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The third order Melnikov function of a quadratic center under quadratic perturbations ☆

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Abstract

We study quadratic perturbations of the integrable system $(1+x)\,dH$, where $H=(x^2+y^2)/2$. We prove that the first three Melnikov functions associated to the perturbed system give rise at most to three limit cycles.

Keywords: Quadratic systems; Limit cycles; Bifurcation; High order Melnikov functions

1. Introduction and statement of the main result

Planar vector fields $\dot{x}=X(x,y)$, $\dot{y}=Y(x,y)$ defined in the real plane, when $X(x,y)=0$ and $Y(x,y)=0$ are arbitrary conics, are usually called quadratic systems. The Hilbert sixteenth problem [5] restricted to them asks for the number and distribution of limit cycles inside this family. It is known that each limit cycle must surround a unique singularity of focus type, that at

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most two nests of limit cycles can coexist and that the following distributions of limit cycles exist: $(0,0)$, $(1,0)$, $(2,0)$, $(3,0)$, $(1,1)$, $(2,1)$ and $(3,1)$, see [1–3,8]. It has been recently proved that $(2,m)$ distribution is only possible for $m \in \{0,1\}$, see [10,11]. It is also generally believed that no more distributions of limit cycles than the ones listed above can exist and so, that quadratic systems have at most four limit cycles. Nevertheless the proof of this assertion turns out to be a very elusive problem. So, nowadays some people pretend to prove this result while other people study different degenerate bifurcations inside quadratic systems to check whether there appear or not more limit cycles. This paper goes in this second direction. We study how many limit cycles can appear in the following quadratic system

$$\begin{aligned} \dot{x} &= -y(1+x) - \varepsilon P(x,y), \\ \dot{y} &= x(1+x) + \varepsilon Q(x,y), \end{aligned} \tag{1}$$

where $\varepsilon > 0$ is a small parameter and P and Q are arbitrary polynomials of degree two given by $P(x,y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2$ and $Q(x,y) = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2$. The unperturbed system (i.e. for $\varepsilon = 0$) has a center at the origin and the first integral $H = (x^2+y^2)/2$ in the region $x^2+y^2 < 1$. Using the energy level $H = h$ as a parameter, we can express the Poincaré map $\mathcal{P}$ of (1) in terms of h and ε. For the corresponding displacement function $d(h,\varepsilon) = \mathcal{P}(h,\varepsilon) - h$ we obtain the following representation as a power series in ε:

$$d(h,\varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \varepsilon^3 M_3(h) + \cdots, \tag{2}$$

which is convergent for small ε. The Melnikov functions $M_k(h)$ are defined for $h \in (0, 1/2)$. Each simple zero $h_0 \in (0,1/2)$ of the first non-vanishing coefficient in (2) corresponds to a limit cycle of (1) emerging from the circle $x^2+y^2 = 2h_0$. We compute these functions by using the algorithm developed in [4,7]. Our main result is:

Theorem 1. *For $i = 1,2,3$, let $M_i(h)$ be the first Melnikov functions associated to system* (1). *Then M_1 has at most* 2 *zeros, taking into account their multiplicities. If $M_1(h) \equiv 0$ then M_2 has also at most* 2 *zeros, taking into account their multiplicities. If $M_1(h) \equiv M_2(h) \equiv 0$ then M_3 has at most* 3 *zeros, taking account their multiplicities, and all these upper bounds are sharp. Moreover, the functions $M_i(h)$, $i = 1,2,3$, can be explicitly obtained from the coefficients of the polynomials P and Q given in* (1) *and are elementary functions of h.*

We have the following corollary:

Corollary 2. *For system* (1) *at most three limit cycles can bifurcate from the set of periodic orbits of the unperturbed system, when considering the expansion of the displacement map* (2) *up to third order in ε. Furthermore this upper bound is reached.*

2. Proof of Theorem 1

We consider the following 1-form

$$\omega = \frac{Q(x,y)}{1+x}\,dx + \frac{P(x,y)}{1+x}\,dy, \tag{3}$$

such that we rewrite (1) in a Pfaffian form

$$dH = \varepsilon\omega.$$

The Melnikov functions will be calculated using the ideas of Françoise [4] and Iliev [7]. For example, the first order Melnikov function is given by

$$M_1(h) = \oint_{H=h} \omega.$$

In order to go further with the calculation of M_1 and, after, to give the integral expression of M_2, we need the relative cohomology decomposition of ω (see [4,6]). We denote

$$\omega_{ij} = \frac{x^i y^j}{1+x}\,dx, \quad \delta_{ij} = \frac{x^i y^j}{1+x}\,dy, \quad 0 \leqslant i + j \leqslant 2,$$

and we give first the decomposition of these forms.

