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Existence of inverse Jacobi multipliers around Hopf points in $\mathbb{R}^3$: Emphasis on the center problem ☆

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ABSTRACT

We consider the center problem at Hopf points of analytic systems in $\mathbb{R}^3$ that has a classical solution in the Lyapunov Center Theorem which is given in terms of an analytic first integral. Here we give a new solution in terms of an analytic inverse Jacobi multiplier V. The existence of a smooth and non-flat inverse Jacobi multiplier around a Hopf point of saddle-focus type is also proved. When studying these problems, we needed to discuss the relation between inverse Jacobi multipliers and center manifolds $\mathcal{W}^c$, in particular to know under what conditions $\mathcal{W}^c \subset V^{-1}(0)$. To illustrate our results, we solve the center problem for the Lü system.

1. Introduction and main results

We consider the analytic three-dimensional system

$$
\begin{aligned}
\dot{x} &= -y + \mathcal{F}_1(x, y, z),\\
\dot{y} &= x + \mathcal{F}_2(x, y, z),\\
\dot{z} &= \lambda z + \mathcal{F}_3(x, y, z),
\end{aligned}
\qquad (1)
$$

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where $\lambda \in \mathbb{R} \setminus \{0\}$, $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) : \mathcal{U} \to \mathbb{R}^3$ is real analytic on the neighborhood of the origin $\mathcal{U} \subset \mathbb{R}^3$ with $\mathcal{F}(0) = 0$ and whose Jacobian matrix $D\mathcal{F}(0) = 0$.

In the sequel $\mathcal{X}$ will denote the associated vector field to system (1), that is,

$$\mathcal{X} = \big(-y + \mathcal{F}_1(x, y, z)\big)\frac{\partial}{\partial x} + \big(x + \mathcal{F}_2(x, y, z)\big)\frac{\partial}{\partial y} + \big(\lambda z + \mathcal{F}_3(x, y, z)\big)\frac{\partial}{\partial z}.$$

Any analytic system $\dot{u} = f(u)$ in $\mathbb{R}^3$ that has the singularity $u = u^*$ which is a *Hopf point* (that is, it possesses two pure imaginary and one nonzero eigenvalues) can be transformed in the form (1) by a translation, an invertible linear change of coordinates and a rescaling of time. For an interested reader, we mention [13] as a classical source for the study of Hopf points in $\mathbb{R}^n$.

One of our aims is to study *the center problem* for (1) at the origin, that is, we wish to decide when the origin is a *center* or not.

The origin is a *center* of (1) if all the orbits on the local center manifold at the origin are periodic, otherwise the origin is a saddle-focus. In this last case, the orbits on the local center manifold at the origin spiral around the origin, hence this singularity is a *focus* for the vector field reduced to the center manifold.

Note that here center in $\mathbb{R}^3$ means *center on the center manifold* and, sometimes, in this paper we will use the term *focus* instead of *saddle-focus* singularity. We remind that a local center manifold at the origin of system (1), denoted $\mathcal{W}^c$, is an invariant surface which is tangent to the (x, y) plane at the origin. More precise, $\mathcal{W}^c = \{z = h(x, y)\colon \text{ for } (x, y) \text{ in a small neighborhood of } (0, 0)\}$ with $h(0, 0) = 0$ and $Dh(0, 0) = 0$. For any $k \geqslant 1$ there exists a C^k local center manifold. The local center manifold need not be unique, but the local flows near the origin on any C^{k+1} center manifold are C^k-conjugate in a neighborhood of the origin. These results can be found in [4,19]. Hence, if the origin is a center (or a focus) on some center manifold, then on any other center manifold the origin is also a center (or a focus).

It seems that the center problem for system (1) is, in fact, a two-dimensional problem, since it coincides with the center problem for the vector field reduced to the center manifold. For this reason we present now two well-known solutions for the center problem in planar systems. The first one is the classical Poincaré–Lyapunov Center Theorem and it is given in terms of a first integral. The second one is the Reeb Criterium and it is given in terms of an inverse integrating factor v (that is, $1/v$ is an integrating factor), see [17,10]. Through this work we will use the following convention: the dots mean higher-order terms in a Taylor expansion. For instance, $f(x, y, z) = z + \cdots$ denotes $f(x, y, z) = z + \mathcal{O}(\|(x, y, z)\|^2)$.

Theorem 1 *(Poincaré–Lyapunov Center Theorem). The planar analytic system*

$$\begin{aligned} \dot{x} &= -y + \mathcal{F}_1(x, y), \\ \dot{y} &= x + \mathcal{F}_2(x, y) \end{aligned} \tag{2}$$

has a center at the origin if and only if it admits a real analytic local first integral of the form $H(x, y) = x^2 + y^2 + \cdots$ *in a neighborhood of the origin in* $\mathbb{R}^2$.

Theorem 2 *(Reeb Criterium). The planar analytic system* (2) *has a center at the origin if and only if it admits a real analytic local inverse integrating factor of the form* $v(x, y) = 1 + \cdots$ *in a neighborhood of the origin in* $\mathbb{R}^2$.

Hence, for those systems (1) for which it is known the existence of an analytic local center manifold at the origin, the center problem has an answer via the Poincaré–Lyapunov Center Theorem or the Reeb Criterium. The following theorem of Lyapunov (which is proved in Chapter 13 of the book [3]) is a classical solution of the center problem in three dimensions which overcomes the difficulty that the center manifold need not be analytic.

Theorem 3 *(Lyapunov Center Theorem). The origin is a center for the analytic system* (1) *if and only if* (1) *admits a real analytic local first integral of the form* $H(x, y, z) = x^2 + y^2 + \cdots$ *in a neighborhood of the origin in* $\mathbb{R}^3$*. Moreover, when there is a center, the local center manifold is unique and analytic.*

One of our main results is another solution to the center problem in $\mathbb{R}^3$, given in terms of *an inverse Jacobi multiplier*, and it can be seen also as an analogous result in $\mathbb{R}^3$ of the Reeb Criterium. Before writing the statement, we define this key notion for a C^1 vector field $\mathcal{Y} = \sum_{i=1}^n f_i(x)\partial_{x_i}$ defined on an open subset D of $\mathbb{R}^n$. A C^1 function $V : D \to \mathbb{R}$ is said to be an *inverse Jacobi last multiplier* of $\mathcal{Y}$ if it is not locally null and it satisfies the linear first-order partial differential equation

$$\mathcal{Y}V = V \operatorname{div}\mathcal{Y},$$

where $\operatorname{div}\mathcal{Y} = \sum_{i=1}^n \partial f_i(x)/\partial x_i$ is the divergence of the vector field $\mathcal{Y}$. For a nice survey on inverse Jacobi multipliers one can see [1].

