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Multiple Hopf bifurcation in $\mathbb{R}^3$ and inverse Jacobi multipliers ☆

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Abstract

In this paper we study the maximum number of limit cycles that can bifurcate from a singular point of saddle-focus type of an analytic, autonomous differential system in $\mathbb{R}^3$ under any analytic perturbation that keeps the location and nature of the singularity. We only consider those foci on center manifolds having associated two nonzero purely imaginary and one nonzero real eigenvalues. Our approach is different from the classical one in the sense that we do not use any center manifold reduction to compute Poincaré–Lyapunov constants. Instead, we study the multiple Hopf bifurcation first doing a Lyapunov–Schmidt reduction to the associated Poincaré map, obtaining in this way an analytic reduced displacement map. Next we prove that the order of this displacement map coincides with the vanishing multiplicity (denoted m) of any locally smooth and non-flat inverse Jacobi multiplier. Finally the cyclicity of the focus is given in terms of m.

MSC: 37G15; 37G10; 34C07

Keywords: Jacobi last multipliers; Multiple Hopf bifurcation; Poincaré map; Limit cycle

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1. Introduction and statement of the results

Let us consider the real analytic autonomous differential system

$$\dot{x} = -y + \mathcal{F}_1(x, y, z), \qquad \dot{y} = x + \mathcal{F}_2(x, y, z), \qquad \dot{z} = \lambda z + \mathcal{F}_3(x, y, z), \tag{1}$$

defined in a neighborhood $U \subset \mathbb{R}^3$ of the origin and with an isolated singular point at the origin. Here, $\lambda \in \mathbb{R}\backslash\{0\}$. We will also define the analytic function $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ in U. It will be assumed that $\mathcal{F}$ only contains nonlinear terms in (1) or equivalently that $\mathcal{F}(0) = 0$ and whose Jacobian matrix $D\mathcal{F}(0) = 0$. We will also denote by $\mathcal{X}_0$ the associated vector field to system (1), that is, $\mathcal{X}_0 = (-y + \mathcal{F}_1(x, y, z))\partial_x + (x + \mathcal{F}_2(x, y, z))\partial_y + (\lambda z + \mathcal{F}_3(x, y, z))\partial_z$.

It is well known that the local dynamics of (1) around the origin on an invariant center manifold can be of two types. We say that the origin is a *center* of (1) if all the orbits on the local center manifold at the origin are periodic. Otherwise the origin is called a *saddle-focus* and the orbits spiral around the origin on any center manifold, hence one has a focus on each two-dimensional center manifold.

A C^1 function $V_0 : U \to \mathbb{R}$ is said to be an *inverse Jacobi multiplier* of $\mathcal{X}_0$ if it is not locally null and it satisfies the linear first order partial differential equation $\mathcal{X}_0 V_0 = V_0 \operatorname{div} \mathcal{X}_0$, where $\operatorname{div} \mathcal{X}_0$ is the divergence of the vector field $\mathcal{X}_0$. For a nice survey on inverse Jacobi multipliers one can see [2] and also [7].

We consider now an analytic perturbation of system (1) of the form

$$\begin{aligned}
\dot{x} &= -y + \mathcal{G}_1(x, y, z; \varepsilon), \\
\dot{y} &= x + \mathcal{G}_2(x, y, z; \varepsilon), \\
\dot{z} &= \lambda z + \mathcal{G}_3(x, y, z; \varepsilon),
\end{aligned} \tag{2}$$

where $\varepsilon \in \mathbb{R}^p$ is a finite dimensional perturbation parameter, that is, $0 < \|\varepsilon\| \ll 1$ and $\mathcal{G}_i(x, y, z; 0) \equiv \mathcal{F}_i(x, y, z)$ for all i. We will assume that the vector field $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ is analytic for both (x, y, z) and ε in a neighborhood of the origin. Additionally we will assume that the position and monodromic nature of the singularity at the origin is not affected by such perturbation imposing that $\mathcal{G}(0, 0, 0; \varepsilon) = 0$ for all ε. More precisely, the allowed perturbation is made so that $D\mathcal{G}(0, 0, 0; \varepsilon) = \operatorname{diag}\{\mu(\varepsilon), \mu(\varepsilon), \nu(\varepsilon)\}$ with analytic functions μ and ν near the origin such that $\mu(0) = \nu(0) = 0$. Equivalently the eigenvalues at the origin of (2) must be $\mu(\varepsilon) \pm i$ and $\lambda + \nu(\varepsilon)$. We associate to the perturbed system (2) the vector field $\mathcal{X}_\varepsilon$.

A limit cycle γ_ε of system (2) bifurcates from the origin if it tends to it (in the Hausdorff distance) as $\varepsilon \to 0$. We will study the existence of periodic orbits of (2) in a neighborhood of $(x, y, z, \varepsilon) = (0, 0, 0, 0)$. A *Hopf bifurcation*, also denoted by Poincaré–Andronov–Hopf bifurcation, is a bifurcation in a neighborhood of an isolated singular point like the origin of system (2). If the stability type of this point changes when ε varies near 0, then this change is usually accompanied with either the appearance or disappearance of a small amplitude periodic orbit close to the equilibrium point. A classical reference for such kind of bifurcation is the textbook [9].

Here we are interested in giving a sharp upper bound for the number of limit cycles which can bifurcate from a saddle-focus at the origin of system (1) under any analytic deformation (2) with a finite number p of parameters and $\|\varepsilon\|$ sufficiently small. In this context, the word sharp means that there exists a perturbation (2) with exactly that number of limit cycles bifurcating from the origin. In other words, this upper bound is realizable. This sharp upper bound is called the *cyclicity* of the origin of system (1) and will be denoted by $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0)$ all along this paper.

The next theorem is our main result.

Theorem 1. *Assume that the origin of (1) is a saddle-focus. Let $V_0(x, y, z)$ be a smooth and non-flat at the origin inverse Jacobi multiplier of the unperturbed analytic system (1). Then up to a multiplicative constant we have $V_0(x, y, z) = z(x^2 + y^2)^n + \cdots$ with $n \geqslant 2$ fixed and where the dots denote higher order terms. Moreover, the cyclicity of the origin in system (1) under any perturbation (2) is* $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0) = n - 1$.

It is worth to point out that Theorem 5 of [3] states that there exists a function V_0 that appears in the hypothesis of Theorem 1.

In order to prove Theorem 1 it is natural to introduce polar coordinates in the following form. We perform the polar blow-up $(x, y, z) \mapsto (\theta, r, w)$ defined by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = rw, \tag{3}$$

bringing system (2) into a system of the form

$$\begin{aligned} \dot{r} &= \mathcal{R}(\theta, r, w; \varepsilon), \\ \dot{\theta} &= 1 + \Theta(\theta, r, w; \varepsilon), \\ \dot{w} &= \lambda w + \mathcal{W}(\theta, r, w; \varepsilon). \end{aligned} \tag{4}$$

We observe that $\Theta(\theta, 0, w; 0) = 0$ so that $\dot{\theta} > 0$ for (r, ε) in a sufficiently small neighborhood of the origin and w in an arbitrary fixed compact set. Therefore, under these conditions we can write system (4) as the following system

$$\frac{dr}{d\theta} = R(\theta, r, w; \varepsilon), \qquad \frac{dw}{d\theta} = \lambda w + W(\theta, r, w; \varepsilon), \tag{5}$$

defined on the cylinder $C = \{(\theta, r, w) \in \mathbb{S}^1 \times \mathbb{R}^2 \text{ with } |r| \text{ sufficiently small}\}$ where $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. We remark that any 2π-periodic solution of (5) corresponds to a periodic orbit of (2) near $(x, y, z) = (0, 0, 0)$ through the transformation (3) and conversely.

