SOME REMARKS ON INVERSE JACOBI MULTIPLIERS AROUND HOPF SINGULARITIES

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Abstract. We study several properties of inverse Jacobi multipliers V around Hopf singularities of analytic vector fields $\mathcal{X}$ in $\mathbb{R}^n$ which are relevant to the study of the local bifurcation of periodic orbits. When $n = 3$ and the singularity is a saddle–focus we show that: (i) any two locally smooth and non–flat linearly independent inverse Jacobi multipliers have the same Taylor expansion; (ii) any smooth and non–flat V has associated exactly one smooth center manifold W^c of $\mathcal{X}$ such that $W^c \subset V^{-1}(0)$. We also study whether the properties of the vanishing set $V^{-1}(0)$ proved in the 3–dimensional case remain valid when $n \geq 4$.

1. Introduction

We consider three–dimensional systems

$$\dot{x} = -y + \mathcal{F}_1(x, y, z), \ \dot{y} = x + \mathcal{F}_2(x, y, z), \ \dot{z} = \lambda\, z + \mathcal{F}_3(x, y, z), \tag{1}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$, $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) : U \to \mathbb{R}^3$ is real analytic on the open neighborhood $U \subset \mathbb{R}^3$ of the origin, and $\mathcal{F}$ satisfies $\mathcal{F}(0) = 0$ and $D\mathcal{F}(0) = 0$. We let $\mathcal{X} = (-y+\mathcal{F}_1(x, y, z))\partial_x + (x+\mathcal{F}_2(x, y, z))\partial_y + (\lambda z+\mathcal{F}_3(x, y, z))\partial_z$ denote the vector field associated to system (1).

The origin is a *Hopf singularity* of system (1) since its associated eigenvalues are $\{\pm i, \lambda\}$ with $i^2 = -1$, see [13]. In a neighborhood of a Hopf point there exists a C^r (local) center manifold for any $r \in \mathbb{N}$, which need not be either unique or analytic. We remind that, when the singularity of (1) restricted to some local center manifold is a center, the local center manifold is unique and analytic. Moreover, when the singularity of (1) restricted to *some* local center manifold is a focus, the same is true for *any* local center manifold. In this case it is said that the singularity is a saddle-focus of (1). A Hopf singularity of an analytic system like (1) is either a *center on the center manifold* or a *saddle–focus*.

A C^1 function $V : U \to \mathbb{R}$ is said to be an *inverse Jacobi multiplier* in U of $\mathcal{X}$, if it is not locally null and it satisfies the linear first order partial differential equation $\mathcal{X}V = V\,\mathrm{div}\mathcal{X}$ in U, where $\mathrm{div}\mathcal{X}$ is the divergence of the vector field $\mathcal{X}$.

In the planar case (i.e. vector fields of the form $F_1(x, y)\partial_x + F_2(x, y)\partial_y$), inverse Jacobi multipliers are called *inverse integrating factors*. The vanishing set of an

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inverse integrating factor is well studied. The fundamental relation between this vanishing set and the location of limit cycles was proved in [12], but many other properties of this set have been found, see [11]. In [7] it is showed that the Poincaré return map associated to a limit cycle can be determined in terms of the inverse integrating factor. See [9] for a survey of the properties of this helpful function.

In the light of these results it looks interesting to study inverse Jacobi multipliers and expect that some of the properties of inverse integrating factors could be generalized to higher dimensions. There are some recent papers in this direction. The relation between inverse Jacobi multipliers of system (1) and center manifolds is studied in [3]. In particular it is discussed under what conditions a local center manifold is included in the vanishing set of an inverse Jacobi multiplier. Moreover, it is given another solution to the center problem in $\mathbb{R}^3$, formulated in terms of an inverse Jacobi multiplier, apart from the classical Lyapunov solution formulated in terms of a first integral.

The *cyclicity* of a saddle–focus singularity at the origin of system (1) is the maximum number of limit cycles that can bifurcate from the origin, considering any analytic perturbation that keeps the location and the monodromic nature of this singularity. In [4] the authors found that the cyclicity of a saddle–focus of system (1) is determined by an inverse Jacobi multiplier of the same system.

One can see [1] for a modern reference of both the classical theory and also new advances about inverse Jacobi multipliers, and [5] for a very recent survey.

This work can be considered as an enlargement of the above mentioned papers [3, 4]. We present here some new properties of inverse Jacobi multipliers. Through this work, whenever we say that an object is "non–flat" we mean "non–flat at the origin". Recall that any smooth and non–flat center manifold has a unique Taylor expansion. In Theorem 2 we show that this property is shared by any smooth and non–flat inverse Jacobi multiplier near a saddle–focus of (1).