The solution to the center problem in $\mathbb{R}^3$ given in terms of an inverse Jacobi multiplier is the following.

Theorem 4. *The analytic system* (1) *has a center at the origin if and only if it admits a local analytic inverse Jacobi multiplier of the form* $V(x, y, z) = z + \cdots$ *in a neighborhood of the origin in* $\mathbb{R}^3$*. Moreover, when such* V *exists, the local analytic center manifold* $\mathcal{W}^c \subset V^{-1}(0)$.

In the planar case ($n = 2$), inverse Jacobi multipliers are called *inverse integrating factors*. In 1996 the first fundamental result relating their vanishing set and the location of limit cycles was proved in [11]. Since than many other properties have been discovered, as one can see in [9,8] and the references therein. In 2009 the problem of existence of inverse integrating factors in a neighborhood of an elementary singularity (as well as of some other nonwandering sets) of some planar analytic vector field was solved in [7]. Our Theorem 4 can be seen also as an answer to the existence problem of an inverse Jacobi multiplier in a neighborhood of a Hopf point of center type in $\mathbb{R}^3$. In the sequel we address the same problem when the singularity is of focus type. The statement of the theorem follows.

Theorem 5. *Assume that the origin is a saddle-focus for the analytic system* (1)*. Then there exists a local* C^∞ *and non-flat inverse Jacobi multiplier of* (1) *having the expression* $V(x, y, z) = z(x^2 + y^2)^k + \cdots$ *for some* $k \geqslant 2$*. Moreover, there is a local* C^∞ *center manifold* $\mathcal{W}^c$ *such that* $\mathcal{W}^c \subset V^{-1}(0)$.

In the center case the Lyapunov Center Theorem assures the existence of an analytic first integral $H(x, y, z) = x^2 + y^2 + \cdots$ while Theorem 4 assures the existence of an analytic inverse Jacobi multiplier $V(x, y, z) = z + \cdots$. It is known that the product between an inverse Jacobi multiplier and a first integral is another inverse Jacobi multiplier. Hence there are analytic inverse Jacobi multipliers at a center of the form $V(x, y, z) = z(x^2 + y^2)^k + \cdots$ for any $k \geqslant 0$. In Proposition 7 we prove that, in both the center and the focus cases, any local C^∞ and non-flat inverse Jacobi multiplier of (1) must have this form.

When proving these theorems we discovered new properties of the vanishing set of inverse Jacobi multipliers of system (1). This set has some nice properties (some of them already noticed in [1]), though it seems that not as many and surprising as the vanishing set of an inverse integrating factor (presented for example in [9]). A consequence of our results in this direction is the fact that, given the expression of a C^∞ and non-flat inverse Jacobi multiplier, one can eventually find the expression of a C^∞ center manifold and, moreover, of an inverse integrating factor of the system reduced to this center manifold. We wrote the word “eventually”, though in the center case this is always true, since we proved that any C^∞ inverse Jacobi multiplier vanishes on the center manifold. But, in the focus case, the situation can be more complicated, as it is discussed in Section 2.

We end up our paper with an illustration of our results on the center problem for the Lü system. This is a three-parametric family of quadratic systems in $\mathbb{R}^3$ that has been studied widely recently since, despite its simplicity, it has a rich dynamical behavior ranging from stable equilibria to periodic and even chaotic oscillations [12]. It was stated as a conjecture in [14], when the parameters belong to some variety L, the two equilibria of the Lü system are of center type. Here we give a surprisingly simple solution to this problem, since we find the expression of an analytic inverse Jacobi multiplier and, consequently, of the analytic center manifold. After completion of this work we learned that the recent work [15] is totally dedicated to solve the same problem.

The paper is organized as follows. In Section 2 we present new properties of inverse Jacobi multipliers, mainly on their relations with the center manifolds. These properties are interesting for themselves, but are also used in the proofs of Theorems 4 and 5 which are contained in Section 3. Section 4 is devoted to the study of the Lü system.

2. Some more properties of inverse Jacobi multipliers

The theory of inverse Jacobi multipliers is presented and developed in [1] from its beginnings in the formal methods of integration of ordinary differential equations, until modern applications in dynamical systems theory. In this section we present new properties of inverse Jacobi multipliers, mainly of their vanishing set.

Having both a Jacobi multiplier and a first integral for a system in $\mathbb{R}^3$, Jacobi found an integrating factor for the system reduced to some invariant two-dimensional surface given as a level surface of the former first integral, see also [1]. But this method gives a trivial (identically zero) inverse integrating factor in the special case that the inverse Jacobi multiplier vanishes on the invariant surface. We give here a procedure to find a nontrivial inverse integrating factor also in this special case. Only the next theorem is given for general vector fields, the other results of this section concern system (1).

Theorem 6. *Let $\mathcal{Y} = f_1(x,y,z)\partial_x + f_2(x,y,z)\partial_y + f_3(x,y,z)\partial_z$ be a smooth vector field defined in an open set $\mathcal{U} \subset \mathbb{R}^3$. Assume that there exists a C^∞ inverse Jacobi multiplier of the form*

$$V(x,y,z) = \big(z - h(x,y)\big)W(x,y,z) \quad \text{with } W\big(x,y,h(x,y)\big) \not\equiv 0. \tag{3}$$

Then $\mathcal{M} = \{(x,y,z) \in \mathcal{U}\colon z = h(x,y)\}$ is an invariant manifold of $\mathcal{Y}$ and

$$v(x,y) = W\big(x,y,h(x,y)\big)$$

is an inverse integrating factor of the reduced vector field $\mathcal{Y}|_{\mathcal{M}} = f_1(x,y,h(x,y))\partial_x + f_2(x,y,h(x,y))\partial_y$.

Proof. Since $\mathcal{Y}V = V \operatorname{div}\mathcal{Y}$, it is clear that $V = 0$ defines an invariant surface of $\mathcal{Y}$. Therefore, $F = z - h(x,y) = 0$ and $W = 0$ are also invariant surfaces of $\mathcal{Y}$ and, in consequence, they have associated smooth cofactors $K(x,y,z)$ and $L(x,y,z)$, respectively. Thus we have $\mathcal{Y}F = KF$ and $\mathcal{Y}W = LW$ and moreover $\operatorname{div}\mathcal{Y} = K + L$.

