LIMIT CYCLES OF A PERTURBED CUBIC POLYNOMIAL DIFFERENTIAL CENTER

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Abstract

In this paper we study the limit cycles of the system $\dot{x} = -y(x+a)(y+b)+\varepsilon P(x,y)$, $\dot{y} = x(x+a)(y+b) + \varepsilon Q(x,y)$ for ε sufficiently small, where $a, b \in \mathbb{R} \setminus \{0\}$, and P, Q are polynomials of degree n. We obtain that $3[(n-1)/2]+4$ if $a \neq b$ and, respectively, $2[(n-1)/2] + 2$ if $a = b$, up to first order in ε, are upper bounds for the number of the limit cycles that bifurcate from the period annulus of the cubic center given by $\varepsilon = 0$. Moreover, there are systems with at least $3[(n-1)/2] + 2$ limit cycles if $a \neq b$ and, respectively, $2[(n-1)/2] + 1$ if $a = b$.

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1 Introduction and statement of the main results

One of the main problems in the qualitative theory of real planar differential systems is the study of their *limit cycles*. Probably, the more classical way to produce limit cycles is by perturbing a system which has a *center*, in such a way that limit cycles *bifurcate* in the perturbed system from some of the periodic orbits of the period annulus of the unperturbed system (see [1, 2, 10]).

Perturbing the linear center by arbitrary polynomials P and Q of degree n, i.e. considering $\dot{x} = -y + \varepsilon\, p(x,y)$, $\dot{y} = x + \varepsilon\, q(x,y)$, we can obtain at most $[(n-1)/2]$ bifurcated limit cycles up to first order in ε, where $[\cdot]$ denotes the integer part function (see [5]). Also it is known that perturbing the quadratic center $\dot{x} = -y(1+x)$, $\dot{y} = x(1+x)$ (note that essentially it is the linear center with a straight line of singular points) inside the polynomial differential systems of degree n we can obtain at most n limit cycles up to first order in ε

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(see [8]). The perturbation of the linear center with a conic of singular points inside the class of all cubic polynomial differential systems has been studied in [6]. The authors of [12] studied the perturbation of the cubic center $\dot{x} = -y(1+x)(2+x)$, $\dot{y} = x(1+x)(2+x)$ inside the polynomial differential systems of degree n and they obtained that $2n+2-(-1)^n$ is an upper bound for the number of limit cycles that bifurcate, up to first order in ε, from the period annulus. We notice that, a lower upper bound can be obtained using the variation of the argument principle, like we used here.

In this paper we are interested in the maximum number of limit cycles that can bifurcate from the period annulus surrounding the origin of the cubic polynomial differential systems of the form

$$\dot{x} = -y(x+a)(y+b), \qquad \dot{y} = x(x+a)(y+b), \tag{1}$$

when we perturb them inside the class of all polynomial differential systems of degree n having the origin as a singular point, that is, we want to study the maximum number of limit cycles for the systems

$$\begin{aligned} \dot{x} &= -y(x+a)(y+b) + \varepsilon P(x,y)\ , \\ \dot{y} &= x(x+a)(y+b) + \varepsilon Q(x,y)\ , \end{aligned} \tag{2}$$

which bifurcate from the period annulus of system (1) up to first order in ε. Here $P, Q \in \mathcal{P}_n(\mathbb{R}^2)$, where $\mathcal{P}_n(\mathbb{R}^2)$ denotes the set of all real polynomials in two variables of degree at most $n \geq 3$ such that $P(0,0) = Q(0,0) = 0$, $a, b \in \mathbb{R} \setminus \{0\}$ and $|\varepsilon|$ a sufficiently small real number.

Note that system (1) is mainly the linear center with two straight lines of singular points. It has the first integral $H(x,y) = x^2+y^2$ and the integrating factor $R(x,y) = \dfrac{1}{(x+a)(y+b)}$. We describe the period annulus of (1) as

$$\Gamma_h \quad : \quad x^2 + y^2 = h, \quad 0 < h < a^2.$$

The periodic orbit Γ_h of (1) is called a generating periodic orbit if the perturbed system (2) has at least one limit cycle which depends continuously on ε for small $|\varepsilon|$ and which tends to Γ_h as $\varepsilon \to 0$. This problem of bifurcation of limit cycles from the periodic annulus can be reduced to the problem of bifurcation of zeros of a real function in the following way (for more details see [1, 2, 10]). We consider the Poincaré return map $r \mapsto \mathcal{P}(r,\varepsilon)$ for system (2) defined on the transversal section given by the positive semiaxis. The corresponding displacement map is defined by $d(r,\varepsilon) = \mathcal{P}(r,\varepsilon) - r$, for $r \in (0,a)$ and $|\varepsilon|$ small. It is known that some $r(\varepsilon)$ is an isolated zero of $d(\cdot,\varepsilon)$ if and only if the trajectory of (2) passing through the point $r(\varepsilon)$ of the positive semiaxis is a limit cycle. Also, it is known that d is an analytic function and, consequently, whenever $d(\cdot,\varepsilon)$ has a zero, it is isolated. In the case that the function $f^0 : (0,a) \to \mathbb{R}$ given by $d(r,\varepsilon) = \varepsilon f^0(r) + O(\varepsilon^2)$ is not identically zero, by the Weierstrass Preparation Theorem, the function $d(\cdot,\varepsilon)$ has at most as many zeros as the function f^0 (counting multiplicities). In our work, we will study only the case that the coefficients of the polynomials P and Q are such that the corresponding function f^0 is not the zero function. This is what is called in the literature, the study of bifurcation of limit cycles up to first order in ε. The function f^0 is called a first–order *Melnikov function* and it is given by the formula

$$f^0(\sqrt{h}) = \oint_{\Gamma_h} \frac{P(x,y)dy - Q(x,y)dx}{(x+a)(y+b)}, \quad h \in (0, a^2). \tag{3}$$

The integral representation (3) is called *Abelian integral* (although the system (1) is not Hamiltonian).

We present now the main result of our work.

Theorem 1 *An upper bound for the number of zeros of the Abelian integral* (3) *and also for the number of limit cycles of system* (2) *that bifurcate from the period annulus of system* (1) *up to first order in* ε *is* $3[(n-1)/2]+4$ *if* $a \neq b$ *and, respectively,* $2[(n-1)/2]+2$ *if* $a = b$*. Moreover, there are systems* (2) *with at least* $3[(n-1)/2]+2$ *limit cycles if* $a \neq b$ *and, respectively,* $2[(n-1)/2]+1$ *if* $a = b$*.*

The structure of the paper is the following. In Section 2 we give the main ideas for the proof of Theorem 1, while Sections 3 and 4 contain the proof of two lemmas used in Section 2. By direct but tedious calculations we will find the exact expression of the Abelian integral (3). In order to give the upper bound for the number of its zeros we use the *variation of the argument principle* (see [7, 11]) . As far as we know this method has been used very few times for studying the zeros of Abelian integrals, see for instance [4, 9, 13].

