}) \to 0$ as $\varepsilon \to 0$.*

In addition we assume that there exists $\alpha_0 \in V$ such that $\widehat{Q}(\alpha_0) = 0$, $\widehat{Q}(\alpha) \neq 0$ for all $\alpha \in \overline{V} \setminus \{\alpha_0\}$ and $d(\widehat{Q}, V) \neq 0$, and we denote $z_0 = S\binom{\alpha_0}{\beta_0(\alpha_0)}$. Moreover we also assume:

(A3) *P is twice differentiable in the points of $\mathcal{Z}$, and for each $i \in \overline{1, k}$ and $z \in \mathcal{Z}$ the Hessian matrix $HP_i(z)$ is symmetric.*
(A4) *There exist $\delta_1 > 0$ and $L_{\widehat{Q}} > 0$ such that*

$$\left\|\widehat{Q}(\alpha_1) - \widehat{Q}(\alpha_2)\right\| \geqslant L_{\widehat{Q}} \|\alpha_1 - \alpha_2\| \quad \textit{for all } \alpha_1, \alpha_2 \in B_{\delta_1}(\alpha_0).$$

(A5) *For $\delta > 0$ sufficiently small we have that*

$$\left\|\pi Q(z_1 + \zeta, \varepsilon) - \pi Q(z_1, 0) - \pi Q(z_2 + \zeta, \varepsilon) + \pi Q(z_2, 0)\right\| \leqslant \tilde{o}(\delta)\|z_1 - z_2\|,$$

for all $z_1, z_2 \in B_\delta(z_0) \cap \mathcal{Z}$, $\varepsilon \in [0, \delta]$ and $\zeta \in B_\delta(0)$.

Then the following conclusion holds.

(C3) *There exists $\delta_2 > 0$ such that for each $\varepsilon \in (0, \varepsilon_1]$ there is exactly one $z_\varepsilon \in B_{\delta_2}(z_0)$ with $F(z_\varepsilon, \varepsilon) = 0$. Moreover $z_\varepsilon \to z_0$ as $\varepsilon \to 0$.*

We note that a map that satisfies (A4) is usually called *dilating map* (cf. [1]).

For proving Theorem 1 we shall use the following version of the Implicit Function Theorem.

Lemma 2. *Let $P \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and let $Q \in C^0(\mathbb{R}^n \times [0, 1], \mathbb{R}^n)$ be locally uniformly Lipschitz with respect to its first variable. Assume that P satisfies the hypotheses* (A1) *and* (A2) *of Theorem* 1*. Then there exist $\delta_0 > 0$, $\varepsilon_0 > 0$ and a function $\beta : \overline{V} \times [0, \varepsilon_0] \to \mathbb{R}^{n-k}$ such that*

(C4) $\pi^{\perp}F\big(S\big(\begin{smallmatrix}\alpha\\ \beta(\alpha,\varepsilon)\end{smallmatrix}\big),\varepsilon\big)=0$ *for all* $\alpha\in\overline{V}$ *and* $\varepsilon\in[0,\varepsilon_0]$.

(C5) $\beta(\alpha,\varepsilon)=\beta_0(\alpha)+\varepsilon\mu(\alpha,\varepsilon)$ *where* $\mu:\overline{V}\times(0,\varepsilon_0]\to\mathbb{R}^{n-k}$ *is bounded. Moreover for any* $\alpha\in\overline{V}$ *and* $\varepsilon\in[0,\varepsilon_0]$, $\beta(\alpha,\varepsilon)$ *is the only zero of* $\pi^{\perp}F\big(S\big(\begin{smallmatrix}\alpha\\ \cdot\end{smallmatrix}\big),\varepsilon\big)$ *in* $B_{\delta_0}(\beta_0(\alpha))$ *and* β *is continuous in* $\overline{V}\times[0,\varepsilon_0]$.

In addition if P *is twice differentiable in the points of* $\mathcal{Z}$, *then*

(C6) *there exists* $L_\mu>0$ *such that* $\|\mu(\alpha_1,\varepsilon)-\mu(\alpha_2,\varepsilon)\|\leqslant L_\mu\|\alpha_1-\alpha_2\|$ *for all* $\alpha_1,\alpha_2\in\overline{V}$ *and* $\varepsilon\in(0,\varepsilon_0]$.

Proof. (C4) Let $\widetilde{F}:\mathbb{R}^k\times\mathbb{R}^{n-k}\times[0,1]\to\mathbb{R}^n$ be defined by

$$\widetilde{F}(\alpha,\beta,\varepsilon)=F\left(S\begin{pmatrix}\alpha\\ \beta\end{pmatrix},\varepsilon\right),$$

and let $\widetilde{P}$, $\widetilde{Q}$ and $\widetilde{\Delta}$ be defined in a similar way. Now the assumptions (A1) and (A2) become $\widetilde{P}(\alpha,\beta_0(\alpha))=0$ and, respectively, the matrix $D\widetilde{P}(\alpha,\beta_0(\alpha))$ has in its upper right corner the null $k\times(n-k)$ matrix and in the lower right corner the $(n-k)\times(n-k)$ invertible matrix $\widetilde{\Delta}(\alpha,\beta_0(\alpha))$ for any $\alpha\in\overline{V}$. Then

$$\widetilde{F}\big(\alpha,\beta_0(\alpha),0\big)=0\quad\text{for any }\alpha\in\overline{V},$$

and

$$\det\big(D_\beta\big(\pi^{\perp}\widetilde{F}\big)\big(\alpha,\beta_0(\alpha),0\big)\big)=\det\big(\widetilde{\Delta}\big(\alpha,\beta_0(\alpha)\big)\big)\neq 0\quad\text{for any }\alpha\in\overline{V}.\tag{5}$$

It follows from (5) that there exists a radius $\delta>0$ such that

$$\pi^{\perp}\widetilde{F}(\alpha,\beta,0)\neq 0\quad\text{for any }\beta\in\overline{B_\delta\big(\beta_0(\alpha)\big)}\backslash\big\{\beta_0(\alpha)\big\},\ \alpha\in\overline{V}.\tag{6}$$

The relations (5) and (6) give (see [18, Theorem 6.3])

$$d\big(\pi^{\perp}\widetilde{F}(\alpha,\cdot,0),B_\delta\big(\beta_0(\alpha)\big)\big)=\operatorname{sign}\big(\det\big(\widetilde{\Delta}\big(\alpha,\beta_0(\alpha)\big)\big)\big)\neq 0,\quad\alpha\in\overline{V}.$$

Hence, by the continuity of the topological degree with respect to parameters (using the compactness of $\overline{V}$) there exists $\varepsilon(\delta)>0$ such that

$$d\big(\pi^{\perp}\widetilde{F}(\alpha,\cdot,\varepsilon),B_\delta\big(\beta_0(\alpha)\big)\big)\neq 0\quad\text{for any }\varepsilon\in\big[0,\varepsilon(\delta)\big],\ \alpha\in\overline{V}.$$

This assures the existence of $\beta(\alpha,\varepsilon)\in B_\delta(\beta_0(\alpha))$ such that conclusion (C4) holds with $\delta_0=\delta$ and $\varepsilon_0=\varepsilon(\delta_0)$.

Without loss of generality we can consider in the sequel that $\varepsilon(\delta)\to 0$ as $\delta\to 0$. The value of the radius δ eventually may decrease in a finite number of steps during this proof (consequently, also the value of $\varepsilon(\delta)$). Sometimes we decrease only the value of $\varepsilon(\delta)$, letting δ maintaining its value. Without explicitly mentioning it, finally, in the statement of the lemma, we replace δ_0 by the least value of the radius δ and ε_0 by $\varepsilon(\delta)$.

(C5) Since P and β_0 are C^1 and $\overline{V}$ is bounded, there exists $\eta>0$ such that the invertible matrix Δ defined by (A2) satisfies $\|\widetilde{\Delta}(\alpha,\beta_0(\alpha))\|\geqslant 2\eta$ for all $\alpha\in\overline{V}$. Using again that P is C^1 and $\widetilde{\Delta}(\alpha,\beta_0(\alpha))=D_\beta(\pi^{\perp}\widetilde{P})(\alpha,\beta_0(\alpha))$, we obtain that the radius $\delta>0$ found before at (C4) can be decreased, if necessary, in such a way that $\|\widetilde{\Delta}(\alpha,\beta_0(\alpha))-D_\beta(\pi^{\perp}\widetilde{P})(\alpha,\beta)\|\leqslant\eta$ for all $\beta\in B_\delta(\beta_0(\alpha))$ and $\alpha\in\overline{V}$. Then $\|D_\beta(\pi^{\perp}\widetilde{P})(\alpha,\beta)\|\geqslant\eta$ for all $\beta\in B_\delta(\beta_0(\alpha))$, $\alpha\in\overline{V}$. Applying the generalized Mean Value Theorem (see [7, Proposition 2.6.5]) to the function $\pi^{\perp}\widetilde{P}(\alpha,\cdot)$, we obtain

$$\big\|\pi^{\perp}\widetilde{P}(\alpha,\beta_1)-\pi^{\perp}\widetilde{P}(\alpha,\beta_2)\big\|\geqslant\eta\|\beta_1-\beta_2\|,\quad\beta_1,\beta_2\in B_\delta\big(\beta_0(\alpha)\big),\ \alpha\in\overline{V}.\tag{7}$$

We take $M_Q > 0$ such that $\|\widetilde{Q}(\alpha, \beta(\alpha,\varepsilon), \varepsilon)\| \leqslant M_Q$ for all $\alpha \in \overline{V}$ and $\varepsilon \in [0, \varepsilon_0]$. Using (7) we obtain for all $\alpha \in \overline{V}$ and $\varepsilon \in [0, \varepsilon(\delta)]$

$$0 = \left\|\pi^\perp \widetilde{P}\big(\alpha, \beta(\alpha,\varepsilon)\big) - \pi^\perp \widetilde{P}\big(\alpha, \beta_0(\alpha)\big) + \varepsilon \pi^\perp \widetilde{Q}\big(\alpha, \beta(\alpha,\varepsilon), \varepsilon\big)\right\|$$
$$\geqslant \eta \left\|\beta(\alpha,\varepsilon) - \beta_0(\alpha)\right\| - \varepsilon M_Q .$$

From these last relations, denoting $m = M_Q/\eta$, we deduce that

$$\left\|\mu(\alpha,\varepsilon)\right\| \leqslant m \quad \text{for all } \alpha \in \overline{V},\ \varepsilon \in \big(0, \varepsilon(\delta)\big]. \tag{8}$$

We choose $L_Q > 0$ such that

$$\left\|\widetilde{Q}(\alpha_2, \beta_2, \varepsilon) - \widetilde{Q}(\alpha_1, \beta_1, \varepsilon)\right\| \leqslant L_Q \big(\|\alpha_2 - \alpha_1\| + \|\beta_2 - \beta_1\|\big), \tag{9}$$

for all $\beta_1, \beta_2 \in B_{\delta_0}(\beta_0(\overline{V}))$, $\alpha_1, \alpha_2 \in \overline{V}$, $\varepsilon \in [0, \varepsilon_0]$. We decrease $\delta > 0$ in such a way that $\eta - \varepsilon L_Q > 0$ for any $\varepsilon \in [0, \varepsilon(\delta)]$.

