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PERIODIC SOLUTIONS OF NONLINEAR PERIODIC DIFFERENTIAL SYSTEMS WITH A SMALL PARAMETER

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Abstract. We deal with nonlinear periodic differential systems depending on a small parameter. The unperturbed system has an invariant manifold of periodic solutions. We provide sufficient conditions in order that some of the periodic orbits of this invariant manifold persist after the perturbation. These conditions are not difficult to check, as we show in some applications. The key tool for proving the main result is the Lyapunov–Schmidt reduction method applied to the Poincaré–Andronov mapping.

1. Introduction.

We consider the problem of bifurcation of T–periodic solutions for a differential system of the form,

$$x'(t) = F_0(t, x) + \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{1}$$

where ε is a small parameter, $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $R : \mathbb{R} \times \Omega \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are C^2 functions, T–periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. One of the main hypotheses is that the unperturbed system

$$x'(t) = F_0(t, x), \tag{2}$$

has a manifold of periodic solutions. This problem was solved before by Malkin (1956) and Roseau (1966) (see [4]). We will give here a new and shorter proof (see Theorem 3.1 and its proof). In addition, we will give a series of corollaries in some particular cases. In order to describe these cases we introduce some notation. We denote the projection onto the first k coordinates by $\pi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ and the one onto the last $(n-k)$ coordinates by $\pi^\perp : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$. For the

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n–dimensional functions F_0 and x we denote $F_0^1 = \pi F_0$, $F_0^2 = \pi^\perp F_0$ and $u = \pi x$, $v = \pi^\perp x$, respectively. We will study the particular situations when the unperturbed system (2) is:

(i) either isochronous, i.e. all its solutions are T–periodic;
(ii) or linear, and it has a k-dimensional manifold of periodic solutions;
(iii) or of the form $u' = F_0^1(t,u)$, $v' = F_0^2(t,u,v)$. Moreover, for all α in some open subset of $\mathbb{R}^k$, the unique solution u_α of $u' = F_0^1(t,u)$ satisfying $u(0) = \alpha$ is T–periodic, and the system $v' = F_0^2(t, u_\alpha(t), v)$ has a unique T–periodic solution.

Case (i) is studied in Section 4. There it is shown that also the classical averaging method for studying periodic solutions can be obtain as a consequence. Case (ii) is considered in Section 5, and, finally, case (iii) in Section 6. Section 2 is dedicated to the main result and its proof. There we use the Lyapunov–Schmidt reduction method for finite dimensional functions, a result that is presented in Section 1. Some remarks are made in Section 7.

The first step in the proof of the main result is to reduce the problem of bifurcation of T–periodic solutions of system (1) to the bifurcation of fixed points of the Poincaré–Andronov mapping, or equivalently, of the zeros of some convenient map $g : D(g) \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ (where $D(g)$ is some open subset of Ω). Since, in general, it is not possible to apply directly the Implicit Function Theorem for the function g, we will use the Lyapunov–Schmidt reduction theory, but not in its general form (like in [3]). This theory here is made simpler by assuming that the Jacobian matrix of $g(\cdot, 0)$ has a particular form. The corresponding hypothesis for the differential system is that some fundamental matrix solution of the linearized system of (2) around each of its periodic solutions has a particular form. But, we will see that this is perfectly suitable for the differential systems considered as examples. The main advantage is that, in this case, the construction of the bifurcation function is easier.

2. **Lyapunov–Schmidt reduction for finite dimensional functions.** The result of this section is known inside the Lyapunov-Schmidt theory, see for instance [3]. Since, in fact, it is a special case of the general theory, we give the proof for completeness. The theorem stated below will be used later in the proof of our main result. We mention that the function f_1 that appears in the following theorem is called the bifurcation function.

Theorem 2.1. *Let* $g : D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$, $\beta_0 : \overline{V} \to \mathbb{R}^{n-k}$ *be* C^2 *functions, where* D *is an open subset of* $\mathbb{R}^n$ *and* V *is an open and bounded subset of* $\mathbb{R}^k$. *We assume that*

(i) $\mathcal{Z} = \left\{ z_\alpha = (\alpha, \beta_0(\alpha)),\ \alpha \in \overline{V} \right\} \subset D$ *and that for each* $z_\alpha \in \mathcal{Z}$, $g(z_\alpha, 0) = 0$;
(ii) *the matrix* $G_\alpha = D_z g(z_\alpha, 0)$ *has in its upper right corner the null* $k \times (n-k)$ *matrix and in the lower right corner the* $(n-k) \times (n-k)$ *matrix* Δ_α, *with* $\det(\Delta_\alpha) \neq 0$.