2 Main ideas for the proof of Theorem 1

A direct calculation of the Abelian integral (3) gives the formula ($h = r^2$)

$$f^0(r) = \int_0^{2\pi} \frac{r\cos\theta P(r\cos\theta, r\sin\theta) + r\sin\theta Q(r\cos\theta, r\sin\theta)}{(a + r\cos\theta)(b + r\sin\theta)} d\theta, \quad r \in (0, a). \tag{4}$$

From now on we will denote

$$N = [(n-1)/2].$$

We notice that, due to symmetry, it is sufficient if, with respect to the real parameters $a, b \in \mathbb{R} \setminus \{0\}$, we study only two cases: $0 < a < b$ and, respectively, $0 < a = b$.

Case $0 < a < b$. The exact expression of the function f^0 is given in the following lemma that will be proved in Section 3.

Lemma 2 *The following system of $3N+3$ linearly independent functions*

$$
\begin{array}{cccc}
1-\dfrac{a}{\sqrt{a^2-r^2}}\ , & 1-\dfrac{b}{\sqrt{b^2-r^2}}\ , & \dfrac{ab-\sqrt{a^2-r^2}\sqrt{b^2-r^2}}{(b^2+a^2-r^2)\sqrt{a^2-r^2}}\ , & r^2, r^4, \dots, r^{2N}, \\
\dfrac{r^2}{\sqrt{a^2-r^2}}\ , \quad \dfrac{r^4}{\sqrt{a^2-r^2}}\ , \dots , \dfrac{r^{2N}}{\sqrt{a^2-r^2}}\ , & & \dfrac{r^2}{\sqrt{b^2-r^2}}\ , \quad \dfrac{r^4}{\sqrt{b^2-r^2}}\ , \dots , \dfrac{r^{2N}}{\sqrt{b^2-r^2}}, &
\end{array} \tag{5}
$$

is a basis of the linear space

$$
\left\{ f^0 \ : \ f^0 \text{ given by (4) where } P, Q \in \mathcal{P}_n(\mathbb{R}^2) \right\}. \tag{6}
$$

In order to give an upper bound for the number of zeros of f^0 in $(0,a)$, we write f^0 as a linear combination of the functions (5) and we obtain that, for each $r \in (0,a)$,

$$
(b^2+a^2-r^2)f^0(r) = \frac{1}{\sqrt{b^2-r^2}}R(r^2) + \frac{1}{\sqrt{a^2-r^2}}S(r^2) + (b^2+a^2-r^2)T(r^2),
$$

where R and S are polynomials of degree $N+1$ and T is a polynomial of degree N. Hence, an upper bound for the number of zeros of f^0 is an upper bound for the number of solutions of the equation obtained by putting $h = r^2$,

$$
R(h)\sqrt{a^2-h} + S(h)\sqrt{b^2-h} + (b^2+a^2-h)T(h)\sqrt{a^2-h}\sqrt{b^2-h} = 0, \quad h \in (0,a^2). \tag{7}
$$

In what follows we will obtain an upper bound for the number of complex roots of equation (7). In order to do this, we need the complex extension of the function defined by the left–hand side of (7). Throughout this paper we consider the following holomorphic branch of the complex square root function

$$
\sqrt{z} = \begin{cases} \sqrt{(|z| + \operatorname{Re} z)/2} + i\sqrt{(|z| - \operatorname{Re} z)/2}, & \text{if } 0 \le \arg z < \pi, \\ \sqrt{(|z| + \operatorname{Re} z)/2} - i\sqrt{(|z| - \operatorname{Re} z)/2}, & \text{if } -\pi < \arg z < 0, \end{cases}
$$

in the domain $\mathbb{C} \setminus \{z \in \mathbb{C} \ : \ \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z \le 0\}$. Now it is easy to see that the complex function

$$
f(z) = R(z)\sqrt{a^2-z} + S(z)\sqrt{b^2-z} + (b^2+a^2-z)T(z)\sqrt{a^2-z}\sqrt{b^2-z} \tag{8}
$$

is holomorphic in the domain $\Omega = \mathbb{C} \setminus \{z \in \mathbb{C} \ : \ \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z \ge a^2\}$. In Section 4 we prove the following lemma.

Lemma 3 *Let $f : \Omega \to \mathbb{C}$ be the complex function defined by (8) and $N_0(f)$ be the number of zeros of f in Ω. Then $N_0(f) \le 3N+5$.*

Since $f^0(0) = 0$, the following relation must hold $\dfrac{1}{b}R(0) + \dfrac{1}{a}S(0) + (b^2+a^2)T(0) = 0$. Hence, also $f(0) = 0$ and we deduce that an upper bound for the number of zeros of $f^0 = f^0(r)$ in $(0,a)$ is $3N+4$.

In order to prove the last part of Theorem 1, it is sufficient to prove that there is a function f^0 in the linear space (6) having at least $3N+2$ simple zeros because, by using the Implicit Function Theorem, one obtains that the displacement map $d(\cdot,\varepsilon)$ has at least $3N+2$ different zeros for $|\varepsilon|$ small enough. We will use Lemma 4.5 from [3] that is stated in the following.

Lemma 4 [3] *Consider $p+1$ linearly independent analytical functions $f_i : U \to \mathbb{R}$, $i = 0,1,...,p$, where $U \subset \mathbb{R}$ is an interval. Suppose that there exists $j \in \{0,1,...,p\}$ such that f_j has constant sign. Then there exist $p+1$ constants C_i, $i = 0,1,...,p$ such that $f(x) = \sum_{i=0}^{p} C_i f_i(x)$ has at least p simple zeros in U.*

In order to apply this lemma, we use again that f^0 is an arbitrary linear combination of the $3N+3$ linearly independent functions (5). All these functions are analytic in $U = (0,a)$ and we can see that there are some of them which are strictly positive on U. Hence, the hypotheses of the above lemma are fulfilled. It follows that there exist coefficients such that f^0 has at least $3N+2$ simple zeros.