Using normal form theory, it is proved in [3] that there exist a smooth and non–flat inverse Jacobi multiplier $\hat{V}$ and a smooth center manifold $\hat{W}^c$ of system (1) such that $\hat{W}^c \subset \hat{V}^{-1}(0)$. In Theorem 4 we improve this property showing that associated to each smooth and non–flat inverse Jacobi multiplier V of (1) there is exactly one smooth center manifold W^c such that $W^c \subset V^{-1}(0)$. Based on these new results, we succeeded to give a shorter proof of some results from [3, 4]. For example, we present here a new proof for the relation between the cyclicity of the saddle–focus at the origin and the vanishing multiplicity of any smooth and non–flat inverse Jacobi multiplier of system (1). In the last section we present a counter–example to support the idea that there is no direct extension of this relation to higher dimensions. Finally we present additional properties of the vanishing set $V^{-1}(0)$ in this higher dimensional setting.

2. New properties of inverse Jacobi multipliers and their zero set

In this section we present several results concerning system (1) that complement those obtained in [4].

Remark 1. In [4] it is proved that, given a system (1), there exists an integer $p \geq 2$ such that any local C^∞ and non–flat inverse Jacobi multiplier of (1) near a saddle–focus at the origin has the following Taylor expansion (up to a multiplicative constant)

$$z(x^2+y^2)^p + \cdots . \tag{2}$$

Here the dots denote higher order terms.

The next theorem is a key point in the determination given in [10] of a basis of the Lie algebra of formal commutators of a formal normal form of system (1) when the origin is a saddle–focus.

Theorem 2. *Assume that the origin is a saddle–focus for system* (1)*. Then any two locally smooth and non–flat linearly independent inverse Jacobi multipliers of* (1) *have the same Taylor expansion at the origin.*

Proof. Let $V(x,y,z)$ and $\bar{V}(x,y,z)$ be two locally smooth and non–flat linearly independent inverse Jacobi multipliers of (1). From equation (2) in Remark 1 we have that $V(x,y,z) = v_{2p+1}(x,y,z) + \sum_{i\geq 2p+2} v_i(x,y,z)$ and $\bar{V}(x,y,z) = v_{2p+1}(x,y,z) + \sum_{i\geq 2p+2} \bar{v}_i(x,y,z)$ where $v_{2p+1}(x,y,z) = z(x^2+y^2)^p$. Then $\hat{V} = V - \bar{V}$ is another smooth and inverse Jacobi multiplier. When we assume that $\hat{V}$ is non–flat, we obtain a contradiction since the order at the origin of $\hat{V}$ is greater than $2p+1$. Hence $\hat{V} = V - \bar{V}$ is flat at the origin. □

Let us consider the following example in order to illustrate the above result. The system

$$\dot{x} = -y - x(x^2+y^2), \quad \dot{y} = x - y(x^2+y^2), \quad \dot{z} = z$$

with a saddle–focus at the origin has the analytic inverse Jacobi multiplier

$$V_0(x,y,z) = z(x^2+y^2)^2$$

and the C^∞ and non–flat inverse Jacobi multipliers (for all $a \in \mathbb{R}^*$)

$$V_a(x,y,z) = \left(z - a\exp\left(-\frac{1}{2(x^2+y^2)}\right)\right)(x^2+y^2)^2.$$

Their diference

$$\hat{V}(x,y,z) = V_0(x,y,z) - V_a(x,y,z) = a\exp\left(-\frac{1}{2(x^2+y^2)}\right)(x^2+y^2)^2$$

is a flat function.

Remark 3. Let V be a smooth and non–flat inverse Jacobi multiplier of (1) around its saddle–focus at the origin, such that there exists a smooth center manifold W^c satisfying $W^c \subset V^{-1}(0)$. In [3] it is proved that such a V and W^c always exists. Actually, [3] shows that in this case there exists a C^∞ function $F(x,y,z)$ such that $F(x,y,h(x,y)) \not\equiv 0$ and the following factorization occurs

$$V(x,y,z) = (z - h(x,y))\,F(x,y,z) \tag{3}$$

where $W^c = \{z = h(x,y)\}$.