Lemma 3. *All the* 1*-forms* ω_{ij} *and* δ_{ij}, *for* $0 \leqslant i + j \leqslant 2$, *can be expressed as follows*:

$$\delta_{01} = \frac{1}{1+x}\,dH - d\big(x - \ln(1+x)\big), \qquad \delta_{10} = dy - \delta_{00}, \qquad \delta_{02} = \frac{y}{1+x}\,dH - \omega_{11},$$

$$\delta_{11} = \frac{x}{1+x}\,dH - d\left(\frac{x^2}{2} - x + \ln(1+x)\right), \qquad \delta_{20} = 2H\delta_{00} - \frac{y}{1+x}\,dH + \omega_{11},$$

and

$$\omega_{00} = d\big(\ln(1+x)\big), \qquad \omega_{10} = d\big(x - \ln(1+x)\big), \qquad \omega_{20} = d\left(\frac{x^2}{2} - x + \ln(1+x)\right),$$

$$\omega_{01} = d(xy) - dy + \frac{y}{1+x}\,dH + (1-2H)\delta_{00} - 2\omega_{11},$$

$$\omega_{02} = 2d\big(H\ln(1+x)\big) - 2\ln(1+x)\,dH - d\left(\frac{x^2}{2} - x + \ln(1+x)\right).$$

Proof. First of all, by definition, we have

$$\delta_{01} = \frac{dy^2}{2(1+x)} = \frac{d(2H - x^2)}{2(1+x)} = \frac{1}{1+x}\,dH - d\big(x - \ln(1+x)\big).$$

In a similar way, we can check one by one the following relations, where we omit some relations whose validity is obvious.

$$\delta_{02} = \frac{y\,d(y^2)}{2(1+x)} = \frac{y}{1+x}\,dH - \frac{xy}{1+x}\,dx = \frac{y}{1+x}\,dH - \omega_{11},$$

$$\delta_{11} = \frac{xy}{1+x}\,dy = \frac{x}{1+x}\,dH - \frac{x^2}{1+x}\,dx,$$

$$\delta_{20} = \frac{2H - y^2}{1+x}\,dy = 2H\delta_{00} - \delta_{02} = 2H\delta_{00} - \frac{y}{1+x}\,dH - \omega_{11},$$

and

$$\omega_{01} = \frac{y}{1+x}\,dx = d(xy) - dy + \frac{y}{1+x}\,dH + (1-2H)\delta_{00}. \qquad \square$$

With the above notations, the 1-form ω given by (3) becomes $\omega = a_{00}\delta_{00} + a_{10}\delta_{10} + a_{01}\delta_{01} + a_{20}\delta_{20} + a_{11}\delta_{11} + a_{02}\delta_{02} + b_{00}\omega_{00} + b_{10}\omega_{10} + b_{01}\omega_{01} + b_{20}\omega_{20} + b_{11}\omega_{11} + b_{02}\omega_{02}$. Replacing these with the expressions given in Lemma 4 and collecting the terms correspondingly, the following result can be found.

Lemma 4. *The* 1*-form* ω *given by* (3) *can be expressed in the following way*

$$\omega = r_1\,dH + dS_1 + N_1, \tag{4}$$

where $r_1 = r_1(x, y)$, $dS_1 = dS_1(x, y, H)$ *and* N_1 *are given as follows*:

$$r_1 = \frac{a_{01} + a_{11}x + (a_{02} - a_{20} + b_{01})y}{1+x} - 2b_{02}\ln(1+x),$$
$$dS_1 = (a_{10} - b_{01})\,dy + b_{01}\,d(xy) + 2b_{02}\,d\big(H\ln(1+x)\big) + \frac{b_{00} + (b_{10} - a_{10})x + (b_{20} - b_{02} - a_{11})x^2}{1+x}\,dx,$$
$$N_1 = (b_{11} + a_{20} - a_{02} - 2b_{01})\omega_{11} + 2(a_{20} - b_{01})H\delta_{00} + (a_{00} - a_{10} + b_{01})\delta_{00}.$$

Using (4) the expression of the first order Melnikov function $M_1(h) = \oint_{H=h}\omega$ follows as

$$M_1(h) = (b_{11} + a_{20} - a_{02} - 2b_{01})J_1(h) + 2(a_{20} - b_{01})hJ_0(h) + (a_{00} - a_{10} + b_{01})J_0(h), \tag{5}$$

where

$$J_0(h) = \oint_{H=h}\delta_{00}, \qquad J_1(h) = \oint_{H=h}\omega_{11}.$$

The explicit expressions of $J_0(h)$ and $J_1(h)$ are

$$J_0(h) = 2\pi\left(1 - \frac{1}{\sqrt{1-2h}}\right), \qquad J_1(h) = 2\pi(1-h) - 2\pi\sqrt{1-2h}.$$

We notice that, for each $z \in (0, 1)$,

$$M_1\big(\big(1 - z^2\big)/2\big) = \frac{1-z}{z}\big(A + Bz + Cz^2\big),$$

where $A = 2\pi(a_{10} - a_{00} - a_{20})$, $B = \pi(b_{11} + a_{20} - a_{02} - 2b_{01})$ and $C = \pi(a_{20} - b_{11} + a_{02})$. Then, the equation $M_1(h) = 0$, $h \in (0, 1/2)$, is equivalent through the change $2h = 1 - z^2$ with $A + Bz + Cz^2 = 0$, $z \in (0, 1)$. Now it is clear that M_1 has at most 2 zeros, taking into account their multiplicities, and there are some coefficients such that M_1 has exactly 2 simple zeros.

