Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 330 (2007) 576–583
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Periodic solutions for functional–differential equations of mixed type

Adriana Buică *, Veronica Ana Ilea

Department of Applied Mathematics, Babeş-Bolyai University, Cluj-Napoca, Romania

Received 27 April 2006
Available online 1 September 2006
Submitted by Steven G. Krantz

Dedicated to Professor Ioan A. Rus on his 70th birth anniversary, with our gratitude

Abstract

In this paper we study the existence of ω-periodic solutions for some functional–differential equations of mixed type. Among the main results are the averaging principle and existence theorems for some equations with homogeneous nonlinearities. We use here the coincidence degree theory of Mawhin.

Keywords: Functional–differential equations of mixed type; Periodic solutions; Coincidence degree

1. Introduction

In this paper we study the existence of periodic solutions for functional–differential equations of mixed type (MFDE, for short) of the form

$$x'(t) = f\big(t, x(t), x\big(t - \tau_1(t)\big), x\big(t + \tau_2(t)\big)\big), \quad t \in \mathbb{R}, \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R}^{3n} \to \mathbb{R}^n$ and $\tau_1, \tau_2 : \mathbb{R} \to \mathbb{R}$ are continuous and ω-periodic in t. For example, if τ_1 and τ_2 are constants and with the same sign, the growth rate of x depends on both forward and backward (advanced and delayed) translates of the argument t. This is what is called of "mixed type." The same phenomenon happened, for example, if $\tau_1(t) = \tau_2(t) = \sin t$. The study of various problems (not only the periodic problem) for this type of equations is

* Corresponding author.
E-mail address: abuica@math.ubbcluj.ro (A. Buică).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.08.009

difficult and, as far as we know, it started recently. An elaborated theory was initiated by Mallet-Paret who, for example, gave in [12] a Fredholm alternative for MFDE. The motivation for this study is that the travelling-wave solutions of a lattice differential equation satisfy a MFDE. Recently, various problems associated to a MFDE (as, for example, the existence of analytic solutions for an equation with analytic data) were studied by Dârzu in [5,6], Rus and Dârzu-Ilea in [14] and Precup in [13]. The authors of [7,9–11] studied the existence of periodic solutions for some MFDE. They use the coincidence degree theory as developed by Gaines and Mawhin in [8]. Here we use the same technique, but our results extend and complement the existing ones. The first main result is the averaging method, the analogous of the very popular method for ordinary differential equations (ODE, for short). Two existence results for periodic solutions of some MFDE with homogeneous nonlinearities are also given. They are extensions of similar results for ODE of Capietto, Mawhin and Zanolin appeared in [4]. Other cases that we studied are the small perturbations of odd MFDE or of autonomous ODE. The paper is organized as follows. In Section 2 we present the definition, some properties and results from the coincidence degree theory that will be used later on. Section 3 contains our main results.

2. The coincidence degree: definition and properties

In this section we present the definition of the coincidence degree and some of its properties, also some continuation theorems, perturbation theorems and some theorems for the case when the nonlinear operator is positively homogeneous. This results appear in [1,2,8].

Let X and Y be two Banach spaces, $L : X \to Y$ be a linear operator, continuous, Fredholm of index zero (meaning that $\operatorname{Im} L$ is a closed subset of Y and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = n < \infty$) and $N : \overline{\Omega} \to Y$ be a completely continuous operator, where Ω is an open and bounded subset of X. We denote $X_1 = \operatorname{Ker} L$ and $Y_2 = \operatorname{Im} L$. We have the decompositions $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. We consider $P : X \to X_1$ and $Q : Y \to Y_1$ linear, continuous projectors (meaning $P^2 = P$ and $Q^2 = Q$) and $J : X_1 \to Y_1$ an isomorphism. The operator $M : \overline{\Omega} \to X$, $M = (L + JP)^{-1}(N + JP)$ is completely continuous and does not depend on the linear operators P, Q and J. Moreover, we have that $Lx = N(x)$ if and only if $x = M(x)$. We assume that

$$Lx \neq N(x), \quad \text{for all } x \in \partial\Omega.$$

By definition, the coincidence degree of operators (L, N) is equal to the Leray–Schauder degree of operator $I - M : \overline{\Omega} \to X$ and is denoted here $d((L, N), \Omega)$. The main properties of the coincidence degree are the following.

Existence. If $d((L, N), \Omega) \neq 0$, then there is $x \in \Omega$ such that $Lx = N(x)$.