Now we define the function $v(x,y) = W(x,y,h(x,y))$ and compute the derivative

$$(\mathcal{Y}|_{\mathcal{M}})v = f_1\frac{\partial v}{\partial x} + f_2\frac{\partial v}{\partial y} = f_1\left(\frac{\partial W}{\partial x} + \frac{\partial W}{\partial z}\frac{\partial h}{\partial x}\right) + f_2\left(\frac{\partial W}{\partial y} + \frac{\partial W}{\partial z}\frac{\partial h}{\partial y}\right) \quad \text{on } \mathcal{M}.$$

Evaluating $\mathcal{Y}F = KF$ on $\mathcal{M}$ we obtain

$$f_1\frac{\partial h}{\partial x} + f_2\frac{\partial h}{\partial y} = f_3 \quad \text{on } \mathcal{M},$$

and therefore

$$(\mathcal{Y}|_{\mathcal{M}})v = f_1\frac{\partial W}{\partial x} + f_2\frac{\partial W}{\partial y} + f_3\frac{\partial W}{\partial z} \quad \text{on } \mathcal{M}.$$

The right-hand side of this expression coincides with the left-hand side of $\mathcal{Y}W = LW$ evaluated on $\mathcal{M}$. Therefore we obtain that

$$(\mathcal{Y}|_{\mathcal{M}})v = LW = Lv \quad \text{on } \mathcal{M}. \tag{4}$$

Taking derivatives with respect to z in $\mathcal{Y}F = KF$ and next evaluating on $\mathcal{M}$ we obtain

$$K = -\frac{\partial f_1}{\partial z}\frac{\partial h}{\partial x} - \frac{\partial f_2}{\partial z}\frac{\partial h}{\partial y} + \frac{\partial f_3}{\partial z} \quad \text{on } \mathcal{M}.$$

Introducing this expression in the identity $L = \operatorname{div}\mathcal{Y} - K$ we obtain

$$L = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_1}{\partial z}\frac{\partial h}{\partial x} + \frac{\partial f_2}{\partial z}\frac{\partial h}{\partial y} = \operatorname{div}(\mathcal{Y}|_{\mathcal{M}}) \quad \text{on } \mathcal{M}.$$

In short, Eq. (4) reduces to $(\mathcal{Y}|_{\mathcal{M}})v = v \operatorname{div}(\mathcal{Y}|_{\mathcal{M}})$ and, therefore, $v(x, y)$ is an inverse integrating factor of the reduced vector field $\mathcal{Y}|_{\mathcal{M}}$. □

In the rest of this section we refer to system (1). We prove now that the lower-order non-null homogeneous polynomial in the Taylor expansion around the origin of some C^∞ inverse Jacobi multiplier cannot be any polynomial.

Proposition 7. *Any local C^∞ and non-flat inverse Jacobi multiplier of* (1) *has the expression $V(x, y, z) = z(x^2 + y^2)^k + \cdots$ for some $k \geqslant 0$.*

Proof. Denote V_m the lower-order non-null homogeneous polynomial of degree $m \geqslant 0$ in the Taylor expansion around the origin of some C^∞ and non-flat inverse Jacobi multiplier V of (1). Then V_m satisfies the linear first-order partial differential equation

$$-y\frac{\partial V_m}{\partial x} + x\frac{\partial V_m}{\partial y} + \lambda z\frac{\partial V_m}{\partial z} = \lambda V_m,$$

whose general solution is $F(x^2 + y^2, ze^{\lambda \arctan \frac{x}{|y|}})e^{-\lambda \arctan \frac{x}{|y|}}$. Therefore, we must have $V_m = z(x^2 + y^2)^k$ where $m = 2k + 1$. □

The special relation between center manifolds and inverse Jacobi multipliers is partially discovered in the next result and further commented at the end of this section. Practically, when the origin of (1) is a center, the situation is completely understood: any C^∞ inverse Jacobi multiplier must vanish on the center manifold. In the case that the origin is a focus, the situation is more delicate.

Theorem 8. *Let V be a local C^∞ inverse Jacobi multiplier of system* (1) *and $\mathcal{W}^c = \{z = h(x, y)\}$ be a C^∞ local center manifold at the origin. Consider the restricted function $V|_{\mathcal{W}^c} : (x, y) \mapsto V(x, y, h(x, y))$. Then, the following holds*:

(i) *$V|_{\mathcal{W}^c}$ is a flat function at the origin.*
(ii) *When $\mathcal{W}^c \subset V^{-1}(0)$, that is, $V|_{\mathcal{W}^c} \equiv 0$, there exists a C^∞ function $W(x, y, z)$ such that $W(x, y, h(x, y)) \not\equiv 0$ and the following factorization occurs $V(x, y, z) = (z - h(x, y))W(x, y, z)$. Moreover, $v(x, y) = W|_{\mathcal{W}^c} = W(x, y, h(x, y))$ is an inverse integrating factor of $\mathcal{X}|_{\mathcal{W}^c}$.*
(iii) *In the case that system* (1) *has a center at the origin we must have that $\mathcal{W}^c \subset V^{-1}(0)$.*

Proof. (i) Since $\mathcal{W}^c = \{z = h(x, y)\}$ is an invariant surface for the vector field $\mathcal{X}$, the following relation holds

$$\frac{\partial h}{\partial x}(-y + \mathcal{F}_1) + \frac{\partial h}{\partial y}(x + \mathcal{F}_2) = \lambda h + \mathcal{F}_3 \quad \text{on } \mathcal{W}^c.$$

This identity, together with $\mathcal{X}V = V \operatorname{div} \mathcal{X}$ is used to show that the function $u(x, y) = V(x, y, h(x, y))$ satisfies the first-order partial differential equation

$$\frac{\partial u}{\partial x}(-y + \mathcal{F}_1) + \frac{\partial u}{\partial y}(x + \mathcal{F}_2) = u(\lambda + \operatorname{div} \mathcal{F}) \quad \text{on } \mathcal{W}^c, \tag{5}$$

where we wrote that $\operatorname{div} \mathcal{X} = \lambda + \operatorname{div} \mathcal{F}$.

