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Bifurcation of Limit Cycles from a Polynomial Degenerate Center

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Abstract

Using Melnikov functions at any order, we provide upper bounds for the maximum number of limit cycles bifurcating from the period annulus of the degenerate center $\dot{x} = -y((x^2 + y^2)/2)^m$ and $\dot{y} = x((x^2 + y^2)/2)^m$ with $m \geq 1$, when we perturb it inside the whole class of polynomial vector fields of degree n. The positive integers m and n are arbitrary. As far as we know there is only one paper that provide a similar result working with Melnikov functions at any order and perturbing the linear center $\dot{x} = -y$, $\dot{y} = x$.

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1 Introduction and statement of the main results

Probably the main problem in the qualitative theory of real planar differential systems is the determination of its limit cycles. A *limit cycle* of a planar differential system was defined by Poincaré [12], as a periodic orbit of the differential system isolated in the set of all periodic orbits. At the end of the 1920s van der Pol [13], Liénard [11] and Andronov [1] proved that a periodic orbit of a self–sustained oscillation occurring in a vacuum tube circuit was a limit cycle as considered by Poincaré. After these works the non-existence, existence, uniqueness and other properties of limit cycles were studied extensively by mathematicians and physicists, and more recently also by chemists, biologists, economists, etc. (see for instance the books [4, 17]).

During 1881–1886 Poincaré defined the notion of a *center* of a real planar differential system, as an isolated singular point having a neighborhood such that all the orbits of this neighborhood are periodic with the unique exception of the singular point. Then one way to produce limit cycles is by perturbing a system which has a center, in such a way that limit cycles bifurcate in the perturbed system from some of the periodic orbits of the unperturbed one [14]. This procedure is effective if one knows the first integral of the unperturbed system, sometime to determine it is a difficult problem, see for instance [5, 6, 8].

In this paper we consider the polynomial differential system

$$\dot{x} = -y\left(\frac{x^2+y^2}{2}\right)^m, \qquad \dot{y} = x\left(\frac{x^2+y^2}{2}\right)^m, \tag{1.1}$$

of degree $2m+1$ having a degenerate center at the origin when m is a positive integer, and we perturb it inside the class of polynomial differential systems of degree $\max\{2m+1, n\}$ given by

$$\dot{x} = -y\left(\frac{x^2+y^2}{2}\right)^m + \sum_{k=1}^{\infty}\varepsilon^k f_k(x,y), \quad \dot{y} = x\left(\frac{x^2+y^2}{2}\right)^m + \sum_{k=1}^{\infty}\varepsilon^k g_k(x,y), \tag{1.2}$$

where f_k and g_k are polynomials of degree n for $k = 1, 2, \ldots$, and $\varepsilon > 0$ is a sufficiently small parameter.

Clearly $H = (x^2+y^2)/2$ is a first integral of the unperturbed system (1.1). Note that H can take values in $[0, \infty)$, and that for every $h \in (0, \infty)$ the circle $H = h$ corresponds to a periodic orbit of the unperturbed system (1.1).

Consider the positive x-half-axis Γ parameterized by h, i.e. the point $(x, 0)$ has the h value $x^2/2$. For $\varepsilon > 0$ sufficiently small we define $\mathcal{P}_\varepsilon : \Gamma \to \Gamma$ as $h \mapsto \mathcal{P}_\varepsilon(h)$, where $\mathcal{P}_\varepsilon(h)$ is the first intersection with Γ in forward time of the orbit of system (1.2) through the point $(x, 0)$. In other words $\mathcal{P}_\varepsilon(h)$ is the so called *first return map* of the perturbed system (1.2) in terms of h and ε. Of course $\mathcal{P}_0(h)$ is the identity map.

The *displacement function* $d(h,\varepsilon) = \mathcal{P}_\varepsilon(h) - h$ has the following representation in power series in ε

$$d(h,\varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \varepsilon^3 M_3(h) + \cdots, \tag{1.3}$$

which is convergent for sufficiently small ε, and where the coefficients $M_k(h)$ are called the *Melnikov functions* defined for $h \geq 0$. Clearly each simple zero $h_0 \in (0,\infty)$ of the first non-vanishing coefficient in (1.3) corresponds to a limit cycle of (1.2).

For studying the limit cycles of a perturbed differential system which bifurcate from the periodic orbits of a center of an unperturbed differential system, there are many papers which study the simple zeros of $M_1(h)$, assuming that it is the first non-vanishing Melnikov function; there are few papers which study the simple zeros of $M_2(h)$, assuming that it is the first non-vanishing Melnikov function; and there are very few papers which study the simple zeros of $M_3(h)$, assuming that it is the first non-vanishing Melnikov function.

As far as we know Iliev in [9] was the first in studying the simple zeros of $M_k(h)$, assuming that it is the first non-vanishing Melnikov function, for an arbitrary k. Since such study perturbing a general center is not possible to do due to the extreme difficulty of the computations that it needs, Iliev does it for the easiest center, the linear one, i.e. the center of system (1.1) with $m = 0$. Here following the ideas of Iliev we shall extend his results to any $m \geq 1$.

