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ORIGINAL RESEARCH

A Note on Forced Oscillations in Differential Equations with Jumping Nonlinearities

A. Buică · J. Llibre · O. Makarenkov

Abstract The goal of this paper is to study bifurcations of asymptotically stable 2π-periodic solutions in the forced asymmetric oscillator $\ddot{u} + \varepsilon c\dot{u} + u + \varepsilon a u^+ = 1 + \varepsilon\lambda \cos t$ by means of a Lipschitz generalization of the second Bogolubov's theorem due to the authors. The small parameter $\varepsilon > 0$ is introduced in such a way that any solution of the system corresponding to $\varepsilon = 0$ is 2π-periodic. We show that exactly one of these solutions whose amplitude is $\frac{\lambda}{\sqrt{a^2+c^2}}$ generates a branch of 2π-periodic solutions when $\varepsilon > 0$ increases. The solutions of this branch are asymptotically stable provided that $c > 0$.

Keywords Asymptotic stability · Periodic solutions · Jumping nonlinearity · Method of averaging

Introduction

The differential equation for the coordinate u of the mass attached via nonlinear spring to an immovable beam drawn at Fig. 1 reads as

$$m\ddot{u} + c\dot{u} + k_1 u + k_2 u^+ = f(t), \tag{1}$$

where f is a force applied to the mass in the vertical direction, see [1,15,11].

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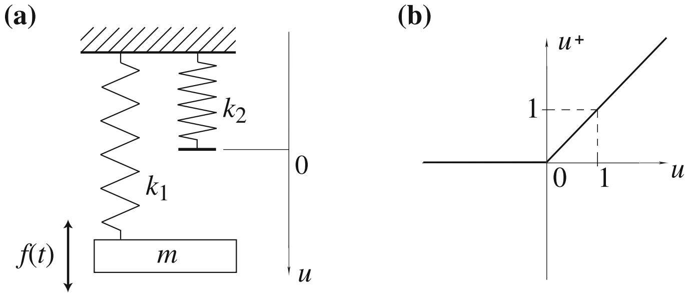

Fig. 1 **a** A driven mass attached to an immovable beam via a spring with piecewise linear stiffness, **b** the jumping nonlinearity $u \mapsto u^+$

The case where Eq. (1) takes the form

$$m\ddot{u} + \varepsilon c_\varepsilon \dot{u} + k_1 u + k_2 u^+ = \varepsilon f(t), \tag{2}$$

where $c_\varepsilon \to 0$ as $\varepsilon \to 0$, the bifurcation of asymptotically stable periodic solutions is studied in Glover–Lazer–McKenna [8]. In the setting where the unperturbed system $m\ddot{u} + k_1 u + k_2 u^+ = 0$ has a T-periodic orbit u_0, these authors related the existence of asymptotically stable T-periodic solutions near $u_0([0, T])$ to the existence of $\alpha \in [0, T]$ such that $\int_0^T \dot{u}_0(\tau) f(\tau - \alpha) d\tau = 0$ and $d = \int_0^T \ddot{u}_0(\tau) f(\tau - \alpha) d\tau > 0$. By the other words, the authors of [8] showed that the conclusion of the second Bogolubov's theorem holds for Eq. (2), even though it is not C^1. We quote this theorem for completeness, see [4].

Second Bogolubov's theorem *Consider the perturbed system*

$$\dot{x} = \varepsilon g(t, x, \varepsilon), \tag{3}$$

where $g \in C^1(\mathbb{R} \times \mathbb{R}^n \times [0, 1], \mathbb{R}^n)$ *is* T*-periodic in the first variable. If* $v_0 \in \mathbb{R}^n$ *is a zero of the bifurcation function*

$$g_0(v) = \int_0^T g(\tau, v, 0)\, d\tau \tag{4}$$

and $\det (g_0)'(v_0) \neq 0$, *then for any* $\varepsilon > 0$ *sufficiently small system* (3) *has a unique* T*-periodic solution* x_ε *such that* $x_\varepsilon(0) \to v_0$ *as* $\varepsilon \to 0$. *If, in addition, all the eigenvalues of the matrix* $(g_0)'(v_0)$ *have negative real part, then* x_ε *is asymptotically stable.*

Note, the change of variables

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

transforms Eq. (2) to the standard form (3) of averaging theory (see a similar example in "Bifurcations of Asymptotically Stable Periodic Solutions in Differential Equations with Jumping Nonlinearities" section).