Replacing in (5) $x = y = 0$ and taking into account that $\lambda \neq 0$ we obtain that $u(0, 0) = 0$. Assume, by contradiction, that $u(x, y)$ is non-flat, and denote by $P_m(x, y)$ the lower-order non-null homogeneous polynomial of degree $m \geqslant 1$ in its Taylor series around the origin. We replace this Taylor series in (5) and, after equating the lower-order terms we obtain that P_m satisfies

$$-y\frac{\partial P_m}{\partial x} + x\frac{\partial P_m}{\partial y} = \lambda P_m. \tag{6}$$

Taking into account that P_m is a homogeneous polynomial of degree m, we have that it is also a solution of the Euler equation

$$x\frac{\partial P_m}{\partial x} + y\frac{\partial P_m}{\partial y} = mP_m. \tag{7}$$

From (6) and (7) we deduce that P_m satisfies

$$(\lambda x + my)\frac{\partial P_m}{\partial x} + (-mx + \lambda y)\frac{\partial P_m}{\partial y} = 0. \tag{8}$$

This means that P_m must be a nontrivial polynomial first integral of the planar linear system $\dot{x} = \lambda x + my$, $\dot{y} = -mx + \lambda y$ which has a focus at the origin. It is known that this is not possible, hence we reached the contradiction and we proved that the function $u = V|_{\mathcal{W}^c}$ is flat at the origin.

(ii) First we prove (ii) in the case that the given local center manifold at the origin is $\mathcal{W}^c = \{z = 0\}$. From the hypothesis we have that $V(x, y, 0) \equiv 0$. Then there exist an integer $m \geqslant 1$ and a smooth and non-flat function W such that $W(x, y, 0) \not\equiv 0$ and $V(x, y, z) = z^m W(x, y, z)$. We proved in Proposition 7 that the lower-order non-null homogeneous polynomial in the Taylor series of V is $z(x^2 + y^2)^k$, hence we must have $m = 1$.

In order to prove the general case, notice that, performing the near identity change of variables $(x, y, z) \mapsto (x, y, Z)$ defined as $Z = z - h(x, y)$, system (1) becomes

$$\begin{aligned} \dot{x} &= -y + \tilde{\mathcal{F}}_1(x, y, Z), \\ \dot{y} &= x + \tilde{\mathcal{F}}_2(x, y, Z), \\ \dot{Z} &= \lambda Z + \tilde{\mathcal{F}}_3(x, y, Z), \end{aligned} \tag{9}$$

where $\tilde{\mathcal{F}}_i$ are nonlinear terms and $\tilde{\mathcal{F}}_3(x, y, 0) \equiv 0$. In this way, the given center manifold of system (1) is transformed into $\{Z = 0\}$, some center manifold of (9).

Given a vector field $\mathcal{X}$ possessing an inverse Jacobi multiplier V and a diffeomorphism ψ such that $\tilde{\mathcal{X}} = \psi_* \mathcal{X}$, we have that $\tilde{V} = J_\psi^{-1} V \circ \psi^{-1}$ is an inverse Jacobi multiplier of $\tilde{\mathcal{X}}$ where J_ψ denotes the Jacobian determinant of ψ (see [1]). In our particular case we have $\psi(x,y,z)=(x,y,Z)$ with $Z=z-h(x,y)$ so that

$$J_\psi=\frac{\partial(x,y,Z)}{\partial(x,y,z)}=\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -h_x & -h_y & 1\end{vmatrix}=1.$$

Hence, the relation between V and the inverse Jacobi multiplier $\tilde{V}(x,y,Z)$ of (9) is $\tilde{V}(x,y,z-h(x,y))=V(x,y,z)$, and the conclusion follows.

The fact that $W(x,y,h(x,y))$ is an inverse integrating factor of $\mathcal{X}|_{\mathcal{W}^c}$ is obtained applying Theorem 6.

(iii) Define the curve $\mathcal{C}=(x,0,h(x,0))\subset\mathcal{W}^c$ with $x\in(0,\epsilon]$ and $\epsilon>0$ sufficiently small. We claim that the curve $\mathcal{C}$ is a transversal section to the flow of system (1) restricted to $\mathcal{W}^c$. To see that, let $u(x)=(1,0,\frac{\partial h}{\partial x}(x,0))$ be a tangent vector to $\mathcal{C}$ at $(x,0,h(x,0))$. The third component of the cross product $u(x)\times\mathcal{X}(x,0,h(x,0))$ is $x+\mathcal{F}_2(x,0,h(x,0))=x+\mathcal{O}(x^2)$ and therefore never vanishes along the curve $\mathcal{C}$. Hence $u(x)$ is not parallel to $\mathcal{X}(x,0,h(x,0))$, proving the claim.

An interesting property of inverse Jacobi multipliers is that they can be computed along the orbits of $\mathcal{X}$. More concretely, let $\phi_t(x,y,z)$ be the flow of system (1) such that $\phi_0(x,y,z)=(x,y,z)$. Using the characteristics method applied to the linear first-order partial differential equation $\mathcal{X}V=V\operatorname{div}\mathcal{X}$ we obtain that

$$V\big(\phi_t\big(x,0,h(x,0)\big)\big)=V\big(x,0,h(x,0)\big)\exp\left(\int_0^t \operatorname{div}\mathcal{X}\big(\phi_s\big(x,0,h(x,0)\big)\big)\,ds\right). \tag{10}$$

Recall that $\operatorname{div}\mathcal{X}=\frac{\partial\mathcal{F}_1}{\partial x}+\frac{\partial\mathcal{F}_2}{\partial y}+\frac{\partial\mathcal{F}_3}{\partial z}+\lambda$.