Case $0 < a = b$. We will follow some of the ideas from the study of the previous case. Now, a basis of the linear space (6) is formed by the following $2N+2$ linearly independent functions

$$\begin{array}{l} 1-\dfrac{a}{\sqrt{a^2-r^2}}\ , \quad \dfrac{r^2}{(2a^2-r^2)\sqrt{a^2-r^2}}\ , \quad r^2, r^4, ..., r^{2N}, \\ \dfrac{r^2}{\sqrt{a^2-r^2}}\ , \quad \dfrac{r^4}{\sqrt{a^2-r^2}}\ , \dots, \ \dfrac{r^{2N}}{\sqrt{a^2-r^2}}, \end{array} \tag{9}$$

In order to give an upper bound for the number of zeros of f^0 in $(0,a)$, we write f^0 as a linear combination of the functions (9) and we obtain that, for each $r \in (0,a)$,

$$(2a^2-r^2)f^0(r) = \frac{1}{\sqrt{a^2-r^2}}S(r^2) + (2a^2-r^2)T(r^2),$$

where S is a polynomial of degree $N+1$ and T is a polynomial of degree N. Hence, an upper bound for the number of zeros of f^0 is an upper bound for the number of solutions of the equation obtained by putting $h = r^2$,

$$S(h) + (2a^2-h)T(h)\sqrt{a^2-h} = 0, \quad h \in (0,a^2). \tag{10}$$

Any solution of (10) is also a solution of

$$S^2(h) = (2a^2-h)^2T^2(h)(a^2-h), \quad h \in (0,a^2), \tag{11}$$

which is a polynomial equation of degree at most $2N+3$. Since $f^0(0) = 0$, we obtain that an upper bound for the number of zeros of $f^0 = f^0(r)$ in $(0,a)$ is $2N+2$. Using Lemma 4, like in the previous case, we also obtain that there exist some function f^0 with at least $2N+1$ simple zeros.

3 Proof of Lemma 2

We write the polynomials P and Q as $P(x,y)=\sum_{k=1}^{n}P_k(x,y)$ and $Q(x,y)=\sum_{k=1}^{n}Q_k(x,y)$, with $P_k(x,y)=\sum_{i+j=k}p_{ij}\,x^iy^j$ and $Q_k(x,y)=\sum_{i+j=k}q_{ij}\,x^iy^j$. We define

$$f_k(\theta)=\cos\theta P_k(\cos\theta,\sin\theta)+\sin\theta Q_k(\cos\theta,\sin\theta)$$

and we write

$$f^0=f^0(r)=\sum_{k=1}^{n}r^{k+1}\int_0^{2\pi}\frac{f_k(\theta)}{(a+r\cos\theta)(b+r\sin\theta)}d\theta,\ \text{ for all } r\in(0,a).$$

Moreover,

$$f^0(r)=\sum_{k=1}^{n}C_k(r)r^{k+1}, \tag{12}$$

where

$$C_k=C_k(r)=\sum_{i+j=k}\left(p_{ij}\,I_{i+1,j}\,(r)+q_{ij}\,I_{i,j+1}\,(r)\right) \tag{13}$$

and

$$I_{p,q}=I_{p,q}(r)=\int_0^{2\pi}\frac{\cos^p\theta\sin^q\theta}{(a+r\cos\theta)(b+r\sin\theta)}\,d\theta\ . \tag{14}$$

From (13), we write $C_k=p_{k,0}I_{k+1,0}+\sum_{i+j=k,j\neq 0}\left(p_{i,j}+q_{i+1,j-1}\right)I_{i+1,j}+q_{0,k}I_{0,k+1}$. Without loss of generality, we rename the coefficients of C_k as

$$C_k(r)=\sum_{j=0}^{k+1}p_{k-j,j}I_{k-j+1,j}(r). \tag{15}$$

Hence the set (6) is the space of all linear combinations of the functions

$$r^{k+1}I_{k-j+1,j}(r),\quad 1\le k\le n,\quad 0\le j\le k+1, \tag{16}$$

or we also say that f^0 is a linear combination with arbitrary coefficients of these functions. In order to prove that (5) is a basis in (6), since they are linearly independent, it is sufficient to show that f^0 is also a linear combination with arbitrary coefficients of (5).

From now on we consider that $n\ge 3$ is an odd number. The case when n is even can be studied in a similar way. Since $N=[(n-1)/2]$ we have that $n=2N+1$ and $N\ge 1$. We also denote,

$$Y_{p,q}=Y_{p,q}(r)=\int_0^{2\pi}\frac{\cos^p\theta\sin^q\theta}{a+r\cos\theta}\,d\theta, \tag{17}$$

$$Z_{p,q} = Z_{p,q}(r) = \int_0^{2\pi} \frac{\cos^p \theta \sin^q \theta}{b + r \sin \theta} \, d\theta. \tag{18}$$

The recurrence relation

$$I_{p,q} = \frac{1}{r} \left(Y_{p,q-1} - b I_{p,q-1} \right), \tag{19}$$

will be used in the following to replace in (12) all the integrals of the form $I_{p,2t+1}$. Thus, for $N = 1$ we have $f^0(r) = r^2 C_1(r) + r^3 C_2(r) + r^4 C_3(r)$ and, using (15) and (19) we write:

$$\begin{aligned}
r^2 C_1 &= p_{0,1} r \left(Y_{1,0} - b I_{1,0} \right) + r^2 \left(p_{1,0} I_{2,0} + p_{-1,2} I_{0,2} \right), \\
r^3 C_2 &= r^2 \left(p_{1,1} Y_{2,0} + p_{-1,3} Y_{0,2} \right) + r^2 \left(-b p_{1,1} I_{2,0} - b p_{-1,3} I_{0,2} \right) + r^3 \left(p_{2,0} I_{3,0} + p_{0,2} I_{1,2} \right), \\
r^4 C_3 &= r^3 \left(p_{2,1} Y_{3,0} + p_{0,3} Y_{1,2} \right) + r^3 \left(-b p_{2,1} I_{3,0} - b p_{0,3} I_{1,2} \right) + r^4 \left(p_{3,0} I_{4,0} + p_{1,2} I_{2,2} + p_{-1,4} I_{0,4} \right).
\end{aligned}$$

Then

$$\begin{aligned}
f^0 = \; & p_{0,1} r \left(Y_{1,0} - b I_{1,0} \right) + r^2 \left(p_{1,1} Y_{2,0} + p_{-1,3} Y_{0,2} \right) + r^3 \left(p_{2,1} Y_{3,0} + p_{0,3} Y_{1,2} \right) + \\
& r^2 \left[(p_{1,0} - b p_{1,1}) I_{2,0} + (p_{-1,2} - b p_{-1,3}) I_{0,2} \right] + r^3 \left[(p_{2,0} - b p_{2,1}) I_{3,0} + (p_{0,2} - b p_{0,3}) I_{1,2} \right] + \\
& r^4 \left(p_{3,0} I_{4,0} + p_{1,2} I_{2,2} + p_{-1,4} I_{0,4} \right).
\end{aligned}$$

It is not difficult to notice that the coefficients of the functions of the form $r^k I_{i,j}$ and $r^k Y_{i,j}$ are independent. For example, since $p_{1,0}$ does not appear anywhere else, we will write as coefficient of $r^2 I_{2,0}$ only $p_{1,0}$ instead of $p_{1,0} - b p_{1,1}$. Hence, for $N = 1$ we can write

$$\begin{aligned}
f^0 = \; & p_{0,1} r \left(Y_{1,0} - b I_{1,0} \right) + r^2 \left(p_{1,1} Y_{2,0} + p_{-1,3} Y_{0,2} \right) + r^3 \left(p_{2,1} Y_{3,0} + p_{0,3} Y_{1,2} \right) + \\
& + r^2 \left(p_{1,0} I_{2,0} + p_{-1,2} I_{0,2} \right) + r^4 \left(p_{3,0} I_{4,0} + p_{1,2} I_{2,2} + p_{-1,4} I_{0,4} \right) + r^3 \left(p_{2,0} I_{3,0} + p_{0,2} I_{1,2} \right).
\end{aligned}$$