Since $J_0(h)$, $hJ_0(h)$ and $J_1(h)$ are linearly independent, $M_1(h) \equiv 0$ if and only if all the coefficients of $J_0(h)$, $hJ_0(h)$ and $J_1(h)$ vanish, namely,

$$b_{01} = a_{20}, \qquad a_{10} = a_{20} + a_{00}, \qquad b_{11} = a_{20} + a_{02}. \tag{6}$$

From now on we assume that

$$M_1(h) = \oint_{H=h}\omega \equiv 0.$$

Then, from (4) and (6) we have the decomposition

$$\omega = r_1\,dH + dS_1,$$

where r_1 and dS_1 are given in Lemma 5. This assures that Assertion 2.1 from [7] holds true. On the basis of this assertion, it is proved in [7] that the second order Melnikov function is given by

$$M_2(h) = \oint_{H=h} r_1\omega.$$

In order to go further with the calculation of M_2 and, after, to give the integral expression of M_3, we need the relative cohomology decomposition of $r_1\omega$. Before stating this result, we make some notations.

$$c_0 = a_{01} - a_{11}, \qquad c_1 = b_{00} - b_{10} + a_{01} + b_{20} - b_{02} - a_{11}, \qquad c_2 = b_{00} - c_1,$$
$$c_3 = b_{20} - b_{02} - a_{11}.$$

Lemma 5. *The following decomposition holds,*

$$r_1\omega = r_2\,dH + dS_2 + N_2, \tag{7}$$

where

$$N_2 = (a_{00}c_0 + a_{02}c_1 + a_{02}c_2 + 2a_{00}b_{02})\delta_{00} + 2(a_{02}b_{02} - a_{20}c_0 - a_{02}c_2 - 2a_{00}b_{02})H\delta_{00} + (a_{02}c_3 + 2a_{20}b_{02} - 2a_{20}c_0 - 2a_{02}c_2 - 4a_{00}b_{02})\omega_{11},$$

and where $r_2 = r_2(x,y)$ and $dS_2 = dS_2(x,y,H)$ are given by the following relations

$$\begin{aligned} r_2 &= r_1^2 + a_{02}a_{20} + (2c_0b_{02} + a_{02}a_{00} - a_{02}a_{20})\frac{1}{1+x} - 2b_{02}^2\big(\ln(1+x)\big)^2 \\ &\quad - 2a_{02}a_{20}\ln(1+x) + 2c_0b_{02}\frac{\ln(1+x)}{1+x} + 2a_{02}b_{02}\frac{y\ln(1+x)}{1+x} \\ &\quad + (c_0a_{20} + c_2a_{02} + 2a_{00}b_{02} + 2a_{02}b_{02})\frac{y}{1+x}, \end{aligned}$$

$$\begin{aligned} dS_2 &= (a_{11}c_2 + c_0c_3 - 2a_{02}a_{20} + a_{02}a_{00})\,dx + (a_{11}c_3 - 2a_{02}a_{20})x\,dx \\ &\quad + (a_{11}c_1 + c_0c_2 - c_0c_3 + a_{00}a_{02} - 2a_{02}a_{20})\frac{1}{1+x}\,dx \\ &\quad + c_0c_1\frac{1}{(1+x)^2}\,dx - 2b_{02}c_2\ln(1+x)\,dx - 2b_{02}c_3x\ln(1+x)\,dx \\ &\quad - 2b_{02}c_1\frac{\ln(1+x)}{1+x}\,dx + 2a_{11}b_{02}\,d\big(H\ln(1+x)\big) - 2b_{02}c_0\,d\left(\frac{H}{1+x}\right) \\ &\quad - 2b_{02}^2\,d\big(H\big(\ln(1+x)\big)^2\big) - a_{02}c_1\,d\left(\frac{y}{1+x}\right) - 2b_{02}a_{00}\,d\big(y\ln(1+x)\big) \\ &\quad - 2b_{02}a_{20}\,d\big(xy\ln(1+x)\big) + (a_{00}a_{11} - a_{02}c_2 - 2a_{00}b_{02})\,dy \\ &\quad - 2a_{02}b_{02}\,d\left(\frac{Hy}{1+x}\right) + (a_{20}a_{11} + a_{20}c_0 + a_{02}c_2 + 2a_{00}b_{02})\,d(xy). \end{aligned}$$

Proof. First we notice that $r_1\omega = r_1^2\,dH + r_1\,dS_1$. We sketch in the sequel how the decomposition of $r_1\,dS_1$ can be obtained.