We consider the function $f_1 : \overline{V} \to \mathbb{R}^k$ *defined by* $f_1(\alpha) = (\partial(\pi g)/\partial\varepsilon)(z_\alpha, 0)$. *If there exists* $a \in V$ *with* $f_1(a) = 0$ *and* $\det((df_1/d\alpha)(a)) \neq 0$, *then there exists* α_ε *such that* $g(z_{\alpha_\varepsilon}, \varepsilon) = 0$ *and* $z_{\alpha_\varepsilon} \to z_a$ *as* $\varepsilon \to 0$.

Proof. We consider the function

$$\pi^\perp g : \mathbb{R}^k \times \mathbb{R}^{n-k} \times [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}^{n-k},\ (\alpha, \beta, \varepsilon) \mapsto \pi^\perp g(\alpha, \beta, \varepsilon).$$

Then, we have $\pi^{\perp} g(z_\alpha, 0) = 0$ and $\left(d\left(\pi^{\perp} g\right)/d\beta\right)(z_\alpha, 0) = \Delta_\alpha$. Since $\det(\Delta_\alpha) \neq 0$, the Implicit Function Theorem implies that, for $|\varepsilon|$ sufficiently small, there exists a function β with $(\alpha, \varepsilon) \mapsto \beta(\alpha, \varepsilon)$ such that

$$\beta(\alpha, 0) = \beta_0(\alpha) \text{ and } \pi^{\perp} g(\alpha, \beta(\alpha, \varepsilon), \varepsilon) = 0.$$

Now we consider the function

$$\delta : \mathbb{R}^k \times [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}^k \text{ given by } \delta(\alpha, \varepsilon) = \pi g(\alpha, \beta(\alpha, \varepsilon), \varepsilon).$$

We have

$$\delta(\alpha, 0) = \pi g(z_\alpha, 0) = 0,$$
$$\delta_\varepsilon(\alpha, 0) = \frac{d\,(\pi g)}{d\beta}(z_\alpha, 0) \cdot \frac{d\beta}{d\varepsilon}(\alpha, 0) + \frac{\partial\,(\pi g)}{\partial \varepsilon}(z_\alpha, 0).$$

Using (ii) we see that $\left(d(\pi g)/d\beta\right)(z_\alpha, 0) = 0_{k\times(n-k)}$, where $0_{k\times(n-k)}$ is the null $k \times (n-k)$ matrix.

Hence, $\delta_\varepsilon(\alpha, 0) = f_1(\alpha)$ and $\delta(\alpha, \varepsilon) = \varepsilon f_1(\alpha) + \varepsilon^2 r(\alpha, \varepsilon)$. By the Implicit Function Theorem, we obtain for $|\varepsilon|$ sufficiently small, the existence of $\alpha(\varepsilon)$ such that $\alpha(0) = a$ and $f_1(\alpha(\varepsilon)) + \varepsilon r(\alpha, \varepsilon) = 0$. Hence, $\delta\left(\alpha(\varepsilon), \varepsilon\right) = 0$ and, moreover, denoting $z_{\alpha_\varepsilon} = (\alpha(\varepsilon), \beta(\alpha(\varepsilon), \varepsilon))$ we have $g(z_{\alpha_\varepsilon}, \varepsilon) = 0$. □

3. **Main theorem.** For $z \in \Omega$ we denote by $x(\cdot, z, \varepsilon) : [0, t_{(z,\varepsilon)}) \to \mathbb{R}^n$ the solution of (1) with $x(0, z, \varepsilon) = z$. From Theorem 8.3 of [1] we deduce that, whenever $t_{(z_0,0)} > T$ for some $z_0 \in \Omega$ there exists a neighborhood of $(z_0, 0)$ in $\Omega \times (-\varepsilon_f, \varepsilon_f)$ such that, for all (z, ε) in this neighborhood, $t_{(z,\varepsilon)} > T$. In this work, one of the main assumptions is the existence of T–periodic solutions of system (1) for $\varepsilon = 0$. Under this assumption there exists an open subset D of Ω and a sufficiently small $\varepsilon_0 > 0$ such that, for all $(z, \varepsilon) \in D \times (-\varepsilon_0, \varepsilon_0)$, the solution $x(\cdot, z, \varepsilon)$ is defined on the interval $[0, T]$. Hence, we can consider the function $f : D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$, given by

$$f(z, \varepsilon) = x(T, z, \varepsilon) - z. \tag{3}$$

Then, every $(z_\varepsilon, \varepsilon)$ such that

$$f(z_\varepsilon, \varepsilon) = 0 \tag{4}$$

provides the periodic solution $x(\cdot, z_\varepsilon, \varepsilon)$ of (1).

The converse is also true, i.e. for every T–periodic solution of (1), if we denote by z_ε its value at $t = 0$ then (4) holds. Then, the problem of finding a T–periodic solution of (1), can be replaced by the problem of finding zeros of the finite–dimensional function $f(\cdot, \varepsilon)$ given by (3).