In fact, given a smooth and non–flat inverse Jacobi multiplier V of (1), there is at most one smooth center manifold W^c of (1) such that $W^c \subset V^{-1}(0)$. To justify this, suppose to the contrary that there are two different center manifolds $W^c = \{z = h(x,y)\}$ and $\hat{W}^c = \{z = \hat{h}(x,y)\}$ of (1) such that $W^c \subset V^{-1}(0)$ and also $\hat{W}^c \subset V^{-1}(0)$. Then, following Remark 3, the factorization $V(x,y,z) = (z - h(x,y))\,(z - \hat{h}(x,y))\hat{F}(x,y,z)$ must hold, which is incompatible with (2).

In the next result we improve this property.

Theorem 4. *Assume that the origin is a saddle–focus for system* (1). *Let V be a locally smooth and non–flat inverse Jacobi multiplier of* (1). *Then there is exactly one smooth center manifold W^c of* (1) *such that $W^c \subset V^{-1}(0)$.*

Proof. First of all we introduce polar coordinates performing the polar blow-up $(x,y,z) \mapsto \phi(x,y,z) = (r,\theta,w)$ defined as $x = r\cos\theta$, $y = r\sin\theta$ and $z = rw$. For r in a sufficiently small neighborhood of the origin, and w in an arbitrary fixed compact set we have $\dot{\theta} > 0$, hence we can write system (1) as the following system

$$\frac{dr}{d\theta} = R(\theta,r,w), \ \frac{dw}{d\theta} = \lambda w + W(\theta,r,w), \tag{4}$$

defined on the cylinder $\{(\theta,r,w) \in \mathbb{S}^1 \times \mathbb{R}\}$ where $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, since the functions R and W are 2π–periodic in θ. Taking into account how inverse Jacobi multipliers are transformed under changes of variables and time rescalings we obtain that

$$\tilde{V}(\theta,r,w) = \frac{V(r\cos\theta, r\sin\theta, rw)}{r^2(1+\Theta(\theta,r,w))}, \tag{5}$$

is an inverse Jacobi multiplier of system (4) in a region with $r \neq 0$. From Remark 1 and equation (5) it follows that $\tilde{V}$ has, up to a multiplicative constant, the Taylor expansion

$$\tilde{V}(\theta,r,w) = w\,r^m \ + \ \mathcal{O}(r^{m+1}) \tag{6}$$

around $r = 0$ where the leading exponent $m = 2p - 1 \geq 3$ is an odd number.

Let $\mathcal{X}$ be the associated vector field to system (1). First of all we claim that any invariant manifold $\mathcal{M} = \{(\theta,r,w) : w = \Omega(\theta,r)\}$ of the pushed forward vector field $\phi_*\mathcal{X}$, given by a smooth function Ω near $r = 0$ which is 2π-periodic in the variable θ and such that $\Omega(\theta,0) = 0$ corresponds to a smooth center manifold of system (1). The claim follows recalling that the polar blow–up $(x,y,z) \mapsto (\theta,r,w)$ defined above brings any smooth center manifold $\{(x,y,z) : z = h(x,y)\}$ of (1) into a smooth invariant manifold $\{(\theta,r,w) : w = r\bar{h}(\theta,r)\}$ of $\phi_*\mathcal{X}$ where $h(r\cos\theta, r\sin\theta) = r^2\bar{h}(\theta,r)$.

Secondly we claim that, given a smooth and non–flat at $r = 0$ inverse Jacobi multiplier $\tilde{V}(\theta,r,w)$ of (4), there is a unique smooth function $\Omega(\theta,r)$ near $r = 0$ which is 2π-periodic in the variable θ such that $\Omega(\theta,0) = 0$ and moreover satisfies

$$\tilde{V}(\theta,r,\Omega(\theta,r)) \equiv 0 \tag{7}$$

for all $\theta \in \mathbb{S}^1$ and for any r near the origin. All the properties of Ω, except its 2π-periodicity, are immediate consequence of the Implicit Function Theorem applied to the function $\tilde{V}(\theta, r, w)/r^m = w + \mathcal{O}(r)$. So we only need to see the periodicity of Ω. Since identity (7) is true for all θ it follows that $\tilde{V}(\theta+2\pi, r, \Omega(\theta+2\pi, r)) \equiv 0$ or equivalently

$$\tilde{V}(\theta, r, \Omega(\theta + 2\pi, r)) \equiv 0 \tag{8}$$

due to the 2π-periodicity in the variable θ of V. Comparing (7) and (8) and taking into account the uniqueness of the function Ω predicted by the Implicit Function Theorem we reach that $\Omega(\theta + 2\pi, r) = \Omega(\theta, r)$ and hence that Ω is 2π-periodic in θ.