Denote by $T(x)$ the time that takes the orbit of (1) starting at the initial point $(x,0,h(x,0))$ to come back to the transversal section $\mathcal{C}$. It is known that $T(x)=2\pi+\mathcal{O}(x)$. This assertion is proved performing the polar change of coordinates to the reduced vector field $\mathcal{X}|_{\mathcal{W}^c}$ that brings the differential equation of the orbits of $\mathcal{X}|_{\mathcal{W}^c}$ to some equation of the form

$$\frac{dr}{d\theta}=\frac{rR(r,\theta)}{1+\Theta(r,\theta)}. \tag{11}$$

Then, denoting by $r(\theta;x)$ the solution of Eq. (11) such that $r(0;x)=x$, the period function $T(x)$ is given by

$$T(x)=\int_0^{2\pi}\frac{d\theta}{1+\Theta(r(\theta;x),\theta)}=\int_0^{2\pi}\big(1+\mathcal{O}(x)\big)\,d\theta=2\pi+\mathcal{O}(x).$$

Taking into account that ϕ_t is a diffeomorphism and using the hypotheses on our field, we obtain that

$$\int_0^{T(x)}\operatorname{div}\mathcal{X}\big(\phi_s\big(x,0,h(x,0)\big)\big)\,ds=\lambda T(x)+\mathcal{O}(x)=2\pi\lambda+\mathcal{O}(x). \tag{12}$$

The orbit of (1) through the point $(x,0,h(x,0))$ of $\mathcal{C}$ is closed, that is,

$$\phi_{T(x)}\big(x,0,h(x,0)\big)=\big(x,0,h(x,0)\big)\quad\text{for all } x\in(0,\varepsilon).$$

We evaluate (10) at $t = T(x)$ and, using (12) we obtain

$$V\big(x, 0, h(x, 0)\big) = V\big(x, 0, h(x, 0)\big)\big(2\pi\lambda + \mathcal{O}(x)\big).$$

This gives that $V(x, 0, h(x, 0)) \equiv 0$ that further, by (10) and using that $\mathcal{C}$ is a transversal section to the flow of system (1), gives that $V(x, y, h(x, y)) \equiv 0$. □

We end this section with further comments on the relation between inverse Jacobi multipliers and center manifolds. We start with an example, the system

$$\dot{x} = -y - x\big(x^2 + y^2\big), \qquad \dot{y} = x - y\big(x^2 + y^2\big), \qquad \dot{z} = -z$$

which has a focus at the origin. This system has the analytic center manifold

$$\mathcal{W}_0^c = \{z = 0\}$$

and the C^∞ flat center manifolds (for all $a \in \mathbb{R}^*$)

$$\mathcal{W}_a^c = \left\{z = a \exp\left(-\frac{1}{2(x^2 + y^2)}\right)\right\}.$$

The system also possesses the analytic inverse Jacobi multiplier

$$V_0(x, y, z) = z\big(x^2 + y^2\big)^2$$

and the C^∞ and non-flat inverse Jacobi multipliers (for all $a \in \mathbb{R}^*$)

$$V_a(x, y, z) = \left(z - a \exp\left(-\frac{1}{2(x^2 + y^2)}\right)\right)\big(x^2 + y^2\big)^2.$$

It is not difficult to check the validity of Theorem 8(i) for each couple of center manifold and inverse Jacobi multiplier of this system. Moreover, we notice that the null set of each of the inverse Jacobi multipliers described above contains one and only one center manifold. Since any linear combination of inverse Jacobi multipliers is again an inverse Jacobi multiplier, it is possible to construct one whose null set does not contain any center manifold. For example,

$$\hat{V}(x, y, z) = V_0(x, y, z) - V_1(x, y, z) = \exp\left(-\frac{1}{2(x^2 + y^2)}\right)\big(x^2 + y^2\big)^2$$

is an inverse Jacobi multiplier of our system, but $\hat{V}^{-1}(0) = \{(0, 0, 0)\}$. However, note that $\hat{V}$ is a flat function at the origin and that each non-flat inverse Jacobi multiplier listed above vanishes on some center manifold. Moreover, note that any center manifold listed above is included in the null set of some inverse Jacobi multiplier. Another remark is that the lower-order homogeneous polynomial in the Taylor expansion around the origin of each inverse Jacobi multiplier listed above is indeed of the form $(x^2 + y^2)^k$, according to Proposition 7, but $k = 2$ is the same constant for all the V's. Of course, it would be interesting to know which of these features maintains in the general case.

3. Proofs of Theorems 4 and 5

We briefly recall some known techniques and results from normal form theory that we will need later.

The book [18] treats in detail what we report below in the special case of the planar center problem. The higher-dimensional context is explained in detail in [3], see also [6] for the 3-dimensional case that we are interested. We use the complex variable $X = x + iy$ and its complex conjugate $Y = \bar{X} = x - iy$ to get the following complexification of family (1):

$$\dot{X} = iX + P(X, Y, z), \qquad \dot{Y} = -iY + Q(X, Y, z), \qquad \dot{z} = \lambda z + R(X, Y, z). \tag{13}$$

System (13) is analytic in $\mathbb{C}^3$ with $P(X, Y, z) = \sum_{p+q+r\geqslant 2} a_{pqr} X^p Y^q z^r$, $Q(X, Y, z) = \sum_{p+q+r\geqslant 2} b_{pqr} X^p Y^q z^r$ and $R(X, Y, z) = \sum_{p+q+r\geqslant 2} c_{pqr} X^p Y^q z^r$. Of course, since $Y = \bar{X}$ we must have $\overline{Q(w, \bar{w}, z)} = P(w, \bar{w}, z)$ all $(w, z) \in \mathbb{C} \times \mathbb{R}$, that is, $b_{qpr} = \bar{a}_{pqr}$. Moreover, c_{pqr} are such that $R(w, \bar{w}, z)$ is real for all $(w, z) \in \mathbb{C} \times \mathbb{R}$. By using normal form theory (see for instance [3]), we can perform a near identity formal change of variables $(X, Y, z) \mapsto (u, v, w)$ that eliminates all nonresonant terms and brings system (13) into the Poincaré formal normal form

$$\dot{u} = iu + u\mathcal{A}(uv), \qquad \dot{v} = -iv + v\mathcal{B}(uv), \qquad \dot{w} = \lambda w + w\mathcal{C}(uv), \tag{14}$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are formal series without independent term. We recall that the monomial $x^p y^q z^r$ in the jth equation of system (13) is *resonant* if (p, q, r) with nonnegative integer components satisfying $p + q + r \geqslant 2$ is a solution of the equation $(p - q)i + \lambda r = \kappa_j$ where $(\kappa_1, \kappa_2, \kappa_3) = (i, -i, \lambda)$. Usually, a normalizing transformation is not unique. In what follows, we call such a transformation *distinguished* normalization if the transformation only contains non-resonant terms. The distinguished normalization is unique.

The next result appears in [6] and gives several characterizations of centers for system (1).