Let $\alpha \in \overline{V}$, $\varepsilon \in [0, \varepsilon(\delta)]$ and assume that $\beta(\alpha,\varepsilon)$ and β_2 are two zeros of $\pi^\perp F\big(S\binom{\alpha}{\cdot}, \varepsilon\big)$ in $B_\delta(\beta_0(\alpha))$. Taking into account (7) and (9), we obtain

$$0 = \left\|\pi^\perp \widetilde{P}(\alpha, \beta_2) - \pi^\perp \widetilde{P}\big(\alpha, \beta(\alpha,\varepsilon)\big) + \varepsilon \pi^\perp \widetilde{Q}(\alpha, \beta_2, \varepsilon) - \varepsilon \pi^\perp \widetilde{Q}\big(\alpha, \beta(\alpha,\varepsilon), \varepsilon\big)\right\|$$
$$\geqslant (\eta - \varepsilon L_Q)\left\|\beta_2 - \beta(\alpha,\varepsilon)\right\|.$$

Since $\eta - \varepsilon L_Q > 0$ for any $\varepsilon \in [0, \varepsilon(\delta)]$ we deduce from this last relation that β_2 and $\beta(\alpha,\varepsilon)$ must coincide.

We prove in the sequel the continuity of the function $\beta : \overline{V} \times [0, \varepsilon(\delta)] \to \mathbb{R}^{n-k}$. Let $(\alpha_1, \varepsilon_1) \in \overline{V} \times [0, \varepsilon(\delta)]$ be fixed and $(\alpha, \varepsilon) \in \overline{V} \times [0, \varepsilon(\delta)]$ be in a small neighborhood of $(\alpha_1, \varepsilon_1)$. Consider $L_P > 0$ such that $\|\widetilde{P}(\alpha_1, \beta) - \widetilde{P}(\alpha, \beta)\| \leqslant L_P \|\alpha_1 - \alpha\|$ for all $\alpha_1, \alpha \in \overline{V}$ and $\beta \in B_{\delta_0}(\beta_0(\overline{V}))$. We diminish $\varepsilon(\delta) > 0$, if necessary, and we consider α so close to α_1 that $\beta(\alpha,\varepsilon) \in B_\delta(\beta_0(\alpha_1))$. Then using (7) and (9) we obtain

$$0 = \left\|\pi^\perp \widetilde{P}\big(\alpha_1, \beta(\alpha_1,\varepsilon_1)\big) - \pi^\perp \widetilde{P}\big(\alpha, \beta(\alpha,\varepsilon)\big)\right.$$
$$\left. + \varepsilon_1 \pi^\perp \widetilde{Q}\big(\alpha_1, \beta(\alpha_1,\varepsilon_1), \varepsilon_1\big) - \varepsilon \pi^\perp \widetilde{Q}\big(\alpha, \beta(\alpha,\varepsilon), \varepsilon\big)\right\|$$
$$\geqslant \eta \left\|\beta(\alpha_1,\varepsilon_1) - \beta(\alpha,\varepsilon)\right\| - L_P \|\alpha_1 - \alpha\|$$
$$- \left\|\varepsilon_1 \pi^\perp \widetilde{Q}\big(\alpha_1, \beta(\alpha_1,\varepsilon_1), \varepsilon_1\big) - \varepsilon \pi^\perp \widetilde{Q}\big(\alpha, \beta(\alpha,\varepsilon), \varepsilon\big)\right\|$$

and

$$-\left\|\varepsilon_1 \pi^\perp \widetilde{Q}\big(\alpha_1, \beta(\alpha_1,\varepsilon_1), \varepsilon_1\big) - \varepsilon \pi^\perp \widetilde{Q}\big(\alpha, \beta(\alpha,\varepsilon), \varepsilon\big)\right\|$$
$$\geqslant -\varepsilon_1 L_Q \|\alpha_1 - \alpha\| - \varepsilon_1 L_Q \left\|\beta(\alpha_1,\varepsilon_1) - \beta(\alpha,\varepsilon)\right\|$$
$$- \left\|\varepsilon_1 \pi^\perp \widetilde{Q}\big(\alpha, \beta(\alpha,\varepsilon), \varepsilon_1\big) - \varepsilon \pi^\perp \widetilde{Q}\big(\alpha, \beta(\alpha,\varepsilon), \varepsilon\big)\right\|.$$

Combining these last two relations we obtain

$$(\eta - \varepsilon_1 L_Q)\left\|\beta(\alpha_1,\varepsilon_1) - \beta(\alpha,\varepsilon)\right\| \leqslant (L_P + \varepsilon_1 L_Q)\|\alpha_1 - \alpha\|$$
$$+ \left\|\varepsilon_1 \pi^\perp \widetilde{Q}\big(\alpha, \beta(\alpha,\varepsilon), \varepsilon_1\big) - \varepsilon \pi^\perp \widetilde{Q}\big(\alpha, \beta(\alpha,\varepsilon), \varepsilon\big)\right\|,$$

from where it follows easily that $\beta(\alpha,\varepsilon) \to \beta(\alpha_1,\varepsilon_1)$ when $(\alpha,\varepsilon) \to (\alpha_1,\varepsilon_1)$.

(C6) We define $\Phi(\alpha,\xi)=\pi^{\perp}\widetilde{P}(\alpha,\beta_0(\alpha)+\xi)$ for all $\alpha\in\overline{V}$ and $\xi\in\mathbb{R}^{n-k}$. From (7) we have that

$$\left\|\Phi(\alpha,\xi_1)-\Phi(\alpha,\xi_2)\right\|\geqslant\eta\|\xi_1-\xi_2\|\quad\text{for all }\alpha\in\overline{V},\ \xi_1,\xi_2\in B_\delta(0).\tag{10}$$

Since $\widetilde{P}(\alpha,\beta_0(\alpha))=0$ for all $\alpha\in\overline{V}$, we have that $\Phi(\alpha,\xi)=\pi^{\perp}\widetilde{P}(\alpha,\beta_0(\alpha)+\xi)-\pi^{\perp}\widetilde{P}(\alpha,\beta_0(\alpha))$ and that

$$\begin{aligned}D_\alpha\Phi(\alpha,\xi)={}&D_\alpha\big(\pi^{\perp}\widetilde{P}\big)\big(\alpha,\beta_0(\alpha)+\xi\big)-D_\alpha\big(\pi^{\perp}\widetilde{P}\big)\big(\alpha,\beta_0(\alpha)\big)\\&+\big[D_\beta\big(\pi^{\perp}\widetilde{P}\big)\big(\alpha,\beta_0(\alpha)+\xi\big)-D_\beta\big(\pi^{\perp}\widetilde{P}\big)\big(\alpha,\beta_0(\alpha)\big)\big]D\beta_0(\alpha).\end{aligned}$$

From this expression, using that $\widetilde{P}$ is twice differentiable in $(\alpha,\beta_0(\alpha))$ and β_0 is C^1, we obtain for some $L_\Phi>0$ that the radius δ can be eventually decreased in a such way that

$$\left\|D_\alpha\Phi(\alpha,\xi)\right\|\leqslant L_\Phi\|\xi\|\quad\text{for all }\alpha\in\overline{V},\ \xi\in B_\delta(0).$$

Hence using the mean value inequality we have

$$\left\|\Phi(\alpha_1,\xi)-\Phi(\alpha_2,\xi)\right\|\leqslant L_\phi\|\xi\|\cdot\|\alpha_1-\alpha_2\|\quad\text{for all }\alpha_1,\alpha_2\in\overline{V},\ \xi\in B_\delta(0).\tag{11}$$

Now we use (10) with $\xi_1=\varepsilon\mu(\alpha_1,\varepsilon)$, $\xi_2=\varepsilon\mu(\alpha_2,\varepsilon)$ diminishing $\varepsilon(\delta)$, if necessary, in order that $\xi_1,\xi_2\in B_\delta(0)$ for all $\alpha_1,\alpha_2\in\overline{V}$ and $\varepsilon\in(0,\varepsilon(\delta)]$. Using also (C5), (8) and (11) we obtain

$$\begin{aligned}\left\|\pi^{\perp}\widetilde{P}\big(\alpha_1,\beta(\alpha_1,\varepsilon)\big)-\pi^{\perp}\widetilde{P}\big(\alpha_2,\beta(\alpha_2,\varepsilon)\big)\right\|&=\left\|\Phi(\alpha_1,\xi_1)-\Phi(\alpha_2,\xi_2)\right\|\\&\geqslant\eta\|\xi_1-\xi_2\|-L_\Phi\|\xi_1\|\cdot\|\alpha_1-\alpha_2\|\\&\geqslant\eta\varepsilon\left\|\mu(\alpha_1,\varepsilon)-\mu(\alpha_2,\varepsilon)\right\|-L_\Phi m\varepsilon\|\alpha_1-\alpha_2\|,\end{aligned}\tag{12}$$

for all $\alpha_1,\alpha_2\in\overline{V}$ and $\varepsilon\in(0,\varepsilon(\delta)]$. Also using (9) we have

$$\begin{aligned}&\left\|\pi^{\perp}\widetilde{Q}\big(\alpha_1,\beta(\alpha_1,\varepsilon),\varepsilon\big)-\pi^{\perp}\widetilde{Q}\big(\alpha_2,\beta(\alpha_2,\varepsilon),\varepsilon\big)\right\|\\&\quad\leqslant\varepsilon L_Q\left\|\mu(\alpha_1,\varepsilon)-\mu(\alpha_2,\varepsilon)\right\|+L_Q(1+L_{\beta_0})\|\alpha_1-\alpha_2\|,\end{aligned}\tag{13}$$

for all $\alpha_1,\alpha_2\in\overline{V}$ and $\varepsilon\in(0,\varepsilon(\delta)]$, where L_{β_0} is the Lipschitz constant of β_0 in $\overline{V}$. By definition of $\beta(\alpha,\varepsilon)$ we have $\pi^{\perp}\widetilde{P}(\alpha_i,\beta(\alpha_i,\varepsilon))+\varepsilon\pi^{\perp}\widetilde{Q}(\alpha_i,\beta(\alpha_i,\varepsilon),\varepsilon)=0$ for $i\in\overline{1,2}$. Using (12) and (13) we obtain

$$0\geqslant\varepsilon[\eta-\varepsilon L_Q]\cdot\left\|\mu(\alpha_1,\varepsilon)-\mu(\alpha_2,\varepsilon)\right\|-\varepsilon\big[L_\Phi m+L_Q(1+L_{\beta_0})\big]\cdot\|\alpha_1-\alpha_2\|,$$

for all $\alpha_1,\alpha_2\in\overline{V}$ and $\varepsilon\in(0,\varepsilon(\delta)]$. Therefore $\mu:\overline{V}\times(0,\varepsilon(\delta)]\to\mathbb{R}^{n-k}$ satisfies (C6) with $L_\mu=[L_\Phi m+L_Q(1+L_{\beta_0})]/[\eta-\varepsilon(\delta)L_Q]$. Hence all the conclusions hold with $\delta_0=\delta$ and $\varepsilon_0=\varepsilon(\delta)$. □

We remark that (C4) and the uniqueness part of (C5) can be obtained by means of the Lipschitz generalization of the Inverse Function Theorem (see e.g. [16, Theorem 5.3.8]), but we provide a different proof because the inequalities (7) and (8) are used for proving the rest of (C5) and (C6).