The main result of this paper is the following.

Theorem 1.1 *Assume that m in system* (1.2) *is a positive integer. Suppose that the first Melnikov function in* (1.3) *which is not identically zero is $M_k(h)$ for some $k \geq 1$. All the zeros of the function $M_k(h)$ will be counted with their multiplicities. Then the following statements hold.*

(a) $M_1(h)$ has at most $[\frac{1}{2}(n-1)]$ positive zeros.

(b) If $n \geq 2m+1$ and $k \geq 2$, then the degree of system (1.2) *is n and $M_k(h)$ has at most $[\frac{1}{2}k(n-1)] + k - 2$ positive zeros.*

(c) If $n \leq 2m$ and $k \geq 2$, then the degree of system (1.2) *is $2m+1$ and $M_k(h)$ has at most $[\frac{1}{2}(n-1)] + (k-1)(m+1) - 1$ positive zeros.*

(d) For $n \geq 2$ the upper bounds given above for $k = 1$, and for $k = 2$ and $n \geq m$ are reached for convenient perturbations f and g in (1.2)*. Moreover for $k = 2$ and $n \leq m-1$ the upper bound given in (c) can be reduced to $[\frac{1}{2}(n-1)] + n$, and this new upper bound is reached.*

(e) For $n = 1$ the number of positive zeros of $M_k(h)$ is always zero.

(f) For $k > 2$ the upper bounds given in statements (b) and (c) are usually not reached. More precisely, the numbers between parentheses in Tables 1 and 2 are the maximum reached upper bounds, which are smaller than the upper bounds given in (b) and (c).

Here and below $[r]$ denotes the entire part of $r \in \mathbb{R}$. The first three statements of Theorem 1 give an upper bound for the number of limit cycles emerging from the period annulus of the center of the unperturbed system. In [10] Iliev studied the case $m = 0$ (i.e. the bifurcations of limit cycles from the harmonic oscillator), and proved that the function $M_k(h)$ has at most $[\frac{1}{2}k(n-1)]$ zeros, counting their multiplicities. In Tables 1 and 2 we provide these upper bounds for different values of k and n fixing $m = 1$ and $m = 2$, respectively.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$\cdots$
$n=1$	0	0	0	0	0	0	$\cdots$
$n=2$	0	1	3(2)	5(2)	7	9	$\cdots$
$n=3$	1	2	4(3)	6	8	10	$\cdots$
$n=4$	1	3	5	8	10	13	$\cdots$
$n=5$	2	4	7	10	13	16	$\cdots$
$n=6$	2	5	8	12	15	19	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n=2\ell$	ℓ	$2\ell-1$	$3\ell-1$	4ℓ	5ℓ	$6\ell+1$	$\cdots$
$n=2\ell+1$	$2\ell-1$	2ℓ	$3\ell+1$	$4\ell+2$	$5\ell+3$	$6\ell+4$	$\cdots$

Table 1: Upper bounds for $m = 1$ and different values of n and k.

A difficult problem that remains open is to determine, for fixed m and n, at which $k_0 = k_0(m, n)$ the number of limit cycles will stabilize, i.e. to determine the order of the Melnikov function $M_{k_0}(h)$ for which all the $M_k(h)$ with $k \geq k_0$ have the same maximum number of isolated zeros. This is equivalent to solving the cyclicity problem for the period annulus, i.e. to provide the maximum number of limit cycles which can bifurcate from the periodic orbits of the center (1.1). In this direction from the results of Bautin [2], for $m = 0$ and $n = 2$, all the functions $M_k(h)$, $k \geq 6$, have at most 3 zeros. Moreover, from the results of Sibirskii [16], for $m = 0$ and $n = 3$ but with f and g homogeneous polynomials, all the functions $M_k(h)$, $k \geq k_0$ for some positive integer k_0, have at most 5 zeros. As far as we know these two mentioned cases are the unique ones for which the maximum number of bifurcated limit cycles is known.

Another difficult problem is to determine for $k \geq 3$ the maximum upper bounds under the assumptions of statements (b) and (c) which are reached. In [10] Iliev studied the case $m = 0$ and proved that the number of zeros given by $M_k(h)$ for $k = 1, 2, 3$ can be reached.

2 Proof of Theorem 1.1

Before proving Theorem 1.1 we need to introduce some previous results and notations.

Denote $\delta = ydx$ and $J(h) = \int_{H=h} \delta$. Note that $J(h) = -2\pi h$. The next lemma and corollary are proved in [10].

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$\cdots$
$n=1$	0	0	0	0	0	0	$\cdots$
$n=2$	0	2	5(3)	8(6)	11	14	$\cdots$
$n=3$	1	3	6(4)	9	12	15	$\cdots$
$n=4$	1	3	6	9	12	15	$\cdots$
$n=5$	2	4	7	10	13	16	$\cdots$
$n=6$	2	5	8	12	15	19	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n=2\ell$	ℓ	$2\ell-1$	$3\ell-1$	4ℓ	5ℓ	$6\ell+1$	$\cdots$
$n=2\ell+1$	$2\ell-1$	2ℓ	$3\ell+1$	$4\ell+2$	$5\ell+3$	$6\ell+4$	$\cdots$

Table 2: Upper bounds for $m=2$ and different values of n and k.