Theorem 9. *The origin is a center for* (1) *if and only if one of the following equivalent statements is verified.*

(i) *System* (1) *admits a formal first integral.*
(ii) *System* (1) *admits a local analytic first integral.*
(iii) *Any normal form* (14) *of system* (1) *satisfies* $\mathcal{A} + \mathcal{B} \equiv 0$.

Statement (iii) of Theorem 9 can be drawn from various parts of [3] or in compact form from [6]. The statement (ii) is just the Lyapunov Center Theorem.

Proof of Theorem 4. Assume that (1) has an analytic local inverse Jacobi multiplier at the origin of the form $V(x, y, z) = z + \cdots$. Using the Implicit Function Theorem for the equation $V(x, y, z) = 0$ we obtain the existence of a unique analytic function $h(x, y)$ defined in a neighborhood of the origin such that $h(0, 0) = 0$ and $V(x, y, h(x, y)) \equiv 0$. Additionally, one has $Dh(0, 0) = 0$. Hence, from the invariance of $V = 0$ under the flow, the definition of the center manifold and Theorem 8 we conclude that $\mathcal{W}^c = \{z = h(x, y)\}$ is an analytic local center manifold at the origin and there exists an analytic function W with $W(x, y, h(x, y)) \not\equiv 0$ such that $V(x, y, z) = (z - h(x, y))W(x, y, z)$. Since $V(x, y, z) = z + \cdots$ we must have $W(0, 0, 0) = 1$. Further we have that $v(x, y) = W(x, y, h(x, y))$ is an analytic inverse integrating factor of $\mathcal{X}|_{\mathcal{W}^c}$ that satisfies $v(0, 0) = 1$. The Reeb Criterium (Theorem 2) assures that the origin is a center for $\mathcal{X}|_{\mathcal{W}^c}$.

Conversely, assume now that (1) has a center at the origin. Then, from Theorem 9, it can be put into the complex normal form (14) with $\mathcal{A} + \mathcal{B} \equiv 0$. In Chapter 5 of [3] it is shown that the condition $\mathcal{A} + \mathcal{B} \equiv 0$ implies that the distinguished normalizing transformation $(X, Y, z) \mapsto (u, v, w)$ that brings system (13) into the normal form (14) is convergent. Moreover, the analytic distinguished normal form (14) is the complexification of a real system, which implies in the center case that $\mathcal{A}$ (and of course $\mathcal{B}$) have only purely imaginary coefficients. This real system can be recovered by making the substitution $u = \xi + i\eta$ and $v = \bar{u} = \xi - i\eta$ in (14) and applying them to $\dot{\xi} = \frac{1}{2}(\dot{u} + \dot{v})$ and $\dot{\eta} = \frac{1}{2i}(\dot{u} - \dot{v})$. Direct computation gives the real analytic center normal form

$$\dot{\xi} = -\eta F\big(\xi^2 + \eta^2\big), \qquad \dot{\eta} = \xi F\big(\xi^2 + \eta^2\big), \qquad \dot{w} = \lambda w + wG\big(\xi^2 + \eta^2\big), \tag{15}$$

where $F(s) = 1 - i\mathcal{A}(s)$ is a real analytic function near the origin such that $F(0) = 1$ and $G = \mathcal{C}$. We emphasize that in the transcendental case of Chapter 13 of the book [3], which corresponds with our center case, it is proved that the diffeomorphism $(x, y, z) \mapsto \Phi(x, y, z) = (\mathrm{Re}(u), \mathrm{Im}(u), w) = (\xi, \eta, w)$ is real analytic. Thus, it possesses an analytic local real inverse. In short we have that systems (1) and (15) are analytically conjugated.

On the other hand, it is straightforward to check that $\hat{V}(\xi, \eta, w) = w$ is an inverse Jacobi multiplier of system (15). Hence, going back to the original real variables (x, y, z) and taking into account how inverse Jacobi multipliers change under changes of variables, we have that system (1) has a local analytic inverse Jacobi multiplier of the form $V(x, y, z) = z + \cdots$. Step by step we have that $\tilde{V}(u, v, w) = w$ is an inverse Jacobi multiplier of (14), $V^*(X, Y, z) = (z + \cdots)/(1 + \cdots) = z + \cdots$ is an inverse Jacobi multiplier of (13) where the numerator and denominator of V^* are the third component of the near identity analytic change $(u, v, w) \mapsto (X, Y, z)$ and its Jacobian determinant, respectively. Finally, $V(x, y, z) = V^*(x + iy, x - iy, z) = z + \cdots$ is the real analytic local inverse Jacobi multiplier of system (1) at the origin.

The final part of the theorem, $\mathcal{W}^c \subset V^{-1}(0)$, is a simple consequence of the Implicit Function Theorem applied to the analytic function $V(x, y, z) = z + \cdots$ near the origin and of the uniqueness of the center manifold in the center situation. $\square$

Proof of Theorem 5. Let the origin be a focus for system (1). Then we do a formal normalizing transformation $(X, Y, z) \mapsto (u, v, w)$ that brings system (13) into the formal normal form (14) which is the complexification of the real formal system

$$\begin{aligned}
\dot{\xi} &= -\eta + \frac{1}{2}\big[(\xi + i\eta)\mathcal{A}\big(\xi^2 + \eta^2\big) + (\xi - i\eta)\mathcal{B}\big(\xi^2 + \eta^2\big)\big],\\
\dot{\eta} &= \xi + \frac{1}{2}\big[(\eta - i\xi)\mathcal{A}\big(\xi^2 + \eta^2\big) + (\eta + i\xi)\mathcal{B}\big(\xi^2 + \eta^2\big)\big],\\
\dot{w} &= \lambda w + w\mathcal{C}\big(\xi^2 + \eta^2\big),
\end{aligned} \tag{16}$$

where $u = \xi + i\eta$ and $v = \bar{u} = \xi - i\eta$. Notice that, since the symmetry conjugation $\overline{\mathcal{B}(s)} = \mathcal{A}(s)$ holds, system (16) is real formal. In short, systems (1) and (16) are formally conjugated.