Let $\Psi(\theta; r_0, w_0; \varepsilon) = (r(\theta; r_0, w_0; \varepsilon), w(\theta; r_0, w_0; \varepsilon))$ denote the solution of (5) with initial condition $\Psi(0; r_0, w_0; \varepsilon) = (r_0, w_0)$. We define the *Poincaré translation map* $\Pi(r_0, w_0; \varepsilon)$ associated to (5) as $\Pi(r_0, w_0; \varepsilon) = \Psi(2\pi; r_0, w_0; \varepsilon)$. We denote the components of the Poincaré map $\Pi = (\Pi_1, \Pi_2)$.

Define now the *displacement map* $d(r_0, w_0; \varepsilon) = \Pi(r_0, w_0; \varepsilon) - \mathrm{Id}(r_0, w_0)$ where Id denotes the identity map and denote its components by $d = (d_1, d_2)$. Now applying a Lyapunov–Schmidt reduction to the displacement map, see Lemma 3, we reduce the problem of looking for zeroes of the displacement map $d(r_0, w_0; \varepsilon)$ around $(r_0, \varepsilon) = (0, 0)$ and with $r_0 > 0$ to the problem of searching zeroes of the analytic *reduced displacement map* $\Delta(r_0; \varepsilon)$ around $(r_0, \varepsilon) = (0, 0)$ with $r_0 > 0$. As far as we know, this approach is new. We define the reduced displacement map of the unperturbed system (1) as $\delta(r_0) = \Delta(r_0; 0)$ and we call k the order at the origin of δ, that is, we can write the Taylor expansion at $r_0 = 0$ of the form $\delta(r_0) = \sum_{i \geqslant k} c_i r_0^i$ with $c_k \neq 0$. Notice that $\delta \not\equiv 0$ because the origin is a saddle-focus rather than a center in (1).

We associate to system (5)$_{\varepsilon=0}$ the vector field $\hat{\mathcal{X}}_0 = \partial_\theta + R(\theta, r, w; 0)\partial_r + (\lambda w + W(\theta, r, w; 0))\partial_w$ defined on the cylinder C. We define an *inverse Jacobi multiplier* $V(\theta, r, w)$ of system (5)$_{\varepsilon=0}$ as a function $V : C \to \mathbb{R}$ of class $\mathcal{C}^1(C)$, which is non-locally null and which satisfies the linear partial differential equation $\hat{\mathcal{X}}_0 V = V \operatorname{div} \hat{\mathcal{X}}_0$. We remark that since $V(\theta, r, w)$ is a continuous function defined over the cylinder C it needs to be 2π-periodic in θ. Thus we have $V(\theta + 2\pi, r, w) = V(\theta, r, w)$.

Throughout the rest of this paper, we will only consider a smooth and non-flat at $r = 0$ inverse Jacobi multiplier $V(\theta, r, w)$ of system (5)$_{\varepsilon=0}$ which comes from a given smooth and non-flat at the origin inverse Jacobi multiplier $V_0(x, y, z)$ of system (1). In Lemma 8 we prove that they are related by the following formula

$$V(\theta, r, w) = \frac{V_0(r\cos\theta, r\sin\theta, rw)}{r^2(1 + \Theta(\theta, r, w; 0))}, \tag{6}$$

where Θ is the function defined in (4). In addition Lemma 8 also proves that the function $V(\theta, r, w)$ defined via (6) has the Taylor series representation

$$V(\theta, r, w) = wr^m + \mathcal{O}\big(r^{m+1}\big), \tag{7}$$

up to a multiplicative constant, and the leading exponent $m \geqslant 1$ is an odd number which we call the *vanishing multiplicity* of $V(\theta, r, w)$ on $r = 0$. Here and throughout this paper $\mathcal{O}(r^{m+1})$ denotes a smooth function of variables (θ, r, w) whose Taylor series around $r = 0$ starts with a term of order $m + 1$ in r.

The next result reveals an important phenomenon that stays behind the understanding of Theorem 1.

Theorem 2. *Assume that the origin of (1) is a saddle-focus. Let $V_0(x, y, z)$ be a smooth and non-flat at the origin inverse Jacobi multiplier of the analytic system (1) and $V(\theta, r, w)$ defined in (6) be the corresponding inverse Jacobi multiplier of system (5)$_{\varepsilon=0}$ having vanishing multiplicity m at $r = 0$. Let k be the order at the origin of the reduced displacement map $\delta(r_0)$ of the unperturbed system (1). Then $m = k \geqslant 3$ and they are odd numbers.*

In order to prove our results, we shall need a fundamental relation between Poincaré maps and inverse Jacobi multipliers of system (5)$_{\varepsilon=0}$. We remark that this relation was proved in a particular case in [5] but we need a generalized version to higher dimensions as it is proved in [4]. In our notation, this fundamental relation is given by

$$V\big(0, \Pi(r_0, w_0)\big) = V(0, r_0, w_0) \det\big(D\Pi(r_0, w_0)\big), \tag{8}$$

where $V(\theta, r, w)$ is an inverse Jacobi multiplier of system (5)$_{\varepsilon=0}$ and $\Pi(r_0, w_0)$ its Poincaré map.

The paper is organized as follows. In Section 2 we perform the Lyapunov–Schmidt reduction of the displacement map $d(r_0, w_0; \varepsilon)$ for a saddle-focus and also prove some of its properties that we will need later. Section 3 is devoted to explain several properties of inverse Jacobi multipliers. In Section 4 we obtain some invariants associated to a change of variables that smoothly flattens a center manifold. Now we are ready to give the proof of the main results in Section 5. Finally, we add Appendix A where we present an example of perturbed system (2) having the maximum number of bifurcating limit cycles and also present a method for computing Poincaré–Lyapunov constants.

2. Lyapunov–Schmidt reduction of the displacement map for a saddle-focus and some of its properties

Let us consider system (5) defined in the cylinder C and in the hypotheses stated in the Introduction. Notice that the functions R and W that define system (5) are analytic, 2π-periodic in θ and satisfy

$$R(\theta, 0, w; \varepsilon) = \frac{\partial R}{\partial r}(\theta, 0, w; 0) = 0, \qquad W(\theta, 0, w; 0) = 0.$$

Moreover, the solution of (5) satisfies

$$r(\theta; 0, w_0; \varepsilon) = 0, \qquad w(\theta; 0, 0; 0) = 0.$$

For further use, we write also here the following expressions of the components of the displacement map $d = (d_1, d_2)$ of system (5),

$$\begin{aligned} d_1(r_0, w_0; \varepsilon) &= r(2\pi; r_0, w_0; \varepsilon) - r_0 \\ &= \int_0^{2\pi} R\big(\theta, r(\theta; r_0, w_0; \varepsilon), w(\theta; r_0, w_0; \varepsilon); \varepsilon\big)\, d\theta, \\ d_2(r_0, w_0; \varepsilon) &= w(2\pi; r_0, w_0; \varepsilon) - w_0 \\ &= \big[\exp(2\pi\lambda) - 1\big] w_0 \\ &\quad + \int_0^{2\pi} \exp\big[(2\pi - \theta)\lambda\big] W\big(\theta, r(\theta; r_0, w_0; \varepsilon), w(\theta; r_0, w_0; \varepsilon); \varepsilon\big)\, d\theta. \end{aligned}$$

These were obtained by the integral equations equivalent with system (5).