$$\begin{aligned} r_1\,dS_1 &= a_{11}\,dS_1 + a_{00}c_0\delta_{00} + c_0c_2\omega_{00} + c_0c_3\omega_{10} + c_0a_{20}\delta_{10} + c_0a_{20}\omega_{01} \\ &\quad + 2c_0b_{02}\frac{H}{(1+x)^2}\,dx + 2c_0b_{02}\frac{\ln(1+x)}{1+x}\,dH + c_0c_1\frac{1}{(1+x)^2}\,dx \\ &\quad + a_{00}a_{02}\delta_{01} + a_{02}c_2\omega_{01} + a_{02}c_3\omega_{11} + a_{02}a_{20}\delta_{11} + a_{02}a_{20}\omega_{02} \\ &\quad + 2a_{02}b_{02}\frac{Hy}{(1+x)^2}\,dx + 2a_{02}b_{02}\frac{y\ln(1+x)}{1+x}\,dH + a_{02}c_1\frac{y}{(1+x)^2}\,dx \end{aligned}$$

$$- 2b_{02}a_{00}\ln(1+x)\,dy - 2b_{02}c_2\ln(1+x)\,dx - 2b_{02}c_3 x\ln(1+x)\,dx$$
$$- 2b_{02}a_{20}\ln(1+x)\,d(xy) - 4b_{02}^2\frac{H\ln(1+x)}{1+x}\,dx - 4b_{02}^2\big(\ln(1+x)\big)^2\,dH$$
$$- 2c_1b_{02}\frac{\ln(1+x)}{1+x}\,dx.$$

In the above relation, we replace all the expressions of these 1-forms given in Lemma 4 and also the following equalities

$$\frac{y}{(1+x)^2}\,dx = \delta_{00} - d\left(\frac{y}{1+x}\right),$$
$$\ln(1+x)\,dy = d\big(y\ln(1+x)\big) - \omega_{01},$$
$$\ln(1+x)\,d(xy) = d\big(xy\ln(1+x)\big) - \omega_{11},$$

and

$$\frac{Hy}{(1+x)^2}\,dx = H\delta_{00} - d\left(\frac{Hy}{1+x}\right) + \frac{y}{1+x}\,dH.$$

Then the decomposition follows by collecting these terms correspondingly. □

Using (7), the expression of the second order Melnikov function, $M_2(h) = \oint_{H=h} r_1\omega$, is given by

$$M_2(h) = (a_{00}c_0 + a_{02}c_1 + a_{02}c_2 + 2a_{00}b_{02})J_0(h)$$
$$+ 2(a_{02}b_{02} - a_{20}c_0 - a_{02}c_2 - 2a_{00}b_{02})hJ_0(h)$$
$$+ (a_{02}c_3 + 2a_{20}b_{02} - 2a_{20}c_0 - 2a_{02}c_2 - 4a_{00}b_{02})J_1(h).$$

It is not difficult to see that the coefficients of $J_0(h)$, $hJ_0(h)$ and $J_1(h)$ involved in the above expression of M_2 are independent and, hence, they can be considered like three arbitrary real numbers. The discussion concerning the number of zeros of M_2 is the same as for M_1. Thus, the statement of Theorem 1 about M_2 is proved.

The relation $M_2(h) \equiv 0$ holds if and only if one of the following three cases holds.

$$a_{02} = a_{20} = a_{00} = 0, \tag{8}$$
$$a_{02} = b_{02} = c_0 = 0, \tag{9}$$
$$a_{02} \neq 0, \qquad a_{02}c_1 = a_{20}c_0 - a_{00}c_0 - a_{02}b_{02},$$
$$a_{02}c_2 = a_{02}b_{02} - c_0a_{20} - 2a_{00}b_{02}, \qquad a_{02}c_3 = -2b_{02}(a_{20} - a_{02}). \tag{10}$$

From now on we assume also that

$$M_2(h) = \oint_{H=h} r_1\omega \equiv 0.$$

Then, from (7), in each of the three cases listed above, we have the decomposition

$$r_1\omega = r_2\,dH + dS_2,$$

where r_2 and dS_2 are given in Lemma 5. Since this decomposition holds true, according to Remark 2.3 from [7], the third order Melnikov function is given by

$$M_3(h) = \oint_{H=h} r_2\omega.$$

Theorem 6. *We assume that* $M_1(h) = M_2(h) \equiv 0$. *Then, when* (8) *or* (9) *holds true,*

$$M_k(h) \equiv 0, \quad k \geqslant 3.$$

When (10) *holds true, the third order Melnikov function has the following general form*:

$$M_3(h) = (\alpha_0 + \beta_0 h)J_0(h) + \alpha_1 J_1(h) + \alpha_2 J_2(h), \tag{11}$$