The generalized theorem of Borsuk type. If Ω is symmetrical in $0 \in \Omega$ and if $N(-x) = -N(x)$ for all $x \in \partial\Omega$, then $d((L, N), \Omega)$ is odd.

Invariance under homotopies. If $H : \overline{\Omega} \times [0, 1] \to Y$ is completely continuous and $Lx \neq H(x, \lambda)$ for all $x \in \partial\Omega$ and $\lambda \in [0, 1]$, then $d((L, H(\cdot, \lambda)), \Omega)$ is constant with respect to λ.

The following results will be also useful in the sequel. With $B_r(0)$ we denote the open ball of the space X centered in the origin and of radius r.

Theorem 2.1. *Suppose that $Lx \neq N(x)$ for each $x \in \partial\Omega$. Then there is $\mu > 0$ such that $\mu \leqslant \inf_{x\in\partial\Omega} \|Lx - N(x)\|$ and for all $N_p : \overline{\Omega} \to Y$ completely continuous with $\sup_{x\in\partial\Omega} \|N_p(x)\| < \mu$, we have $d((L, N + N_p), \Omega) = d((L, N), \Omega)$.*

Theorem 2.2. *Let $\varepsilon > 0$ and $N = N(\cdot, \varepsilon)$ such as $N(x, \varepsilon) \to N(x, 0)$ when $\varepsilon \searrow 0$ is uniformly with respect to $x \in \partial\Omega$. Suppose that $QN(x, 0) \neq 0$ for all $x \in \partial\Omega \cap X_1$. Then for $\varepsilon > 0$ sufficiently small, $Lx \neq \varepsilon N(x, \varepsilon)$ for all $x \in \partial\Omega$ and*

$$d\big(\big(L, \varepsilon N(\cdot, \varepsilon)\big), \Omega\big) = d_B\big(J^{-1}QN(\cdot, 0), \Omega \cap X_1, 0\big).$$

Theorem 2.3. (i) *If N is positively homogeneous of order $\alpha > 1$ (i.e., $N(rx) = r^\alpha N(x)$ for all $r > 0$, $x \in X$) and $QN(y) \neq 0$ for $y \in X_1$, with $|y| = 1$, then for r sufficiently small, $d((L, N), B_r(0)) = d_B(J^{-1}QN, B_r(0) \cap X_1, 0)$.*

(ii) *If, moreover, N_p is positively homogeneous of order $\beta > \alpha$, then for r sufficiently small, $d((L, N + N_p), B_r(0)) = d((L, N), B_r(0))$.*

Theorem 2.4. (i) *If N is positively homogeneous of order $0 < \alpha < 1$ and $QN(y) \neq 0$ for $y \in X_1$, with $|y| = 1$, then for r sufficiently large, $d((L, N), B_r(0)) = d_B(J^{-1}QN, B_r(0) \cap X_1, 0)$.*

(ii) *If, moreover, N_p is positively homogeneous of order $0 < \beta < \alpha$, then for r sufficiently large, $d((L, N + N_p), B_r(0)) = d((L, N), B_r(0))$.*

3. Periodic solutions for functional–differential equations of mixed type

This section contains our main results. In the first part we show how to choose appropriate function spaces and operators in order to transform the periodic problem for a MFDE into an operator equation suitable to apply the coincidence degree theory.

The periodic problem like a coincidence-type operator equation. Throughout this paper, C_ω is the Banach space defined by

$$C_\omega = \big\{x : \mathbb{R} \to \mathbb{R}^n \;\big|\; \text{continuous and } \omega\text{-periodic}\big\}$$

and endowed with the supremum norm $\|x\|_\infty = \sup_{t\in[0,\omega]} |x(t)|$. We define the operator $F : C_\omega \to C_\omega$ by

$$F(x)(t) = f\big(t, x(t), x\big(t - \tau_1(t)\big), x\big(t + \tau_2(t)\big)\big), \quad \text{for all } t \in \mathbb{R} \text{ and } x \in C_\omega. \tag{2}$$

It is not difficult to see that the operator F is well defined, continuous and bounded. We remind here that F is bounded if $F(M)$ is bounded for any bounded subset M of C_ω.