It can be proved inductively, in a similar way, that for all $N \geq 1$,

$$\begin{aligned}
f^0 = \; & p_{0,1} r \left(Y_{1,0} - b I_{1,0} \right) + \\
& \sum_{s=1}^{N} \sum_{i=0}^{s} p_{2s-2i-1,2i+1} r^{2s} Y_{2s-2i,2i} + \sum_{s=1}^{N} \sum_{i=0}^{s} p_{2s-2i,2i+1} r^{2s+1} Y_{2s-2i+1,2i} + \\
& \sum_{s=1}^{N+1} \sum_{i=0}^{s} p_{2s-2i-1,2i} r^{2s} I_{2s-2i,2i} + \sum_{s=1}^{N} \sum_{i=0}^{s} p_{2s-2i,2i} r^{2s+1} I_{2s-2i+1,2i},
\end{aligned} \tag{20}$$

and the coefficients are independent.

For each $q \geq 2$ even, by replacing $(\sin^2 \theta)^{q/2} = (1 - \cos^2 \theta)^{q/2}$ in the formula (17) one obtains $Y_{p,q} = \sum_{s=0}^{q/2} (-1)^s \begin{pmatrix} q/2 \\ s \end{pmatrix} Y_{p+2s,0}$. Using this, we replace $Y_{2s-2i,2i}$ in the first sum appearing in f^0 and obtain:

$$\sum_{s=1}^{N} r^{2s} \sum_{i=0}^{s} p_{2s-2i-1,2i+1} Y_{2s-2i,2i} = Y_{0,0} \left(\sum_{s=1}^{N} c_{s,0} r^{2s} \right) + \sum_{t=1}^{N} r^{2t} Y_{2t,0} \left(\sum_{s=t}^{N} c_{s,t} r^{2(s-t)} \right),$$

where

$$c_{s,t} = \sum_{j=s-t}^{s} (-1)^{j-s+t} p_{2s-2j-1,2j+1} \left(\begin{array}{c} j \\ j-s+t \end{array} \right), \quad 1 \le s \le N, \ 0 \le t \le s. \tag{21}$$

It is possible to proceed in a similar way for the remaining sums appearing in f^0 and finally we obtain

$$f^0 = p_{0,1} r \left(Y_{1,0} - b I_{1,0} \right) + I_{0,0} Q_0(r^2) + r I_{1,0} Q_1(r^2) + Y_{0,0} P_0(r^2) + r Y_{1,0} P_1(r^2) + \sum_{t=1}^{N} \left[r^{2t} Y_{2t,0} P_{2t}(r^2) + r^{2t+1} Y_{2t+1,0} P_{2t+1}(r^2) + r^{2t} I_{2t,0} Q_{2t}(r^2) + r^{2t+1} I_{2t+1,0} Q_{2t+1}(r^2) \right], \tag{22}$$

where P_{2t}, P_{2t+1}, Q_{2t+1} are real polynomials of degree $N-t$, Q_{2t} is a real polynomial of degree $N+1-t$ for each $t = 0, ..., N$. Moreover, the polynomials P_0, P_1, Q_0 and Q_1 have no free term. In order to see that the coefficients of the functions of the form $r^k I_{i,j}$ and $r^k Y_{i,j}$ in (22) are independent, we go back to (20) and first notice that each sum appearing there has a different set of coefficients. Then it is sufficient if we study only the first sum, for example, whose new coefficients are $c_{s,t}$ and they satisfy (21). We notice that $p_{2t-1,2s-2t+1}$ appears in $c_{s,t}$ but not in c_{s^*,t^*} for $s^* \le s$ and $t^* < t$. From this we deduce that the coefficients $c_{s,t}$ can be taken as arbitrary real numbers.

In the next step we use the recurrences

$$I_{p,0} = \frac{1}{r} \left(Z_{p-1,0} - a I_{p-1,0} \right), \tag{23}$$

$$Y_{p,0} = \frac{1}{r} \left(2\pi m_{p-1} - a Y_{p-1,0} \right), \tag{24}$$

where $m_p = \dfrac{(p-1)!!}{p!!}$ for $p \ge 1$ and $m_0 = 1$ (with $(2t)!!$ is denoted the product of all even natural numbers less or equal with $2t$, and analogously for $(2t-1)!!$). Using (23) and (24) we replace all the integrals $I_{2t+1,0}$ and $Y_{2t+1,0}$ in (22). We obtain the expression

$$\begin{aligned} f^0 = \ & p_{0,1} \left(2\pi - a Y_{0,0} - b Z_{0,0} + ab I_{0,0} \right) + \left[2\pi P_1(r^2) + 2\pi \sum_{t=1}^{N} r^{2t} m_{2t} P_{2t+1}(r^2) \right] + \\ & Y_{0,0} \left[P_0(r^2) - a P_1(r^2) \right] + \sum_{t=1}^{N} r^{2t} Y_{2t,0} \left[P_{2t}(r^2) - a P_{2t+1}(r^2) \right] + \\ & Z_{0,0} Q_1(r^2) + I_{0,0} \left[Q_0(r^2) - a Q_1(r^2) \right] + \\ & \sum_{t=1}^{N} r^{2t} Z_{2t,0} Q_{2t+1}(r^2) + \sum_{t=1}^{N} r^{2t} I_{2t,0} \left[Q_{2t}(r^2) - a Q_{2t+1}(r^2) \right]. \end{aligned}$$

Since the coefficients of the polynomials involved in this expression are arbitrary, we can write

$$
\begin{aligned}
f^0 = \quad & p_{0,1}\left(2\pi - aY_{0,0} - bZ_{0,0} + abI_{0,0}\right) + P_1(r^2) + Y_{0,0}P_0(r^2) + \sum_{t=1}^{N} r^{2t} Y_{2t,0} P_{2t}(r^2) + \\
& Z_{0,0}Q_1(r^2) + I_{0,0}Q_0(r^2) + \sum_{t=1}^{N} r^{2t} Z_{2t,0} Q_{2t+1}(r^2) + \sum_{t=1}^{N} r^{2t} I_{2t,0} Q_{2t}(r^2),
\end{aligned} \tag{25}
$$

where P_i and Q_i are new polynomials with arbitrary coefficients, like above.