Proof of Theorem 1. Let δ_0, ε_0, $\beta(\alpha,\varepsilon)$ and $\mu(\alpha,\varepsilon)$ be as in Lemma 2. We consider the notations $\widetilde{F}$, $\widetilde{P}$ and $\widetilde{Q}$ like in the proof of Lemma 2.

(C1) Let the sequences $(z_m)_{m\geqslant 1}$ from $\mathbb{R}^n$ and $(\varepsilon_m)_{m\geqslant 1}$ from $[0,1]$ be such that $z_m \to z_0 \in \mathcal{Z}$, $\varepsilon_m \to 0$ as $m \to \infty$ and $F(z_m, \varepsilon_m) = 0$ for any $m \geqslant 1$. We define $\alpha_0 \in \mathbb{R}^k$, the sequences $(\alpha_m)_{m\geqslant 1}$ from $\mathbb{R}^k$ and $(\beta_m)_{m\geqslant 1}$ from $\mathbb{R}^{n-k}$ by $z_0 = S\binom{\alpha_0}{\beta_0(\alpha_0)}$ and $z_m = S\binom{\alpha_m}{\beta_m}$. Then we have that $\alpha_0 = \lim_{m\to\infty} \alpha_m$, $\beta_0(\alpha_0) = \lim_{m\to\infty} \beta_m$ and there exists $m_0 \in \mathbb{N}$ such that $\beta_m \in B_{\delta_0}(\beta_0(\alpha_m))$ and $\varepsilon_m \in [0, \varepsilon_0]$ for all $m \geqslant m_0$. Therefore, since $F(z_m, \varepsilon_m) = 0$, Lemma 2 implies $\beta_m = \beta(\alpha_m, \varepsilon_m)$ for any $m \geqslant m_0$. Since $\pi \widetilde{P}(\alpha_m, \beta_0(\alpha_m)) = 0$ and $D_\beta(\pi \widetilde{P})(\alpha_m, \beta_0(\alpha_m)) = 0$, we obtain that $\lim_{m\to\infty} \frac{1}{\varepsilon_m} \pi \widetilde{P}(\alpha_m, \beta(\alpha_m, \varepsilon_m)) = 0$. Hence

$$
\begin{aligned}
0 &= \lim_{m\to\infty} \frac{1}{\varepsilon_m} \pi \widetilde{F}\big(\alpha_m, \beta(\alpha_m, \varepsilon_m), \varepsilon_m\big) \\
&= \lim_{m\to\infty} \left[\frac{1}{\varepsilon_m} \pi \widetilde{P}\big(\alpha_m, \beta(\alpha_m, \varepsilon_m)\big) + \pi \widetilde{Q}\big(\alpha_m, \beta(\alpha_m, \varepsilon_m), \varepsilon_m\big) \right] = \widehat{Q}(\alpha_0)
\end{aligned}
$$

from where (C1) follows.

(C2) Using (C4) of Lemma 2, we note that it is enough to prove the existence of at least one zero in V of the function $\alpha \mapsto \pi \widetilde{F}(\alpha, \beta(\alpha, \varepsilon), \varepsilon)$ for each $\varepsilon \in (0, \varepsilon_1]$ where ε_1 with $0 < \varepsilon_1 \leqslant \varepsilon_0$ has to be found. This will follow from the claim that the Brouwer topological degree $d(\frac{1}{\varepsilon} \pi \widetilde{F}(\cdot, \beta(\cdot, \varepsilon), \varepsilon), V) \neq 0$ for $\varepsilon \in (0, \varepsilon_1]$. Now we prove this claim. Since $\beta(\alpha, \varepsilon) = \beta_0(\alpha) + \varepsilon \mu(\alpha, \varepsilon)$ with $\mu : \overline{V} \times (0, \varepsilon_0] \to \mathbb{R}^{n-k}$ a bounded function, $\pi \widetilde{P}(\alpha, \beta_0(\alpha)) = 0$ and $D_\beta(\pi \widetilde{P})(\alpha, \beta_0(\alpha)) = 0$, we have

$$
\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \pi \widetilde{P}\big(\alpha, \beta(\alpha, \varepsilon)\big) = 0.
$$

Therefore

$$
\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \pi \widetilde{F}\big(\alpha, \beta(\alpha, \varepsilon), \varepsilon\big) = \lim_{\varepsilon\to 0} \left[\frac{1}{\varepsilon} \pi \widetilde{P}\big(\alpha, \beta(\alpha, \varepsilon)\big) + \pi \widetilde{Q}\big(\alpha, \beta(\alpha, \varepsilon), \varepsilon\big) \right] = \widehat{Q}(\alpha).
$$

Using the continuity of the Brouwer degree with respect to the parameter ε, and taking into account that, by hypothesis, $d(\widehat{Q}, V) \neq 0$, for each $\varepsilon \in (0, \varepsilon_1]$ there exists $\varepsilon_1 > 0$ sufficiently small such that

$$
d\left(\frac{1}{\varepsilon} \pi \widetilde{F}\big(\cdot, \beta(\cdot, \varepsilon), \varepsilon\big), V \right) = d(\widehat{Q}, V) \neq 0.
$$

Hence the claim is proved. Then for each $\varepsilon \in (0, \varepsilon_1]$ there exists $\alpha_\varepsilon \in V$ such that $\pi \widetilde{F}(\alpha_\varepsilon, \beta(\alpha_\varepsilon, \varepsilon), \varepsilon) = 0$ and, moreover, using also (C4) of Lemma 2, we have that $\widetilde{F}(\alpha_\varepsilon, \beta(\alpha_\varepsilon, \varepsilon), \varepsilon) = 0$. Denoting $z_\varepsilon = S\binom{\alpha_\varepsilon}{\beta(\alpha_\varepsilon, \varepsilon)}$ we have that $F(z_\varepsilon, \varepsilon) = 0$. From the definitions of z_ε and $\mathcal{Z}$, and the continuity of β, it follows easily that $\rho(z_\varepsilon, \mathcal{Z}) \to 0$ as $\varepsilon \to 0$.

(C3) Since $\alpha_0 \in V$ is an isolated zero of $\widehat{Q}$, applying the topological degree arguments like in (C2) for V that shrinks to $\{\alpha_0\}$, we obtain the existence of α_ε such that $\alpha_\varepsilon \to \alpha_0$ as $\varepsilon \to 0$, and $\pi \widetilde{F}(\alpha_\varepsilon, \beta(\alpha_\varepsilon, \varepsilon), \varepsilon) = 0$ for any $\varepsilon \in (0, \varepsilon_1]$. Hence $z_\varepsilon = S\binom{\alpha_\varepsilon}{\beta(\alpha_\varepsilon, \varepsilon)}$ and $z_0 = S\binom{\alpha_0}{\beta_0(\alpha_0)} \in \mathcal{Z}$ are such that $F(z_\varepsilon, \varepsilon) = 0$ and $z_\varepsilon \to z_0$ as $\varepsilon \to 0$.

In order to prove that z_ε is the unique zero of $F(\cdot, \varepsilon)$ in a neighborhood of z_0, we define

$$
r_1(\alpha, \varepsilon) = \frac{1}{\varepsilon} \pi \widetilde{P}\big(\alpha, \beta(\alpha, \varepsilon)\big), \qquad r_2(\alpha, \varepsilon) = \pi \widetilde{Q}\big(\alpha, \beta(\alpha, \varepsilon), \varepsilon\big) - \pi \widetilde{Q}\big(\alpha, \beta_0(\alpha), 0\big),
$$

for all $\alpha \in \overline{V}$ and $\varepsilon \in (0, \varepsilon_1]$, and we study the Lipschitz properties with respect to α of these two functions.

Since $\widetilde{P}(\alpha, \beta_0(\alpha)) = 0$ for all $\alpha \in \overline{V}$, by taking the derivative with respect to α we obtain

$$
D_\alpha(\pi \widetilde{P})\big(\alpha, \beta_0(\alpha)\big) + D_\beta(\pi \widetilde{P})\big(\alpha, \beta_0(\alpha)\big) D\beta_0(\alpha) = 0 \quad \text{for all } \alpha \in \overline{V}. \tag{14}
$$

Assumption (A2) assures that $D_\beta(\pi\widetilde{P})(\alpha,\beta_0(\alpha))=0$ for all $\alpha\in\overline{V}$. Taking the derivative with respect to α, we have

$$D_{\beta\alpha}(\pi\widetilde{P})\big(\alpha,\beta_0(\alpha)\big)+D_{\beta\beta}(\pi\widetilde{P})\big(\alpha,\beta_0(\alpha)\big)D\beta_0(\alpha)=0\quad\text{for any }\alpha\in\overline{V}. \tag{15}$$

For any $\alpha\in\overline{V}$ and $\xi\in\mathbb{R}^{n-k}$ we define $\Phi(\alpha,\xi)=\pi\widetilde{P}(\alpha,\beta_0(\alpha)+\xi)$. Taking into account the relations (14) and (15) and that, by hypothesis (A3) we have that $D_{\beta\alpha}(\pi\widetilde{P})(\alpha,\beta_0(\alpha))=D_{\alpha\beta}(\pi\widetilde{P})(\alpha,\beta_0(\alpha))$, we obtain

$$\begin{aligned}D_\alpha\Phi(\alpha,\xi)=\;&D_\alpha(\pi\widetilde{P})\big(\alpha,\beta_0(\alpha)+\xi\big)+D_\beta(\pi\widetilde{P})\big(\alpha,\beta_0(\alpha)+\xi\big)D\beta_0(\alpha)\\&-D_\alpha(\pi\widetilde{P})\big(\alpha,\beta_0(\alpha)\big)-D_\beta(\pi\widetilde{P})\big(\alpha,\beta_0(\alpha)\big)D\beta_0(\alpha)\\&-D_{\alpha\beta}(\pi\widetilde{P})\big(\alpha,\beta_0(\alpha)\big)\xi-D_{\beta\beta}(\pi\widetilde{P})\big(\alpha,\beta_0(\alpha)\big)D\beta_0(\alpha)\xi.\end{aligned}$$

From this last equality, using that $D_\alpha(\pi\widetilde{P})$ and, respectively, $D_\beta(\pi\widetilde{P})$ are differentiable at $(\alpha,\beta_0(\alpha))$, we deduce that $D_\alpha\Phi(\alpha,\xi)=o(\xi)$ for all $\alpha\in\overline{V}$ and $\xi\in\mathbb{R}^{n-k}$ with $\|\xi\|$ sufficiently small. Hence the mean value inequality assures that

$$\big\|\Phi(\alpha_1,\xi)-\Phi(\alpha_2,\xi)\big\|\leqslant o(\xi)\|\alpha_1-\alpha_2\|\quad\text{for all }\alpha_1,\alpha_2\in\overline{V}.$$