In the next section of the paper we discuss a general class of Lipschitz systems [which includes, in particular, Eq. (2)] for which the conclusion of the second Bogolubov's theorem holds. The Lipschitz analogue of the second Bogolyubov's theorem (Theorem 1) is then applied to the asymmetric oscillator

$$\ddot{u} + \varepsilon c\dot{u} + u + \varepsilon a u^+ = \varepsilon \lambda \cos t \tag{5}$$

in “Bifurcations of Asymptotically Stable Periodic Solutions in Differential Equations with Jumping Nonlinearities” section, where we obtain (Theorem 2) explicit conditions for the coefficients c, a and λ that guarantee the bifurcation of a branch of asymptotically stable 2π-periodic solutions.

Our Theorem 2 complements the previous studies. Indeed, Eq. (5) is formally different from (2), so the result from [8] cannot be readily applied. The existence and stability of 2π-periodic solutions of (1) are also discussed in Lazer–McKenna [10] and Fabry [7]. However, it is assumed in [10] that the amplitude of the forcing term f is sufficiently large, while the authors of [7] address those periodic solutions whose amplitude tends to $+\infty$ as a suitable small parameter $\varepsilon > 0$ approaches 0. A degree theoretic approach is developed in [12]. See our survey [11] for a broad analysis of the research around equations of type (5). Extending the range of conclusions about the dynamics of (5) is important as this equation occurs in a variety of applications, e.g. offshore structures [15], resonant screening [16], drilling [6] and others (see [5]).

Lipschitz Generalization of the Second Bogolubov’s Theorem

Throughout the paper $\Omega \subset \mathbb{R}^k$ is some open set. For any $\delta > 0$ we denote $B_\delta(v_0) = \left\{v \in \mathbb{R}^k : \|v - v_0\| \le \delta\right\}$. For any set $M \subset [0, T]$ measurable in the sense of Lebesgue we denote by $\mathrm{mes}(M)$ the Lebesgue measure of M. We proved the following result in [5].

Theorem 1 *Let $g \in C^0(\mathbb{R} \times \Omega \times [0, 1], \mathbb{R}^k)$. Let g_0 be the averaging function given by (4) and consider $v_0 \in \Omega$ such that $g_0(v_0) = 0$. Assume that:*

(i) For some $L > 0$ we have that $\|g(t, v_1, \varepsilon) - g(t, v_2, \varepsilon)\| \le L\,\|v_1 - v_2\|$ for any $t \in [0, T],\ v_1, v_2 \in \Omega,\ \varepsilon \in [0, 1]$;
(ii) given any $\gamma > 0$ there exist $\delta > 0$ and $M \subset [0, T]$ measurable in the sense of Lebesgue with $\mathrm{mes}(M) < \gamma$ such that for every $v \in B_\delta(v_0)$, $t \in [0, T] \setminus M$ and $\varepsilon \in [0, \delta]$ we have that $g(t, \cdot, \varepsilon)$ is differentiable at v and $\|g'_v(t, v, \varepsilon) - g'_v(t, v_0, 0)\| \le \gamma$;
(iii) g_0 is continuously differentiable in a neighborhood of v_0 and the real parts of all the eigenvalues of $(g_0)'(v_0)$ are negative.

Then there exists $\delta_1 > 0$ such that for every $\varepsilon \in (0, \delta_1]$, system (3) has exactly one T-periodic solution x_ε with $x_\varepsilon(0) \in B_{\delta_1}(v_0)$. Moreover the solution x_ε is asymptotically stable and $x_\varepsilon(0) \to v_0$ as $\varepsilon \to 0$.

We briefly outline the proof of this theorem (see [5, Theorem 2.5] for details). To prove Theorem 1 we represent the Poincaré map P_ε of (3) as

$$P_\varepsilon(v) = v + \varepsilon \int_0^T g(\tau, x(\tau, v, \varepsilon), \varepsilon)d\tau,$$

where $x(\cdot, v, \varepsilon)$ is the solution x of (3) with the initial condition $x(0) = v$. We then show that condition (iii) ensures that the map

$$\overline{P}_\varepsilon(v) = v + \varepsilon \int_0^T g(\tau, v, 0)d\tau$$

contracts in a neighborhood of v_0, which, in combination with (i) and (ii), allows to conclude that P_ε contracts in a neighborhood of v_0 too. The later is known to be equivalent to the existence of such a T-periodic solution to (3) which originates in the above-mentioned neighborhood of v_0 and which attracts all other solutions of (3) that originate in this neighborhood. Thus the statement of Theorem 1.