Using the recurrences (23) and (24) we obtain for each even $p \geq 2$ that

$$
r^p I_{p,0} = a^{p-1}\left(aI_{0,0} - Z_{0,0}\right) - \left(a^{p-3} Z_{2,0} r^2 + \dots + aZ_{p-2,0} r^{p-2}\right), \tag{26}
$$

$$
r^p Y_{p,0} = a^{p-1}\left(aY_{0,0} - 2\pi\right) - 2\pi\left(a^{p-3} m_2 r^2 + \dots + a m_{p-2} r^{p-2}\right). \tag{27}
$$

Now we use them for replacing $I_{2t,0}$ and $Y_{2t,0}$ in (25). One can easily obtain

$$
\begin{aligned}
f^0(r) = \quad & p_{0,1}\left(2\pi - aY_{0,0}\right) + p_{1,0}\left(aI_{0,0} - Z_{0,0}\right) + P_1(r^2) + Y_{0,0}P_0(r^2) + \\
& Z_{0,0}Q_1(r^2) + I_{0,0}Q_0(r^2) + \sum_{t=1}^{N} r^{2t} Z_{2t,0} Q_{2t+1}(r^2),
\end{aligned} \tag{28}
$$

where P_i and Q_i are again new polynomials with arbitrary coefficients.

Using recursively that for each even $p \geq 2$ we have

$$
Z_{p,0} = -\frac{1}{r^2}\left((b^2 - r^2) Z_{p-2,0} - 2\pi b m_{p-2}\right),
$$

we obtain

$$
(-1)^{p/2} r^p Z_{p,0} = \rho^{p-2}\left(\rho^2 Z_{0,0} - 2\pi b\right) + 2\pi b\left(\rho^{p-4} m_2 r^2 - \dots + (-1)^{p/2} m_{p-2} r^{p-2}\right) \tag{29}
$$

where $\rho = \sqrt{b^2 - r^2}$. After replacing $Z_{2t,0}$ with formula (29) in (28), we have

$$
\begin{aligned}
f^0(r) = \quad & p_{0,1}\left(2\pi - aY_{0,0}\right) + p_{1,0}\left(aI_{0,0} - Z_{0,0}\right) + p_{0,0}\left(2\pi - bZ_{0,0}\right) + \\
& P_1(r^2) + Y_{0,0}P_0(r^2) + Z_{0,0}Q_1(r^2) + I_{0,0}Q_0(r^2),
\end{aligned} \tag{30}
$$

where $p_{0,1}, p_{1,0}, p_{0,0}$ are arbitrary real numbers, P_1, P_0, Q_1, Q_0 are new polynomials with arbitrary coefficients but without free term, the first three polynomials have degree N and the last one has degree $N + 1$.

Using the Residue Theorem [7, 11] we find the formulas

$$
I_{0,0}(r) = 2\pi \frac{a\sqrt{a^2 - r^2} + b\sqrt{b^2 - r^2}}{(b^2 + a^2 - r^2)\sqrt{a^2 - r^2}\sqrt{b^2 - r^2}}, \tag{31}
$$

$$
Y_{0,0}(r) = 2\pi \frac{1}{\sqrt{a^2 - r^2}}, \tag{32}
$$

$$
Z_{0,0}(r) = 2\pi \frac{1}{\sqrt{b^2 - r^2}}. \tag{33}
$$

It follows that

$$r^2 I_{0,0}(r) = (b^2 + a^2) I_{0,0}(r) - a Z_{0,0}(r) - b Y_{0,0}(r).$$

We use this formula in a recursive way to find $r^4 I_{0,0}(r)$, ..., $r^{2N+2} I_{0,0}(r)$ and obtain

$$r^{2k+2} I_{0,0}(r) = (b^2 + a^2)^k r^2 I_{0,0}(r) + Z_{0,0} Q_2(r^2) + Y_{0,0} P_2(r^2),$$

where P_2 and Q_2 are polynomials of degree k without constant term. Now we notice that, moreover,

$$r^2 I_{0,0}(r) = \frac{b}{a}\left(2\pi - a Y_{0,0}\right) + \frac{b^2 + a^2}{a}\left(a I_{0,0} - Z_{0,0}\right) - \frac{b}{a}\left(2\pi - b Z_{0,0}\right).$$

We replace these last two expressions in (30) and obtain

$$\begin{aligned} f^0(r) = \; & p_{0,1}\left(2\pi - a Y_{0,0}\right) + p_{1,0}\left(a I_{0,0} - Z_{0,0}\right) + p_{0,0}\left(2\pi - b Z_{0,0}\right) + \\ & P_1(r^2) + Y_{0,0} P_0(r^2) + Z_{0,0} Q_1(r^2), \end{aligned} \tag{34}$$

where $p_{0,1}, p_{1,0}, p_{0,0}, P_1, P_0, Q_1$ are new, but with the same qualities as before. We use again the expressions of $I_{0,0}$, $Y_{0,0}$ and $Z_{0,0}$ given by (31), (32) and (33), respectively, and obtain that, indeed, f^0 is an arbitrary linear combination of the functions (5). Hence, Lemma 2 is proved.

4 Proof of Lemma 3

In the beginning of this section we state some useful results of complex analysis, see [7, 11]. A continuous function $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$ is called a *path* in $\mathbb{C} \setminus \{0\}$. *The index (or winding number)* of the path γ in $\mathbb{C} \setminus \{0\}$ with respect to 0 is defined by

$$w(\gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z}.$$

The next theorem is known as the *variation of the argument principle.*

Theorem 5 *Let G be a Jordan closed curve and we denote by D its interior. Let f be a holomorphic function in a neighborhood of $\overline{D}$ and such that it has no zeros on G. We denote $N_0(f)$ the number of zeros of f in D. Then*

$$N_0(f) = w(f(G), 0) = \frac{1}{2\pi i} \int_{f(G)} \frac{dz}{z} = \frac{1}{2\pi i} \int_G \frac{f'(z)}{f(z)} dz.$$

Another useful result is the following.