Lemma 2.1 *Any polynomial one-form ω of degree s can be expressed as*

$$\omega = dQ(x,y) + q(x,y)dH + \alpha(H), \tag{2.1}$$

where $Q(x,y)$ and $q(x,y)$ are polynomials of degree $s+1$ and $s-1$ respectively and $\alpha(h)$ is a polynomial of degree $[\frac{1}{2}(s-1)]$.

Corollary 2.1 *Any integral $I(h) = \int_{H=h}\omega$ of a polynomial one-form of degree d has at most $[\frac{1}{2}(d-1)]$ isolated zeros in $(0,\infty)$.*

We next write (1.2) in the form

$$H^m dH + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots = 0, \tag{2.2}$$

where $\omega_k = g_k(x,y)dx - f_k(x,y)dy$ with $\deg f_k \le n$, and $\deg g_k \le n$.

The Melnikov functions will be computed using the ideas of Françoise [7], Roussarie [15] and Iliev [9, 10]. We summarize their results in the next proposition.

Proposition 2.1 *Denoting $\Omega_1 = \dfrac{\omega_1}{H^m}$ we have that*

$$M_1(h) = \int_{H=h}\Omega_1\,. \tag{2.3}$$

Assume that for some $k \ge 2$, $M_1(h) = \ldots = M_{k-1}(h) \equiv 0$ in (1.3). Then

$$M_k(h) = \int_{H=h}\Omega_k, \tag{2.4}$$

where $\Omega_k = \dfrac{\omega_k}{H^m} + \displaystyle\sum_{i=1}^{k-1} r_i \frac{\omega_{k-i}}{H^m}$, and the functions r_i are determined successively from the representations $\Omega_i = dS_i + r_i dH$, $1 \le i \le k-1$.

The following result will be used several times in the proof of Lemma 2.3. Its proof follows by direct computations.

Lemma 2.2 *We consider a polynomial one-form expressed as* $\omega = dQ(x,y) + q(x,y)dH + \alpha(H)\delta$. *Then the one-form* $\Omega = \dfrac{\omega}{H^l}$ *can be expressed as*

$$\Omega = \frac{\omega}{H^l} = d\left(\frac{Q(x,y)}{H^l}\right) + r(x,y)dH + \frac{\alpha(H)}{H^l},$$

where

$$r(x,y) = \frac{1}{H^{l+1}}(lQ(x,y) + q(x,y)H)\,.$$

Since for the calculation of the Melnikov functions, as it is stated in Proposition 2.1, one needs the decomposition of the one–forms Ω_i, the following result will be crucial in the proof of our main result the Theorem 1.1.

Lemma 2.3 *The following statements hold.*

(a) $\omega_1 = H^m\Omega_1 = dQ_1 + q_1dH + \alpha_1(H)\delta$ *where* Q_1 *and* q_1 *are polynomials of degrees* $n+1$ *and* $n-1$ *respectively and* $\alpha_1(h)$ *is a polynomial of degree* $[\frac{1}{2}(n-1)]$.

(b) Assume that for some $k \geq 2$, $M_1(h) = \ldots = M_{k-1}(h) \equiv 0$. *Then*

$$\tilde{\omega}_k = H^{km+k-1}\Omega_k$$

is a polynomial one-form of degree $\max\{kn+k-1,\, n+2(k-1)(m+1)\}$ *and it can be expressed as*

$$\tilde{\omega}_k = d\left(Q_kH + c_kQ_1^k\right) + q_kdH + H\alpha_k(H),$$

where Q_k *and* q_k *are polynomials of degree* $\max\{kn+k-2,\, n+2(k-1)(m+1)-1\}$ *and* $\alpha_k(h)$ *is a polynomial of degree* $[\frac{1}{2}\max\{kn+k-1,\, n+2(k-1)(m+1)\} - \frac{3}{2}]$. *Here* $c_k = \prod_{i=1}^{k}(im-m+i)/i$.

Proof. Statement (a) follows directly from Lemma 2.1. We shall prove statement (b) by induction. Suppose that $k=2$. Then $M_1(h) \equiv 0$, and consequently $\alpha_1(h) \equiv 0$. Therefore, from statement (a), $\omega_1 = dQ_1 + q_1dH$, and by Lemma 2.2 we have

$$\Omega_1 = \frac{dQ_1}{H^m} + \frac{q_1}{H^m}dH = d\left(\frac{Q_1}{H^m}\right) + r_1dH,$$

where

$$r_1 = \frac{1}{H^{m+1}}(mQ_1 + q_1H). \tag{2.5}$$

We note that $H^{m+1}r_1$ is a polynomial of degree $n+1$. Using Proposition 2.1 we obtain

$$\Omega_2 = \frac{\omega_2}{H^m} + r_1\frac{\omega_1}{H^m}.$$

Then

$$\tilde{\omega}_2 = H^{2m+1}\Omega_2 = H^{m+1}\omega_2 + (mQ_1 + q_1 H)\omega_1,$$

and we can easily see that this is a polynomial one-form of degree $\max\{2n+1,\, n+2m+2\}$. Further we have

$$\tilde{\omega}_2 = H(H^m\omega_2 + q_1\omega_1) + mQ_1 dQ_1 + mQ_1 q_1 dH\,. \tag{2.6}$$