where

$$J_2(h) = \oint_{H=h} \frac{xy\ln(1+x)}{(1+x)^2}\,dx \tag{12}$$

and

$$\begin{aligned}
\alpha_0 &= 2a_{00}a_{02}^2 - 6b_{02}^2a_{00} - 4a_{02}^2a_{20} + a_{00}^2a_{02} - 3c_0b_{02}a_{00} + a_{02}a_{20}a_{00},\\
\beta_0 &= -2\big(3a_{02}a_{20}a_{00} + a_{00}a_{02}^2 - 6b_{02}^2a_{00} - a_{02}a_{20}^2 - 2a_{02}^2a_{20} + 3a_{02}a_{11}b_{02}\\
&\quad + 6c_0a_{02}b_{02} - 3a_{02}b_{02}a_{20} - 3a_{02}b_{02}a_{00} - 3b_{02}c_0a_{20}\big),\\
\alpha_1 &= -2\big(-6b_{02}^2a_{00} + a_{00}a_{02}^2 + 3b_{02}^2a_{20} - a_{02}^2a_{20} - 2a_{02}a_{20}^2 - 3b_{02}^2a_{02} + 3a_{02}a_{11}b_{02}\\
&\quad + 6c_0a_{02}b_{02} + 3a_{02}a_{20}a_{00} - 3a_{02}b_{02}a_{20} - 3a_{02}b_{02}a_{00} - 3b_{02}c_0a_{20}\big),\\
\alpha_2 &= -2a_{02}b_{02}(-c_0 - a_{11} + a_{20} + a_{00}).
\end{aligned}$$

Proof. We denote $s_2 = r_2 - r_1^2$ and we write $r_2\omega = r_1^2\omega + s_2\omega = r_1r_2\,dH + r_1\,dS_2 + s_2r_1\,dH + s_2\,dS_1$. Then, the third order Melnikov function can be calculated as

$$M_3(h) = \oint_{H=h} r_2\omega = \oint_{H=h} r_1\,dS_2 + s_2\,dS_1.$$

Now we notice that we can write

$$\begin{aligned}
r_1(x,y) &= f_1(x) + g_1(x)y, & dS_1 &= F_1(x)\,dx + dG_1(x,H) + d\big(R_1(x)y\big)\\
s_2(x,y) &= f_2(x) + g_2(x)y, & dS_2 &= F_2(x)\,dx + dG_2(x,H) + d\big(R_2(x,H)y\big),
\end{aligned}$$

where

$$\begin{aligned}
f_1(x) &= \frac{a_{01} + a_{11}x}{1+x} - 2b_{02}\ln(1+x), \qquad g_1(x) = \frac{(a_{02} - a_{20} + b_{01})}{1+x},\\
R_1(x) &= (a_{10} - b_{01}) + b_{01}x, \qquad G_1(x,H) = 2b_{02}H\ln(1+x),\\
F_1(x) &= \frac{b_{00} + (b_{10} - a_{10})x + (b_{20} - b_{02} - a_{11})x^2}{1+x},
\end{aligned}$$

and

$$f_2(x) = a_{02}a_{20} + (2c_0b_{02} + a_{02}a_{00} - a_{02}a_{20})\frac{1}{1+x} - 2b_{02}^2\big(\ln(1+x)\big)^2$$
$$- 2a_{02}a_{20}\ln(1+x) + 2c_0b_{02}\frac{\ln(1+x)}{1+x},$$
$$g_2(x) = 2a_{02}b_{02}\frac{\ln(1+x)}{1+x} + (c_0a_{20} + c_2a_{02} + 2a_{00}b_{02} + 2a_{02}b_{02})\frac{1}{1+x},$$
$$F_2(x) = (a_{11}c_2 + c_0c_3 - 2a_{02}a_{20} + a_{02}a_{00}) + (a_{11}c_3 - 2a_{02}a_{20})x$$
$$+ (a_{11}c_1 + c_0c_2 - c_0c_3 + a_{00}a_{02} - 2a_{02}a_{20})\frac{1}{1+x}$$
$$+ c_0c_1\frac{1}{(1+x)^2} - 2b_{02}c_2\ln(1+x) - 2b_{02}c_3x\ln(1+x)$$
$$- 2b_{02}c_1\frac{\ln(1+x)}{1+x},$$
$$G_2(x,H) = 2a_{11}b_{02}H\ln(1+x) - 2b_{02}c_0\frac{H}{1+x} - 2b_{02}^2H\big(\ln(1+x)\big)^2,$$
$$R_2(x,H) = -a_{02}c_1\frac{1}{1+x} - 2b_{02}a_{00}\ln(1+x) - 2b_{02}a_{20}x\ln(1+x)$$
$$+ (a_{00}a_{11} - a_{02}c_2 - 2a_{00}b_{02}) - 2a_{02}b_{02}\frac{H}{1+x}$$
$$+ (a_{20}a_{11} + a_{20}c_0 + a_{02}c_2 + 2a_{00}b_{02})x.$$