Of course from (7) follows the factorization $\tilde{V}(\theta, r, w) = r^m(w + \mathcal{O}(r)) = r^m(w - \Omega(\theta, r))F(\theta, r)$ implying that $\{(\theta, r, w) : w = \Omega(\theta, r)\}$ is a smooth invariant manifold of $\phi_* \mathcal{X}$ that corresponds to a smooth center manifold $\{(x, y, z) : z = h(x, y)\}$ of system (1). Therefore the smooth and non–flat inverse Jacobi multiplier $V(x, y, z)$ of (1) associated to $\tilde{V}(\theta, r, w)$ via (5) factorizes like $V(x, y, z) = (z - h(x, y))\, F(x, y, z)$.

In summary, we have shown that given a locally smooth and non–flat inverse Jacobi multiplier V of (1) there is exactly one smooth center manifold $W^c = \{z = h(x, y)\}$ of (1) such that $W^c \subset V^{-1}(0)$. □

In [3] is proved a stronger version of the following result, which is independent of the nature of the singularity. Based on Theorem 4 we give now a shorter proof of it.

Proposition 5. *Assume that the origin is a saddle–focus for system* (1)*. Let V be a local C^∞ and non–flat inverse Jacobi multiplier of* (1) *and $W^c = \{z = h(x, y)\}$ be a C^∞ local center manifold at the origin. Then the restricted function $V|_{W^c}$: $(x, y) \mapsto V(x, y, h(x, y))$ is flat at the origin.*

Proof. By Theorem 4, there is a unique smooth center manifold $W_0^c = \{z = h_0(x, y)\}$ of (1) such that $W_0^c \subset V^{-1}(0)$, that is, $V(x, y, h_0(x, y)) \equiv 0$. Therefore the factorization $V(x, y, z) = (z - h_0(x, y))F(x, y, z)$ holds with F smooth at the origin. Hence $V(x, y, h(x, y)) = (h(x, y) - h_0(x, y))F(x, y, h(x, y))$ which is clearly flat at $(x, y) = (0, 0)$ because of the property of center manifolds stating that $h(x, y) - h_0(x, y)$ is flat at the origin. □

3. The cyclicity problem

We remark that any 2π–periodic solution of (4) corresponds to a periodic orbit of (1) near $(x, y, z) = (0, 0, 0)$ and conversely.

Let $\Psi(\theta; r_0, w_0) = (r(\theta; r_0, w_0), w(\theta; r_0, w_0))$ be the solution of system (4) with initial condition $\Psi(0; r_0, w_0) = (r_0, w_0)$. We define the *Poincaré translation map* $\Pi(r_0, w_0)$ associated to (4) as $\Pi(r_0, w_0) = (r(2\pi; r_0, w_0), w(2\pi; r_0, w_0))$. We define now the *displacement map* $d(r_0, w_0) = (d_1(r_0, w_0), d_2(r_0, w_0)) = \Pi(r_0, w_0) - \mathrm{Id}(r_0, w_0)$ where Id denotes the identity map. Doing a Lyapunov–Schmidt reduction to the Poincaré map, in [4] it is proved that there exists a unique analytic function $\bar{w}(r_0)$ defined near $r_0 = 0$ such that $\bar{w}(0) = 0$ and $d_2(r_0, \bar{w}(r_0)) = 0$. Thus, consider the analytic *reduced displacement map* $\delta(r_0)$ defined as

$$\delta(r_0) := d_1(r_0, \bar{w}(r_0)) \tag{9}$$

and reduce the problem of looking for zeros of the displacement map $d(r_0, w_0)$ around $r_0 = 0$ with $r_0 > 0$ to the problem of searching for zeros of the reduced displacement map $\delta(r_0)$ around $r_0 = 0$ with $r_0 > 0$. Writing the Taylor expansion $\delta(r_0) = \sum_{i \geq k} c_i r_0^i$ with $c_k \neq 0$, we say that k is the order at the origin of δ. In [4] it is showed that $k \geq 3$ is odd and the maximum number of limit cycles that can bifurcate from a saddle–focus of system (1) under any analytic perturbation that keeps the location and monodromic nature of the singularity is $(k-1)/2$. More precisely, the cyclicity of a saddle–focus of system (1) is studied in [4] without using any center manifold reduction, thus without computing Poincaré–Lyapunov constants of $\mathcal{X}|W^c$. The following example was one of the original motivations that led us to write the work [4]. It suggests that the cyclicity can be given in terms of the order at the origin of a C^∞ and non–flat inverse Jacobi multiplier.