Since $H^m\omega_2 + q_1\omega_1$ is a polynomial one-form of degree $\max\{2n-1,\, n+2m\}$, from Lemma 2.1 follows that it can be expressed as $dQ_2 + \tilde{q}_2 dH + \alpha_2(H)$. where Q_2 and $\tilde{q}_2$ are polynomials of degrees $\max\{2n-1,\, n+2m\}+1$ and $\max\{2n-1,\, n+2m\}-1$, respectively, and $\alpha_2(H)$ is a polynomial of degree $[\frac{1}{2}\max\{2n-1,\, n+2m\} - \frac{1}{2}]$. Hence, denoting $q_2 = -Q_2 + H\tilde{q}_2 + mQ_1 q_1$ we have

$$\tilde{\omega}_2 = d\left(Q_2 H + \frac{m}{2}Q_1^2\right) + q_2 dH + H\alpha_2(H).$$

Indeed Q_2, q_2 and $\alpha_2(H)$ are polynomials of degrees as the ones given in the statement (b) of this lemma for $k = 2$.

Now, given $k \geq 2$ we assume that the statement (b) is true for all $2 \leq i \leq k$, and we prove it for $k+1$. We have that $M_1(h) = \cdots = M_k(h) \equiv 0$. Then $\omega_1 = dQ_1 + q_1 dH$, $H^{m+1}r_1 = mQ_1 + q_1 H$ and, for each $2 \leq i \leq k$, $\tilde{\omega}_i = d\left(Q_i H + c_i Q_1^i\right) + q_i dH$. Further, using also Lemma 2.2,

$$\Omega_i = \frac{\tilde{\omega}_i}{H^{im+i-1}} = d\left(\frac{Q_i H + c_i Q_1^i}{H^{im+i-1}}\right) + r_i dH,$$

where $H^{im+i}r_i = (im+i-1)(Q_i H + c_i Q_1^i) + q_i H$ for each $2 \leq i \leq k$. We note that for each $1 \leq i \leq k$ the polynomial $H^{im+i}r_i$ is of degree $\max\{in+i,\, n+2(i-1)(m+1)+1\}$. Using Proposition 2.1 we obtain

$$\Omega_{k+1} = \frac{\omega_{k+1}}{H^m} + \sum_{i=1}^{k} r_i \frac{\omega_{k+1-i}}{H^m}\,.$$

Then

$$\tilde{\omega}_{k+1} = H^{(k+1)m+k}\Omega_{k+1} = H^{km+k}\omega_{k+1} + \sum_{i=1}^{k} H^{(k-i)m+k-i} \cdot H^{im+i}r_i \cdot \omega_{k+1-i}$$

is a polynomial one-form. The degree of $H^{km+k}\omega_{k+1}$ is $n + 2k(m+1)$ while, for $1 \leq i \leq k$ the degree of $H^{(k-i)m+k-i} \cdot H^{im+i}r_i \cdot \omega_{k+1-i}$ is $\max\{n+2k(m+1)+i(n-2m-1),\, n+2k(m+1)+n-2m-1\}$. Then the degree of $\tilde{\omega}_{k+1}$ is $\max\{n+2k(m+1)+k(n-2m-1),\, n+2k(m+1)\} = \max\{(k+1)n+k,\, n+2k(m+1)\}$. From the expression of $\tilde{\omega}_{k+1}$ we can see that it is a polynomial one-form of degree $\max\{(k+1)n+k,\, n+2k(m+1)\}$. We have that there exists a polynomial one–form

Ω such that $\tilde{\omega}_{k+1} = H\Omega + H^{km+k} r_k \omega_1$. Consequently Ω is a polynomial one-form of degree $\deg \tilde{\omega}_{k+1} - 2$. Then we have

$$\begin{aligned}\tilde{\omega}_{k+1} = &\ H\Omega + \big((km+k-1)(Q_k H + c_k Q_1^k) + q_k H\big)\,\omega_1 \\ = &\ H\left(\Omega + (km+k-1)Q_k\omega_1 + q_k\omega_1\right) + \\ &\ (km+k-1)c_k Q_1^k\left(dQ_1 + q_1 dH\right).\end{aligned}$$

The degree of the polynomial one-form $\Omega + (km+k-1)Q_k\omega_1 + q_k\omega_1$ is $\max\{(k+1)n+k,\ n+2k(m+1)\} - 2$, hence it can be expressed as $dQ_{k+1} + \tilde{q}_{k+1}dH + \alpha_{k+1}(H)$, where Q_{k+1} and $\tilde{q}_{k+1}$ are polynomials of degrees $\max\{(k+1)n+k,\ n+2k(m+1)\}-1$ and $\max\{(k+1)n+k,\ n+2k(m+1)\}-3$ respectively, and $\alpha_{k+1}(h)$ is a polynomial of degree

$$\left[\frac{1}{2}\max\{(k+1)n+k,\ n+2k(m+1)\} - \frac{3}{2}\right].$$

Hence denoting $q_{k+1} = -Q_{k+1} + H\tilde{q}_{k+1} + (km+k-1)c_k Q_1^k q_1$ we have that $\tilde{\omega}_{k+1}$ can be expressed as in the statement of the lemma.