Combining in an elementary way all the above relations, one can obtain some of the following results of this section. For these kind of proofs we will omit the details.

Now we will apply a Lyapunov–Schmidt reduction to the displacement map.

Lemma 3. *Let $d(r_0, w_0; \varepsilon)$ be the displacement map of system (5). Then, there exists a unique analytic function $\bar{w}(r_0, \varepsilon)$ defined near $(r_0, \varepsilon) = (0, 0)$ such that $\bar{w}(0, 0) = 0$ and $d_2(r_0, \bar{w}(r_0, \varepsilon); \varepsilon) \equiv 0$.*

Proof. Using the relations written in the beginning of this section one can find that

$$d_2(0, 0, 0) = 0 \quad \text{and} \quad \frac{\partial d_2}{\partial w_0}(0, 0; 0) = \exp(2\pi\lambda) - 1 \neq 0,$$

since $\lambda \neq 0$. Hence, we can apply the Implicit Function Theorem to conclude the existence of a unique analytic function $\bar{w}(r_0, \varepsilon)$ defined near $(r_0, \varepsilon) = (0, 0)$ such that $\bar{w}(0, 0) = 0$ and $d_2(r_0, \bar{w}(r_0, \varepsilon); \varepsilon) \equiv 0$. □

Lemma 4. *The displacement map $d = (d_1, d_2)$ and the Poincaré map $\Pi = (\Pi_1, \Pi_2)$ satisfy the following relations.*

$$\frac{\partial d_1}{\partial r_0}(0, w_0; 0) = 0, \qquad d_2(0, w_0; 0) = \left[\exp(2\pi\lambda) - 1\right] w_0,$$

$$\Pi_1(0, w_0; 0) = 0, \qquad \frac{\partial \Pi_1}{\partial r_0}(0, w_0; 0) = 1, \qquad \Pi_2(0, w_0; 0) = \exp(2\pi\lambda) w_0, \tag{9}$$

$$\frac{\partial \Pi_1}{\partial w_0}(0, w_0; 0) = 0, \qquad \frac{\partial \Pi_2}{\partial w_0}(0, w_0; 0) = \exp(2\pi\lambda). \tag{10}$$

Lemma 5. *Assume that a center manifold of (1) at the origin is given by $\mathcal{W}^c(0) = \{z = 0\}$. Then, the following holds*:

$$\Pi_2(r_0, 0; 0) = 0, \qquad \frac{\partial \Pi_2}{\partial r_0}(r_0, 0; 0) = 0. \tag{11}$$

Proof. Since $\mathcal{W}^c(0) = \{z = 0\}$, in coordinates (θ, r, w) the center manifold becomes $\mathcal{W}^c(0) = \{w = 0\}$. The flow-invariance of $\mathcal{W}^c(0)$ gives $w(\theta; r_0, 0; 0) = 0$ and therefore $d_2(r_0, 0; 0) = 0$ or, in other words, $\bar{w}(r_0, 0) = 0$. Hence, since $\Pi_2(r_0, w_0; 0) = w_0 + d_2(r_0, w_0; 0)$, we get (11). □

Lemma 6. *The determinant* $\det(D\Pi(r_0, w_0; 0))$ *of the linear part of the Poincaré map of system* $(5)_{\varepsilon=0}$ *satisfies*

$$\det\bigl(D\Pi(0, w_0; 0)\bigr) = \exp(2\pi\lambda),$$
$$\det\bigl(D\Pi(r_0, 0; 0)\bigr) = \exp(2\pi\lambda) + \mathcal{O}(r_0). \tag{12}$$

If in addition $\mathcal{W}^c(0) = \{z = 0\}$ is a center manifold of (1) then

$$\det\bigl(D\Pi(r_0, 0; 0)\bigr) = \left[1 + k\hat{\delta}(0) r_0^{k-1} + \mathcal{O}\bigl(r_0^k\bigr)\right] \frac{\partial \Pi_2}{\partial w_0}(r_0, 0; 0), \tag{13}$$

where the reduced displacement map $\delta(r_0) = r_0^k \hat{\delta}(r_0)$ with $\hat{\delta}(0) \neq 0$.

Proof. By definition, we have

$$\det\bigl(D\Pi(r_0, w_0; 0)\bigr) = \begin{vmatrix} \frac{\partial \Pi_1}{\partial r_0}(r_0, w_0; 0) & \frac{\partial \Pi_1}{\partial w_0}(r_0, w_0; 0) \\ \frac{\partial \Pi_2}{\partial r_0}(r_0, w_0; 0) & \frac{\partial \Pi_2}{\partial w_0}(r_0, w_0; 0) \end{vmatrix}.$$

Taking into account the second equation in (9) and (10) we have

$$\det\bigl(D\Pi(0, w_0; 0)\bigr) = \begin{vmatrix} 1 & 0 \\ \frac{\partial \Pi_2}{\partial r_0}(r_0, w_0; 0) & \exp(2\pi\lambda) \end{vmatrix} = \exp(2\pi\lambda) \neq 0.$$

Finally, taking into account the second equation in (10) we have $\frac{\partial \Pi_2}{\partial w_0}(0,0;0) = \exp(2\pi\lambda)$ and therefore Eq. (12) $\frac{\partial \Pi_2}{\partial w_0}(r_0,0;0) = \exp(2\pi\lambda) + \mathcal{O}(r_0)$ follows.

Assume now that $\mathcal{W}^c(0) = \{z=0\}$ is a center manifold of (1). Then $\bar{w}(r_0,0) = 0$ and therefore the reduced displacement map has the following expression $\delta(r_0) = d_1(r_0,0;0)$. This implies that

$$\Pi_1(r_0,0;0) = r_0 + \delta(r_0) = r_0 + r_0^k \hat{\delta}(r_0),$$

where $\hat{\delta}(0) \neq 0$. Using this expression of $\Pi_1(r_0,0;0)$ together with (11) we have

$$\begin{aligned}
\det\bigl(D\Pi(r_0,0;0)\bigr) &= \begin{vmatrix} \frac{\partial \Pi_1}{\partial r_0}(r_0,0;0) & \frac{\partial \Pi_1}{\partial w_0}(r_0,0;0) \\ \frac{\partial \Pi_2}{\partial r_0}(r_0,0;0) & \frac{\partial \Pi_2}{\partial w_0}(r_0,0;0) \end{vmatrix} \\
&= \begin{vmatrix} 1 + k r_0^{k-1}\hat{\delta}(r_0) + r^k \hat{\delta}'(r_0) & \frac{\partial \Pi_1}{\partial w_0}(r_0,0;0) \\ 0 & \frac{\partial \Pi_2}{\partial w_0}(r_0,0;0) \end{vmatrix} \\
&= \bigl[1 + k\hat{\delta}(0) r_0^{k-1} + \mathcal{O}\bigl(r_0^k\bigr)\bigr] \frac{\partial \Pi_2}{\partial w_0}(r_0,0;0),
\end{aligned}$$

proving (13). Thus, the lemma follows. □

Remark 7. We want to emphasize that all the results in this section remain valid (only replacing the word "analytic" by "smooth and non-flat" in Lemma 3) in case that system (1) is not analytic and it is only smooth and non-flat. This fact will be used later in the proof of Theorem 2.