With the above notations, the ω-periodic problem for Eq. (1) can be written in the following form:

$$x' = F(x), \quad x \in C_\omega. \tag{3}$$

According to the notations from the introductory part we consider the following spaces of functions:

$$X = C_\omega, \qquad Y = \big\{y : \mathbb{R} \to \mathbb{R}^n \text{ continuous, with } y(t + \omega) = y(t) + y(\omega),\ t \in \mathbb{R}\big\}.$$

Both X and Y are Banach spaces endowed with the norm $\|\cdot\|_\infty$. Now we define a linear operator L and a nonlinear operator N by

$$L, N : X \to Y, \quad Lx(t) = x(t) - x(0), \quad N(x)(t) = \int_0^t F(x)(s)\,ds. \tag{4}$$

We can easily note that problem (3) can be written under the form of an operator equation of coincidence type

$$Lx = N(x), \quad x \in X.$$

We shall see that for this equation we can apply the coincidence degree theory. The linear operator L is continuous, and the kernel and its image are the following $X_1 = \operatorname{Ker} L = \{x \in X\colon x \text{ a constant function}\}$, $Y_2 = \operatorname{Im} L = \{y \in Y\colon\ y(\omega) = 0\}$. The space Y can be decomposed as $Y = Y_1 \oplus Y_2$, where $Y_1 = \{y \in Y\colon\ y(t) = ct,\ t \in \mathbb{R},\ c \in \mathbb{R}^n\}$. Observing that Y_2 is closed in Y and $\dim X_1 = \operatorname{codim} Y_2 = n$, it follows that *$L$ is a linear Fredholm operator of index zero*.

Since the operator F is bounded, and the convergence in the Banach spaces X and Y is the uniform convergence on the interval $[0,\omega]$, the Arzela–Ascoli theorem implies that the *nonlinear operator N is completely continuous*.

We define the isomorphism $J : X_1 \to Y_1$ by $J(c) = ct/\omega$ and the projectors $P : X \to X_1$ and $Q : Y \to Y_1$ by $Px = x(0)$ and $Qy(t) = y(\omega)t/\omega$, respectively. In order to apply the coincidence degree theory, the following formula for the restriction to the finite dimensional space X_1 of the operator $J^{-1}QN : X \to X_1$ is useful:

$$J^{-1}QN\big|_{X_1}(c) = \int_0^\omega f(s,c,c,c)\,ds. \tag{5}$$

It is also interesting to observe that the operator $M = (L + JP)^{-1}(N + JP) : X \to X$ is given by the formula

$$M(x)(t) = x(0) + \int_0^t F(x)(s)\,ds + \frac{\omega - t}{\omega}\int_0^\omega F(x)(s)\,ds.$$

We remind that the fixed points of the operator M are solutions of the coincidence type equation $Lx = N(x)$, and hence, are ω-periodic solutions of (1).

Main results. The first main result is the averaging method for MFDE.

Theorem 3.1. *We consider the functional–differential equation*

$$x'(t) = \varepsilon f\big(t, x(t), x\big(t - \tau_1(t)\big), x\big(t + \tau_2(t)\big), \varepsilon\big), \tag{6}$$

where $f : \mathbb{R} \times \mathbb{R}^{3n} \times [0,\varepsilon_1] \to \mathbb{R}$ a continuous function and ω-periodic in the first argument, $\tau_1, \tau_2 : \mathbb{R} \to \mathbb{R}$ are continuous and ω-periodic and $\varepsilon_1 > 0$. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$g(c) = \int_0^\omega f(t,c,c,c,0)\,dt. \tag{7}$$

We assume that there exists $a \in \mathbb{R}^n$ and a neighborhood V of a, such that $g(a) = 0$, $g(c) \neq 0$ for all $c \in \overline{V} \setminus \{a\}$ and $d_B(g, V, 0) \neq 0$.

Then, for $\varepsilon > 0$ sufficiently small, there exist an ω-periodic solution of Eq. (6), denoted $\varphi(\cdot, \varepsilon)$, that also satisfies $\varphi(\cdot, \varepsilon) \to a$ when $\varepsilon \to 0$.

Proof. In the case of Eq. (6) the operators F and N depend on the parameter ε and we underline this fact by denoting $F(x, \varepsilon)(t) = f(t, x(t), x(t - \tau_1(t)), x(t + \tau_2(t)), \varepsilon)$ and $N(x, \varepsilon)(t) = \int_0^t F(x, \varepsilon)(s)\,ds$. Then the ω-periodic problem for Eq. (6) is equivalent with

$$Lx = \varepsilon N(x, \varepsilon), \quad x \in X. \tag{8}$$

Let $\Omega = \{x \in X\colon\ x(t) \in V \text{ for all } t \in \mathbb{R}\}$, that is an open and bounded subset of space X. We have that $\overline{\Omega} = \{x \in X\colon\ x(t) \in \overline{V} \text{ for all } t \in \mathbb{R}\}$ and $\partial\Omega = \{x \in X\colon \text{ there exist } t \in \mathbb{R} \text{ with } x(t) \in \partial V\}$.