Proof. [Proof of Theorem 1.1] (a) Using (2.3) we see that $M_1(h) = h^{-m}\int_{H=h}\omega_1$. From the decomposition of ω_1 given in statement (a) of Lemma 2.3 we finally obtain that $M_1(h) = h^{-m}\alpha_1(h)J(h)$, where the polynomial $\alpha_1(h)$ has degree $[\frac{1}{2}(n-1)]$ and $J(h) = -2\pi h$. Hence $M_1(h)$ has at most $[\frac{1}{2}(n-1)]$ positive zeros. This proves statement (a) of Theorem 1.1.

It is easy to see that the degree of system (1.2) is $\max\{n,\ 2m+1\}$. Of course we have that the degree is n if $n \geq 2m+1$, and is equal to $2m+1$ if $n \leq 2m$. Using (2.4) we see that $M_k(h) = \int_{H=h}\Omega_k$. From the notation and the decomposition given in statement (b) of Lemma 2.3 we finally obtain that $M_k(h) = h^{-km-k+1}\tilde{\omega}_k = h^{-km-k+1}h\alpha_k(h)J(h)$, where the polynomial $\alpha_k(h)$ has degree $[\frac{1}{2}\max\{kn+k-1,\ n+2(k-1)(m+1)\} - \frac{3}{2}]$. The expression of this degree can be written as $[\max\{\frac{1}{2}k(n-1)+k-2,\ \frac{1}{2}(n-1)+(k-1)(m+1)-1\}]$ and, further $[\frac{1}{2}k(n-1)]+k-2$ if $n \geq 2m+1$, or $[\frac{1}{2}(n-1)]+(k-1)(m+1)-1$ if $n \leq 2m$. It is clear now that these numbers are upper bounds for the number of positive zeros of $M_k(h)$. Therefore statements (b) and (c) of Theorem 1.1 are proved.

Assume that $n \geq 2$. To obtain the result for $k=1$ we take, like in [10], $\omega_1 = \alpha(H)\delta$ in (2.2) where the polynomial $\alpha(h)$ of degree $[\frac{1}{2}(n-1)]$ has only real positive roots. Then $M_1(h) = \int_{H=h}\Omega_1 = h^{-m}\int\omega_1 = h^{-m}\alpha(h)J(h)$ has as many zeros as in statement (a).

We prove now the result for $k=2$. Then $M_1(h) \equiv 0$ and, consequently, $\alpha_1(h) \equiv 0$. Therefore from Lemma 2.3 we have $\omega_1 = dQ_1 + q_1 dH$ where Q_1 and q_1 are polynomials of degrees $n+1$ and $n-1$, respectively. Then $q_1 dQ_1$ is a polynomial one–form of degree $2n-1$. Using Lemma 2.1 we obtain that $\int_{H=h} q_1 dQ_1 = \beta(h)J(h)$, where $\beta(h)$ is a polynomial of degree $n-1$. In particular choosing

$$\begin{aligned}&Q_1(x) = a_0 + a_1x + \ldots + a_{n+1}x^{n+1},\\ &q_1(x,y) = \big(b_0 + b_1x + \ldots + b_{n-2}x^{n-2}\big)\,y,\end{aligned}$$

we obtain that

$$q_1 dQ_1 = \left(A_0 + A_1 x + A_2 x^2 + ... + A_{2n-3} x^{2n-3} + A_{2n-2} x^{2n-2}\right) y dx.$$

Now we use the following equalities proved in [10]

$$\int_{H=h} x^{2i+1} y dx = 0 \quad \text{and} \quad \int_{H=h} x^{2i} y dx = \tilde{c}_i h^i J(h),$$

where $\tilde{c}_i = (2i-1)!!/(i+1)!$, and we obtain that $\int_{H=h} q_1 dQ_1 = \beta(h) J(h)$ with $\beta(h)$ a polynomial of degree $n-1$ with arbitrary coefficients.