3. Several properties of inverse Jacobi multipliers

Lemma 8. *Let $V_0(x,y,z)$ be an inverse Jacobi multiplier of system (1) defined in a neighborhood of the origin. Then, an inverse Jacobi multiplier $V(\theta,r,w)$ of system $(5)_{\varepsilon=0}$ in $r \neq 0$ is given by*

$$V(\theta,r,w) = \frac{V_0(r\cos\theta, r\sin\theta, rw)}{r^2(1 + \Theta(\theta,r,w;0))}.$$

In addition if $V_0(x,y,z)$ is smooth and non-flat at the origin and this is a saddle-focus of (1) then there exists an odd integer $m \geqslant 1$ such that

$$V(\theta,r,w) = wr^m + \mathcal{O}\bigl(r^{m+1}\bigr), \tag{14}$$

up to a multiplicative constant.

Proof. First of all, we calculate the Jacobian determinant of the polar blow-up $(x,y,z) \mapsto (\theta,r,w)$ defined in (3). We obtain that

$$\frac{\partial(x,y,z)}{\partial(\theta,r,w)} = \begin{vmatrix} -r\sin\theta & \cos\theta & 0 \\ r\cos\theta & \sin\theta & 0 \\ 0 & w & r \end{vmatrix} = -r^2.$$

Then, we have that the function $V_0(r\cos\theta, r\sin\theta, rw)/r^2$ is an inverse Jacobi multiplier for system $(4)_{\varepsilon=0}$. Finally, since the relation between systems $(4)_{\varepsilon=0}$ and $(5)_{\varepsilon=0}$ is just the time rescaling $t \to \theta$ with $\dot{\theta} = 1 + \Theta(\theta, r, w; 0)$, the first part of the lemma follows.

Recalling Proposition 7 of [3] we know that when $V_0(x, y, z)$ is smooth and non-flat at the origin then $V_0(x, y, z) = z(x^2 + y^2)^n + \cdots$ with $n \geqslant 0$. Actually, we claim that in fact $n \geqslant 1$, hence we get $m = 2n - 1 \geqslant 1$ finishing the proof. In order to prove the claim we assume by contradiction that $n = 0$, hence $V_0(x, y, z) = z + \cdots$. In Theorem 4 of [3] it is proved that the existence of an analytic inverse Jacobi multiplier of the form $z + \cdots$ of system (1) implies that the origin must be a center. Repeating verbatim the arguments in the first paragraph of the proof of Theorem 4 of [3] only replacing the word "analytic" by "smooth" one can check that the origin must be a center also in our case. This contradicts the hypothesis that the origin is a saddle-focus. □

In order to motivate why we need to introduce some flat terms in the expressions of the Jacobi multipliers, we present an example which already appeared in [3]. The system

$$\dot{x} = -y - x\left(x^2 + y^2\right), \qquad \dot{y} = x - y\left(x^2 + y^2\right), \qquad \dot{z} = -z \tag{15}$$

has a saddle-focus at the origin and the following 1-parameter family of C^∞ and non-flat at the origin inverse Jacobi multipliers

$$V_0(x, y, z) = \left(z - a\exp\left(-\frac{1}{2(x^2 + y^2)}\right)\right)\left(x^2 + y^2\right)^2 = z\left(x^2 + y^2\right)^2 + aF_0(x, y)$$

for all $a \in \mathbb{R}$. The corresponding inverse Jacobi multipliers in polar coordinates (θ, r, w) become

$$V(\theta, r, w) = r^2\left(rw - a\exp\left(-\frac{1}{2r^2}\right)\right) = wr^3 + aF(r).$$

Notice that the function F_0 is flat at $(x, y, z) = (0, 0, 0)$ while F is flat at $r = 0$. Hence the Taylor expansion of V_0 at $(x, y, z) = (0, 0, 0)$ is the same for all the real parameters a. The same occurs for the Taylor expansion of V at $r = 0$. Of course, we wonder whether this is always the case.

Lemma 9. *Assume the origin is a saddle-focus of (1). Let $V(\theta, r, w)$ be an inverse Jacobi multiplier of system $(5)_{\varepsilon=0}$ coming from a smooth and non-flat at the origin inverse Jacobi multiplier $V_0(x, y, z)$ of system (1). Then $V(\theta, r, w)$ is smooth and non-flat at $r = 0$ and has a Taylor expansion (14). Moreover either*

(i) $V(0, r, w) = wr^m V_m(r, w) + F(r, w)$ *or*
(ii) $V(0, r, w) = wr^m V_m(r, w) + r^{m+m'} V_{m'}(r) + F(r, w)$,

where $m' \geqslant 1$ is an integer number, $V_m(0, 0) \neq 0$, $V_{m'}(0) \neq 0$ and $F(r, w)$ is a flat function at $r = 0$. In addition, when a center manifold of (1) is $\mathcal{W}^c(0) = \{z = 0\}$ then option (i) *holds.*

Proof. The fact that $V(\theta, r, w)$ is smooth and non-flat at $r = 0$ and has a Taylor expansion (14) follows easily from Lemma 8. Since $V(0, r, w) = wr^m + \mathcal{O}(r^{m+1})$, the two decompositions of

$V(0, r, w)$ given in the lemma are obvious. We observe that it is necessary to include a flat function F at $r = 0$ as example (15) shows. The unique difference between both decompositions consists in whether after removing the flat terms F the remaining expression of $V(0, r, w)$ vanishes or not at $w = 0$.

To prove the last statement of the lemma, we recall that in [3] it is proved that the restricted function $V_0|_{\mathcal{W}^c} : (x, y) \mapsto f(x, y) = V_0(x, y, h(x, y))$ is a flat function at $(x, y) = (0, 0)$ for any smooth center manifold $\mathcal{W}^c(0) = \{z = h(x, y)\}$ of (1). Now we claim that the function $g(\theta, r) = f(r\cos\theta, r\sin\theta)$ is flat at $r = 0$. Indeed, one can easily obtain that $\frac{\partial^i g}{\partial r^i}(\theta, 0) = 0$ for all i provided that all the partial derivatives of $f(x, y)$ at the origin vanish.