Proposition 6 *Let γ and γ_1 be two paths in $\mathbb{C}\setminus\{0\}$ such that*

$$|\gamma(t)-\gamma_1(t)|\leq|\gamma_1(t)| \quad \textit{for all } t\in[0,1]. \tag{35}$$

Then, connecting the points $P=\gamma(0)$ with $P_1=\gamma_1(0)$, and $Q=\gamma(1)$ with $Q_1=\gamma_1(1)$ by a segment having these endpoints, we obtain a closed curve that does not contain the origin inside. Moreover,

$$w(\gamma,0)=w(\gamma_1,0)+w(PP_1,0)-w(QQ_1,0). \tag{36}$$

Proof. We define the continuous function $\phi:[0,1]\times[0,1]\to\mathbb{C}$ by $\phi(t,\lambda)=\gamma_1(t)+\lambda\left(\gamma(t)-\gamma_1(t)\right)$. It is not difficult to see that the relation (35) assures that ϕ has values in $\mathbb{C}\setminus\{0\}$. This means that any straight segment connecting $\gamma(t)$ with $\gamma_1(t)$ does not contain the origin. Hence, indeed, the closed curve obtained by connecting with segments the endpoints of the curves γ and γ_1, does not contain the origin in the interior. This implies that the index of this curve is 0 and from this we obtain (36). □

In the case that γ is piecewise smooth, using that $\ln z=\ln|z|+i\arg z$, the index can be calculated with the formula

$$w(\gamma,0)=\frac{1}{2\pi i}\int_\gamma\frac{dz}{z}=\frac{1}{2\pi i}\ln\frac{|\gamma(1)|}{|\gamma(0)|}+\frac{1}{2\pi}\Delta\arg\gamma, \tag{37}$$

where $\Delta\arg\gamma$ denotes the increasing of the argument on the curve γ. If, moreover, γ is a closed curve then we have

$$w(\gamma,0)=\frac{1}{2\pi}\Delta\arg\gamma.$$

Proof of Lemma 3. The zeros of the function f are among the zeros of some polynomial of degree $4N+8$. This can be seen after noticing that any zero of f also satisfies

$$R^2(z)(a^2-z)+2R(z)S(z)\sqrt{a^2-z}\sqrt{b^2-z}+S^2(z)(b^2-z)=(b^2+a^2-z)^2T^2(z)(a^2-z)(b^2-z)$$

and, moreover,

$$\begin{aligned}&4R^2(z)S^2(z)(a^2-z)(b^2-z)=\\&\left[(b^2+a^2-z)^2T^2(z)(a^2-z)(b^2-z)-R^2(z)(a^2-z)-S^2(z)(b^2-z)\right]^2.\end{aligned}$$

Then $N_0(f)\leq 4N+8$, but we will use only that $N_0(f)$ is finite in order to choose a closed curve whose interior is included in Ω and contains all the zeros of f.

From now on ρ will denote a sufficiently large positive constant and ε a sufficiently small positive constant. We denote by C_ρ the circle centered at the origin and having the radius ρ and consider the points $A,A'\in C_\rho$, $A=(x_A,\varepsilon)$, $A'=(x_A,-\varepsilon)$, $B=(a^2,\varepsilon)$ and $B'=(a^2,-\varepsilon)$. By $C_{\rho,\varepsilon}$ we denote the curve obtained by removing the arc AA' from the circle C_ρ, and by C_ε we denote the arc BB' from the circle with center at $(a^2, 0)$ and radius ε (the one contained in Ω). The segments that join A and B, respectively B' and A', are denoted by L_+^ε, respectively L_-^ε, and we denote also $I_{\rho,\varepsilon} = L_+^\varepsilon \cup C_\varepsilon \cup L_-^\varepsilon$. Now we define the following closed curve in the complex plane,

$$G = G_{\rho,\varepsilon} = C_{\rho,\varepsilon} \cup I_{\rho,\varepsilon}, \tag{38}$$

and denote its interior by D. The counterclockwise orientation is considered on G. Since $N_0(f)$ is finite, ρ is sufficiently large and ε is sufficiently small, all the zeros of f are in D. We apply Theorem 5 and deduce that

$$N_0(f) = \frac{1}{2\pi i}\oint_{f(G)} \frac{dz}{z}. \tag{39}$$

We denote by

$$Z_1 = \frac{1}{2\pi i}\int_{f(C_{\rho,\varepsilon})} \frac{dz}{z}, \qquad Z_2 = \frac{1}{2\pi i}\int_{f(I_{\rho,\varepsilon})} \frac{dz}{z}, \tag{40}$$

and we have that $N_0(f) = Z_1 + Z_2$.

In order to estimate Z_1 we use that there exist $\alpha_0 \in \mathbb{C}$ and an integer $m \geq 0$ such that $m/2 \leq N + 2$ and

$$\lim_{|z|\to\infty} \frac{f(z)}{z^{m/2}} = \alpha_0. \tag{41}$$

Then the following inequality also holds,

$$|f(z) - \alpha_0 z^{m/2}| \leq |\alpha_0 z^{m/2}| \quad \text{for } z \in C_{\rho,\varepsilon}.$$

The hypotheses of Proposition 6 are fulfilled for the curves $f(C_{\rho,\varepsilon})$ and $g(C_{\rho,\varepsilon})$, where we denoted $g : \mathbb{C} \to \mathbb{C}$, $g(z) = \alpha_0 z^{m/2}$. Here $P = f(A)$ and $Q = f(A')$, respectively $P_1 = g(A)$ and $Q_1 = g(A')$ are conjugate. From this we have that $|P| = |Q|$, $|P_1| = |Q_1|$ and the angles $\sphericalangle POP_1$ and $\sphericalangle QOQ_1$ are equal. Using relation (41) we get that $f(A)/g(A) \to 1$ as $\rho \to \infty$. The argument of $f(A)/g(A)$ is equal to the measure of the angle $\sphericalangle POP_1$ and we have that it tends to 0 when $\rho \to \infty$. These facts assure that, by applying the formula (37), $w(PP_1, 0) - w(QQ_1, 0) = O(1/\rho)$, where $O(1/\rho)$ denotes some function that goes to 0 when $\rho \to \infty$. A direct calculation gives that

$$w(g(C_{\rho,\varepsilon}), 0) = \frac{1}{2\pi i}\int_{C_{\rho,\varepsilon}} \frac{g'(z)}{g(z)} dz = \frac{m}{2} + O(\varepsilon),$$

where $O(\varepsilon)$ denotes some function that goes to 0 as $\varepsilon \to 0$. Now we replace all these in (36) of Proposition 6 and obtain the following estimation

$$Z_1 \leq N + 2 + O(\varepsilon) + O(1/\rho). \tag{42}$$

In order to continue our analysis we need to consider the functions $\gamma_+, \gamma_- : [a^2, \infty) \to \mathbb{C}$ given by $\gamma_+(x) = \lim_{\varepsilon \searrow 0} f(x + i\varepsilon)$ and $\gamma_-(x) = \lim_{\varepsilon \searrow 0} f(x - i\varepsilon)$ for all $x \geq a^2$. A direct calculation gives for $\gamma_+(x)$ the expressions

$$iR(x)\sqrt{x - a^2} + S(x)\sqrt{b^2 - x} + i(b^2 + a^2 - x)T(x)\sqrt{x - a^2}\sqrt{b^2 - x}, \quad \text{for } a^2 \leq x \leq b^2, \tag{43}$$
$$iR(x)\sqrt{x - a^2} + iS(x)\sqrt{x - b^2} - (b^2 + a^2 - x)T(x)\sqrt{x - a^2}\sqrt{x - b^2}, \quad \text{for } x > b^2,$$

and $\gamma_-(x) = \overline{\gamma_+(x)}$ for all $x \geq a^2$. In the case that $\gamma_+(x) \neq 0$ for all $x \in [a^2, \infty)$ we will be able to give an estimation for Z_2. We will call this *Case* 1. Otherwise we need to write the function f as $f(z) = h(z)f_1(z)$ such that f and f_1 have the same zeros and f_1 fits in *Case* 1.