In addition, system (16) possesses the following real formal inverse Jacobi multiplier

$$V^*(\xi, \eta, w) = w\big(\xi^2 + \eta^2\big)\big(\mathcal{A}\big(\xi^2 + \eta^2\big) + \mathcal{B}\big(\xi^2 + \eta^2\big)\big),$$

with $\mathcal{A}(\xi^2 + \eta^2) + \mathcal{B}(\xi^2 + \eta^2) \not\equiv 0$. Hence, $\mathcal{A}(s) + \mathcal{B}(s) = \alpha_m s^m + O(s^{m+1})$ with $\alpha_m \neq 0$ and $m \geqslant 1$ a positive integer. Undoing the formal change of coordinates we recover that

$$V(x, y, z) = \frac{(z + \cdots)(x^2 + y^2 + \cdots)(\alpha_m (x^2 + y^2)^m + \cdots)}{1 + \cdots}. \tag{17}$$

Thus, up to multiplicative constants, we get that $V(x, y, z) = z(x^2 + y^2)^{m+1} + \cdots$ is a formal inverse Jacobi multiplier of system (1).

Now, we can use Borel's Theorem, see for instance [16], to ensure the existence of a smooth function whose Taylor expansion at the origin is just the above formal series representing $V(x, y, z)$. Of course this smooth function is not unique due to the possible addition of flat terms and need not be an inverse Jacobi multiplier (a solution of the partial differential equation $\mathcal{X}V = V \operatorname{div}\mathcal{X}$). Moreover, it is clear that C^∞-conjugacy between vector fields implies formal conjugacy but the contrary is not always satisfied (one exception are the hyperbolic singularities from the Sternberg–Chen Theorem, see [5]). The question to clarify in what cases, aside from hyperbolic, formal conjugacy implies C^∞-conjugacy is very important and depends on the nature of the flow restricted to the center manifold $\mathcal{W}^c$. In [2], it is studied the case that interests us: $\dim \mathcal{W}^c = 2$ due to a nonzero pair of purely imaginary eigenvalues. The conclusion of [2] is that only the focus case ($\mathrm{Re}(\mathcal{A}) \not\equiv 0$) satisfies that any vector field which is formally conjugate to it is necessarily C^∞-conjugate to it. Hence, we can ensure the existence of a C^∞ and non-flat inverse Jacobi multiplier of system (1) of the form $V(x, y, z) = z(x^2 + y^2)^k + \cdots$ for some $k \geqslant 2$.

Looking at the form (17) of V, we notice that it contains a factor of the form $(z + \cdots)$. Applying the Implicit Function Theorem around the origin to this factor, we obtain the existence of a C^∞ invariant surface $z = h(x, y)$ which is tangent to the origin. Hence, this must be a C^∞ local center manifold $\mathcal{W}^c \subset V^{-1}(0)$. □

Remark 10. Observe that the proof of Theorem 5 depends crucially on theorems proved in [2] and [3]. We remark that there are more recent and much stronger results in normal form theory, see for instance the excellent survey of Stolovitch [20]. In essence, instead of the formal normal form (16) one can use a quasi-analytic normal form of some Gevrey type, thus avoiding the use of Sternberg-like type arguments.

4. The Lü system

For practical reasons, we give the following theorem as a direct consequence of Theorem 4. We denote by $\nabla V(u)$ the line vector of first-order partial derivatives of V calculated in u.

Corollary 11. *Consider the analytic three-dimensional system $\dot{u} = f(u)$ which has a Hopf point at $u = u^* \in \mathbb{R}^3$. This system has a center at $u = u^*$ if and only if it admits a local analytic inverse Jacobi multiplier V at u^* with $\nabla V(u^*) \neq 0$.*

Proof. Using a linear invertible change of coordinates $\xi = P(u - u^*)$ and a time scaling, system $\dot{u} = f(u)$ is transformed in a system of the form (1) whose linear part is in Jordan form. If we denote by V an inverse Jacobi multiplier of system $\dot{u} = f(u)$, then

$$\tilde{V}(\xi) = (\det P) V\left(P^{-1}\xi + u^*\right)$$

is an inverse Jacobi multiplier of the transformed system. The converse is also true. In particular, we deduce that

$$\nabla \tilde{V}(0) = (\det P) \nabla V\left(u^*\right) P^{-1}. \tag{18}$$

We have the following equivalences. System $\dot{u} = f(u)$ has a center at $u = u^*$ $\Leftrightarrow$ the transformed system of the form (1) has a center at $\xi = 0$ $\Leftrightarrow$ it admits a local analytic inverse Jacobi multiplier $\tilde{V}$ such that $\nabla \tilde{V}(0) \neq 0$ $\Leftrightarrow$ system $\dot{u} = f(u)$ admits a local analytic inverse Jacobi multiplier V such that $\nabla V(u^*) \neq 0$. In order to establish these equivalences we used Theorem 4 and relation (18). □

Theorem 12. *Consider the 3-parametric Lü family given by the following quadratic system in $\mathbb{R}^3$*

$$\dot{x} = a(y - x), \qquad \dot{y} = cy - xz, \qquad \dot{z} = -bz + xy, \tag{19}$$

with parameters $(a, b, c) \in \mathbb{R}^3$. The singularities $(\pm\sqrt{bc}, \pm\sqrt{bc}, c)$ are centers if and only if $(a, b, c) \in L$ where the center variety L is the straight line $L = \{(a, b, c) \in \mathbb{R}^3\colon\ a \neq 0,\ b = 2a,\ c = a\}$ of the parameter space.

Moreover, when $(a,b,c) \in L$*,* $V(x,y,z) = x^2 - 2az$ *is a global inverse Jacobi multiplier,* $\{V(x,y,z) = 0\}$ *is a global center manifold for both singularities and the system reduced to the center manifold is Hamiltonian with Hamiltonian function* $H(x,y) = axy - \frac{a}{2}y^2 - \frac{1}{8a}x^4$*.*

Theorem 12 is just an application of Corollary 11 and gives a positive answer to a conjecture formulated in the paper [14] about the classification of centers in the Lü system. Before proving Theorem 12, we summarize the known results on the Lü system (19). Despite its simplicity, in [12] it is showed that system (19) has a rich dynamical behavior, ranging from stable equilibrium points to periodic and even chaotic oscillations, depending on the parameter values (a,b,c). System (19) has the equilibrium point $(0,0,0)$, which exists for any parameter values and, for $bc > 0$ it also has the symmetric equilibria $Q_\pm = (\pm\sqrt{bc}, \pm\sqrt{bc}, c)$. The stability and degenerate Hopf bifurcation which occur at the equilibria $Q_\pm$ up to codimension three of the Lü system are studied in the article [14]. Notice that system (19) is invariant under the discrete symmetry $(x,y,z) \to (-x,-y,z)$. So the stability of the equilibrium Q_- can be obtained from the stability of Q_+.