We shall apply Theorem 2.2 for Eq. (8) and the set Ω. For this we need to identify the operators $QN(\cdot, 0)|_{X_1}$ and $J^{-1}QN(\cdot, 0)|_{X_1}$. From the formula (5), identifying X_1 with R^n we find that $J^{-1}QN(\cdot, 0)|_{X_1} = g$, the function given by the relation (7). So $QN(\cdot, 0)|_{X_1} = Jg$, where we remind that J is a linear isomorphism between X_1 and Y_1.

From all these here we note that the hypotheses of Theorem 2.2 are fulfilled. Moreover, we have that $d_B(J^{-1}QN(\cdot, 0), \Omega \cap X_1, 0) = d_B(g, V, 0) \neq 0$. Then, for ε sufficiently small, $d((L, \varepsilon N(\cdot, \varepsilon)), \Omega, 0) \neq 0$, fact that assures that there exist a solution of Eq. (8) in Ω. We denote this solution with $\varphi(\cdot, \varepsilon)$, this being of course an ω-periodic solution of Eq. (6) that has the property that $\varphi(t, \varepsilon) \in V$ for any $t \in \mathbb{R}$.

Now, we note that, instead of V, we can take a neighborhood of a (like in [3]), $V_\mu \subset V$ such that V_μ shrinks to $\{a\}$ when $\mu \to 0$. This implies that the corresponding set Ω_μ is a neighborhood of the constant function a, such as the diameter Ω_μ is sufficiently small when $\mu \to 0$. So, for ε sufficiently small, Eq. (6) has an ω-periodic solution, $\varphi(\cdot, \varepsilon) \in \Omega_\mu$. Hence, we can choose the solution such that $\varphi(\cdot, \varepsilon) \to a$ when $\varepsilon \to 0$. $\square$

The next results consider the cases of small perturbations of on odd MFDE and of an autonomous ODE, respectively.

Theorem 3.2. *We consider the functional–differential equation*

$$\begin{aligned} x'(t) = {} & f\big(t, x(t), x\big(t - \tau_1(t)\big), x\big(t + \tau_2(t)\big)\big) \\ & + \varepsilon f_p\big(t, x(t), x\big(t - \tau_1(t)\big), x\big(t + \tau_2(t)\big)\big), \end{aligned} \tag{9}$$

where $f, f_p : \mathbb{R} \times \mathbb{R}^{3n} \times [0, \varepsilon_1] \to \mathbb{R}^n$ are continuous functions and ω-periodic in the first argument, $\tau_1, \tau_2 : \mathbb{R} \to \mathbb{R}$ are continuous and ω-periodic and $\varepsilon > 0$. We assume in addition that f is odd, i.e., $f(t, -u, -v, -w) = -f(t, u, v, w)$ for all $t \in \mathbb{R}$ and any $(u, v, w) \in \mathbb{R}^{3n}$.

Then, for any ε sufficiently small, there exists an ω-periodic solution of Eq. (9).

Proof. We denote $F_p(x)(t) = f_p(t, x(t), x(t - \tau_1(t)), x(t + \tau_2(t)))$ and $N_p(x)(t) = \int_0^t F_p(x)(s)\,ds$. Then the ω-periodic problem for Eq. (9) is equivalent to

$$Lx = N(x) + \varepsilon N_p(x), \quad x \in X. \tag{10}$$

Let $\Omega = B_r(0)$ be the ball centered in the origin 0 and with radius $r > 0$ from the Banach space X. Since f is an odd function, the same is true for the operator N. We apply the generalized theorem of Borsuk type [8] and deduce that $d((L, N), B_r(0))$ is odd. It is obvious that

$$d\big((L, N), B_r(0)\big) \neq 0. \tag{11}$$

Now, for ε small enough we apply Theorem 2.1 for Eq. (10) in $B_r(0)$ and obtain that

$$d\big((L, N+\varepsilon N_p), B_r(0)\big) = d\big((L, N), B_r(0)\big). \tag{12}$$