We denote the linearization of (2) by

$$y' = P(t, z)y, \tag{5}$$

where

$$P(t, z) = D_x F_0(t, x(t, z, 0)), \tag{6}$$

and let $Y(\cdot, z)$ be some fundamental matrix solution of (5).

The next theorem is our main result. Various consequences of it will be given in the next sections. In the proof we apply Theorem 2.1 to the function (3) after a suitable change of coordinates.

Theorem 3.1. *Let $\beta_0 : \overline{V} \to \mathbb{R}^{n-k}$ be a C^2 function, where $V \subset \mathbb{R}^k$ is open and bounded. We assume that*

(i) *$\mathcal{Z} = \left\{z_\alpha = (\alpha, \beta_0(\alpha)),\ \alpha \in \overline{V}\right\} \subset D$ and that for each $z_\alpha \in \mathcal{Z}$, the unique solution x_α of (2) with $x(0) = z_\alpha$, is T–periodic;*

(ii) *for each* $z_\alpha \in \mathcal{Z}$, *there exists a fundamental matrix solution of (5),* $Y_\alpha(t) = Y(t, z_\alpha)$ *such that the matrix* $Y_\alpha^{-1}(0) - Y_\alpha^{-1}(T)$ *has in the upper right corner the null* $k \times (n-k)$ *matrix, while in the lower right corner has the* $(n-k) \times (n-k)$ *matrix* Δ_α, *with* $\det(\Delta_\alpha) \neq 0$.

We consider the function $f_1 : \overline{V} \to \mathbb{R}^k$ *given by*

$$f_1(\alpha) = \pi \int_0^T Y_\alpha^{-1}(t) F_1(t, x_\alpha(t)) dt. \tag{7}$$

If there exists $a \in V$ *with* $f_1(a) = 0$ *and* $\det\left(\left(df_1/d\alpha\right)(a)\right) \neq 0$, *then there exists a* T*–periodic solution* $\varphi(\cdot, \varepsilon)$ *of system (1) such that* $\varphi(0, \varepsilon) \to z_a$ *as* $\varepsilon \to 0$.

Proof. We need to study the zeros of the function (3), or, equivalently, of

$$g(z, \varepsilon) = Y^{-1}(T, z) f(z, \varepsilon).$$

We have that $g(z_\alpha, 0) = 0$, because $x(\cdot, z_\alpha, 0)$ is T–periodic, and we shall prove that

$$G_\alpha = \frac{dg}{dz}(z_\alpha, 0) = Y_\alpha^{-1}(0) - Y_\alpha^{-1}(T). \tag{8}$$

For this we need to know $(\partial x/\partial z)(\cdot, z, 0)$. Since it is the matrix solution of (5) with $(\partial x/\partial z)(0, z, 0) = I_n$, we have that $(\partial x/\partial z)(t, z, 0) = Y(t, z) Y^{-1}(0, z)$. Moreover,

$$\frac{df}{dz}(z, 0) = \frac{\partial x}{\partial z}(T, z, 0) - I_n = Y(T, z) Y^{-1}(0, z) - I_n$$

and

$$\frac{dg}{dz}(z, 0) = Y^{-1}(0, z) - Y^{-1}(T, z) + \left(\frac{\partial Y^{-1}}{\partial z_1}(T, z) f(z, 0), ..., \frac{\partial Y^{-1}}{\partial z_n}(T, z) f(z, 0) \right),$$

which, for $z_\alpha \in \mathcal{Z}$, reduces to (8).

We have

$$\frac{\partial g}{\partial \varepsilon}(z, 0) = Y^{-1}(T, z) \frac{\partial x}{\partial \varepsilon}(T, z, 0).$$

The function $(\partial x/\partial \varepsilon)(\cdot, z, 0)$ is the unique solution of the IVP

$$y' = D_x F_0(t, x(t, z, 0)) y + F_1(t, x(t, z, 0)), \; y(0) = 0.$$

Hence,

$$\frac{\partial x}{\partial \varepsilon}(t, z, 0) = Y(t, z) \int_0^t Y^{-1}(s, z) F_1(s, x(s, z, 0)) ds.$$

Now, we have

$$\frac{\partial g}{\partial \varepsilon}(z, 0) = \int_0^T Y^{-1}(s, z) F_1(s, x(s, z, 0)) ds,$$

Hence,

$$\frac{\partial (\pi g)}{\partial \varepsilon}(z_\alpha, 0) = f_1(\alpha),$$

where f_1 is given by (7). Applying Theorem 2.1, there exists $\alpha_\varepsilon \in V$ such that $g(z_{\alpha_\varepsilon}, \varepsilon) = 0$ and, further, $f(z_{\alpha_\varepsilon}, \varepsilon) = 0$, which assures that $\varphi(\cdot, \varepsilon) = x(\cdot, z_{\alpha_\varepsilon}, \varepsilon)$ is a T–periodic solution of (1). □