In the last inequality we replace $\xi=\varepsilon\mu(\alpha_1,\varepsilon)$ (where μ is given by Lemma 2). We use that $D_\xi\Phi(\alpha,0)=D_\beta\pi\widetilde{P}(\alpha,\beta_0(\alpha))=0$ for any $\alpha\in\overline{V}$, and that μ is Lipschitz with respect to $\alpha\in\overline{V}$. Then we obtain, considering that ε_1 is small enough, for all $\varepsilon\in(0,\varepsilon_1]$

$$\big\|\Phi\big(\alpha_1,\varepsilon\mu(\alpha_1,\varepsilon)\big)-\Phi\big(\alpha_2,\varepsilon\mu(\alpha_2,\varepsilon)\big)\big\|\leqslant o(\varepsilon)\|\alpha_1-\alpha_2\|\quad\text{for all }\alpha_1,\alpha_2\in V.$$

Now coming back to our notations and recalling that $\beta(\alpha,\varepsilon)=\beta_0(\alpha)+\varepsilon\mu(\alpha,\varepsilon)$, we obtain for $\varepsilon\in(0,\varepsilon_1]$

$$\big\|r_1(\alpha_1,\varepsilon)-r_1(\alpha_2,\varepsilon)\big\|\leqslant\frac{o(\varepsilon)}{\varepsilon}\|\alpha_1-\alpha_2\|\quad\text{for all }\alpha_1,\alpha_2\in\overline{V}. \tag{16}$$

We will prove that a similar relation holds for the function r_2. First we note that the hypothesis (A5) and the fact that Q is locally uniformly Lipschitz with respect to the first variable imply that

$$\begin{aligned}&\big\|\pi Q(z_1+\zeta_1,\varepsilon)-\pi Q(z_1,0)-\pi Q(z_2+\zeta_2,\varepsilon)+\pi Q(z_2,0)\big\|\\&\quad\leqslant\tilde{o}(\delta)\|z_1-z_2\|+L_Q\|\zeta_1-\zeta_2\|,\end{aligned} \tag{17}$$

for all $z_1,z_2\in B_\delta(z_0)\cap\mathcal{Z}$, $\varepsilon\in[0,\delta]$ and $\zeta_1,\zeta_2\in B_\delta(0)$. We diminish $\delta_1>0$ given in (A4) and $\varepsilon_1>0$ in such a way that $\delta_1\leqslant\delta$, $\varepsilon_1\leqslant\delta$, $S\binom{\alpha}{\beta_0(\alpha)}\in B_\delta(z_0)$ and $S\binom{0_{k\times1}}{\varepsilon\mu(\alpha,\varepsilon)}\in B_\delta(0)$ for any $\alpha\in B_{\delta_1}(\alpha_0)$, $\varepsilon\in(0,\varepsilon_1]$. Replacing $z_i=S\binom{\alpha_i}{\beta_0(\alpha_i)}$, $\zeta_i=S\binom{0_{k\times1}}{\varepsilon\mu(\alpha_i,\varepsilon)}$, $i\in\overline{1,2}$, in (17) we obtain that

$$\big\|r_2(\alpha_1,\varepsilon)-r_2(\alpha_2,\varepsilon)\big\|\leqslant\tilde{o}(\delta)\big(\|\alpha_1-\alpha_2\|+\big\|\beta_0(\alpha_1)-\beta_0(\alpha_2)\big\|\big)+\varepsilon L_Q\big\|\mu(\alpha_1,\varepsilon)-\mu(\alpha_2,\varepsilon)\big\|,$$

for all $\alpha_1,\alpha_2\in B_{\delta_1}(\alpha_0)$ and $\varepsilon\in(0,\varepsilon_1]$. By hypothesis, β_0 is C^1 in $\overline{V}$ and, by Lemma 2 (conclusion (C6)), $(\alpha,\varepsilon)\mapsto\mu(\alpha,\varepsilon)$ is Lipschitz with respect to $\alpha\in\overline{V}$ (with a Lipschitz constant that does not depend on ε). Hence for $\delta_1,\varepsilon_1\leqslant\delta$ small enough,

$$\big\|r_2(\alpha_1,\varepsilon)-r_2(\alpha_2,\varepsilon)\big\|\leqslant\tilde{o}(\delta)\|\alpha_1-\alpha_2\|,\quad\alpha_1,\alpha_2\in B_{\delta_1}(\alpha_0),\ \varepsilon\in(0,\varepsilon_1]. \tag{18}$$

Therefore we have proved that r_1 and r_2 satisfy the Lipschitz conditions (16) and, respectively, (18). In what follows we define some constant $\delta_2 > 0$, and after we prove that it is the one that satisfies the requirements of (C3).

We diminish $\delta_1 > 0$ in such a way that there exists $\delta_3 > 0$ such that $\delta_3 \leqslant \delta_0$ and $B_{\delta_3}(\beta_0(\alpha_0)) \subset \bigcap_{\alpha \in B_{\delta_1}(\alpha_0)} B_{\delta_0}(\beta_0(\alpha))$. We choose $\delta_2 > 0$ so small that $S^{-1}(B_{\delta_2}(z_0)) \subset B_{\delta_1}(\alpha_0) \times B_{\delta_3}(\beta_0(\alpha_0))$. We diminish $\varepsilon_1 > 0$, if necessary, such that $z_\varepsilon \in B_{\delta_2}(z_0)$ for any $\varepsilon \in (0, \varepsilon_1]$. For any $\varepsilon \in (0, \varepsilon_1]$ we claim that z_ε is the only zero of $F(\cdot, \varepsilon)$ in $B_{\delta_2}(z_0)$. Assume by contradiction that there exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that z_{ε_2} and z_2 are two different zeros of $F(\cdot, \varepsilon_2)$ in $B_{\delta_2}(z_0)$. Denoting $\alpha_2 = \pi S^{-1} z_2$ and $\beta_2 = \pi^\perp S^{-1} z_2$ we have that $\beta_2 \in B_{\delta_0}(\beta_0(\alpha_2))$. By (C5) of Lemma 2, since β_2 is a zero of $\pi^\perp F\big(S\big(\begin{smallmatrix}\alpha_2\\ \cdot\end{smallmatrix}\big), \varepsilon_2\big)$ (using the notations introduced before, $\pi^\perp \widetilde{F}(\alpha_2, \cdot, \varepsilon_2)$), we must have that $\beta_2 = \beta(\alpha_2, \varepsilon_2)$. Therefore α_{ε_2} and α_2 are two different zeros of $\pi \widetilde{F}(\cdot, \beta(\cdot, \varepsilon_2), \varepsilon_2)$ in $B_{\delta_1}(\alpha_0)$. We have the identity

$$\frac{1}{\varepsilon} \pi \widetilde{F}\big(\alpha, \beta(\alpha, \varepsilon), \varepsilon\big) = \widehat{Q}(\alpha) + r_1(\alpha, \varepsilon) + r_2(\alpha, \varepsilon) \quad \text{for all } \alpha \in \overline{V},\ \varepsilon \in (0, \varepsilon_1].$$

We denote $r(\alpha, \varepsilon) = r_1(\alpha, \varepsilon) + r_2(\alpha, \varepsilon)$. Then assumption (A4), properties (16) and (18) give

$$0 = \big\| \widehat{Q}(\alpha_{\varepsilon_2}) - \widehat{Q}(\alpha_2) + r(\alpha_{\varepsilon_2}, \varepsilon_2) - r(\alpha_2, \varepsilon_2) \big\| \geqslant \big(L_{\widehat{Q}} - o(\varepsilon_2)/\varepsilon_2 - \tilde{o}(\delta)\big) \|\alpha_{\varepsilon_2} - \alpha_2\|.$$

Since $\varepsilon_1 > 0$ and $\delta > 0$ are sufficiently small and $0 < \varepsilon_2 \leqslant \varepsilon_1$, the constant $(L_{\widehat{Q}} - o(\varepsilon_2)/\varepsilon_2 - \tilde{o}(\delta))$ must be positive and, consequently, α_{ε_2} and α_2 must coincide. Hence also z_{ε_2} and z_2 must coincide and we conclude the proof. □

4. Bifurcation of *T*-periodic solutions from *T*-periodic families in differential equations with nonsmooth Lipschitz right-hand sides

In this section we provide a perturbation result for the T-periodic differential system

$$\dot{x} = f(t, x) + \varepsilon g(t, x, \varepsilon), \tag{19}$$

where $f \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $g \in C^0(\mathbb{R} \times \mathbb{R}^n \times [0, 1], \mathbb{R}^n)$ are T-periodic in the first variable and g is locally uniformly Lipschitz with respect to its second variable. For $z \in \mathbb{R}^n$ we denote by $x(\cdot, z, \varepsilon)$ the solution of (19) such that $x(0, z, \varepsilon) = z$. We consider the situation when the unperturbed system

$$\dot{x} = f(t, x) \tag{20}$$

has a nondegenerate (in a sense that will be precised below) family of T-periodic solutions and prove the existence and uniqueness of T-periodic solutions to (19) that emanate from this family.

The main tool for the proof of our main result is Theorem 1. We will show that the assumptions of Theorem 1 can be expressed in terms of the function g and of the solutions of the linear differential system

$$\dot{y} = D_x f\big(t, x(t, z, 0)\big) y. \tag{21}$$

The main result of this paper is Theorem 7 below, which is a Lipschitz analogue of the perturbation results by Malkin [21], Loud [19] and Rhouma and Chicone [25]. The next theorem is an important milestone step towards the proof of Theorem 7 and it can also be of independent interest.

Theorem 3. *Assume that $f \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $g \in C^0(\mathbb{R} \times \mathbb{R}^n \times [0, 1], \mathbb{R}^n)$ are T-periodic in the first variable, and that g is locally uniformly Lipschitz with respect to the second variable. Suppose that the unperturbed system* (20) *satisfies the following conditions.*

(A6) *There exist an invertible $n \times n$ real matrix S, an open ball $V \subset \mathbb{R}^k$ with $k \leqslant n$, and a C^1 function $\beta_0 : \overline{V} \to \mathbb{R}^{n-k}$ such that any point of the set $\mathcal{Z} = \bigcup_{\alpha \in \overline{V}} \{S\binom{\alpha}{\beta_0(\alpha)}\}$ is the initial condition of a T-periodic solution of* (20)*.*

(A7) *For each $z \in \mathcal{Z}$ there exists a fundamental matrix solution $Y(\cdot, z)$ of* (21) *such that $Y(0, z)$ is C^1 with respect to z and such that the matrix $(Y^{-1}(0, z) - Y^{-1}(T, z))S$ has in the upper right corner the null $k \times (n-k)$ matrix, while in the lower right corner has the $(n-k) \times (n-k)$ matrix $\Delta(z)$ with $\det(\Delta(z)) \neq 0$.*

We define the function $G : \overline{V} \to \mathbb{R}^k$ by

$$G(\alpha) = \pi \int_0^T Y^{-1}\left(t, S\begin{pmatrix}\alpha \\ \beta_0(\alpha)\end{pmatrix}\right) g\left(t, x\left(t, S\begin{pmatrix}\alpha \\ \beta_0(\alpha)\end{pmatrix}, 0\right), 0\right) dt. \tag{22}$$

Then the following statements hold.