We take in (2.2) $\omega_2 = \alpha(H)\delta$ with $\alpha(h)$ an arbitrary polynomial of degree $[\frac{1}{2}(n-1)]$ and $\omega_1 = dQ_1(x) + q_1(x,y)dH$, where $q_1(x,y)$ and $Q_1(x)$ are as above. Then $M_1(h) \equiv 0$ and, from (2.6),

$$\begin{aligned} M_2(h) &= \int_{H=h} \Omega_2 = \frac{1}{h^{2m+1}} \int_{H=h} \tilde{\omega}_2 \\ &= \frac{h}{h^{2m+1}} \int_{H=h} (H^m \omega_2 + q_1 \omega_1) \\ &= \frac{h}{h^{2m+1}} \left(h^m \alpha(h) + \beta(h)\right) J(h). \end{aligned}$$

One can see that when $n \geq 2m+1$ the polynomial $h^m\alpha(h) + \beta(h)$ has degree $n-1$ and it has arbitrary coefficients, while when $m \leq n \leq 2m$ the same polynomial has degree $[\frac{1}{2}(n-1)] + m$ and also has arbitrary coefficients. Hence in these cases the upper bounds given in (b) and (c) for the number of zeros of $M_2(h)$ can be reached.

When $n \leq m-1$ the polynomial $h^m\alpha(h) + \beta(h)$ has degree $[\frac{1}{2}(n-1)] + m$, but it does not have the monomials $h^n, \ldots, h^{m-1}$. Hence it has only $[\frac{1}{2}(n-1)] + n + 1$ monomials. By the generalized Descartes Theorem (see the Appendix) an upper bound for the number of its positive zeros is $[\frac{1}{2}(n-1)] + n$, and there are polynomials having such number of zeros. This completes the proof of statement (d).

The fact that $M_1(h)$ has no zeros when $n = 1$ follows directly from statement (a). In order to count the number of zeros of $M_k(h)$ for $k \geq 2$ in the special case $n = 1$ we need to find suitable decompositions of the one–forms Ω_k, other than the ones given in statement (b) of Lemma 2.3. Note that it is known that these decompositions are not unique.

By Lemma 2.1 we have that any polynomial one–form ω of degree 1 can be written as $\omega = dQ + qdH + \alpha\delta$, where Q is a quadratic polynomial and q and α are real numbers. From here we deduce that $\int_{H=h} \omega$ cannot have isolated positive zeros. It is also clear that, since q is a constant, ω can be written as $\omega = dQ + \alpha\delta$, where Q is a quadratic polynomial and α is a constant.

For each polynomial one–form ω_i, $i \geq 1$, that appears in (2.2) we write

$$\omega_i = dQ_i + \alpha_i,$$

where Q_i is a quadratic polynomial and α_i is a constant. In the following the functions r_i are the ones defined in Proposition 2.1.

We claim that for each $k \geq 1$ such that $M_1(h) = \ldots = M_k(h) \equiv 0$ we have

$$r_k = \sum_{j=1}^{k} \frac{c_j}{H^{jm+j}} \sum_{l_1+\ldots+l_j=k} Q_{l_1} Q_{l_2} \ldots Q_{l_j}, \tag{2.7}$$

where $c_j = \prod_{l=1}^{j} (lm + l - 1)/l$.

Then $\alpha_1 = \ldots = \alpha_k = 0$ and $\omega_i = dQ_i$ for each $1 \leq i \leq k$. Hence

$$\begin{aligned} \sum_{i=1}^{k} r_i \omega_{k+1-i} &= \sum_{i=1}^{k} \sum_{j=1}^{i} \frac{c_j}{H^{jm+j}} \sum_{l_1+\ldots+l_j=i} Q_{l_1} Q_{l_2} \ldots Q_{l_j} \cdot dQ_{k+1-i} \\ &= \sum_{j=1}^{k} \frac{c_j}{H^{jm+j}} \sum_{i=j}^{k} \sum_{l_1+\ldots+l_j=i} Q_{l_1} Q_{l_2} \ldots Q_{l_j} \cdot dQ_{k+1-i} \, . \end{aligned}$$

For each fixed j we have that

$$\sum_{i=j}^{k} \sum_{l_1+\ldots+l_j=i} Q_{l_1} Q_{l_2} \ldots Q_{l_j} \cdot dQ_{k+1-i} =$$

$$\frac{1}{j+1} d \left(\sum_{l_1+\ldots+l_{j+1}=k+1} Q_{l_1} Q_{l_2} \ldots Q_{l_{j+1}} \right).$$

Hence

$$\sum_{i=1}^{k} r_i \omega_{k+1-i} = \sum_{j=1}^{k} \frac{c_j}{(j+1)H^{jm+j}} d \left(\sum_{l_1+\ldots+l_{j+1}=k+1} Q_{l_1} Q_{l_2} \ldots Q_{l_{j+1}} \right) \tag{2.8}$$

and, consequently

$$\int_{H=h} \sum_{i=1}^{k} r_i \omega_{k+1-i} = 0.$$

It is known from Proposition 2.1 that $M_{k+1}(h) = \int_{H=h} \Omega_{k+1}$, where

$$\Omega_{k+1} = \frac{\omega_{k+1}}{H^m} + \frac{1}{H^m} \sum_{i=1}^{k} r_i \omega_{k+1-i}. \tag{2.9}$$

Then $M_{k+1}(h) = h^{-m} \int_{H=h} \omega_{k+1}$ and it cannot have any positive zero. Statement (e) is proved.