Example 6. Consider a special system (1) decoupled in the form

$$\dot{x} = -y + \mathcal{F}_1(x, y), \;\; \dot{y} = x + \mathcal{F}_2(x, y), \;\; \dot{z} = \lambda z + z\mathcal{F}_3(x, y), \tag{10}$$

and having a saddle–focus at the origin. Due to the nature of the singularity we recall from the results of [6] and [8] that there exists a smooth and non–flat inverse integrating factor $v(x, y)$ of the planar subsystem $\dot{x} = -y + \mathcal{F}_1(x, y)$, $\dot{y} = x + \mathcal{F}_2(x, y)$ in a neighborhood of its focus at the origin.

Taking into account that $\mathcal{F}_3$ does not depend on z, it is easy to see that $V(x, y, z) = zv(x, y)$ is an inverse Jacobi multiplier of (10). Moreover, in the variables (θ, r, w), the associated system (4) on the cylinder has the inverse Jacobi multiplier

$$\tilde{V}(\theta, r, w) = \frac{V(r\cos\theta, r\sin\theta, rw)}{r^2\,\dot{\theta}} = \frac{r\,w\,v(r\cos\theta, r\sin\theta)}{r^2\,\dot{\theta}} = w\,\hat{v}(\theta, r).$$

From the results of [8] it follows that $\hat{v}(\theta, r) = \hat{v}_m(\theta) r^m + O(r^{m+1})$ with $\hat{v}_m(\theta) \neq 0$ for all $\theta \in [0, 2\pi)$ and $m \in \mathbb{Z}^+$ is odd.

Now we claim that m coincides with the order at the origin of the associated reduced displacement map $\delta(r_0)$. To see this, first we notice that the center manifold at the origin of system (10) is $W^c = \{z = 0\}$ and the orbits of (10) on W^c spiral around the origin. Thus, since the initial conditions $(r_0, w_0) = (r_0, 0)$ corresponds with initial conditions $(x_0, y_0, z_0) = (x_0, y_0, 0)$ on W^c, it is clear that $d_2(r_0, 0) \equiv 0$. Hence $\bar{w}(r_0) \equiv 0$. This implies that the reduced displacement map $\delta(r_0) = d_1(r_0, 0)$ coincides with the displacement map associated to the focus at the origin of the planar system $\dot{x} = -y + \mathcal{F}_1(x, y)$, $\dot{y} = x + \mathcal{F}_2(x, y)$. Hence $\delta(r_0) = \delta_m r_0^m + O(r^{m+1})$ with $\delta_m \neq 0$ from the results in the plane of [8] and the claim is proved.

In what follows we present a different proof of Theorem 2 of [4]. This new proof is based on Theorem 4 proved in Section 2 and the result in the plane stated in Theorem 1 of [8].

Theorem 7. *Assume that the origin is a saddle–focus for system* (1) *and let* $k \geq 3$ *be the order at the origin of its associated reduced displacement map* $\delta(r_0) = \sum_{i \geq k} c_i r_0^i$ *with* $c_k \neq 0$. *Let* $V(x,y,z) = z(x^2+y^2)^p + \cdots$ *with* $p \geq 2$ *be any smooth and non–flat inverse Jacobi multiplier of* (1). *Then* $k = 2p-1$.

Proof. From Theorem 4, there is a unique smooth center manifold $W_0^c = \{z = h_0(x,y)\}$ of (1) such that $V(x,y,h_0(x,y)) \equiv 0$ or equivalently

$$V(x,y,z) = (z - h_0(x,y))F(x,y,z) \tag{11}$$

for some smooth function F at the origin. Notice that

$$F(x,y,z) = (x^2+y^2)^p + \cdots \tag{12}$$

due to (2) and (11) and taking into account that the order at the origin of $h_0(x,y)$ is at least 2.

Theorem 8 from [4] assures that the restricted function $v(x,y) = F|_{W_0^c} = F(x,y,h_0(x,y))$ is an inverse integrating factor of the smooth restricted vector field $\mathcal{X}|W_0^c$. Hence v is smooth, and from (12) we deduce that it is non–flat. In fact,

$$v(x,y) = (x^2+y^2)^p + \cdots \, . \tag{13}$$

Denote $\Delta(r_0)$ the displacement map associated to the focus at the origin of the planar vector field $\mathcal{X}|W_0^c$. We claim that the order of Δ is k.

At this point we want to remark that the results stated in [8] about smooth and non–flat inverse integrating factors of analytic planar vector fields around non-degenerate foci remain true if we change the analytic planar vector field with a smooth planar vector field having a non–flat displacement map at the focus.

Therefore, using (13) and the results in [8], we deduce that $k = 2p-1$ finishing the proof. It remains to justify the claim.