Then

$$\oint_{H=h} r_1\,dS_2 = \oint_{H=h} f_1(x)F_2(x)\,dx + f_1(x)\,dG_2(x,H) + f_1(x)\,d\big(R_2(x,H)y\big)$$
$$+ g_1(x)yF_2(x)\,dx + g_1(x)y\,dG_2(x,H) + g_1(x)y\,d\big(R_2(x,H)y\big)$$
$$= -\oint_{H=h} f_1'(x)R_2(x,H)y\,dx + g_1(x)F_2(x)y\,dx + g_1(x)y\,dG_2(x,H)$$

and, analogously,

$$\oint_{H=h} s_2\,dS_1 = -\oint_{H=h} f_2'(x)R_1(x,H)y\,dx + g_2(x)F_1(x)y\,dx + g_2(x)y\,dG_1(x,H).$$

Taking into account the conditions between the coefficients that guarantee that $M_1(h) \equiv M_2(h) \equiv 0$ and the notations (7) we make the following substitutions:

$$b_{01} = a_{20}, \qquad a_{10} = a_{20} + a_{00}, \qquad b_{11} = a_{20} + a_{02}, \qquad a_{01} = c_0 + a_{11},$$
$$b_{10} = b_{00} + a_{01} + b_{20} - b_{02} - a_{11} - c_1, \qquad b_{00} = c_2 + c_1, \qquad b_{20} = c_3 + b_{02} + a_{11}$$

and, moreover,

Case (i): $a_{02} = a_{20} = a_{00} = 0$,

Case (ii): $a_{02} = b_{02} = c_0 = 0$,

Case (iii): $c_1 = (a_{20}c_0 - a_{00}c_0 - a_{02}b_{02})/a_{02}$,
$c_2 = (a_{02}b_{02} - c_0a_{20} - 2a_{00}b_{02})/a_{02}$, $\qquad c_3 = -2b_{02}(a_{20} - a_{02})/a_{02}$.

After these substitutions, all the remaining coefficients are independent.

In Case (i) we have $g_1 \equiv g_2 \equiv R_1 \equiv R_2 \equiv 0$, while in Case (ii) we have $r_1 = a_{11}$ and $s_2 \equiv 0$. From all these we deduce that, in both cases, $M_3(h) \equiv 0$. Moreover, going further with the procedure of Françoise [4] and Iliev [7] for finding higher order Melnikov functions (described for example in Remark 2.3 from [7]), it can be seen that all the Melnikov functions vanish in these cases.

In Case (iii), the expression of M_3 is found as linear combination of $J_2(h)$ and the integrals of the following 1-forms:

$$\omega_{11}, \quad \omega_{01}, \quad \frac{y}{(1+x)^2}\,dx, \quad \frac{xy}{(1+x)^2}\,dx, \quad \frac{yH}{(1+x)^2}\,dx, \quad \frac{x^2y}{(1+x)^2}\,dx.$$

Using the following relations

$$\oint\limits_{H=h} \omega_{11} = J_1(h), \qquad \oint\limits_{H=h} \omega_{01} = (1-2h)J_0(h) - 2J_1(h),$$

$$\oint\limits_{H=h} \frac{y}{(1+x)^2}\,dx = J_0(h), \qquad \oint\limits_{H=h} \frac{xy}{(1+x)^2}\,dx = -2hJ_0(h) - 2J_1(h),$$

$$\oint\limits_{H=h} \frac{x^2y}{(1+x)^2}\,dx = 2hJ_0(h) + 3J_1(h),$$

the expression (11) is obtained. □

The function J_2 given in the integral form (12) can be expressed in terms of elementary functions as

$$J_2(h) = \frac{2\pi}{\sqrt{1-2h}}\left[2h - \left(1+\sqrt{1-2h}\right)^2 \ln\frac{1+\sqrt{1-2h}}{2} + 4(1-h)\ln\sqrt{1-2h}\right]. \tag{13}$$

Since the method of calculating J_2 is not a standard one, we will present it in Appendix A at the end of the paper. It remains to study the number of zeros of M_3. Through the change $2h = 1 - z^2$, the equation

$$M_3(h) = 0, \quad h \in (0, 1/2),$$

is equivalent with

$$A + Bz + Cz^2 = \frac{1}{2\pi}\frac{z}{1-z}J_2\left(\left(1-z^2\right)/2\right), \quad z \in (0,1),$$

where $A = (2\alpha_0 + \beta_0)/2\alpha_2$, $B = -\alpha_1/2\alpha_2$ and $C = (-\beta_0 + \alpha_1)/2\alpha_2$.