From Lemma 8 we know that V and V_0 are related via

$$V(0, r, w) = \frac{V_0(r, 0, rw)}{r^2(1 + \Theta(0, r, w; 0))}.$$

Assume now that a center manifold of (1) is $\mathcal{W}^c(0) = \{z = 0\}$. Therefore $V_0(x, y, 0)$ is flat at the origin and hence $V_0(r\cos\theta, r\sin\theta, 0)$ is flat at $r = 0$ for any θ. Taking $\theta = 0$ we get in particular that $V_0(r, 0, 0)$ is flat at $r = 0$ and therefore

$$V(0, r, 0) = \frac{V_0(r, 0, 0)}{r^2(1 + \Theta(0, r, 0; 0))}$$

is flat at $r = 0$. This last property of $V(0, r, 0)$ is only satisfied by the expression (i) in the lemma. □

4. Smoothly flattening a center manifold

We remind that a local center manifold $\mathcal{W}^c(0)$ at the origin of system (1) is an invariant surface which is tangent to the (x, y) plane at the origin. More precise, $\mathcal{W}^c(0) = \{z = h(x, y)$: for (x, y) in a small neighborhood of $(0, 0)\}$ with $h(0, 0) = 0$ and $Dh(0, 0) = 0$. It is known that for any $k \geqslant 1$ there exists a C^k local center manifold and, moreover, in [3] it is proved that there exists a C^∞ local center manifold around a saddle-focus of (1). In this paper we will work only with smooth local center manifolds. The polar blow-up $(x, y, z) \mapsto (\theta, r, w)$ defined by (3) brings any smooth center manifold $\mathcal{W}^c(0) = \{(x, y, z)\colon z = h(x, y)\}$ of (1) into a smooth invariant manifold $\{(\theta, r, w)\colon w = r\bar{h}(\theta, r)\}$ of $(5)_{\varepsilon=0}$ where $h(r\cos\theta, r\sin\theta) = r^2\bar{h}(\theta, r)$.

Lemma 10. *Let $\mathcal{W}^c(0) = \{z = h(x, y)\}$ be a smooth local center manifold at the origin of system (1), and $\bar{h}$ be such that $h(r\cos\theta, r\sin\theta) = r^2\bar{h}(\theta, r)$. Then either the function $\bar{h}(0, r)$ is flat at $r = 0$, or its order at $r = 0$ is $m' - 1$. The integer $m' \geqslant 1$ is defined in Lemma 9.*

Proof. Let $V(\theta, r, w)$ and $V_0(x, y, z)$ be like in Lemma 9. From [3], $f(x, y) = V_0(x, y, h(x, y))$ is flat at $(x, y) = (0, 0)$. Hence the function $g(\theta, r) = f(r\cos\theta, r\sin\theta)$ is flat at $r = 0$. In particular we have that $g(0, r) = V_0(r, 0, r^2\bar{h}(0, r))$ is flat at $r = 0$.

On the other hand using Lemma 8, and further evaluating at $w = r\bar{h}(\theta, r)$ and $\theta = 0$ we must have that

$$V\big(0, r, r\bar{h}(0, r)\big) = \frac{V_0(r, 0, r^2\bar{h}(0, r))}{r^2(1 + \Theta(0, r, r\bar{h}(0, r); 0))}$$

which is flat at $r = 0$. If we evaluate (i) or (ii) of Lemma 9 at $w = r\bar{h}(\theta, r)$, since $V(0, r, r\bar{h}(0, r))$ is flat at $r = 0$, we obtain the conclusion. □

Let $\mathcal{W}^c(0) = \{z = h(x, y)\}$ be a smooth local center manifold at the origin of system (1). As a consequence of the flow-invariance of $W^c(0)$, it is clear that doing the near identity smooth change of variables

$$(x, y, z) \mapsto (x, y, Z) \quad \text{where } Z = z - h(x, y), \tag{16}$$

system (1) becomes

$$\begin{aligned} \dot{x} &= -y + \tilde{\mathcal{F}}_1(x, y, Z), \\ \dot{y} &= x + \tilde{\mathcal{F}}_2(x, y, Z), \\ \dot{Z} &= \lambda Z + \tilde{\mathcal{F}}_3(x, y, Z), \end{aligned} \tag{17}$$

where $\tilde{\mathcal{F}}_i$ are nonlinear terms and $\tilde{\mathcal{F}}_3(x, y, 0) \equiv 0$, that is, a center manifold of (17) is $W^c(0) = \{Z = 0\}$. Notice however that system (17) will be only C^∞ although the original system (1) is analytic. The rest of this section is devoted to show several invariants associated to the change of variables (16).

Lemma 11. *Let $V_0(x, y, z)$ be a smooth and non-flat at the origin inverse Jacobi multiplier of (1) having order $2n + 1$ at the origin. Then the corresponding inverse Jacobi multiplier $\tilde{V}_0(x, y, Z)$ of (17) has the same order at the origin.*

Proof. Clearly the change of variables (16) is symplectic because its Jacobian determinant is

$$\frac{\partial(x, y, Z)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\partial h}{\partial x} & -\frac{\partial h}{\partial y} & 1 \end{vmatrix} = 1.$$

Therefore we have $\tilde{V}_0(x, y, Z) = V_0(x, y, Z + h(x, y))$. Finally, using both that $V_0(x, y, z) = z(x^2 + y^2)^n + \cdots$ and that $h(0, 0) = 0$ and $Dh(0, 0) = 0$ leads to $\tilde{V}_0(x, y, Z) = Z(x^2 + y^2)^n + \cdots$ and the proof is completed. □

To obtain a new invariant associated to the change of variables (16), we shall use an argument that relates the conjugation between T-periodic systems and its associated Poincaré maps, see for instance Lemma 8 of [8].

Lemma 12. *(See [8].) Two real C^k with $1 \leqslant k \leqslant \infty$ (resp. analytic), T-periodic systems are C^k (resp. analytically) equivalent if and only if their Poincaré maps are C^k (resp. analytically) conjugate.*

Lemma 13. *The smooth change of variables (16) that brings system (1) into system (17) keeps invariant the order at the origin k of the corresponding reduced displacements maps $\delta(r_0)$ and $\tilde{\delta}(r_0)$ of systems (1) and (17), respectively.*

Proof. Let $\Psi(\theta; r_0, w_0)$ be the solution of the 2π-periodic system $(5)_{\varepsilon=0}$ with initial condition $\Psi(0; r_0, w_0) = (r_0, w_0)$ and $\tilde{\Psi}(\theta; r_0, w_0)$ be the solution with initial condition $\tilde{\Psi}(0; r_0, w_0) = (r_0, w_0)$ associated to the 2π-periodic system analogous to system $(5)_{\varepsilon=0}$ but corresponding to system (17) via the polar blow-up $(x, y, Z) \mapsto (\theta, r, w)$ defined by $x = r\cos\theta$, $y = r\sin\theta$ and $Z = rw$. We have associated to each 2π-periodic system their Poincaré maps $\Pi(r_0, w_0) = \Psi(2\pi; r_0, w_0)$ and $\tilde{\Pi}(r_0, w_0) = \tilde{\Psi}(2\pi; r_0, w_0)$. From the proof of the necessity part of Lemma 12, it follows that if there is a θ dependent C^k diffeomorphism $H_\theta(r, w)$ which carries the solutions $\Psi(\theta; r_0, w_0)$ into the solutions $\tilde{\Psi}(\theta; H_\theta(r_0, w_0))$, that is, $\tilde{\Psi}(\theta; H_0(r_0, w_0)) = H_\theta(\Psi(\theta; r_0, w_0))$ and $H_\theta(r, w)$ is 2π-periodic in θ, that is, $H_\theta(r, w) = H_{\theta+2\pi}(r, w)$, then $H_0(r, w)$ is a C^k diffeomorphism having the property that conjugates their Poincaré maps

$$\tilde{\Pi} \circ H_0(r_0, w_0) = H_0 \circ \Pi(r_0, w_0). \tag{18}$$

In our particular case, since $w = Z/r$, we have

$$H_\theta(r, w) = \left(r, w - \frac{h(r\cos\theta, r\sin\theta)}{r}\right) = \big(r, w - r\bar{h}(r\cos\theta, r\sin\theta)\big),$$