Combining (11) and (12), it follows that $d((L, N+\varepsilon N_p), B_r(0)) \neq 0$, for ε sufficiently small. This assures the existence of at least one solution of Eq. (10) in $B_r(0)$. $\square$

Theorem 3.3. *We consider the functional–differential equation*

$$x'(t) = f_0\big(x(t)\big) + \varepsilon f_p\big(t, x(t), x\big(t-\tau_1(t)\big), x\big(t+\tau_2(t)\big)\big), \tag{13}$$

where $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ and $f_p : \mathbb{R}\times\mathbb{R}^{3n} \to \mathbb{R}^n$ are continuous functions, f_p is ω-periodic in the first argument, and $\varepsilon > 0$. We assume that there exists V an open and bounded subset of $\mathbb{R}^n$, such that $f_0(c) \neq 0$ for all $c \in \partial V$ and $d_B(f_0, V, 0) \neq 0$. Moreover, we assume that equation $x' = f_0(x)$ has no periodic solutions such that $x(t) \in \partial V$ for some $t \in \mathbb{R}$.

Then, for any ε sufficiently small, there exists an ω-periodic solution of Eq. (13).

Proof. We denote $F_0(x)(t) = f_0(x(t))$ and $N_0(x)(t) = \int_0^t F_0(x)(s)\,ds$. With the above notations the ω-periodic problem for Eq. (13) is equivalent to

$$Lx = N_0(x) + \varepsilon N_p(x), \quad x \in X. \tag{14}$$

We choose $\Omega = \{x \in X\colon x(t) \in V \text{ for all } t \in \mathbb{R}\}$ that is an open and bounded subset of the Banach space X. In the proof of Theorem 2.1 we described also its boundary. The fact that equation $x' = f_0(x)$ has no periodic solutions such that $x(t) \in \partial V$ for some $t \in \mathbb{R}$ assures that $Lx \neq N_0(x)$ for all $x \in \partial\Omega$. For ε sufficiently small we apply Theorem 2.1 for Eq. (14). Hence

$$d\big((L, N_0+\varepsilon N_p), \Omega\big) = d\big((L, N_0), \Omega\big). \tag{15}$$

Also, we have $d((L, N_0), \Omega) = (-1)^n d_B(f_0, V, 0)$ [4]. Since $d_B(f, V, 0) \neq 0$, using also (15), we have

$$d\big((L, N_0+\varepsilon N_p), \Omega\big) \neq 0.$$

This implies, like before, that there exists an ω-periodic solution of Eq. (13). $\square$

The next two theorems treat the cases of MFDE with homogeneous nonlinearities and they are extensions of some results of Capietto, Mawhin and Zanolin [4] given for the ODE case. We give the proof only for the first one, since for the second one is similar. We use Theorems 2.3 and 2.4 (see also [1]), that are abstract results on how to compute the coincidence degree for homogeneous nonlinear operators.

Theorem 3.4. *We consider the functional–differential equation*

$$x'(t) = f\big(t, x(t), x\big(t-\tau_1(t)\big), x\big(t+\tau_2(t)\big)\big) + p(t), \tag{16}$$

where $f : \mathbb{R}\times\mathbb{R}^{3n} \to \mathbb{R}^n$ and $p : \mathbb{R} \to \mathbb{R}^n$ are continuous functions, ω-periodic in the first argument. We assume that f is positively homogeneous of order $\alpha < 1$, i.e., $f(t, ru, rv, rw) = r^\alpha f(t, u, v, w)$ for any $r > 0$, $t \in \mathbb{R}$, $(u, v, w) \in \mathbb{R}^{3n}$.

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$g(c) = \int_0^\omega f(t, c, c, c, 0)\,dt. \tag{17}$$

We assume, moreover, that $g(c) \neq 0$ *for all* $c \in \mathbb{R}^n \setminus \{0\}$ *and* $d_B(g, V, 0) \neq 0$ *for some open neighborhood* V *of* 0.

Then there exists an ω*-periodic solution of Eq.* (16).