We denote

$$f(z) = \frac{1}{2\pi}\frac{z}{1-z}J_2\left(\left(1-z^2\right)/2\right),$$

and $g(z) = f(z) - A - Bz - Cz^2$ such that we need to study the number of zeros of the function g. We have

$$f(z) = 1 + z - \frac{(1+z)^2}{1-z}\ln\frac{1+z}{2} + 2\frac{1+z^2}{1-z}\ln z$$

and $g'''(z) = f'''(z) = \frac{24}{(1-z)^4} P(z)$, where

$$P(z) = \frac{z^6 - z^5 - 19z^4 + 7z^3 + 22z^2 - 14z + 4}{24z^3(z+1)} - \ln\frac{z+1}{2z}.$$

After noticing that $\lim_{z\to 0} P(z) = \infty$, $P(1) = 0$ and that for all $z \in (0,1)$,

$$P'(z) = \frac{(z^4 + 4z^3 + 8z^2 + 12z + 6)(z-1)^3}{12z^4(z+1)^2} < 0,$$

we deduce that, for all $z \in (0,1)$, $P(z) > 0$ and, as a consequence, $g'''(z) > 0$. By applying the Rolle's rule we have that g has at most 3 zeros, taking into account their multiplicities. Hence, M_3 has also at most 3 zeros, taking into account their multiplicities, as we wanted to prove.

We consider now the system (1) with the following coefficients $a_{00} = 0$, $a_{10} = 3/4$, $a_{01} = -\sqrt{3759}/358 + 3/4$, $a_{20} = 3/4$, $a_{11} = -3\sqrt{3759}/1253 + 3/4$, $a_{02} = -1$, $b_{00} = 0$, $b_{10} = -\sqrt{3759}/1432$, $b_{01} = 3/4$, $b_{20} = 75\sqrt{3759}/716 + 3/4$, $b_{11} = -1/4$ and $b_{02} = \sqrt{3759}/42$. By direct calculations, it can be seen that relations (6) and (10) hold, i.e. $M_1(h) = M_2(h) \equiv 0$. The coefficients from the expression (11) of M_3 are $\alpha_0 = -3$, $\beta_0 = 0$, $\alpha_1 = -25$, $\alpha_2 = 1/2$ and, moreover, $A = -6$, $B = 25$, $C = -25$. It can be easily seen that $g(0.01) < 0$, $g(0.1) > 0$, $g(0.4) < 0$ and $g(0.8) > 0$. Hence, g has at least 3 zeros. Since we have proved before that g has at most 3 zeros, it follows that it has exactly 3 zeros. Then, the corresponding M_3 has exactly simple 3 zeros and the theorem follows.

Appendix A

This appendix is devoted to proving that the function $J_2(h)$ defined in (12) is given by (13), i.e. that

$$J_2(h) = \oint\limits_{H=h} \frac{xy\ln(1+x)}{(1+x)^2}\,dx = \frac{2\pi}{\sqrt{1-2h}}\left[2h - \left(1+\sqrt{1-2h}\right)^2 \ln\frac{1+\sqrt{1-2h}}{2} + 4(1-h)\ln\sqrt{1-2h}\right]. \tag{A.1}$$

First denote for each $-1 < r < 1$,

$$J(r) = \int_0^{2\pi} \frac{\sin^2\theta\cos\theta\ln(1+r\cos\theta)}{(1+r\cos\theta)^2}\,d\theta.$$

Once the expression of J is known, we calculate J_2 as

$$J_2(h) = -2h\sqrt{2h}\,J\left(\sqrt{2h}\right), \quad \text{for } 0 < h < 1/2.$$

In order to find J we will find first the expression of

$$F_k(r) = \int_0^{2\pi} \frac{\cos^k\theta\ln(1+r\cos\theta)}{1+r\cos\theta}\,d\theta,$$

for $k = 0$ and $k = 2$. We will prove that for each $-1 < r < 1$,

$$\text{(i)}\quad F_0(r) = \frac{2\pi}{\sqrt{1-r^2}} \ln \frac{2(1-r^2)}{1+\sqrt{1-r^2}},$$

$$\text{(ii)}\quad F_2(r) = \frac{2\pi}{r^2}\bigl(1-\sqrt{1-r^2}\bigr) - \frac{2\pi}{r^2} \ln \frac{1+\sqrt{1-r^2}}{2} + \frac{2\pi}{r^2\sqrt{1-r^2}} \ln \frac{2(1-r^2)}{1+\sqrt{1-r^2}}.$$

Proof of (i): Mainly, we use the Poisson's formula [9]. A function f that is harmonic in the unit disk of the complex plain can be calculated using only its values on the boundary of the disk according to the formula:

$$f\bigl(\rho e^{it}\bigr) = \frac{1}{2\pi} \int_0^{2\pi} f\bigl(e^{i\theta}\bigr) \frac{1-\rho^2}{|1-\rho e^{it} e^{-i\theta}|^2}\, d\theta,$$

for all $0 < \rho < 1$ and $0 \leqslant t < 2\pi$. For $t = 0$ we have

$$\int_0^{2\pi} \frac{f(e^{i\theta})}{1+\rho^2 - 2\rho\cos\theta}\, d\theta = 2\pi \frac{f(\rho)}{1-\rho^2}.$$

Now let $f_s(\rho e^{it}) = \ln|1 - s\rho e^{it}|$ which is a harmonic function in the unit disk of the complex plain for each fixed $0 < s < 1$ and we write the above equality for this function:

$$\int_0^{2\pi} \frac{\ln(1+s^2 - 2s\cos\theta)}{1+\rho^2 - 2\rho\cos\theta}\, d\theta = 2\pi \frac{\ln(1-s\rho)^2}{1-\rho^2}.$$