where we have defined $\bar{h}$ as before, that is such that $h(r\cos\theta, r\sin\theta) = r^2\bar{h}(\theta, r)$. Notice that $H_\theta(r, w)$ is a 2π-periodic in θ C^k diffeomorphism because $\bar{h} \in C^k$ and the inverse of H_θ is $H_\theta^{-1}(r, w) = (r, w + r\bar{h}(\theta, r))$ which is also C^k. Thus we have

$$\begin{aligned}\tilde{\Pi} \circ H_0(r_0, w_0) &= \tilde{\Pi}\big(r_0, w_0 - r_0\bar{h}(0, r_0)\big)\\ &= \big(r_0 + \tilde{d}_1\big(r_0, w_0 - r_0\bar{h}(0, r_0)\big), w_0 - r_0\bar{h}(0, r_0) + \tilde{d}_2\big(r_0, w_0 - r_0\bar{h}(0, r_0)\big)\big)\end{aligned}$$

and

$$\begin{aligned}H_0 \circ \Pi(r_0, w_0) &= H_0\big(r_0 + d_1(r_0, w_0), w_0 + d_2(r_0, w_0)\big)\\ &= \big(r_0 + d_1(r_0, w_0), w_0 + d_2(r_0, w_0) - \big(r_0 + d_1(r_0, w_0)\big)\bar{h}\big(0, r_0 + d_1(r_0, w_0)\big)\big).\end{aligned}$$

Introducing these expressions in the conjugation relation (18) and equating its components we have $\tilde{d}_1(r_0, w_0 - r_0\bar{h}(0, r_0)) = d_1(r_0, w_0)$ and $\tilde{d}_2(r_0, w_0 - r_0\bar{h}(0, r_0)) = d_2(r_0, w_0) - (r_0 + d_1(r_0, w_0))\bar{h}(0, r_0 + d_1(r_0, w_0)) + r_0\bar{h}(0, r_0)$. Evaluating these equations at $w_0 = \bar{w}(r_0)$ and recalling that by definition $d_2(r_0, \bar{w}(r_0)) = 0$ and $\delta(r_0) = d_1(r_0, \bar{w}(r_0))$ gives

$$\tilde{d}_1\big(r_0, \bar{w}(r_0) - r_0\bar{h}(0, r_0)\big) = \delta(r_0), \tag{19}$$

$$\tilde{d}_2\big(r_0, \bar{w}(r_0) - r_0\bar{h}(0, r_0)\big) = -\big(r_0 + \delta(r_0)\big)\bar{h}\big(0, r_0 + \delta(r_0)\big) + r_0\bar{h}(0, r_0). \tag{20}$$

We want to show that the order at the origin of the reduced maps $\delta(r_0)$ and $\tilde{\delta}(r_0) = \tilde{d}_1(r_0, 0)$ coincides. The center case is trivial because $\delta(r_0) = \tilde{\delta}(r_0) \equiv 0$ which implies that $\bar{w}(r_0) = r_0\bar{h}(0, r_0)$ from (19). The saddle-focus case is a bit more involved as shown below.

First we will see that the order of the right-hand side of (20) is $k + m' - 1$. To do that we denote $f(r_0) := -r_0\bar{h}(0, r_0)$, where we know from Lemma 10 that f has order m' at the origin, and the right-hand side of (20) can be written as

$$f\big(r_0+\delta(r_0)\big)-f(r_0)=\delta(r_0)f'(r_0)+O\big(\delta^2(r_0)\big),$$

hence its order at the origin is indeed $k+m'-1$.

Now we shall prove that the order at the origin of the left-hand side of (20) must be equal to the order at the origin of the function $g(r_0):=\bar{w}(r_0)-r_0\bar{h}(0,r_0)$. This is because we have the expansion of left-hand side of (20)

$$\tilde{d}_2\big(r_0,g(r_0)\big)=\tilde{d}_2(r_0,0)+g(r_0)\frac{\partial\tilde{d}_2}{\partial w}(r_0,0)+O\big(g^2(r_0)\big).$$

Since $\tilde{d}_2(r_0,0)=0$ and $\frac{\partial\tilde{d}_2}{\partial w}(0,0)=e^{2\pi\lambda}-1\neq 0$ from the second equation in (10) it is clear that the claim holds.

From the above discussions we deduce that the order at the origin of $g(r_0)$ is $k+m'-1$. Having this in mind we now want to establish the order at the origin of $\tilde{d}_1(r_0,g(r_0))-\tilde{d}_1(r_0,0)$. We have again an expansion of the form

$$\tilde{d}_1\big(r_0,g(r_0)\big)-\tilde{d}_1(r_0,0)=g(r_0)\frac{\partial\tilde{d}_1}{\partial w}(r_0,0)+O\big(g^2(r_0)\big).$$

Since $\frac{\partial\tilde{d}_1}{\partial w}(0,0)=0$ due to the first equation in (10), the order at the origin of the right-hand side of the above equality is at least $k+m'$. Now we get

$$\delta(r_0)-\tilde{\delta}(r_0)=\delta(r_0)-\tilde{d}_1(r_0,0)=\tilde{d}_1\big(r_0,g(r_0)\big)-\tilde{d}_1(r_0,0),$$

where the last equality follows from (19). Hence we deduce that the order at the origin of $\delta(r_0)-\tilde{\delta}(r_0)$ is at least $k+m'$. Since $m'\geqslant 1$ by definition we conclude that the order at the origin of $\tilde{\delta}(r_0)$ is also k, as the order of $\delta(r_0)$. $\square$

5. Proofs of the main results

Proof of Theorem 2. If $\{z=0\}$ is not a center manifold at the origin of system (1), we perform the smooth and non-flat change of variables $(x,y,z)\mapsto(x,y,Z)$ defined in (16) such that system (1) becomes (17) and therefore having a center manifold $\mathcal{W}^c(0)=\{Z=0\}$.

Let us denote by $V(\theta,r,w)$ the smooth and non-flat at $r=0$ inverse Jacobi multiplier of the corresponding smooth system $(5)_{\varepsilon=0}$ associated to (17). Taking into account Lemma 11 we know that $V(\theta,r,w)$ also has vanishing multiplicity m at $r=0$. All these facts together imply, from Lemma 9, that $V(0,r,w)=wr^mV_m(r,w)+F(r,w)$ with $V_m(0,0)\neq 0$ and F a flat function at $r=0$.