It remains to prove the above claim. We will do this by induction. For $k = 1$ formula (2.7) becomes $r_1 = \dfrac{m}{H^{m+1}} Q_1$. In order to see that this formula is valid we write

$$\Omega_1 = \frac{\omega_1}{H^m} = \frac{dQ_1}{H^m} = d\left(\frac{Q_1}{H^m}\right) + \frac{m}{H^{m+1}} Q_1 dH.$$

We assume now that the claim is true for k and we prove it for $k+1$. By the induction assumptions we have that $M_1(h) = \ldots = M_{k+1}(h) \equiv 0$, and that for each $1 \le i \le k$ the function r_i is given by formula (2.7). In order to prove that formula (2.7) is valid also for the function r_{k+1}, we need the decomposition of the one-form Ω_{k+1} given by (2.9). Now we have that $\omega_{k+1} = dQ_{k+1}$. Replacing (2.8) in (2.9) one can find that $\Omega_{k+1} = dS_{k+1} + r_{k+1}dH$, where $S_{k+1} = \sum_{j=1}^{k+1} \frac{c_{j-1}}{jH^{jm+j-1}} \sum_{l_1+\ldots+l_j=k+1} Q_{l_1}Q_{l_2}\ldots Q_{l_j}$, and r_{k+1} is given by formula (2.7). The claim is proved.

We shall prove that for $m=1$, $n=2$ and $k=3$ the upper bound 3 provided in statement (c) of Theorem 1 cannot be reached because the maximum upper bound reached is 2 as we shall prove in what follows. This is indicated in Table 1 like 3(2) in position $n=2$ and $k=3$. The other sharp upper bounds provided in Tables 1 and 2 indicated also between parentheses can be proved in a similar way.

We consider the system

$$\begin{aligned}\dot{x} &= -y((x^2+y^2)/2)^m + \sum_{i=1}^{3} \varepsilon^i f_i(x,y) + \mathcal{O}(\varepsilon^4),\\ \dot{y} &= x((x^2+y^2)/2)^m + \sum_{i=1}^{3} \varepsilon^i f_i(x,y) + \mathcal{O}(\varepsilon^4),\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}f_i(x,y) &= a_{00}^{(i)} + a_{10}^{(i)}x + a_{01}^{(i)}y + a_{20}^{(i)}x^2 + a_{11}^{(i)}xy + a_{02}^{(i)}y^2,\\ g_i(x,y) &= b_{00}^{(i)} + b_{10}^{(i)}x + b_{01}^{(i)}y + b_{20}^{(i)}x^2 + b_{11}^{(i)}xy + b_{02}^{(i)}y^2.\end{aligned}$$

Taking polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, system (2.10) takes the form

$$\begin{aligned}\dot{r} &= \varepsilon\, R_1(r,\theta) + \varepsilon^2 R_2(r,\theta) + \varepsilon^3 R_3(r,\theta) + \mathcal{O}(\varepsilon^4),\\ \dot{\theta} &= r^2 + \varepsilon\, F_1(r,\theta)/r + \varepsilon^2 F_2(r,\theta)/r + \varepsilon^3 F_3(r,\theta)/r + \mathcal{O}(\varepsilon^4),\end{aligned}$$

where

$$\begin{aligned}R_i(r,\theta) &= \cos\theta\, f_i(r\cos\theta, r\sin\theta) + \sin\theta\, g_i(r\cos\theta, r\sin\theta),\\ F_i(r,\theta) &= \cos\theta\, g_i(r\cos\theta, r\sin\theta) - \sin\theta\, f_i(r\cos\theta, r\sin\theta).\end{aligned}$$

Now taking θ as independent variable we obtain the differential equation

$$\frac{dr}{d\theta} = \frac{\varepsilon\, R_1(r,\theta) + \varepsilon^2 R_2(r,\theta) + \varepsilon^3 R_3(r,\theta)}{r^2 + \varepsilon\, F_1(r,\theta)/r + \varepsilon^2 F_2(r,\theta)/r + \varepsilon^3 F_3(r,\theta)/r} + \mathcal{O}(\varepsilon^4). \tag{2.11}$$

We expand the solution $r(\theta)$ into the form $r(\theta) = r_0 + \sum_{i=1}^{3} \varepsilon^i r_i(\theta) + \mathcal{O}(\varepsilon^4)$, with the initial condition $r(0) = r_0$, i.e. $r_i(0) = 0$ for $i \ge 1$. Introducing this solution $r(\theta)$ in (2.11) and solving recursively the differential equations obtained for different powers of ε we compute $r_i(\theta)$ for $1 \le i \le 3$. In order that the solution $r(\theta)$ be 2π–periodic, we must force that $r_i(2\pi) = 0$ for $1 \le i \le 3$. We obtain for $r_1(2\pi)$ and $r_2(2\pi)$ respectively

$$r_1(2\pi) = \pi(b_{01}^{(1)} + a_{10}^{(1)})/r_0, \qquad r_2(2\pi) = \pi(b_0 + b_1 r_0^2)/(4r_0^3),$$

where

$$b_0 = 8a_{00}^{(1)}b_{02}^{(1)} + 4a_{00}^{(1)}a_{11}^{(1)} - 4b_{00}^{(1)}b_{11}^{(1)} - 8b_{00}^{(1)}a_{20}^{(1)},$$
$$b_1 = 2a_{02}^{(1)}b_{02}^{(1)} + a_{02}^{(1)}a_{11}^{(1)} - b_{02}^{(1)}b_{11}^{(1)} + a_{11}^{(1)}a_{20}^{(1)} - b_{11}^{(1)}b_{20}^{(1)} - 2a_{20}^{(1)}b_{20}^{(1)} + 4b_{01}^{(2)} + 4a_{10}^{(2)}.$$