(C7) *For any sequences $(\varphi_m)_{m \geqslant 1}$ from $C^0(\mathbb{R}, \mathbb{R}^n)$ and $(\varepsilon_m)_{m \geqslant 1}$ from $[0, 1]$ such that $\varphi_m(0) \to z_0 \in \mathcal{Z}$, $\varepsilon_m \to 0$ as $m \to \infty$ and φ_m is a T-periodic solution of* (19) *with $\varepsilon = \varepsilon_m$ for any $m \geqslant 1$, we have that $G(\pi S^{-1} z_0) = 0$.*

(C8) *If $G(\alpha) \neq 0$ for any $\alpha \in \partial V$ and $d(G, V) \neq 0$, then there exists $\varepsilon_1 > 0$ sufficiently small such that for each $\varepsilon \in (0, \varepsilon_1]$ there is at least one T-periodic solution φ_ε of system* (19) *such that $\rho(\varphi_\varepsilon(0), \mathcal{Z}) \to 0$ as $\varepsilon \to 0$.*

In addition we assume that there exists $\alpha_0 \in V$ such that $G(\alpha_0) = 0$, $G(\alpha) \neq 0$ for all $\alpha \in \overline{V} \setminus \{\alpha_0\}$ and $d(G, V) \neq 0$, and we denote $z_0 = S\binom{\alpha_0}{\beta_0(\alpha_0)}$. Moreover we also assume:

(A8) *There exist $\delta_1 > 0$ and $L_G > 0$ such that*

$$\left\|G(\alpha_1) - G(\alpha_2)\right\| \geqslant L_G \|\alpha_1 - \alpha_2\|, \quad \text{for all } \alpha_1, \alpha_2 \in B_{\delta_1}(\alpha_0).$$

(A9) *For $\delta > 0$ sufficiently small there exists $M_\delta \subset [0, T]$ Lebesgue measurable with $mes(M_\delta) = \tilde{o}(\delta)$ such that*

$$\left\|g(t, z_1 + \zeta, \varepsilon) - g(t, z_1, 0) - g(t, z_2 + \zeta, \varepsilon) + g(t, z_2, 0)\right\| \leqslant \tilde{o}(\delta)\|z_1 - z_2\|,$$

for all $t \in [0, T] \setminus M_\delta$ and for all $z_1, z_2 \in B_\delta(z_0)$, $\varepsilon \in [0, \delta]$ and $\zeta \in B_\delta(0)$.

Then the following conclusion holds.

(C9) *There exists $\delta_2 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1]$, φ_ε is the only T-periodic solution of* (19) *with initial condition in $B_{\delta_2}(z_0)$. Moreover $\varphi_\varepsilon(0) \to z_0$ as $\varepsilon \to 0$.*

To prove the theorem we need three preliminary lemmas that are interesting by themselves. For example, in Lemma 5 we prove the existence of the derivative (in $\varepsilon = 0$) with respect to some parameter denoted ε of the solution of some initial value problem without assuming that the system is C^1. We also study the properties of this derivative.

Lemma 4. *Let $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and K_1, K_2 be compact subsets of $\mathbb{R}^n$. Then the following inequality holds for all $x_1^0, x_2^0 \in K_1$, $y_1, y_2 \in K_2$ and $\varepsilon \in [0, 1]$.*

$$\left\|f\left(x_1^0 + \varepsilon y_1\right) - f\left(x_1^0\right) - f\left(x_2^0 + \varepsilon y_2\right) + f\left(x_2^0\right)\right\| \leqslant O(\varepsilon)\left\|x_1^0 - x_2^0\right\| + O(\varepsilon)\|y_1 - y_2\|. \tag{23}$$

In addition for $m > 0$ sufficiently small and $u_1, u_2, v_1, v_2 \in B_m(0) \subset \mathbb{R}^n$ we have

$$\begin{aligned}&\big\| f\big(x_1^0+v_1+\varepsilon y_1^0+\varepsilon u_1\big)-f\big(x_1^0+v_1\big)-\varepsilon f'\big(x_1^0\big)y_1^0\\&\quad -f\big(x_2^0+v_2+\varepsilon y_2^0+\varepsilon u_2\big)+f\big(x_2^0+v_2\big)+\varepsilon f'\big(x_2^0\big)y_2^0\big\|\\&\leqslant \big[o(\varepsilon)+\varepsilon O(m)\big]\big\|x_1^0-x_2^0\big\|+O(\varepsilon)\|v_1-v_2\|\\&\quad +\big[o(\varepsilon)+\varepsilon O(m)\big]\big\|y_1^0-y_2^0\big\|+O(\varepsilon)\|u_1-u_2\|.\end{aligned}\tag{24}$$

Proof. We define $\Phi(x^0,y,\varepsilon)=f(x^0+\varepsilon y)-f(x^0)$ for all $x^0\in\overline{\mathrm{co}}K_1$, $y\in\overline{\mathrm{co}}K_2$ and $\varepsilon\in[0,1]$. Relation (23) follows from the mean value inequality applied to Φ_i with $i\in\overline{1,n}$ and the following estimations:

$$\begin{aligned}&\frac{\partial\Phi_i}{\partial x^0}\big(x^0,y,\varepsilon\big)=(f_i)'\big(x^0+\varepsilon y\big)-(f_i)'\big(x^0\big)=O(\varepsilon)\quad\text{and}\\&\frac{\partial\Phi_i}{\partial y}\big(x^0,y,\varepsilon\big)=\varepsilon(f_i)'\big(x^0+\varepsilon y\big)=O(\varepsilon).\end{aligned}$$

In order to prove relation (24) we define

$$\Phi\big(x^0,v,y^0,u,\varepsilon\big)=f\big(x^0+v+\varepsilon y^0+\varepsilon u\big)-f\big(x^0+v\big)-\varepsilon f'\big(x^0\big)y^0,$$

for all $x^0\in\overline{\mathrm{co}}K_1$, $y^0\in\overline{\mathrm{co}}K_2$, $u,v\in B_m(0)$ and $\varepsilon\in[0,1]$. We apply again the mean value inequality to the components Φ_i, $i\in\overline{1,n}$, using the following estimations:

$$\begin{aligned}\frac{\partial\Phi_i}{\partial x^0}\big(x^0,v,y^0,u,\varepsilon\big)&=(f_i)'\big(x^0+v+\varepsilon y^0+\varepsilon u\big)-(f_i)'\big(x^0+v\big)-\varepsilon(f_i)''\big(x^0\big)y^0\\&=o(\varepsilon)+\varepsilon(f_i)''\big(x^0+v\big)u+\varepsilon\big[(f_i)''\big(x^0+v\big)-(f_i)''\big(x^0\big)\big]y^0\\&=o(\varepsilon)+\varepsilon O(m)+\varepsilon o(m)/m=o(\varepsilon)+\varepsilon O(m),\\\frac{\partial\Phi}{\partial v}\big(x^0,v,y^0,u,\varepsilon\big)&=(f_i)'\big(x^0+v+\varepsilon y^0+\varepsilon u\big)-(f_i)'\big(x^0+v\big)=O(\varepsilon),\\\frac{\partial\Phi_i}{\partial y^0}\big(x^0,v,y^0,u,\varepsilon\big)&=\varepsilon(f_i)'\big(x^0+v+\varepsilon y^0+\varepsilon u\big)-\varepsilon(f_i)'\big(x^0\big)\\&=\varepsilon(f_i)'\big(x^0+v+\varepsilon y^0+\varepsilon u\big)-\varepsilon(f_i)'\big(x^0+v\big)+\varepsilon(f_i)'\big(x^0+v\big)-\varepsilon(f_i)'\big(x^0\big)\\&=o(\varepsilon)+\varepsilon O(m),\\\frac{\partial\Phi}{\partial u}\big(x^0,v,y^0,u,\varepsilon\big)&=\varepsilon(f_i)'\big(x^0+v+\varepsilon y^0+\varepsilon u\big)=O(\varepsilon).\qquad\square\end{aligned}$$

Lemma 5. *We consider $f\in C^2(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ and $g\in C^0(\mathbb{R}\times\mathbb{R}^n\times[0,1],\mathbb{R}^n)$ a locally uniformly Lipschitz function with respect to the second variable. For $z\in\mathbb{R}^n$ and $\varepsilon\in[0,1]$, we denote by $x(\cdot,z,\varepsilon)$ the unique solution of*

$$\dot{x}=f(t,x)+\varepsilon g(t,x,\varepsilon),\quad x(0)=z,$$

and by $y(t,z,\varepsilon)=[x(t,z,\varepsilon)-x(t,z,0)]/\varepsilon$ (here $\varepsilon\neq0$). We assume that for a given $T>0$ there exist a compact set $K\subset\mathbb{R}^n$ with nonempty interior and $\delta>0$ such that $x(t,z,\varepsilon)$ is well defined for all $t\in[0,T]$, $z\in K$ and $\varepsilon\in[0,\delta]$. Then the following statements hold.

(C10) *There exists $y(t,z,0)=\lim_{\varepsilon\to0}y(t,z,\varepsilon)$ being the solution of the initial value problem*

$$\dot{y}(t)=D_xf\big(t,x(t,z,0)\big)y+g\big(t,x(t,z,0),0\big),\quad y(0)=0.$$

The above limit holds uniformly with respect to $(t,z)\in[0,T]\times K$.

(C11) *The functions* $x, y : [0, T] \times K \times [0, \delta] \to \mathbb{R}^n$ *are continuous and uniformly Lipschitz with respect to their second variable.*

(C12) *In addition if there exists* $z_0 \in \operatorname{int}(K)$ *such that assumption* (A9) *of Theorem* 3 *holds with the same small* $\delta > 0$ *as above, then*

$$\left\| y(t, z_1 + \zeta, \varepsilon) - y(t, z_1, 0) - y(t, z_2 + \zeta, \varepsilon) + y(t, z_2, 0)\right\| \leqslant \tilde{o}(\delta)\|z_1 - z_2\|,$$

for all $t \in [0, T]$, $z_1, z_2 \in B_\delta(z_0)$, $\varepsilon \in [0, \delta]$ *and* $\zeta \in B_\delta(0)$.