Taking $s = \rho$ in this last formula, we have

$$\int_0^{2\pi} \frac{\ln(1+\rho^2 - 2\rho\cos\theta)}{1+\rho^2 - 2\rho\cos\theta}\, d\theta = 4\pi \frac{\ln(1-\rho^2)}{1-\rho^2}.$$

Using this we obtain

$$F_0\left(-\frac{2\rho}{1+\rho^2}\right) = 4\pi \frac{(1+\rho^2)}{1-\rho^2} \ln\bigl(1-\rho^2\bigr) - 2\pi \frac{(1+\rho^2)}{1-\rho^2} \ln\bigl(1+\rho^2\bigr),$$

where we have also used that

$$\int_0^{2\pi} \frac{1}{1+\rho^2 - 2\rho\cos\theta}\, d\theta = 2\pi \frac{1}{1-\rho^2}.$$

Then (i) follows now for $-1 < r < 0$ and, using that it is an even function, also for $-1 < r < 1$.

Proof of (ii): Denote

$$f(r) = \int_0^{2\pi} \cos\theta \ln(1 + r\cos\theta)\, d\theta \quad \text{and} \quad g(r) = \int_0^{2\pi} \ln(1 + r\cos\theta)\, d\theta.$$

Then

$$F_2(r) = \frac{1}{r} f(r) - \frac{1}{r^2} g(r) + \frac{1}{r^2} F_0(r).$$

In order to find f and g we notice that

$$f'(r) = \int_0^{2\pi} \frac{\cos^2\theta}{1 + r\cos\theta}\, d\theta = \frac{2\pi}{r^2\sqrt{1-r^2}}\big(1 - \sqrt{1-r^2}\big),$$

$$g'(r) = \int_0^{2\pi} \frac{\cos\theta}{1 + r\cos\theta}\, d\theta = -rf'(r).$$

Then, taking also into account that $f(0) = g(0) = 0$ we obtain

$$f(r) = \frac{2\pi}{r}\big(1 - \sqrt{1-r^2}\big) \quad \text{and} \quad g(r) = 2\pi \ln\frac{1 + \sqrt{1-r^2}}{2},$$

and (ii) follows.

The last step in finding the expression of J is noticing that

$$J(r) = F_2'(r) - F_0'(r) + h(r),$$

where

$$h(r) = \int_0^{2\pi} \frac{\sin^2\theta\cos\theta}{(1 + r\cos\theta)^2}\, d\theta = -\frac{2\pi}{r^3\sqrt{1-r^2}}\big(1 - \sqrt{1-r^2}\big)^2.$$

Hence we obtain

$$J(r) = -\frac{2\pi}{r^3\sqrt{1-r^2}}\left[r^2 - \big(1 + \sqrt{1-r^2}\big)^2 \ln\frac{1+\sqrt{1-r^2}}{2} + 2\big(2 - r^2\big)\ln\sqrt{1-r^2}\right].$$

From the above formula the expression (A.1) follows.

References

[1] N.N. Bautin, On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type, Mat. Sb. 30 (72) (1952) 181–196 (in Russian); Transl. Amer. Math. Soc. 100 (1) (1954) 397–413 (in English).

[2] L. Chen, M. Wang, On relative locations and the number of limit cycles for quadratic systems, Acta Math. Sinica 22 (1979) 751–758 (in Chinese).

[3] W.A. Coppel, A survey of quadratic systems, J. Differential Equations 2 (1966) 293–304.

[4] J.-P. Françoise, Successive derivatives of a first return map, application to the study of quadratic vector fields, Ergodic Theory Dynam. Systems 16 (1996) 87–96.

[5] D. Hilbert, Mathematical problems, Transl. Bull. Amer. Math. Soc. 8 (1902) 437–479; Bull. Amer. Math. Soc. (N.S.) 37 (2000) 407–436.

[6] I.D. Iliev, Perturbations of quadratic centers, Bull. Sci. Math. 122 (1998) 107–161.

[7] I.D. Iliev, On second order bifurcations of limit cycles, J. London Math. Soc. 58 (1998) 353–366.

[8] S. Shi, A concrete example of the existence of four limit cycles for plane quadratic systems, Sci. Sinica 11 (1979) 1051–1056 (in Chinese); Sci. Sinica 23 (1980) 153–158 (in English).

[9] V.S. Vladimirov, The Equations of Mathematical Physics, Mir, Moscow, 1984.

[10] P. Zhang, On the distribution and number of limit cycles for quadratic system with two foci, Acta Math. Sinica 44 (2001) 37–44 (in Chinese).

[11] P. Zhang, On the distribution and number of limit cycles for quadratic systems with two foci, Qual. Theory Dyn. Syst. 3 (2002) 437–463.