As a consequence of the invariance of W_0^c with respect to the flow of $\mathcal{X}$, it is clear that doing the near identity smooth change of variables

$$(x,y,z) \mapsto (x,y,Z) \quad \text{where } Z = z - h_0(x,y), \tag{14}$$

the analytic vector field $\mathcal{X}$ is pulled back into a smooth vector field $\tilde{\mathcal{X}}$ having a saddle–focus at the origin with the associated center manifold $\{Z = 0\}$. More precisely $\tilde{\mathcal{X}} = (-y + \tilde{\mathcal{F}}_1(x,y,Z))\partial_x + (x + \tilde{\mathcal{F}}_2(x,y,Z))\partial_y + (\lambda Z + \tilde{\mathcal{F}}_3(x,y,Z))\partial_Z$ where $\tilde{\mathcal{F}}_i$ are nonlinear terms and $\tilde{\mathcal{F}}_3(x,y,0) \equiv 0$. Therefore the Lyapunov–Schmidt reduction of the associated displacement map $\tilde{d}(r_0,w_0) = (\tilde{d}_1(r_0,w_0), \tilde{d}_2(r_0,w_0))$ of $\tilde{\mathcal{X}}$ is trivial and produces the reduced displacement map $\tilde{\delta}(r_0) = \tilde{d}_1(r_0,0)$ which is clearly smooth. In [4] it is proved that the change of variables (14) keeps invariant the order of the reduced displacement maps $\delta(r_0)$ and $\tilde{\delta}(r_0)$ of $\mathcal{X}$ and $\tilde{\mathcal{X}}$, respectively. Hence $\tilde{\delta}(r_0)$ has order k. But one can see that $\tilde{\delta}(r_0)$ is the displacement map of $\tilde{\mathcal{X}}|_{\{Z=0\}}$. In addition, $\mathcal{X}|W_0^c = \tilde{\mathcal{X}}|_{\{Z=0\}}$. Then $\Delta = \tilde{\delta}$ and the claim is proved. □

EXAMPLE 8. The next example shows how Theorem 7 works. The planar polynomial system $\dot{x} = -y + xf_s(x,y)$, $\dot{y} = x + yf_s(x,y)$, where f_s is an homogeneous polynomial of degree s has a center at the origin if s is odd. Taking $f_2(x,y) = ax^2+bxy+cy^2$ it follows that the origin is a focus if and only if $a+c \neq 0$. Moreover, it always has the inverse integrating factor $v(x,y) = (x^2+y^2)^2$. The system in $\mathbb{R}^3$ defined by

$$\begin{aligned} \dot{x} &= -y + x(ax^2+bxy+cy^2), \\ \dot{y} &= x + y(ax^2+bxy+cy^2), \\ \dot{z} &= \lambda z + z(c_1 x + c_2 y), \end{aligned}$$

has the inverse Jacobi multiplier $V(x,y,z) = zv(x,y) = z(x^2+y^2)^2$. Then $p=2$. Taking polar coordinates (θ, r, w) we get $d_1(r_0,w_0) = (a+c)\pi r_0^3 + \mathcal{O}(r_0^4)$, $d_2(r_0,w_0) = (\exp(2\pi\lambda)-1)w_0 - (a+c)\pi\exp(2\pi\lambda)w_0 r_0^2 + \mathcal{O}(r_0^3)$ and $\bar{w}(r_0) \equiv 0$. Finally, $\delta(r_0) = (a+c)\pi r_0^3 + \mathcal{O}(r_0^4)$. Then $k=3$. This example validates Theorem 7 since, indeed, $k = 2p-1$.

4. Extension to higher dimensions

The cyclicity of a saddle–focus of system (1) in $\mathbb{R}^3$ can be obtained from the knowledge of the order at the origin of any locally smooth and non–flat inverse Jacobi multiplier. Moreover, the vanishing set $V^{-1}(0)$ of any smooth inverse Jacobi multiplier always contains the center manifold W^c when the origin of system (1) is a center, see [3]. In the saddle–focus case we have Theorem 4 relating the sets $V^{-1}(0)$ and W^c. So the natural question that arises is whether or not these properties of inverse Jacobi multipliers of (1) are valid to higher dimensions. We have a partial answer to this question.

First we define a higher dimensional version of system (1). We consider the family of n–dimensional systems

$$\dot{\xi} = C\,\xi + \mathcal{F}(\xi), \tag{15}$$

with space state variable $\xi = (x,y,z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$ and where $\mathcal{F}$ is a local analytic function defined near the origin satisfying $\mathcal{F}(0) = 0$ and whose Jacobian matrix $D\mathcal{F}(0) = 0$. Moreover, C is a constant square matrix of order n having the block diagonal representation

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where the $(n-2)\times(n-2)$ matrix B has no eigenvalues on the imaginary axis. In particular, B is non-singular, that is $\det B \neq 0$.