Proof. (C10) We define $\tilde{f}(t, z, \varepsilon) = \frac{f(t,x(t,z,\varepsilon)) - f(t,x(t,z,0))}{x(t,z,\varepsilon) - x(t,z,0)}$ for $\varepsilon \neq 0$ and $\tilde{f}(t, z, 0) = D_x f(t, x(t, z, 0))$. In this way we obtain the continuous function $\tilde{f} : [0, T] \times K \times [0, \delta] \to \mathbb{R}^n$. For $\varepsilon \neq 0$, using the definitions of $x(t, z, \varepsilon)$ and $y(t, z, \varepsilon)$ we deduce immediately that $y(0, z, \varepsilon) = 0$ and also that

$$\dot{y}(t, z, \varepsilon) = \tilde{f}\big(t, x(t, z, \varepsilon)\big) y(t, z, \varepsilon) + g\big(t, x(t, z, \varepsilon), \varepsilon\big). \tag{25}$$

Passing to the limit as $\varepsilon \to 0$, we obtain that $y(\cdot, z, 0)$ is the solution of the given initial value problem. Hence (25) holds also for $\varepsilon = 0$. Since the right-hand side of (25) is given by a continuous function, we have that the limit $y(t, z, 0) = \lim_{\varepsilon \to 0} y(t, z, \varepsilon)$ holds uniformly with respect to $(t, z) \in [0, T] \times K$.

(C11) The facts that the functions $x, y : [0, T] \times K \times [0, \delta] \to \mathbb{R}^n$ are continuous, and that x is Lipschitz with respect to its second variable can be obtained as a corollary of the general theorem on the dependence of the solutions of an ordinary differential equation on the parameters (see [2, Lemma 8.2]).

It remains to prove that $y : [0, T] \times K \times [0, \delta] \to \mathbb{R}^n$ is uniformly Lipschitz with respect to its second variable.

There exist compact subsets K_1 and K_2 of $\mathbb{R}^n$ such that $x(t, z, \varepsilon) \in K_1$ and $y(t, z, \varepsilon) \in K_2$ for all $(t, z, \varepsilon) \in [0, T] \times K \times [0, \delta]$.

Moreover the representation $x(s, z, \varepsilon) = x(s, z, 0) + \varepsilon y(s, z, \varepsilon)$ allows to use Lemma 4, relation (23) with $x_1^0 = x(s, z_1, 0)$, $x_2^0 = x(s, z_2, 0)$, $y_1 = y(s, z_1, \varepsilon)$, $y_2 = y(s, z_2, \varepsilon)$ in order to obtain

$$\begin{aligned} &\left\| f\big(t, x(t, z_1, \varepsilon)\big) - f\big(t, x(t, z_1, 0)\big) - f\big(t, x(t, z_2, \varepsilon)\big) + f\big(t, x(t, z_2, 0)\big)\right\| \\ &\quad \leqslant O(\varepsilon)\left\|x(t, z_1, 0) - x(t, z_2, 0)\right\| + O(\varepsilon)\left\| y(t, z_1, \varepsilon) - y(t, z_2, \varepsilon)\right\|, \end{aligned}$$

for all $t \in [0, T]$, $z \in K$ and $\varepsilon \in [0, \delta]$. This last inequality and the fact that g is locally uniformly Lipschitz, used together with the representation

$$y(t, z, \varepsilon) = \frac{1}{\varepsilon}\int_0^t \left[f\big(s, x(s, z, \varepsilon)\big) - f\big(s, x(s, z, 0)\big)\right] ds + \int_0^t g\big(s, x(s, z, \varepsilon), \varepsilon\big)\, ds,$$

imply that

$$\begin{aligned} \left\| y(t, z_1, \varepsilon) - y(t, z_2, \varepsilon)\right\| &\leqslant \tilde{o}(\delta)\int_0^t \left\| y(s, z_1, \varepsilon) - y(s, z_2, \varepsilon)\right\| ds \\ &\quad + \tilde{o}(\delta)\int_0^t \left\|x(s, z_1, \varepsilon) - x(s, z_2, \varepsilon)\right\| ds, \end{aligned}$$

for all $t \in [0, T]$, $z_1, z_2 \in K$ and $\varepsilon \in [0, \delta]$.

We use now the fact that the function $x(t,z,\varepsilon)$ is Lipschitz with respect to z and we deduce

$$\left\| y(t,z_1,\varepsilon)-y(t,z_2,\varepsilon)\right\| \leqslant \tilde{o}(\delta)\|z_1-z_2\|+\tilde{o}(\delta)\int_0^t \left\| y(s,z_1,\varepsilon)-y(s,z_2,\varepsilon)\right\| ds.$$

Applying Grönwall lemma (see [11, Lemma 6.2] or [8, Ch. 2, Lemma § 11]) we finally have for all $t\in[0,T]$, $z_1,z_2\in K$, $\varepsilon\in[0,\delta]$, $\|y(t,z_1,\varepsilon)-y(t,z_2,\varepsilon)\|\leqslant\tilde{o}(\delta)\|z_1-z_2\|$.

(C12) First we note that assumption (A9) of Theorem 3 and the fact that g is locally uniformly Lipschitz with respect to the second variable assure that the following relation holds

$$\begin{aligned}&\left\| g(t,z_1+\zeta_1,\varepsilon)-g(t,z_1,0)-g(t,z_2+\zeta_2,\varepsilon)+g(t,z_2,0)\right\| \\ &\quad\leqslant \tilde{o}(\delta)\|z_1-z_2\|+\tilde{o}(\delta)\|\zeta_1-\zeta_2\|,\end{aligned}\tag{26}$$

for all $t\in[0,T]\setminus M_\delta$, $z_1,z_2\in B_\delta(z_0)$, $\varepsilon\in[0,\delta]$ and $\zeta_1,\zeta_2\in B_\delta(0)$. We introduce the notations $v(t,z,\zeta)=x(t,z+\zeta,0)-x(t,z,0)$, $\tilde{\zeta}(s,z,\zeta,\varepsilon)=v(s,z,\zeta)+\varepsilon y(s,z+\zeta,\varepsilon)$ and $u(t,z,\zeta,\varepsilon)=y(t,z+\zeta,\varepsilon)-y(t,z,0)$. Since the function $x(\cdot,\cdot,0)$ is C^1, v is Lipschitz with respect to z on $[0,T]\times K\times B_\delta(0)$ with some constant $\tilde{o}(\delta)$, we have

$$\begin{aligned}u(t,z,\zeta,\varepsilon)&=y(t,z+\zeta,\varepsilon)-y(t,z,0)\\ &=\frac{1}{\varepsilon}\int_0^t\left[f\big(s,x(s,z+\zeta,\varepsilon)\big)-f\big(s,x(s,z+\zeta,0)\big)-\varepsilon D_x f\big(s,x(s,z,0)\big)y(s,z,0)\right]ds\\ &\quad+\int_0^t\left[g\big(s,x(s,z+\zeta,\varepsilon),\varepsilon\big)-g\big(s,x(s,z,0),0\big)\right]ds.\end{aligned}$$

Our aim is to estimate a Lipschitz constant with respect to z of the function u on $[0,T]\times B_\delta(z_0)\times B_\delta(0)\times[0,\delta]$. We will apply Lemma 4, relation (26), the fact that g is locally uniformly Lipschitz, and using the following decompositions and estimations that hold for $(s,z,\zeta,\varepsilon)\in[0,T]\times B_\delta(z_0)\times B_\delta(0)\times[0,\delta]$,

$$x(s,z+\zeta,\varepsilon)=x(s,z,0)+v(s,z,\zeta)+\varepsilon y(s,z,0)+\varepsilon u(s,z,\zeta,\varepsilon),$$

$$x(s,z+\zeta,0)=x(s,z,0)+v(s,z,\zeta),$$

$$x(s,z+\zeta,\varepsilon)=x(s,z,0)+\tilde{\zeta}(s,z,\zeta,\varepsilon),$$

$$\left\|v(t,z,\zeta)\right\|\leqslant\tilde{o}(\delta),\qquad\left\|u(t,z,\zeta,\varepsilon)\right\|\leqslant\tilde{o}(\delta),\qquad\left\|\tilde{\zeta}(s,z,\zeta,\varepsilon)\right\|\leqslant\delta\tilde{o}(\delta),$$

we obtain

$$\begin{aligned}&\left\|u(t,z_1,\zeta,\varepsilon)-u(t,z_2,\zeta,\varepsilon)\right\|\\ &\quad\leqslant\frac{1}{\varepsilon}\int_0^t\left[o(\varepsilon)+\varepsilon\tilde{o}(\delta)\right]\left\|x(s,z_1,0)-x(s,z_2,0)\right\|+O(\varepsilon)\left\|v(s,z_1,\zeta)-v(s,z_2,\zeta)\right\|\\ &\qquad+\left[o(\varepsilon)+\varepsilon\tilde{o}(\delta)\right]\left\|y(s,z_1,0)-y(s,z_2,0)\right\|+O(\varepsilon)\left\|u(s,z_1,\zeta,\varepsilon)-u(s,z_2,\zeta,\varepsilon)\right\|ds\end{aligned}$$

$$+ \int\limits_{(0,t)\setminus M_\delta} \tilde{o}(\delta)\left\|x(s,z_1,0)-x(s,z_2,0)\right\|$$
$$+ \tilde{o}(\delta)\left\|\tilde{\zeta}(s,z_1,\zeta,\varepsilon)-\tilde{\zeta}(s,z_2,\zeta,\varepsilon)\right\| ds + \tilde{o}(\delta)\|z_1-z_2\|.$$

Now we use that some Lipschitz constants with respect to z for the functions x and y on $[0,T]\times B_\delta(z_0)\times[0,\delta]$ are $\tilde{o}(\delta)$, while for the functions v on $[0,T]\times B_\delta(z_0)\times[0,\delta]$ and $\tilde{\zeta}$ on $[0,T]\times B_\delta(z_0)\times B_\delta(0)\times[0,\delta]$ are $\tilde{o}(\delta)$, and finally we obtain that

$$\left\|u(t,z_1,\zeta,\varepsilon)-u(t,z_2,\zeta,\varepsilon)\right\| \leqslant \tilde{o}(\delta)\|z_1-z_2\| + \tilde{o}(\delta)\int_0^t \left\|u(t,z_1,\zeta,\varepsilon)-u(t,z_2,\zeta,\varepsilon)\right\| ds.$$

The conclusion follows after applying the Grönwall inequality. □

Lemma 6. *We consider the C^1 function Y acting from $\mathbb{R}^n$ into the space of $n\times n$ matrices, the C^2 function $\widetilde{P}:\mathbb{R}^n\to\mathbb{R}^n$ and $z_*\in\mathbb{R}^n$ such that $\widetilde{P}(z_*)=0$. We denote $P:\mathbb{R}^n\to\mathbb{R}^n$ the C^1 function given by $P(z)=Y(z)\widetilde{P}(z)$ for all $z\in\mathbb{R}^n$. Then $DP(z_*)=Y(z_*)D\widetilde{P}(z_*)$, P is twice differentiable in z_* and, for each $i\in\overline{1,n}$, the Hessian matrix $HP_i(z_*)$ is symmetric.*

Proof. We have $DP(z)=(\frac{\partial Y}{\partial z_1}(z)\widetilde{P}(z),\ldots,\frac{\partial Y}{\partial z_n}(z)\widetilde{P}(z))+Y(z)D\widetilde{P}(z)$ for all $z\in\mathbb{R}^n$. From this it follows the formula for $DP(z_*)$ since $\widetilde{P}(z_*)=0$.