Proof. Let $\Omega = B_r(0)$, where $B_r(0)$ is the ball centered in the origin and of radius r, and V is $B_r(0) \cap X_1$ after identifying a constant function with the corresponding real number. We apply Theorem 2.4(i). From the fact that f is positively homogeneous of order α, it follows that N is positively homogeneous of order α. Like we have seen before, $J^{-1}QN(\cdot, 0)|_{X_1} = g$. We notice that the Brouwer degree of g is constant with respect to any open and bounded set V that contains the origin. Then, by Theorem 2.4(i), for r sufficiently large, $d((L, N), B_r(0)) = d_B(J^{-1}QN, B_r(0) \cap X_1, 0) = d_B(g, V, 0)$. It follows that

$$d\big((L, N), B_r(0)\big) \neq 0.$$

We denote $N_p(x)(t) = \int_0^t p(s)\,ds$. Hence, the ω-periodic problem for Eq. (16) is equivalent to

$$Lx = N(x) + N_p(x), \quad \text{for all } x \in X. \tag{18}$$

Also, we notice that N_p is positively homogeneous of order $\beta = 0$. Now we apply Theorem 2.4(ii) with $0 = \beta < \alpha < 1$ and deduce that, for r sufficiently large we have

$$d\big((L, N + N_p), B_r(0)\big) = d\big((L, N), B_r(0)\big) \neq 0.$$

Thus Eq. (18) has a solution in $\Omega = B_r(0)$, so there exists an ω-periodic solution of Eq. (16). □

Theorem 3.5. *We consider the functional–differential equation*

$$x'(t) = f\big(t, x(t), x\big(t - \tau_1(t)\big), x\big(t + \tau_2(t)\big)\big) + \varepsilon p(t), \tag{19}$$

where $f : \mathbb{R} \times \mathbb{R}^{3n} \to \mathbb{R}^n$ *and* $p : \mathbb{R} \to \mathbb{R}$ *are continuous functions,* ω*-periodic in the first argument. We assume that* f *is positively homogeneous of order* $\alpha > 1$.

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ *be defined by* (17). *We assume, moreover, that* $g(c) \neq 0$ *for all* $c \in \mathbb{R}^n \setminus \{0\}$ *and* $d_B(g, V, 0) \neq 0$ *for some open neighborhood* V *of* 0.

Then, for $\varepsilon > 0$ *sufficiently small, there is an* ω*-periodic solution of Eq.* (19).

Acknowledgment

We thank Professor Ioan A. Rus and Professor Radu Precup for valuable discussions.

References

[1] A. Buică, Contributions to coincidence degree theory of some homogeneous operators, Pure Math. Appl. 11 (2000) 39–47.

[2] A. Buică, Principii de coincidenţă şi aplicaţii (Coincidence Principles and Applications), Presa Universitară Clujeană, Cluj-Napoca, 2001 (in Romanian).

[3] A. Buică, J. Llibre, Averaging methods for finding periodic orbits via Brouwer degree, Bull. Sci. Math. 128 (2004) 7–22.

[4] A. Capietto, J. Mawhin, F. Zanolin, Continuation theorems for periodic perturbations of autonomous systems, Trans. Amer. Math. Soc. 329 (1992) 41–72.

[5] V.A. Dârzu, Wheeler–Feynman problem on compact interval, Studia Univ. Babeş-Bolyai Math. 47 (2002) 43–46.

[6] V.A. Dârzu, Data dependence for functional–differential equations of mixed type, Mathematica (Cluj) 46 (69) (2004) 6–66.

[7] X. Fu, S. Zhang, Periodic solutions for differential equations at resonance with unbounded nonlinearities, Nonlinear Anal. 52 (2003) 755–767.
[8] R.E. Gaines, J. Mawhin, The Coincidence Degree and Nonlinear Differential Equations, Lecture Notes in Math., vol. 568, Springer, Berlin, 1977.
[9] B. Liu, Periodic solutions of a nonlinear second-order differential equation with deviating argument, J. Math. Anal. Appl. 309 (2005) 313–321.
[10] Y. Liu, P. Yang, W. Ge, Periodic solutions of higher-order delay differential equations, Nonlinear Anal. 63 (2005) 136–152.
[11] S. Ma, Z. Wang, J. Yu, An abstract existence theorem at resonance and its applications, J. Differential Equations 145 (1998) 274–294.
[12] J. Mallet-Paret, The Fredholm alternative for functional–differential equations of mixed type, J. Dynam. Differential Equations 11 (1999) 1–46.
[13] R. Precup, Some existence results for differential equations with both retarded and advanced arguments, Mathematica 44 (67) (2002) 31–38.
[14] I.A. Rus, V.A. Dârzu-Ilea, First order functional–differential equations with both advanced and retarded arguments, Fixed Point Theory 5 (2004) 103–115.