Let $V(\xi)$ be a locally smooth and non–flat inverse Jacobi multiplier of system (15) having the Taylor expansion around the origin $V(\xi) = \sum_{j\geq q} V_j(\xi)$ with V_j homogeneous polynomials of degree j and $V_q(\xi) \not\equiv 0$. Thus $q \in \mathbb{N}$ is the order of V at the origin. We will see that, in general, when $n \geq 4$ the order q is not unique, contrary to what happens in the case $n = 3$.

Proposition 9. *Assume that the origin is a saddle–focus for system* (15) *with* $n \geq 4$. *Let* $V_i(\xi)$ *be two different smooth and non–flat inverse Jacobi multipliers of* (15) *having order* q_i *at the origin with* $i = 1, 2$. *Then* q_1 *and* q_2 *need not be equal.*

Proof. We present a counter-example. Consider the special system (15) in $\mathbb{R}^4$ defined as

$$\dot{x} = -y + \mathcal{F}_1(x, y), \ \dot{y} = x + \mathcal{F}_2(x, y) \ , \ \dot{z}_1 = \lambda_1 z_1, \ \dot{z}_2 = \lambda_2 z_2. \tag{16}$$

We assume that the origin is a focus for the planar subsystem $\dot{x} = -y + \mathcal{F}_1(x, y)$, $\dot{y} = x + \mathcal{F}_2(x, y)$. Then this subsystem has a smooth and non–flat inverse integrating factor $v(x, y)$ in a neighborhood of the origin. On the other hand, the linear subsystem $\dot{z}_1 = \lambda_1 z_1$, $\dot{z}_2 = \lambda_2 z_2$, has the inverse integrating factor $v_1(z) = z_1 z_2$. Furthermore, it has the polynomial first integral $H(z) = z_1^{n_1} z_2^{n_2}$ provided that the eigenvalues λ_1 and λ_2 are resonant, that is, $n_1\lambda_1 + n_2\lambda_2 = 0$ for some nonnegative integers n_1 and n_2 such that $n_1 + n_2 \geq 1$. In this way, taking $\lambda_1 = n_2$ and $\lambda_2 = -n_1$, the linear subsystem has another inverse integrating factor $v_2(z) = v_1(z)H(z) = z_1^{1+n_1} z_2^{1+n_2}$. Thus it is easy to see that $V_i(x, y, z) = v(x, y)\, v_i(z)$ for $i = 1, 2$ are two different smooth and non–flat inverse Jacobi multipliers of (16) having different orders at the origin. □

From Proposition 9 we conclude that there is no direct extension of Theorem 7 to higher dimensions.

The following system in $\mathbb{R}^n$ is such that the vanishing set of an inverse Jacobi multiplier around the origin contains the local center manifold. Consider the following decoupled system (15)

$$\dot{x} = -y + \mathcal{F}_1(x, y), \ \dot{y} = x + \mathcal{F}_2(x, y), \ \dot{z} = Bz + \mathcal{F}_3(z). \tag{17}$$

with $(x, y) \in \mathbb{R}^2$ and $z \in \mathbb{R}^{n-2}$. It is easy to see that if $v(x, y)$ is an inverse integrating factor of the planar subsystem $\dot{x} = -y + \mathcal{F}_1(x, y)$, $\dot{y} = x + \mathcal{F}_2(x, y)$ and $V_1(z)$ is an inverse Jacobi multiplier of the other subsystem $\dot{z} = Bz + \mathcal{F}_3(z)$, then $V(x, y, z) = v(x, y)\, V_1(z)$ is an inverse Jacobi multiplier of (17). Considering the simple case $\mathcal{F}_3(z) \equiv 0$, the center manifold at the origin of system (17) is $W^c = \{z = 0\}$. Moreover, assuming $B = \mathrm{diag}\{\lambda_1, \ldots, \lambda_{n-2}\}$ we have $V_1(z) = \prod_{i=1}^{n-2} z_i$. Hence $W^c \subset V^{-1}(0)$.

We show that, when the extra condition $\mathrm{Tr}(B) \neq 0$ holds, additional properties are derived for system (15).

First we generalize statement (iii) of Theorem 8 in [3] related to centers of (1).