Bifurcations of Asymptotically Stable Periodic Solutions in Differential Equations with Jumping Nonlinearities

In this section we apply Theorem 1 to studying the bifurcation of asymptotically stable 2π-periodic solutions in Eq. (5). A function u is a solution of (5) if and only if $(z_1, z_2) = (u, \dot{u})$ is a solution of the system

$$\begin{aligned} \dot{z}_1 &= z_2\,, \\ \dot{z}_2 &= -z_1 + \varepsilon[-az_1^+ - cz_2 + \lambda\cos t]\,. \end{aligned} \tag{6}$$

After the change of variables

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

system (6) takes the form

$$\begin{aligned} \dot{x}_1 &= \varepsilon\sin t\left[a(x_1\cos t + x_2\sin t)^+ + c(-x_1\sin t + x_2\cos t) - \lambda\cos t\right], \\ \dot{x}_2 &= \varepsilon\cos t\left[-a(x_1\cos t + x_2\sin t)^+ + c(x_1\sin t - x_2\cos t) + \lambda\cos t\right]. \end{aligned} \tag{7}$$

The corresponding averaging function g_0, calculated according to the formula (4), is

$$g_0(x_1, x_2) = \begin{pmatrix} -\pi c & \pi a/2 \\ -\pi a/2 & -\pi c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \pi\lambda \end{pmatrix}.$$

It can be easily checked that the unique zero of g_0 is

$$\left(\frac{2a\lambda}{a^2+4c^2}, \frac{4c\lambda}{a^2+4c^2}\right)$$

and

$$\text{the eigenvalues of } (g_0)' \text{ are } -\pi c \pm i\pi a\,. \tag{8}$$

The amplitude of this zero is

$$A = \frac{2|\lambda|}{\sqrt{a^2+4c^2}}. \tag{9}$$

To apply Theorem 1 it remains to prove the following proposition.

Proposition 1 *Let $v_0 \in \mathbb{R}^2 \setminus \{0\}$. Then the right hand side of (7) satisfies (ii) for any $c, a, \lambda \in \mathbb{R}$.*

Proof Let $[v]_i$ be the i-th component of the vector $v \in \mathbb{R}^2$. Let $g(t, v) = ([v]_1\cos t + [v]_2\sin t)^+$ and notice that it is enough to prove that $g : [0, 2\pi] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies (ii). Define $\theta(v) = \arctan(-[v]_1/[v]_2)$, if $[v_0]_2 \neq 0$, and put

$$\theta(v) = \begin{cases} \arctan(-[v]_1/[v]_2) & \text{if } [v_0]_1[v]_2 < 0, \\ \pi/2 & \text{if } v = v_0, \\ \arctan(-[v]_1/[v]_2) + \pi & \text{if } [v_0]_1[v]_2 > 0, \end{cases}$$

if $[v_0]_2 = 0$. In any case notice that the function $v \mapsto \theta(v)$ is continuous in every sufficiently small neighborhood of v_0. Fix $\gamma > 0$. Let M be the union of the interval M_1 centered in $\theta(v_0)$ (when $\theta(v_0) < 0$, take $\theta(v_0) + 2\pi$ instead of $\theta(v_0)$) and of the interval M_2 centered in $\theta(v_0) + \pi$, each of length $\gamma/2$. Take $\delta > 0$ such that $\theta(v) \in M_1$ for all $v \in B_\delta(v_0)$. Of course, also $\theta(v) + \pi \in M_2$ for all $v \in B_\delta(v_0)$. This implies that for fixed $t \in [0, 2\pi] \setminus M$, $[v]_1 \cos t + [v]_2 \sin t$ has constant sign for all $v \in B_\delta(v_0)$. Therefore, $g(t, \cdot)$ is differentiable and ${g'}_v(t, v) = {g'}_v(t, v_0)$ for all $v \in B_\delta(v_0)$. Hence (ii) is fulfilled. □

The result of this section can be now summarized as follows.

Theorem 2 *Assume that* $c > 0$ *and* $A = 2|\lambda|/\sqrt{a^2 + 4c^2} \neq 0$ *and take arbitrary* $0 < \delta < R$. *Then for each* $\varepsilon > 0$ *sufficiently small, Eq. (5) has an asymptotically stable* 2π*-periodic solution whose amplitude goes to A as* $\varepsilon \to 0$. *Moreover, (5) doesn't have* 2π*-periodic solutions with amplitudes in*

$$(\delta, R] \setminus (A - \delta, A + \delta)\,. \tag{10}$$

Proof The hypotheses (i) of Theorem 1 is immediate to verify, (ii) is proved in Proposition 1 and (iii) follows from (8). Hence, the existence of a unique branch of asymptotically stable 2π-periodic solutions whose amplitudes approache A as $\varepsilon \to 0$ follows from Theorem 1.