When $c = (a+b)/3$ and $ab > 0$, Q_+ is a Hopf point of system (19) because it possesses one real nonzero eigenvalue and a pair of purely imaginary eigenvalues. In this case, the first Poincaré–Lyapunov constant of Q_+ is different from zero if and only if $(a-5b)(2a-b) \neq 0$. When $a - 5b = 0$, in [14] it is proved that the second Poincaré–Lyapunov constant is different from zero. Finally, when $2a - b = 0$, it is showed that the second and third Poincaré–Lyapunov constants vanish. Based on this last result, in [14] the following conjecture is given: In the straight line $L = \{(a,b,c) \in \mathbb{R}^3\colon\ a \neq 0,\ b = 2a,\ c = a\}$ of the parameter space, the equilibria $Q_\pm$ are centers of the Lü system.

Proof of Theorem 12. Taking into account the above discussion, it remained to prove that when $(a,b,c) \in L$ each equilibrium point $Q_\pm = (\pm|a|\sqrt{2}, \pm|a|\sqrt{2}, a)$ is a center. So, let $(a,b,c) \in L$. It is easy to check that the polynomial function $V(x,y,z) = x^2 - 2az$ is a solution of the first-order partial differential equation

$$a(y-x)\frac{\partial V}{\partial x} + (ay - xz)\frac{\partial V}{\partial y} + (-2az + xy)\frac{\partial V}{\partial z} = -2aV,$$

hence indeed it is an inverse Jacobi multiplier of (19). We have that $\nabla V(Q_\pm) = (\pm 2|a|\sqrt{2}, 0, -2a)$ which is not the null vector. Applying Corollary 11 we deduce that $Q_\pm$ are both centers.

We denote by $\mathcal{W}^c$ the center manifold at Q_+ (or Q_-). Based on Theorem 8 we must have that $\mathcal{W}^c \subset V^{-1}(0)$. Since this is a geometric property, it does not depend on whether the system is in the Jordan form or not. Hence $\mathcal{W}^c = x^2 - 2az = 0$ is indeed a global center manifold for both singularities $Q_\pm$. Applying Theorem 6 we deduce that $v(x,y) = 1$ is an inverse integrating factor for the Lü system reduced to $\mathcal{W}^c$, hence this planar system is Hamiltonian. Of course, these calculations are simple enough to be checked directly, without using the theoretical results that are, in fact, validated in this way.

The reduced system is $\dot{x} = a(y-x)$, $\dot{y} = ay - \frac{1}{2a}x^3$ with the Hamiltonian given in the statement of the theorem. □

Remark 13. We notice that the authors of [15] performed two successive changes of coordinates to the Lü system. For the final system they found an algebraic center manifold and proved that the system reduced to this center manifold has a polynomial Hamiltonian.

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References

[1] L.R. Berrone, H. Giacomini, Inverse Jacobi multipliers, Rend. Circ. Mat. Palermo (2) 52 (2003) 77–130.
[2] G.R. Belitskii, Smooth equivalence of germs of vector fields with one zero or a pair of purely imaginary eigenvalues, Funct. Anal. Appl. 20 (4) (1986) 253–259.
[3] Y.N. Bibikov, Local Theory of Nonlinear Analytic Ordinary Differential Equations, Lecture Notes in Math., vol. 702, Springer-Verlag, New York, 1979.
[4] A. Burchard, B. Deng, K. Lu, Smooth conjugacy of center manifolds, Proc. Roy. Soc. Edinburgh Sect. A 120 (1992) 61–77.
[5] K. Chen, Equivalence and decomposition of vector fields about an elementary critical point, Amer. J. Math. 85 (1963) 693–722.
[6] V.F. Edneral, A. Mahdi, V.G. Romanovski, D.S. Shafer, The center problem on a center manifold in $\mathbb{R}^3$, Nonlinear Anal. 75 (2012) 2614–2622.
[7] A. Enciso, D. Peralta-Salas, Existence and vanishing set of inverse integrating factors for analytic vector fields, Bull. Lond. Math. Soc. 41 (2009) 1112–1124.
[8] I.A. García, H. Giacomini, M. Grau, The inverse integrating factor and the Poincaré map, Trans. Amer. Math. Soc. 362 (2010) 3591–3612.
[9] I.A. García, M. Grau, A survey on the inverse integrating factor, Qual. Theory Dyn. Syst. 9 (2010) 115–166.
[10] I.A. García, S. Maza, A new approach to center conditions for simple analytic monodromic singularities, J. Differential Equations 248 (2010) 363–380.
[11] H. Giacomini, J. Llibre, M. Viano, On the nonexistence, existence, and uniqueness of limit cycles, Nonlinearity 9 (1996) 501–516.
[12] J. Lü, J. Chen, A new chaotic attractor coined, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 12 (2002) 659–661.
[13] J.E. Marsden, M. McCracken, The Hopf Bifurcation and Its Applications, Appl. Math. Sci., vol. 19, Springer-Verlag, New York, 1976.
[14] L.F. Mello, S.F. Coelho, Degenerate Hopf bifurcations in the Lü system, Phys. Lett. A 373 (2009) 1116–1120.
[15] A. Mahdi, C. Pessoa, D.S. Shafer, Centers on center manifolds in the Lü system, Phys. Lett. A 375 (2011) 3509–3511.
[16] R. Narasimhan, Analysis on Real and Complex Manifolds, North-Holland Math. Library, vol. 35, North-Holland, Amsterdam, 1985, Reprint of the 1973 edition.
[17] G. Reeb, Sur certaines propiétés topologiques des variétés feuilletées, in: W.T. Wu, G. Reeb (Eds.), Sur les espaces fibrés et les variétés feuilletées, Tome XI, in: Actualités Sci. Indust., vol. 1183, Hermann et Cie, Paris, 1952.
[18] V.G. Romanovski, D.S. Shafer, The Center and Cyclicity Problems: A Computational Algebra Approach, Birkhäuser Boston, Inc., Boston, MA, 2009.
[19] J. Sijbrand, Properties of center manifolds, Trans. Amer. Math. Soc. 289 (1985) 431–469.
[20] L. Stolovitch, Progress in normal form theory, Nonlinearity 22 (2009) R77–R99.