Proposition 10. *Consider system* (15) *with* $\mathrm{Tr}(B) \neq 0$. *Assume that the origin is a center on the center manifold* W^c *of system* (15). *Let* V *be a local* C^1 *inverse Jacobi multiplier of* (15). *Then* $W^c \subset V^{-1}(0)$.

Proof. The proof follows the same lines as those of the proof of the analogous result in [3], so we only outline it. Denote $\mathcal{X}$ the vector field associated to (15).

Since the origin is a center of (15), it is known that there is a unique local center manifold W^c and this is the graph of an analytic function $z = h(x,y)$. See for example [2]. The curve $\mathcal{C} = \{(x, 0, h(x, 0)) \quad : \quad x \in (0, \epsilon]\} \subset W^c$ with $\epsilon > 0$ sufficiently small, is a transversal section to the orbits of $\mathcal{X}|W^c$. By hypothesis, these orbits are closed.

Let $\phi_t(x, y, z)$ be the solution of (15) with initial condition $\phi_0(x, y, z) = (x, y, z)$. Then, using the characteristics method, one obtains an expression for V along the orbits of $\mathcal{X}$ as

$$
\begin{aligned}
V(\phi_t(x, 0, h(x, 0))) &= V(x, 0, h(x, 0)) \times \\
&\quad \exp\left(\int_0^t \mathrm{div}\mathcal{X}(\phi_s(x, 0, h(x, 0)))\, ds\right).
\end{aligned} \tag{18}
$$

See [1] for a proof of this formula. For each $x \in (0, \epsilon]$, let $T(x) > 0$ be the minimal period of $\phi_t(x, 0, h(x, 0))$. Thus $\phi_{T(x)}(x, 0, h(x, 0)) = (x, 0, h(x, 0))$ for all $x \in (0, \epsilon]$. Since ϕ_t is a diffeomorphism we get that

$$
\int_0^{T(x)} \mathrm{div}\mathcal{X}(\phi_s(x, 0, h(x, 0)))\, ds = \mathrm{Tr}(B)\, T(x) + \mathcal{O}(x) = 2\pi \mathrm{Tr}(B) + \mathcal{O}(x), \tag{19}
$$

where we have used that $\mathrm{div}\mathcal{X} = \mathrm{Tr}(B) + \cdots$ and that $T(x) = 2\pi + \mathcal{O}(x)$. Evaluating (18) at $t = T(x)$ we have

$$
V(x, 0, h(x, 0)) = V(x, 0, h(x, 0)) \times \exp\left(2\pi \mathrm{Tr}(B) + \mathcal{O}(x)\right)
$$

which gives $V(x, 0, h(x, 0)) \equiv 0$ since $\mathrm{Tr}(B) \neq 0$. Finally, using (18) and the fact that $\mathcal{C}$ is a transversal section leads $V(x, y, h(x, y)) \equiv 0$ finishing the proof. $\square$

It is known from [1] that, in general, a limit cycle γ need not be contained in the zero set of inverse Jacobi multipliers well defined in a neighborhood of γ. We will see that the small amplitude limit cycles of (15) with $\mathrm{Tr}(B) \neq 0$ have this property.

Proposition 11. *Consider system* (15) *with* $\mathrm{Tr}(B) \neq 0$. *Let* γ *be a limit cycle around the origin of system* (15) *with sufficiently small amplitude. Let* V *be a* C^1 *inverse Jacobi multiplier of* (15) *defined in a neighborhood of the origin. Then,* $\gamma \subset V^{-1}(0)$.

Proof. Let $\mathcal{X}$ be the associated vector field of system (15). We have $\mathrm{div}\mathcal{X}(0) = \mathrm{Tr}(B) \neq 0$. Thus, from the continuity of the function $\mathrm{div}\mathcal{X}$, there is a ball $B_r(0)$ of radius r sufficiently small with center at the origin such that $\mathrm{div}\mathcal{X}(x, y, z) \neq 0$ for all $(x, y, z) \in B_r(0)$. Hence when the T–periodic limit cycle γ satisfies $\gamma \subset B_r(0)$, we get that

$$
\Delta(\gamma) := \int_0^T \mathrm{div}\mathcal{X} \circ \gamma(t)\ dt \neq 0.
$$

Since $\Delta(\gamma) \neq 0$, from the results of [1], we obtain that $\gamma \subset V^{-1}(0)$ finishing the proof. $\square$

Remark 12. After the completion of this work we found that Prof. X. Zhang has a recent preprint entitled "Inverse Jacobian multipliers and Hopf bifurcation on center manifolds" where several results of [3] and [4] have been generalized to the higher dimensional context.

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