To prove that none of 2π-periodic solutions of (5) have amplitudes within (10), we recall that the initial conditions of 2π-periodic solutions of (5) must converge to a zero of the averaging function g_0 as $\varepsilon \to 0$, see Buică–Llibre–Makarenkov [2, Theorem 7 (C13)] (same result under a formal assumption of analiticity was proved in Makarenkov–Ortega [14, Lemma 2]). This completes the proof because we earlier noticed that the only zero of g_0 is that of the amplitute A. □

Theorem 2 allows deriving the curves of the dependence of the amplitudes of asymptotically stable 2π-periodic oscillations in (5) upon the parameters, that we have drawn in Fig. 2. In particular, one can see that, for any fixed $\lambda \in \mathbb{R} \setminus \{0\}$, the amplitude tends to $+\infty$ as $\sqrt{a^2 + 4c^2} \to 0$.

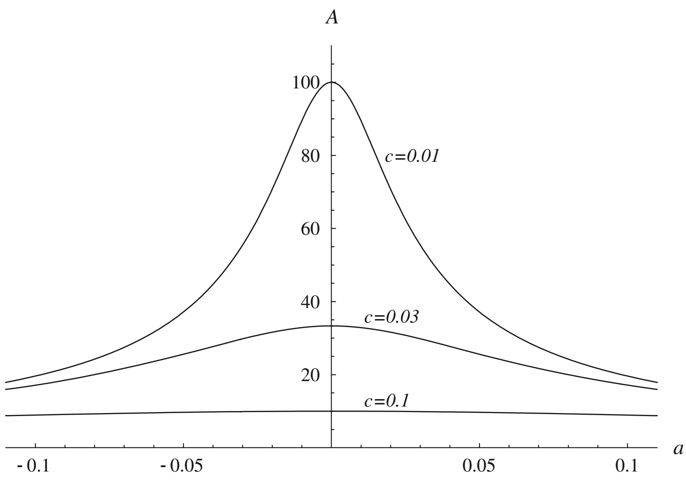

Fig. 2 The curves of dependence of the amplitude of asymptotically stable 2π-periodic oscillations in (5) upon the parameter $a \in \mathbb{R}$ drawn for fixed $\lambda = 1$ and varying values of c

Finally, we note that the case where the period of the perturbation in (5) deviates from π slightly (i.e. when we have a detuning, as in the classical Van der Pol oscillator) can be approached over Theorem 2 too. Indeed, the change of the variables

$$v(t) = u((1+\gamma\varepsilon)t)$$

brings the equation with detuning in time

$$\ddot{u} + \varepsilon c\dot{u} + u + \varepsilon a u^{+} = \varepsilon\lambda \cos\frac{t}{1+\gamma\varepsilon}$$

to the equation with detuning in the rest of the coefficients

$$\ddot{v} + \varepsilon c(1+\varepsilon\gamma)\dot{v} + (1+\varepsilon\gamma)^2 v + \varepsilon a(1+\varepsilon\gamma)^2 v^{+} = \varepsilon\lambda(1+\varepsilon\gamma)^2 \cos t. \tag{11}$$

Literally same arguments as in "Bifurcations of Asymptotically Stable Periodic Solutions in Differential Equations with Jumping Nonlinearitiessection" apply to investigate asymptotically stable π-periodic oscillations of Eq. (11). The only difference is that (11) gives a simple additional term $-\gamma z_1$ in the square brackets of system (6), thus formula (9) will contain the parameter γ now.

Theorem 1 can be also used for establishing stable resonance oscillations in the case where the unperturbed oscillator is Hamiltonian, e.g. when (5) is of more generic form

$$\ddot{u} + \varepsilon c\dot{u} + \sin u + \varepsilon a u^{+} = \varepsilon\lambda \cos t$$

or

$$\ddot{u} + \sin u = \varepsilon F(t, u, \dot{u}, \varepsilon), \tag{12}$$

where F is continuous and piecewise smooth in a suitable sense (see Makarenkov [13]). This can be done alone the same lines as the classical Second Bogolyubov's theorem is used for establishing stable resonance oscillations in mechanical oscillators (12) with smooth F, see Greenspan–Holmes [9] or Burd [3].

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