Case 1. $\gamma_+(x) \neq 0$ for all $x \in [a^2, \infty)$. For the curve $f(L_+^\varepsilon)$ we have the parametrization $x \in [a^2, \rho] \mapsto f(x + i\varepsilon)$, while the parametrization $x \in [a^2, \rho] \mapsto f(x - i\varepsilon)$ is good for the (oriented) curve $-f(L_-^\varepsilon)$. Since γ_+ is continuous on $[a^2, \rho]$, $\gamma_+(x) \neq 0$ for all $x \in [a^2, \rho]$ and the convergence $f(x + i\varepsilon) \to \gamma_+(x)$ as $\varepsilon \searrow 0$ is uniform on $[a^2, \rho]$, we have

$$|f(x + i\varepsilon) - \gamma_+(x)| < |\gamma_+(x)| \quad \text{for all } x \in [a^2, \rho].$$

The hypotheses of Proposition 6 are fulfilled for the curves $f(\cdot + i\varepsilon)$ and γ_+ and, using also that the endpoints are ε–closed, we obtain the relation

$$w\left(f(L_+^\varepsilon), 0\right) = w\left(\gamma_+, 0\right) + O(\varepsilon).$$

Analogously,

$$w\left(f(L_-^\varepsilon), 0\right) = -w\left(\gamma_-, 0\right) + O(\varepsilon).$$

If we write $\gamma_+(x) = r(x)\exp(i\theta(x))$ then $\gamma_-(x) = r(x)\exp(-i\theta(x))$ and, using the formula (37) we have

$$w(\gamma_+, 0) = \frac{1}{2\pi i} \ln \frac{r(\rho)}{r(a^2)} + \frac{1}{2\pi}[\theta(\rho) - \theta(a^2)],$$

and

$$w(\gamma_-, 0) = \frac{1}{2\pi i} \ln \frac{r(\rho)}{r(a^2)} - \frac{1}{2\pi}[\theta(\rho) - \theta(a^2)].$$

Since

$$Z_2 = w\left(f(L_+^\varepsilon), 0\right) + w\left(f(L_-^\varepsilon), 0\right) + \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{f'(z)}{f(z)}dz,$$

using the above relations we obtain

$$Z_2 = \frac{1}{\pi}\left[\theta(\rho) - \theta(a^2)\right] + O(\varepsilon), \tag{44}$$

where $[\theta(\rho) - \theta(a^2)]$ is the variation of the argument on the curve $\gamma_+\left([a^2, \rho]\right)$. From the formula (43) we have that the number of zeros of $\operatorname{Re}\gamma_+$ has the upper bound $2N + 3$.

We notice that in the case that T has degree less then N we can provide a lower upper bound. It is also useful to notice that the starting point of the curve $\gamma_+([a^2, \rho])$ is on the real axis. About its ending point we can say that either is very close to the real axis, i.e. $\operatorname{Im}\gamma_+(\rho)/\operatorname{Re}\gamma_+(\rho) = O(1/\rho)$, or to the imaginary one. This last case can happen only if T has degree less then N. From all these we deduce that

$$|\theta(\rho) - \theta(a^2)| \leq (2N + 3)\pi + O(1/\rho).$$

Hence,

$$|Z_2| \leq 2N + 3 + O(1/\rho) + O(\varepsilon),$$

and we obtain the conclusion of the lemma by using also (42).

Case 2. There exists some $x^* \in [a^2, \infty)$ such that $\gamma_+(x^*) = 0$.

For each such $x^* \in (a^2, b^2) \cup (b^2, \infty)$, we consider $h^*(z) = (z - x^*)^{k_*}$, where k_* is the multiplicity of x^* as zero of γ_+. Note that γ_+ is analytic on $(a^2, b^2) \cup (b^2, \infty)$.

If $x^* = a^2$ then we take $h^*(z) = (\sqrt{a^2 - z})^{k_a}$, where $k_a = 2\min\{k_{a1}, k_{a2} + 1/2\}$, k_{a1} being the multiplicity of a^2 as zero of S and k_{a2} being the multiplicity of a^2 as zero of the function $x \mapsto R(x) + (b^2 + a^2 - x)T(x)\sqrt{b^2 - x}$. A necessary condition for the fact that a^2 is a zero of γ_+ is that $k_{a1} \geq 1$.

If $x^* = b^2$ then we take $h^*(z) = (\sqrt{b^2 - z})^{k_b}$, where $k_b = 2\min\{k_{b1} + 1/2,\ k_{b2},\ k_{b3} + 1/2\}$, k_{b1} being the multiplicity of b^2 as zero of S, k_{b2} the multiplicity of b^2 as zero of R and k_{b3} the multiplicity of b^2 as zero of T. A necessary condition for the fact that b^2 is a zero of γ_+ is that $k_{b2} \geq 1$.

Clearly γ_+ has finitely many zeros in $[a^2, \infty)$. So, we can choose ρ sufficiently large in order that all the zeros of γ_+ in $[a^2, \infty)$ are contained in $[a^2, \rho]$. We consider a function h as the product of all the functions h^* defined as before for each $x^* \in [a^2, \infty)$ zero of γ_+, and another function f_1 by

$$f_1(z) = \frac{f(z)}{h(z)}.$$

Both h and f_1 are holomorphic in Ω, and the number of zeros of f in D is equal to the number of zeros of f_1 in D, i.e.

$$N_0(f) = N_0(f_1) = \frac{1}{2\pi i} \oint_{f_1(G)} \frac{dz}{z}. \tag{45}$$

We denote by

$$Y_1 = \frac{1}{2\pi i} \int_{f_1(C_{\rho,\varepsilon})} \frac{dz}{z}, \qquad Y_2 = \frac{1}{2\pi i} \int_{f_1(I_{\rho,\varepsilon})} \frac{dz}{z}, \tag{46}$$

and we have that $N_0(f_1) = Y_1 + Y_2$.