Let $\Pi(r_0,w_0)=(\Pi_1(r_0,w_0),\Pi_2(r_0,w_0))$ be the Poincaré map of the corresponding smooth system $(5)_{\varepsilon=0}$ associated to (17). We recall that $\Pi_2(r_0,0)=0$ and $\frac{\partial\Pi_2}{\partial w_0}(0,0)\neq 0$, which implies that we can write

$$\Pi_2(r_0,w_0)=w_0\hat{\Pi}_2(r_0,w_0)\quad\text{with } \hat{\Pi}_2(0,0)\neq 0. \tag{21}$$

We use the fundamental relation (8), that is,

$$V\big(0,\Pi_1(r_0,w_0),\Pi_2(r_0,w_0)\big)=V(0,r_0,w_0)\det\big(D\Pi(r_0,w_0)\big),$$

where we have used the 2π-periodicity of V. Substituting here the expression of V we obtain

$$\Pi_2\Pi_1^m V_m(\Pi_1,\Pi_2)+F(\Pi_1,\Pi_2)=\big[w_0r_0^m V_m(r_0,w_0)+F(r_0,w_0)\big]\det\big(D\Pi(r_0,w_0)\big).$$

We want to remove from this equation the flat terms at $r=0$ which are only contained in $F(r_0,w_0)$ and $F(\Pi_1(r_0,w_0),\Pi_2(r_0,w_0))$. Therefore, taking the Taylor series at $r_0=0$ in both members we obtain the following identity of Taylor series

$$\mathcal{J}^\infty\big(\Pi_2\Pi_1^m V_m(\Pi_1,\Pi_2)\big)=\mathcal{J}^\infty\big(w_0r_0^m V_m(r_0,w_0)\det\big(D\Pi(r_0,w_0)\big)\big).$$

Here $\mathcal{J}^\infty$ denotes the infinite jet. Inserting here (21) we obtain

$$\mathcal{J}^\infty\big(\hat\Pi_2\Pi_1^m V_m(\Pi_1,w_0\hat\Pi_2)\big)=\mathcal{J}^\infty\big(r_0^m V_m(r_0,w_0)\det\big(D\Pi(r_0,w_0)\big)\big). \tag{22}$$

We will use (22) for $w_0=0$. For this reason we write the following relations that rely also on the fact that (17) possesses a center manifold $\mathcal{W}^c(0)=\{Z=0\}$.

$$\Pi_1(r_0,0)=r_0+\delta(r_0)=r_0+c_kr_0^k+\mathcal{O}\big(r_0^{k+1}\big),\quad \text{with } c_k\neq 0,$$

$$V_m\big(\Pi_1(r_0,0),0\big)=V_m\big(r_0+\delta(r_0),0\big)=V_m\big(r_0+\mathcal{O}\big(r_0^k\big),0\big)=V_m(r_0,0)+\mathcal{O}\big(r_0^k\big),$$

where we used the Mean Value Theorem in order to obtain the last equality. We also have

$$\big[\Pi_1(r_0,0)\big]^m=r_0^m\big(1+mc_kr_0^{k-1}+\mathcal{O}\big(r_0^k\big)\big),$$

$$\det\big(D\Pi(r_0,0)\big)=\big[1+kc_kr_0^{k-1}+\mathcal{O}\big(r_0^k\big)\big]\hat\Pi_2(r_0,0),$$

where the last equation comes from Lemma 6 and (21). Evaluating (22) at $w_0=0$ and using the above properties we obtain

$$\begin{aligned}&\mathcal{J}^\infty\big(\big[1+mc_kr_0^{k-1}+\mathcal{O}\big(r_0^k\big)\big]\big[V_m(r_0,0)+\mathcal{O}\big(r_0^k\big)\big]\big)\\&\quad=\mathcal{J}^\infty\big(V_m(r_0,0)\big[1+kc_kr_0^{k-1}+\mathcal{O}\big(r_0^k\big)\big]\big),\end{aligned} \tag{23}$$

where we have removed in both sides the common factors $\hat\Pi_2(r_0,0)=\exp(2\pi\lambda)+\mathcal{O}(r_0)$ and r_0^m. Equating the coefficients of r_0^{k-1} in both sides of (23) yields $mc_kV_m(0,0)=kc_kV_m(0,0)$ which implies that $m=k$ since $c_kV_m(0,0)\neq 0$.

It only remains to prove that $k\geqslant 3$. We note that, since $m=k$, the vanishing multiplicity m is the same for any inverse Jacobi multiplier V. Theorem 5 of [3] states the existence of one smooth and non-flat at the origin inverse Jacobi multiplier $V_0^*(x,y,z)=z(x^2+y^2)^{n^*}+\cdots$ with $n^*\geqslant 2$, hence the vanishing multiplicity of the corresponding V^* is $m=2n^*-1\geqslant 3$. □

Proof of Theorem 1. From Proposition 7 of [3] it follows that for an arbitrary V_0 with the qualities stated in the hypothesis, there exists $n\geqslant 0$ such that $V_0(x,y,z)=z(x^2+y^2)^n+\cdots$. Denote by $V(\theta,r,w)$ the inverse Jacobi multiplier of system $(5)_{\varepsilon=0}$ which comes from V_0. Due to (6), $m=2n-1$. From Theorem 2 we know that $m\geqslant 3$, hence $n\geqslant 2$.

Now we are interested in obtaining the maximum number of limit cycles that can bifurcate from the saddle-focus at the origin in the perturbed system (2) with $\|\varepsilon\|$ sufficiently small. We recall that any 2π-periodic solution of (5) corresponds to a periodic orbit of (2) near the origin and conversely. As we already explained in the Introduction, these periodic solutions are in correspondence with the zeroes of analytic reduced displacement map $\Delta(r_0;\varepsilon)$ around $(r_0,\varepsilon)=(0,0)$ with $r_0>0$.

Since $\Delta(r_0;0)=\delta(r_0)=\sum_{i\geqslant k}c_i r_0^i$ with $c_k\neq 0$, from the Weierstrass Preparation Theorem, it follows that the number of zeros of $\Delta(r_0;\varepsilon)$ near $(r_0,\varepsilon)=(0,0)$ is at most k.

But system (5) is invariant under the discrete symmetry which maps $(r,\theta,w)\mapsto(-r,\theta+\pi,-w)$. This symmetry is inherited by the symmetries of the polar blow-up (3) which cause that the function defining (5) satisfies $R(\theta+\pi,-r,-w;\varepsilon)=-R(\theta,r,w;\varepsilon)$ and $W(\theta+\pi,-r,-w;\varepsilon)=-W(\theta,r,w;\varepsilon)$. In consequence if $\theta\mapsto(r(\theta;r_0,w_0;\varepsilon),w(\theta;r_0,w_0;\varepsilon))$ is a solution of (5), then so is the function $\theta\mapsto(-r(\theta+\pi;r_0,w_0;\mu),-w(\theta+\pi;r_0,w_0;\mu))$. In addition, $\{r=0\}$ is an invariant cylinder of (5), hence the orbits of (5) do not cross transversally the cylinder $\{r=0\}$. Putting all these facts together, one consequence of the above symmetry is that the zeroes of $\Delta(r_0;\varepsilon)$ near $(r_0,\varepsilon)=(0,0)$ appear in pairs of opposite signs except the trivial one $r_0=0$. Hence we conclude that the maximum number of limit cycles (associated with the zeros with $r_0>0$) that can bifurcate from the origin in (2) with $\|\varepsilon\|$ sufficiently small is $(k-1)/2=n-1$.

Finally, in order to see that the cyclicity $\mathrm{Cycl}(\mathcal{X}_\varepsilon,0)=n-1$ we must show an example of system (2) having exactly $n-1$ limit cycles bifurcating from the origin. The example is given in Section A.1 of Appendix A. □

Appendix A

A.1. Cyclicity of the origin of the vector field $\mathcal{X}_\varepsilon$

We provide an example of a perturbation of system $(5)_{\varepsilon=0}$, with k limit cycles bifurcating from $\gamma_0=\{(\theta,0,0)\colon \theta\in[0,2\pi)\}=\{r=0\}\cap\{w=0\}$, whose transformation to cartesian coordinates (x,y,z) gives a perturbation of (1) with exactly $\ell:=(k-1)/2$ limit cycles bifurcating from the origin, i.e., $\mathrm{Cycl}(\mathcal{X}_\varepsilon,0)=(k-1)/2$. We remark that in the planar case this kind of perturbations was performed in [6].