From the vanishing of $r_1(2\pi)$ we get $b_{01}^{(1)} = -a_{10}^{(1)}$, and from the vanishing of $r_2(2\pi)$ we have

$$a_{11}^{(1)} = (-2a_{00}^{(1)}b_{02}^{(1)} + b_{00}^{(1)}b_{11}^{(1)} + 2b_{00}^{(1)}a_{20}^{(1)})/a_{00}^{(1)}, \text{ with } a_{00}^{(1)} \neq 0,$$
$$a_{10}^{(2)} = -\tfrac{1}{4}(2a_{02}^{(1)}b_{02}^{(1)} + a_{02}^{(1)}a_{11}^{(1)} - b_{02}^{(1)}b_{11}^{(1)} + a_{11}^{(1)}a_{20}^{(1)} - b_{11}^{(1)}b_{20}^{(1)} - 2a_{20}^{(1)}b_{20}^{(1)} + 4b_{01}^{(2)}),$$

Finally we have that $r_3(2\pi)$ takes the form

$$r_3(2\pi) = c_0 + c_1 r_0^2 + c_2 r_0^4,$$

where

$$\begin{aligned}
c_0 = {} & 12a_{00}^{(1)}(a_{00}^{(1)}b_{00}^{(1)}a_{01}^{(1)} + {a_{00}^{(1)}}^2 a_{10}^{(1)} - {b_{00}^{(1)}}^2 a_{10}^{(1)} + a_{00}^{(1)}b_{00}^{(1)}b_{10}^{(1)})(b_{11}^{(1)} + 2a_{20}^{(1)}),\\
c_1 = {} & 24{a_{00}^{(1)}}^3(a_{11}^{(2)} + 2b_{02}^{(2)}) + 8a_{10}^{(1)}{b_{00}^{(1)}}^2(2a_{20}^{(1)} + b_{11}^{(1)})^2 - a_{00}^{(1)}b_{00}^{(1)}(2a_{20}^{(1)} + b_{11}^{(1)})\\
& (-24a_{00}^{(2)} - 7a_{01}^{(1)}a_{02}^{(1)} + 5a_{01}^{(1)}a_{20}^{(1)} - a_{02}^{(1)}b_{10}^{(1)} + 11a_{20}^{(1)}b_{10}^{(1)} + 4(a_{01}^{(1)} + b_{10}^{(1)})b_{11}^{(1)} +\\
& 4a_{10}^{(1)}(3b_{02}^{(1)} + b_{20}^{(1)})) + {a_{00}^{(1)}}^2(-24b_{00}^{(1)}(2a_{20}^{(2)} + b_{11}^{(2)}) + (2a_{20}^{(1)} + b_{11}^{(1)})(4a_{02}^{(1)}a_{10}^{(1)} -\\
& 24b_{00}^{(2)} - 3a_{01}^{(1)}b_{02}^{(1)} + 3b_{02}^{(1)}b_{10}^{(1)} - 4a_{10}^{(1)}(a_{20}^{(1)} + 2b_{11}^{(1)}) + a_{01}^{(1)}b_{20}^{(1)} + 7b_{10}^{(1)}b_{20}^{(1)})),\\
c_2 = {} & 6a_{00}^{(1)}(b_{00}^{(1)}(b_{11}^{(1)} + 2a_{20}^{(1)})(a_{02}^{(2)} + a_{20}^{(1)}) + a_{00}^{(1)}(a_{02}^{(1)}(2b_{02}^{(2)} + a_{11}^{(2)}) -\\
& (b_{02}^{(1)} + b_{20}^{(1)})(b_{11}^{(2)} + 2a_{20}^{(2)}) + a_{20}^{(1)}(a_{11}^{(2)} - 2b_{20}^{(2)}) - b_{11}^{(1)}(b_{02}^{(2)} + b_{20}^{(2)}) +\\
& 4(b_{01}^{(3)} + a_{10}^{(3)}))).
\end{aligned}$$

Hence we have that at most 2 limit cycles can bifurcate from the period annulus of system (2.10). This completes the proof of statement (f).

3 The appendix

We recall the Descartes Theorem about the number of zeros of a real polynomial (for a proof see for instance [3]).

Descartes Theorem *Consider the real polynomial $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \cdots + a_{i_r}x^{i_r}$ with $0 \le i_1 < i_2 < \cdots < i_r$ and $a_{i_j} \neq 0$ real constants for $j \in \{1, 2, \cdots, r\}$. When $a_{i_j}a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is m, then $p(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly $r-1$ positive real roots.*

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