In order to prove that P is twice differentiable in z_*, taking into account the above expression of DP, it is enough to prove that for each $i\in\overline{1,n}$, $z\mapsto\frac{\partial Y}{\partial z_i}(z)\widetilde{P}(z)$ and $z\mapsto Y(z)D\widetilde{P}(z)$ are differentiable in z_*. The last map is C^1, hence it remains to prove the differentiability only for the first one. We fix $i\in\overline{1,n}$. From the relation

$$\begin{aligned}&\frac{\partial Y}{\partial z_i}(z_*+h)\widetilde{P}(z_*+h)-\frac{\partial Y}{\partial z_i}(z_*)\widetilde{P}(z_*)\\&\quad=\frac{\partial Y}{\partial z_i}(z_*+h)\big(\widetilde{P}(z_*+h)-\widetilde{P}(z_*)\big)=\frac{\partial Y}{\partial z_i}(z_*+h)D\widetilde{P}(z_*)+o(h),\end{aligned}$$

we deduce that $z\mapsto\frac{\partial Y}{\partial z_i}(z)\widetilde{P}(z)$ is differentiable in z_* and that

$$D\left(\frac{\partial Y}{\partial z_i}\cdot\widetilde{P}\right)(z_*)=\frac{\partial Y}{\partial z_i}(z_*)D\widetilde{P}(z_*).$$

In order to prove that the Hessian matrix $HP_i(z_*)$ is symmetric, for every $j,k\in\{1,\ldots,n\}$ we must prove that

$$\frac{\partial^2 P_i}{\partial z_j\partial z_k}(z_*)=\frac{\partial^2 P_i}{\partial z_k\partial z_j}(z_*).$$

We denote by $Y_i(z)$ the i-th row of the $n\times n$ matrix $Y(z)$. For all $z\in\mathbb{R}^n$ we have

$$\frac{\partial P_i}{\partial z_j}(z)=Y_i(z)\frac{\partial\widetilde{P}}{\partial z_j}(z)+\frac{\partial Y_i}{\partial z_j}(z)\widetilde{P}(z).$$

Then

$$\frac{\partial^2 P_i}{\partial z_j \partial z_k}(z_*) = \frac{\partial Y_i}{\partial z_k}(z_*)\frac{\partial \widetilde{P}}{\partial z_j}(z_*) + Y_i(z_*)\frac{\partial^2 \widetilde{P}}{\partial z_j \partial z_k}(z_*) + \frac{\partial Y_i}{\partial z_j}(z_*)\frac{\partial \widetilde{P}}{\partial z_k}(z_*).$$

Since $\widetilde{P}$ is C^2 it is easy to check the symmetry of this last relation with respect to (j,k). □

Proof of Theorem 3. We need to study the zeros of the function $z \mapsto x(T,z,\varepsilon) - z$, or equivalently of

$$F(z,\varepsilon) = Y^{-1}(T,z)\big(x(T,z,\varepsilon) - z\big).$$

The function F is well defined at least for any z in some small neighborhood of $\mathcal{Z}$ and any $\varepsilon \geqslant 0$ sufficiently small. We will apply Theorem 1. We denote

$$P(z) = Y^{-1}(T,z)\big(x(T,z,0) - z\big), \qquad Q(z,\varepsilon) = Y^{-1}(T,z)y(T,z,\varepsilon),$$

where $y(t,z,\varepsilon) = [x(t,z,\varepsilon) - x(t,z,0)]/\varepsilon$, like in Lemma 5. Hence $F(z,\varepsilon) = P(z) + \varepsilon Q(z,\varepsilon)$.

The fact that f is C^2 assures that the function $z \mapsto x(T,z,0)$ is also C^2 (see [24, Ch. 4, § 24]). Since (see [8, Ch. III, Lemma § 12]) $(Y^{-1}(\cdot,z))^*$ is a fundamental matrix solution of the system

$$\dot{u} = -\big(D_x f\big(t, x(t,z,0), 0\big)\big)^* u,$$

and f is C^2, we have that the matrix function $(t,z) \mapsto (Y^{-1}(t,z))^*$ is C^1. Therefore the matrix function $(t,z) \mapsto Y^{-1}(t,z)$, and consequently also the function P are C^1.

By Lemma 5 we now conclude that Q is continuous, locally uniformly Lipschitz with respect to z, and

$$Q(z,0) = \int_0^T Y^{-1}(s,z)g\big(s, x(s,z,0), 0\big)\,ds. \tag{27}$$

Since, by our hypothesis (A6), $x(\cdot,z,0)$ is T-periodic for all $z \in \mathcal{Z}$ we have that $x(T,z,0) - z = 0$ for all $z \in \mathcal{Z}$, and consequently $P(z) = 0$ for all $z \in \mathcal{Z}$. This means that hypothesis (A1) of Theorem 1 holds. Moreover applying Lemma 6 we have that

$$DP(z) = Y^{-1}(T,z)\left(\frac{\partial x}{\partial z}(T,z,0) - I_{n\times n}\right) \quad \text{for any } z \in \mathcal{Z},$$

and P satisfies hypothesis (A3) of Theorem 1. But $(\partial x/\partial z)(\cdot,z,0)$ is the normalized fundamental matrix of the linearized system (21) (see [17, Theorem 2.1]). Therefore $(\partial x/\partial z)(t,z,0) = Y(t,z)Y^{-1}(0,z)$, and we can write

$$DP(z) = Y^{-1}(0,z) - Y^{-1}(T,z) \quad \text{for any } z \in \mathcal{Z}. \tag{28}$$

Using our hypothesis (A7) we see that also assumption (A2) of Theorem 1 is satisfied. From the definition of G and relation (27) we have that

$$G(\alpha) = \pi Q\left(S\begin{pmatrix}\alpha \\ \beta_0(\alpha)\end{pmatrix}, 0\right).$$

That is, the function denoted in Theorem 1 by $\widehat{Q}$ is here G, and it satisfies the hypotheses of Theorem 1. Moreover, note that when G satisfies (A8) then assumption (A4) of Theorem 1 is fulfilled.

(C7) follows from (C1) of Theorem 1.

(C8) follows from (C2) of Theorem 1.

(C9) In order to prove the uniqueness of the T-periodic solution, it remains only to check (A5) of Theorem 1. For doing this we show that the function $(z,\zeta,\varepsilon)\in B_\delta(z_0)\times B_\delta(0)\times[0,\delta]\mapsto Q(z+\zeta,\varepsilon)-Q(z,0)$ is Lipschitz with respect to z with some constant $\tilde{o}(\delta)$. We write

$$Q(z+\zeta,\varepsilon)-Q(z,0)=Y^{-1}(T,z+\zeta)\big[y(T,z+\zeta,\varepsilon)-y(T,z,0)\big]+\big[Y^{-1}(T,z+\zeta)-Y^{-1}(T,z)\big]y(T,z,0).$$

It is known that for proving that a sum of two functions is Lipschitz with some constant of order $\tilde{o}(\delta)$, it is enough to prove that each function is Lipschitz with such constant; while in order to prove that a product of two functions is Lipschitz with some constant $\tilde{o}(\delta)$, it is sufficient to prove that both functions are Lipschitz and only one of them is bounded by some constant $\tilde{o}(\delta)$ and Lipschitz with respect to z with some constant $\tilde{o}(\delta)$.

By Lemma 5 we know that the function $z\in B_\delta(z_0)\mapsto y(T,z,0)$ is Lipschitz. The fact that $z\mapsto Y^{-1}(T,z)$ is C^1 assures that $(z,\zeta)\in B_\delta(z_0)\times B_\delta(0)\mapsto Y^{-1}(T,z+\zeta)$ is Lipschitz with respect to z.

From Lemma 5 we have that the function

$$(z,\zeta,\varepsilon)\in B_\delta(z_0)\times B_\delta(0)\times[0,\delta]\mapsto y(T,z+\zeta,\varepsilon)-y(T,z,0)$$

is bounded by some constant $\tilde{o}(\delta)$ and Lipschitz with some constant $\tilde{o}(\delta)$. Since $z\mapsto Y^{-1}(T,z)$ is C^1, the same is true for the function

$$(z,\zeta)\in B_\delta(z_0)\times B_\delta(0)\mapsto \big[Y^{-1}(T,z+\zeta)-Y^{-1}(T,z)\big].$$

Hence Q satisfies (A5) of Theorem 1 and the conclusion holds. □

By using Theorem 3 we can finally prove the following Lipschitz analogue of the results by Malkin [21], Loud [19] and Rhouma and Chicone [25].

Theorem 7. *Assume that $f\in C^2(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ and $g\in C^0(\mathbb{R}\times\mathbb{R}^n\times[0,1],\mathbb{R}^n)$ are T-periodic in the first variable, and that g is locally uniformly Lipschitz with respect to the second variable. Assume that the unperturbed system* (20) *satisfies the following conditions.*

(A10) *There exist an open ball $U\subset\mathbb{R}^k$ with $k\leqslant n$ and a function $\xi\in C^1(\overline{U},\mathbb{R}^n)$ such that for any $h\in\overline{U}$ the $n\times k$ matrix $D\xi(h)$ has rank k and $\xi(h)$ is the initial condition of a T-periodic solution of* (20)*.*

(A11) *For each $h\in\overline{U}$ the linear system* (21) *with $z=\xi(h)$ has the Floquet multiplier $+1$ with the geometric multiplicity equal to k.*

Let $u_1(\cdot,h),\dots,u_k(\cdot,h)$ be linearly independent T-periodic solutions of the adjoint linear system

$$\dot{u}=-\big(D_xf\big(t,x\big(t,\xi(h),0\big)\big)\big)^*u, \tag{29}$$

such that $u_1(0,h),\dots,u_k(0,h)$ are C^1 with respect to h and define the function $M:\overline{U}\to\mathbb{R}^k$ (called the Malkin's bifurcation function) by

$$M(h)=\int_0^T\begin{pmatrix}\langle u_1(s,h),g(s,x(s,\xi(h),0),0)\rangle\\ \cdots\\ \langle u_k(s,h),g(s,x(s,\xi(h),0),0)\rangle\end{pmatrix}ds.$$

Then the following statements hold.

(C13) *For any sequences* $(\varphi_m)_{m\geqslant 1}$ *from* $C^0(\mathbb{R},\mathbb{R}^n)$ *and* $(\varepsilon_m)_{m\geqslant 1}$ *from* $[0,1]$ *such that* $\varphi_m(0)\to\xi(h_0)\in\xi(\overline{U})$, $\varepsilon_m\to 0$ *as* $m\to\infty$ *and* φ_m *is a* T*-periodic solution of* (19) *with* $\varepsilon=\varepsilon_m$, *we have that* $M(h_0)=0$.