We consider the associated system $(4)_{\varepsilon=0}$ from which $(5)_{\varepsilon=0}$ comes from and we perturb it in the following 1-parameter way:

$$\begin{aligned}
\dot r &= \mathcal{R}(\theta,r,w;0)+\sum_{i=0}^{\ell-1}\varepsilon^{\ell-i}a_i r^{2i+1},\\
\dot\theta &= 1+\Theta(\theta,r,w;0),\\
\dot w &= \mathcal{W}(\theta,r,w;0),
\end{aligned}\tag{24}$$

with the convention that if $\ell=0$ no perturbation term is taken. The real constant ε is the perturbation parameter and therefore $|\varepsilon|\ll 1$. The a_i, $i=0,1,2,\dots,\ell-1$, are real constants to be chosen in such a way that the reduced displacement map $\Delta(r_0;\varepsilon)$ associated to the system

$$\frac{dr}{d\theta}=\frac{\mathcal{R}(\theta,r,w;0)+\sum_{i=0}^{\ell-1}\varepsilon^{\ell-i}a_i r^{2i+1}}{1+\Theta(\theta,r,w;0)}=R(\theta,r,w;0)+\frac{\sum_{i=0}^{\ell-1}\varepsilon^{\ell-i}a_i r^{2i+1}}{1+\Theta(\theta,r,w;0)},$$
$$\frac{dw}{d\theta}=\frac{\mathcal{W}(\theta,r,w;0)}{1+\Theta(\theta,r,w;0)}=\lambda w+W(\theta,r,w;0), \tag{25}$$

has $2\ell+1$ real zeroes; ℓ of them positive. Since $\Theta(\theta,0,w;0)=0$, we remark that (25) is an analytic perturbation in a neighborhood of $r=0$ of system $(5)_{\varepsilon=0}$. The proof of the fact that this choice of a_i can be done is analogous to the one described in [1], pp. 254–259. More precisely, the exponent of the leading term of the reduced displacement function $\delta(r_0)=\Delta(r_0;0)$ of system $(5)_{\varepsilon=0}$ is k and the considered perturbation (25) produces that $\Delta(r_0;\varepsilon)$ has all the monomials of odd powers of r_0 up to order k. The coefficient of each monomial, for ε sufficiently small, is dominated by one of the constants a_i.

Undoing the change to polar coordinates, system (25) gives rise to an analytic system (2) with ℓ limit cycles bifurcating from the origin. This system is of the form

$$\dot{x}=-y+\mathcal{F}_1(x,y,z)+xK(x,y;\varepsilon),$$
$$\dot{y}=x+\mathcal{F}_2(x,y,z)+yK(x,y;\varepsilon),$$
$$\dot{z}=\lambda z+\mathcal{F}_3(x,y,z)+zK(x,y;\varepsilon),$$

where $K(x,y;\varepsilon)=\sum_{i=0}^{\ell-1}\varepsilon^{\ell-i}a_i(x^2+y^2)^i$.

A.2. An algorithm to compute Poincaré–Lyapunov constants

Let us define $\Psi(\theta;r_0,w_0;\varepsilon)=(r(\theta;r_0,w_0;\varepsilon),w(\theta;r_0,w_0;\varepsilon))$ to be the solution associated to (5) with initial condition $\Psi(0;r_0,w_0;\varepsilon)=(r_0,w_0)$ and denote by $\Delta(r_0;\varepsilon)$ its reduced displacement map. We have $\Delta(0;\varepsilon)=0$. Moreover, near $r_0=0$ we have the following Taylor series $\Delta(r_0;\varepsilon)=\sum_{i\geqslant 1}c_i(\varepsilon)r_0^i$, where $c_i(\varepsilon)$ are called *Poincaré–Lyapunov* constants. If for certain $\varepsilon^*\in\mathbb{R}^p$ one has $c_i(\varepsilon^*)=0$ for all $i\geqslant 1$, then the origin of system (2) with $\varepsilon=\varepsilon^*$ is a center and in this case the reduced displacement map becomes $\Delta(r_0;\varepsilon^*)\equiv 0$. Otherwise, when $\Delta(r_0;\varepsilon^*)\not\equiv 0$ the origin of system (2) with $\varepsilon=\varepsilon^*$ is a saddle-focus.

To distinguish between a center and a focus is a classical difficult problem in the qualitative theory of ordinary differential equations. In the planar case this problem goes back to the 19th century and until now it has been object of an intensive research. We recall that, essentially, the main difficulty of the center problem is that it cannot be solved by using the blow-up technique to characterize the local phase portrait near an isolated singular point of a planar vector field.

The values of the Poincaré–Lyapunov constants $c_i(\varepsilon)$ can be determined in a recursive way, although many computations are involved. The method consists in several steps: first of all we take the Taylor series of $\Psi(\theta;r_0,w_0;\varepsilon)$ near $r_0=0$

$$r(\theta;r_0,w_0;\varepsilon)=r_0\sum_{i\geqslant 0}R_i(\theta;w_0;\varepsilon)r_0^i,$$
$$w(\theta;r_0,w_0;\varepsilon)=\sum_{i\geqslant 0}W_i(\theta;w_0;\varepsilon)r_0^i,$$

where the coefficients satisfy the initial conditions $R_0(0; w_0; \varepsilon) = 1$, $W_0(0; w_0; \varepsilon) = w_0$ and $r_i(0; w_0; \varepsilon) = W_i(0; w_0; \varepsilon) = 0$ for all $i \geqslant 1$. The coefficients $R_i(\theta; w_0; \varepsilon)$ and $W_i(\theta; w_0; \varepsilon)$ are uniquely determined as solutions of linear Cauchy problems which come from equating the same powers of r_0 after imposing this Taylor series to be a solution of (5).

In this way, the displacement map $d(r_0, w_0; \varepsilon) = (d_1(r_0, w_0; \varepsilon), d_2(r_0, w_0; \varepsilon))$ admits the following Taylor series

$$d_1(r_0, w_0; \varepsilon) = r(2\pi; r_0, w_0; \varepsilon) - r_0 = r_0 \sum_{i \geqslant 0} R_i(2\pi; w_0; \varepsilon) r_0^i - r_0,$$

$$d_2(r_0, w_0; \varepsilon) = w(2\pi; r_0, w_0; \varepsilon) - w_0 = \sum_{i \geqslant 0} W_i(2\pi; w_0; \varepsilon) r_0^i - w_0.$$

In the next step we propose another Taylor series for the function $\bar{w}(r_0, \varepsilon)$ in the form $\bar{w}(r_0, \varepsilon) = \sum_{i \geqslant 1} \bar{w}_i(\varepsilon) r_0^i$ and calculate its coefficients $\bar{w}_i(\varepsilon)$ imposing $d_2(r_0, \bar{w}(r_0, \varepsilon); \varepsilon) \equiv 0$. Finally, we can obtain the Taylor series of the reduced displacement map

$$\Delta(r_0; \varepsilon) := d_1\big(r_0, \bar{w}(r_0, \varepsilon); \varepsilon\big) = \sum_{i \geqslant 1} c_i(\varepsilon) r_0^i,$$

whose coefficients $c_i(\varepsilon)$ are the *Poincaré–Lyapunov constants*.

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