Since $f(z) = h(z)f_1(z)$ and h is the product of the functions h^*, the integral Z_1 defined by the formula (40) can be written as

$$Z_1 = Y_1 + \sum_{x^*} \frac{1}{2\pi i} \int_{C_{\rho,\varepsilon}} \frac{(h^*)'(z)}{h^*(z)} dz.$$

Moreover, the expression $(h^*)'/h^*$ is either $k_*/(z-x^*)$, or $k_a/\left(2(z-a^2)\right)$, or $k_b/\left(2(z-b^2)\right)$. In particular, we notice that $(h^*)'/h^*$ is continuous on C_ρ. Then, we can write

$$\frac{1}{2\pi i} \int_{C_{\rho,\varepsilon}} \frac{(h^*)'(z)}{h^*(z)} dz = \frac{1}{2\pi i} \oint_{C_\rho} \frac{(h^*)'(z)}{h^*(z)} dz + O(\varepsilon) = k_* \left(\text{ or } \frac{k_a}{2} \text{ or } \frac{k_b}{2} \right) + O(\varepsilon).$$

We denote with k the sum with respect to all zeros of γ_+ of all positive numbers of the form k_*, $k_a/2$, $k_b/2$. Using also the estimation (42) for Z_1, we deduce that

$$Y_1 \leq N + 2 - k + O(\varepsilon). \tag{47}$$

In order to give an estimation for Y_2, we define $h_+(x) = \lim_{\varepsilon \searrow 0} h(x + i\varepsilon)$, $h_-(x) = \lim_{\varepsilon \searrow 0} h(x - i\varepsilon)$, $\beta_+(x) = \lim_{\varepsilon \searrow 0} f_1(x + i\varepsilon)$ and $\beta_-(x) = \lim_{\varepsilon \searrow 0} f_1(x - i\varepsilon)$ for all $x \geq a^2$. It is easy to check that $h_-(x) = \overline{h_+(x)}$ and, as a consequence, $\beta_-(x) = \overline{\beta_+(x)}$. In what follows, we will justify that β_+ is continuous and it has no zeros in the interval $[a^2, \infty)$. Since $\beta_+(x) = \gamma_+(x)/h_+(x)$ for all $x \in [a^2, \infty)$ it is sufficient to study this function in the points x^*, the zeros of γ_+ which, in fact, are also the zeros of h_+. We will do here an analysis only at the point $x^* = b^2$ in the case $k_b = 1$, i.e. the corresponding factor of h is $h^*(z) = \sqrt{b^2 - z}$. Since the other factors of h do not influence our analysis, for simplicity, we write $h(z) = \sqrt{b^2 - z}$. We have that

$$h_+(x) = \begin{cases} \sqrt{b^2 - x} & \text{for } a^2 \leq x \leq b^2, \\ i\sqrt{x - b^2} & \text{for } x > b^2, \end{cases}$$

and, using formula (43), we obtain that $\beta_+(x)$ has the expressions

$$\begin{aligned} & i\frac{R(x)}{\sqrt{b^2 - x}}\sqrt{x - a^2} + S(x) + i(b^2 + a^2 - x)T(x)\sqrt{x - a^2}, \quad \text{for } a^2 \leq x \leq b^2, \\ & \frac{R(x)}{\sqrt{x - b^2}}\sqrt{x - a^2} + S(x) + i(b^2 + a^2 - x)T(x)\sqrt{x - a^2}, \quad \text{for } x > b^2. \end{aligned} \tag{48}$$

From the fact that b^2 is a zero of R (we already emphasized that this is a necessary condition from the fact that b^2 is a zero of γ_+) we obtain that β_+ is continuous at b^2. We have that $\beta_+(b^2) \neq 0$ because $k_b = 1$, which assures that b^2 is not a zero for both polynomials S and T.

The discussions for the other zeros with their multiplicities are similar, but we will not write them here. Hence, β_+ is continuous and it has no zeros in the interval $[a^2, \infty)$. This assures that, by repeating the arguments performed for obtaining the formula (44), we also obtain that

$$Y_2 = \frac{1}{\pi}\left[\theta(\rho) - \theta(a^2)\right] + O(\varepsilon),$$

where $[\theta(\rho) - \theta(a^2)]$ is the variation of the argument on the curve $\beta_+([a^2, \rho])$. We claim that

$$|\theta(\rho) - \theta(a^2)| \leq (2N + 3)\pi + O(1/\rho), \tag{49}$$

that gives

$$|Y_2| \leq 2N + 3 + O(\varepsilon) + O(1/\rho),$$

and, moreover, using the estimation (47) for Y_1 and that $N_0(f) = Y_1 + Y_2$, we obtain that $N_0(f) \leq 3N + 5$.

In order to justify the claim (49), again we consider only the case when $h(z) = \sqrt{b^2 - z}$ and, consequently, $k = 1/2$ and β_+ is given by (48). Therefore, the number of zeros of $\mathrm{Re}\beta_+$ in the interval $[a^2, b^2]$ has the upper bound $N + 1$, while that of $\mathrm{Im}\beta_+$ in the interval $[b^2, \rho]$ has the upper bound $N + 1$. The starting point, $\beta_+(a^2)$, is on the real axis. From all these facts we can deduce that, indeed, the increasing of the argument satisfies (49). The other cases can be treated in a similar manner. □

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References

[1] T.R. Blows, L.M. Perko, Bifurcation of limit cycles from centers and separatrix cycles of planar analytic systems, SIAM Rev. 36(1994) 341–376.

[2] S.N. Chow, C. Li, D. Wang, Normal forms and bifurcation of planar vector fields, Cambridge University Press, 1994.

[3] B. Coll, A. Gasull, R. Prohens, Bifurcation of limit cycles from two families of centers, Dyn. Contin. Discrete Impuls. Syst., in press.

[4] A. Gasull, W. Li, J. Llibre, Z. Zhang, Chebyshev property of complete elliptic integrals and its application to Abelian integrals, Pacific J. Math. 202(2002) 341–361.

[5] H. Giacomini, J. Llibre, M. Viano, On the nonexistence, existence and uniqueness of limit cyles, Nonlinearity 9(1996) 501–516.

[6] J. Giné, J. Llibre, Limit cycles of cubic polynomial vector fields via averaging theory, Preprint, 2005.

[7] P. Hamburg, P. Mocanu, N. Negoescu, Analiză matematică. Funcţii complexe (Mathematical analysis. Complex functions, in romanian), Editura Didactică şi Pedagogică, Bucureşti, 1982.

[8] J. Llibre, J.S. Pérez del Río, J. A. Rodríguez, Averaging analysis of a perturbated quadratic center, Nonlinear Anal. 46(2001) 45–51.

[9] G.S. Petrov, On the nonoscillation of elliptic integrals, Funct. Anal. Appl. 31 (1997) 262–265.

[10] L.S. Pontryagin, Über Autoschwingungssysteme, die den hamiltonshen nahe liegen, Physikalische Zeitschrift der Sowjetunion, Band 6, Heft(1–2) (1934) 25–28.

[11] M. Rao, H. Stetkær, Complex Analysis. An invitation, World Scientific, 1991.

[12] G. Xiang and M. Han, Global bifurcation of limit cycles in a family of polynomial systems, J. Math. Anal. Appl. 295(2004) 633-644.

[13] Y. Zhao, S. Zhu, Higher order Melnikov function for a quartic Hamiltonian with cuspidal loop, Discrete Contin. Dyn. Syst. 8(2002) 995–1018.