(C14) *If* $M(h)\neq 0$ *for any* $h\in\partial U$ *and* $d(M,U)\neq 0$, *then there exists* $\varepsilon_1>0$ *sufficiently small such that for each* $\varepsilon\in(0,\varepsilon_1]$ *there is at least one* T*-periodic solution* φ_ε *of system* (19) *such that* $\rho(\varphi_\varepsilon(0),\xi(\overline{U}))\to 0$ *as* $\varepsilon\to 0$.

In addition we assume that there exists $h_0\in U$ *such that* $M(h_0)=0$, $M(h)\neq 0$ *for all* $h\in\overline{U}\setminus\{h_0\}$ *and* $d(M,U)\neq 0$. *Moreover we assume that hypothesis* (A9) *of Theorem* 3 *holds with* $z_0=\xi(h_0)$ *and that*:

(A12) *There exist* $\delta_1>0$ *and* $L_M>0$ *such that*

$$\left\|M(h_1)-M(h_2)\right\|\geqslant L_M\|h_1-h_2\|,\quad \text{for all } h_1,h_2\in B_{\delta_1}(h_0).$$

Then the following conclusion holds.

(C15) *There exists* $\delta_2>0$ *such that for any* $\varepsilon\in(0,\varepsilon_1]$, φ_ε *is the only* T*-periodic solution of* (19) *with initial condition in* $B_{\delta_2}(z_0)$. *Moreover* $\varphi_\varepsilon(0)\to\xi(h_0)$ *as* $\varepsilon\to 0$.

Remark 1. The existence of k linearly independent T-periodic solutions of the adjoint linear system (29) follows by hypothesis (A10) (see e.g. [8, Ch. III, § 23, Theorem 2]). Indeed, we have that $y_i(t,h)=D_z x(t,\xi(h),0)D_{h_i}\xi(h)$ for $i\in\overline{1,k}$ are solutions of (21) and they are linearly independent on the base of (A10). The assertion follows by the fact that a linear system and its adjoint have the same number of linearly independent solutions. Moreover, hypothesis (A11) assures that there is no other T-periodic solution to (21) linearly independent of these.

Remark 2. When the function g is of class C^1 and the zero h_0 of M is simple, all the hypotheses on g of the above theorem and the hypothesis (A12) on M are automatically satisfied.

Proof of Theorem 7. We apply Theorem 3. For the moment we describe the set $\mathcal{Z}$ that appear in hypothesis (A6) as $\mathcal{Z}=\bigcup_{h\in\overline{U}}\{\xi(h)\}$. First we find the matrix S such that hypothesis (A7) holds. In order to achieve this, for each $z\in\mathcal{Z}$ we denote by $U(t,z)$ some fundamental matrix solution of (29) that has in its first k columns the T-periodic solutions $u_1,\ldots,u_k$ and such that $z\mapsto U(0,z)$ is C^1. Then the first k columns of the matrix $U(0,z)-U(T,z)$ are null vectors. The matrix $Y(t,z)$ such that $Y^{-1}(t,z)=[U(t,z)]^*$ is a fundamental matrix solution of (21), i.e. of the system ($z=\xi(h)\in\mathcal{Z}$)

$$\dot{y}=D_x f\big(t,x\big(t,\xi(h),0\big)\big)y. \tag{30}$$

Then the first k lines of the matrix $Y^{-1}(0,z)-Y^{-1}(T,z)$ are null vectors. Since the Floquet multiplier 1 of (21) has geometric multiplicity k we have that the matrix $Y^{-1}(0,z)-Y^{-1}(T,z)$ has range $n-k$. Hence this matrix has $n-k$ linearly independent columns. We claim that there exists an invertible matrix S such that the matrix $(Y^{-1}(0,z)-Y^{-1}(T,z))S$ has in the first k lines null vectors and in the lower right corner some $(n-k)\times(n-k)$ invertible matrix $\Delta(z)$. With this we prove that (A7) holds. In order to justify the claim we note first that whatever the matrix S would be, the first k lines of $(Y^{-1}(0,z)-Y^{-1}(T,z))S$ are null vectors. Now we choose an invertible matrix S such that its last $(n-k)$ columns are vectors of the form

$$e_i=\begin{pmatrix}0_{(i-1)\times 1}\\ 1\\ 0_{(n-i)\times 1}\end{pmatrix},\quad i\in\overline{1,n},$$

distributed in such a way that the $n-k$ linearly independent columns of $Y^{-1}(0,z)-Y^{-1}(T,z)$ become the last $n-k$ columns of $(Y^{-1}(0,z)-Y^{-1}(T,z))S$. Now it is easy to see that the $(n-k)\times(n-k)$ matrix from the lower right corner of $(Y^{-1}(0,z)-Y^{-1}(T,z))S$ is invertible.

Now we come back to prove (A6). By taking the derivative with respect to $h \in U$ of $\dot{x}(t,\xi(h)) = f(t,x(t,\xi(h)))$ we obtain that $D_\xi x(\cdot,\xi(h)) \cdot D\xi(h)$ is a matrix solution for (30). But $x(\cdot,\xi(h))$ is T-periodic for any $h \in U$, therefore $D_\xi x(\cdot,\xi(h)) \cdot D\xi(h)$ is T-periodic. This fact assures that each column of $D\xi(h)$ is the initial condition of some T-periodic solution of (30) and these T-periodic solutions are the columns of $Y(t,\xi(h))Y^{-1}(0,\xi(h))D\xi(h)$. Then $Y(T,\xi(h))Y^{-1}(0,\xi(h))D\xi(h) = D\xi(h)$, that further gives

$$\left[Y^{-1}\big(0,\xi(h)\big) - Y^{-1}\big(T,\xi(h)\big)\right] S S^{-1} D\xi(h) = 0.$$

Hence the columns of $S^{-1}D\xi(h)$ belong to the kernel of

$$\left[Y^{-1}\big(0,\xi(h)\big) - Y^{-1}\big(T,\xi(h)\big)\right] S.$$

Since (A7) holds we have that the kernel of $[Y^{-1}(0,\xi(h)) - Y^{-1}(T,\xi(h))]S$ contains vectors whose last $n-k$ components are null. We deduce that there exists some $k \times k$ matrix, denoted by Ψ, such that

$$S^{-1}D\xi(h) = \begin{pmatrix} \Psi \\ 0_{(n-k)\times k} \end{pmatrix}. \tag{31}$$

Since by the assumption (A10) the matrix $D\xi(h)$ has rank k and S^{-1} is invertible, we have that the matrix $S^{-1}D\xi(h)$ should also have rank k, that is only possible if

$$\det \Psi \neq 0. \tag{32}$$

We fix some $h_* \in U$ and we denote $\alpha_* = \pi S^{-1}\xi(h_*)$. Using (31) and (32), and applying the Implicit Function Theorem we have that there exist an open ball, neighborhood of α_*, denoted $V \subset \mathbb{R}^k$, and a C^1 function $\tilde{h} : \overline{V} \to U$ such that

$$\pi S^{-1}\xi\big(\tilde{h}(\alpha)\big) = \alpha \quad \text{for any } \alpha \in \overline{V}. \tag{33}$$

Now we define the C^1 function $\beta_0 : \overline{V} \to \mathbb{R}^{n-k}$ as $\beta_0(\alpha) = \pi^\perp S^{-1}\xi(\tilde{h}(\alpha))$. Note that $S\big(\begin{smallmatrix}\alpha \\ \beta_0(\alpha)\end{smallmatrix}\big) = \xi(\tilde{h}(\alpha))$. Hence the assumption (A6) of Theorem 3 is satisfied with S, V and β_0 defined as above.

The bifurcation function G defined in Theorem 3 can be written using our notations as

$$G(\alpha) = \pi \int_0^T Y^{-1}\big(s,\xi\big(\tilde{h}(\alpha)\big)\big) g\big(s, x\big(s,\xi\big(\tilde{h}(\alpha)\big),0\big)\big)\, ds. \tag{34}$$

Since $Y^{-1}(s,\xi(h)) = [U(t,h)]^*$ (see the beginning of the proof) we have that in the first k lines of $Y^{-1}(s,\xi(h))$ there are the vectors $(u_1(s,h))^*, \ldots, (u_n(s,h))^*$ and so

$$G(\alpha) = \int_0^T \begin{pmatrix} \langle u_1(s,\tilde{h}(\alpha)), g(s,x(s,\xi(\tilde{h}(\alpha)),0),0)\rangle \\ \cdots \\ \langle u_k(s,\tilde{h}(\alpha)), g(s,x(s,\xi(\tilde{h}(\alpha)),0),0)\rangle \end{pmatrix} ds.$$

From here one can see that there is the following relation between G and the Malkin bifurcation function M,

$$G(\alpha) = M\big(\tilde{h}(\alpha)\big) \quad \text{for any } \alpha \in \overline{V}. \tag{35}$$

(C13) follows from (C7) of Theorem 3.

(C14) Without loss of generality (we can diminish U, if necessary) we can consider that $\tilde{h}$ is a homeomorphism from V onto U and taking into account that C is an invertible matrix by [18, Theorem 26.4] we have

$$\deg(G, V) = \deg(M, U).$$

Thus (C14) follows applying conclusion (C8) of Theorem 3.

(C15) We need only to prove assumption (A8) of Theorem 3 provided that our hypothesis (A12) holds. First, taking the derivative of (33) with respect to α and using (31), we obtain that $D\tilde{h}(\alpha_*) = \Psi^{-1}$, hence it is invertible and, moreover, $L_h = \|D\tilde{h}(\alpha_*)\|/2 \neq 0$. We have that there exists $\delta > 0$ sufficiently small such that $\|D\tilde{h}(\alpha) - D\tilde{h}(\alpha_*)\| \leqslant L_h$ for all $\alpha \in B_\delta(\alpha_*)$. Using the generalized Mean Value Theorem (see [7, Proposition 2.6.5]), we have that

$$\left\|\tilde{h}(\alpha_1) - \tilde{h}(\alpha_2)\right\| \geqslant L_h\|\alpha_1 - \alpha_2\| \quad \text{for all } \alpha_1, \alpha_2 \in B_\delta(\alpha_0).$$

Since C is invertible, M satisfies (A10) and (35), we deduce that G satisfies hypothesis (A8) of Theorem 3. □

5. Conclusions

In this paper we provide a perturbation result about the unique response of a normally nondegenerate family of periodic solutions to a Lipschitz perturbation g. Despite possible nondifferentiability of g we succeeded to construct suitable projectors that reduced the dimension of the analysis to the dimension of this family. This result suggests that, under the conditions of Theorem 7, the manifold of the fixed points of the Poincaré map of the unperturbed system transforms into an invariant manifold of the Poincaré map of the perturbed system. This fact is well known for smooth differential equations (see e.g. Wiggins [28]), but we do not know whether or not the latter is completely correct in the case where g is only Lipschitz. Understanding the stability of the aforementioned invariant manifold could allow to access asymptotic stability of periodic solutions given by Theorem 7.

Another question that this paper raises is whether assumption (A9) implies continuous differentiability of the bifurcation function $\widehat{Q}$, so that (A12) is just the requirement for the derivative of the bifurcation function M to have all its eigenvalues in the left half-plane. This is the case in particular examples, but in general the answer is unknown to us.

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