4. **Case (i): Perturbations of an isochronous system and the first order averaging method.** In this section we assume that there exists an open set V with $\overline{V} \subset D$ and such that for each $z \in \overline{V}$, $x(\cdot, z, 0)$ is T–periodic (we recall that $x(\cdot, z, 0)$ is the solution of the unperturbed system (2) with $x(0) = z$). An answer to the problem of bifurcation of T–periodic solutions from $x(\cdot, z, 0)$ is given in the following result. It is obtained as a consequence of Theorem 3.1 by considering $k = n$.

Corollary 1. (Perturbations of an isochronous system) *We assume that there exists an open set V with $\overline{V} \subset D$ and such that for each $z \in \overline{V}$, $x(\cdot, z, 0)$ is T–periodic and we consider the function $f_1 : \overline{V} \to \mathbb{R}^n$ given by*

$$f_1(z) = \int_0^T Y^{-1}(t, z) F_1(t, x(t, z, 0))dt. \tag{9}$$

If there exists $a \in V$ with $f_1(a) = 0$ and $\det\left(\left(df_1/d\alpha\right)(a)\right) \neq 0$, then there exists a T–periodic solution $\varphi(\cdot, \varepsilon)$ of system (1) such that $\varphi(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

A particular case is when F_0 is identically zero, i.e. the system (2) becomes $x' = 0$ and hence all its solutions are constant, $x(t, z, 0) = z$ for all $t \in \mathbb{R}$. Of course, the linearized system is the same, and we take as its fundamental matrix solution $Y(t, z) = I_n$, the identity matrix, for all $t \in \mathbb{R}$ and $z \in \overline{V}$. It is easy to see now that the well known averaging method (see, for example [9, 2]) is obtained as consequence of the above Corollary.

Corollary 2. (The first order averaging method) *We assume that $F_0(t, x)$ is identically zero and we consider the function $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ given by*

$$f_1(z) = \int_0^T F_1(t, z)dt. \tag{10}$$

If there exists $a \in V$ with $f_1(a) = 0$ and $\det\left(\left(df_1/d\alpha\right)(a)\right) \neq 0$, then there exists a T–periodic solution $\varphi(\cdot, \varepsilon)$ of system (1) such that $\varphi(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

5. **Case (ii): Perturbations of a linear system.** In this section we consider the system (1) with $F_0(t, x) = P(t)x + q(t)$, i.e. the unperturbed system (2) is the linear system $x' = P(t)x + q(t)$. Before stating the main result as a consequence of Theorem 3.1, we need two lemmas from linear systems theory.

Lemma 5.1. *Let $P : \mathbb{R} \to \mathcal{M}_n$ be a continuous and T–periodic function and consider the system*

$$y' = P(t)y. \tag{11}$$

The following statements are equivalent:

(i) *the system (11) has k T–periodic linearly independent solutions.*
(ii) *there exists a fundamental matrix of solutions, $Y(t)$, of (11) such that $Y^{-1}(t)$ has in its first k lines only T–periodic functions.*

Proof. We consider the adjoint system

$$y' = -P^*(t)y, \tag{12}$$

where $P^*(t)$ is the transpose matrix of $P(t)$. A nonsingular $n \times n$ matrix $Y(t)$ is a fundamental matrix solution for (11) if and only if $Y_a(t) = \left(Y^{-1}(t)\right)^*$ is a fundamental matrix for (12) (Lemma 7.1 page 62, [5]).

The systems (11) and (12) have the same number of linearly independent T–periodic solutions (Lemma 1.3 page 410 [5]). Hence, (i) is equivalent to the fact that

(12) has k T–periodic linearly independent solutions. Moreover, this is equivalent to the existence of some fundamental matrix of solutions for (12), denoted Y_a, that has in the first k columns only T–periodic functions. Further, using that $Y^{-1}(t) = Y_a^*(t)$, this is equivalent to (ii). □

Lemma 5.2. *Let $P : \mathbb{R} \to \mathcal{M}_n$ and $q : \mathbb{R} \to \mathbb{R}^n$ be continuous and T–periodic functions. We assume that the system (11) has k T–periodic linearly independent solutions and we denote by $Y(t)$ its fundamental matrix of solutions as given by Lemma 5.1 (ii). In addition, we assume that*

(i) $\pi \int_0^T Y^{-1}(s)q(s)ds = 0$,
(ii) $\det(\Delta) \neq 0$, *where Δ is the $(n-k) \times (n-k)$ matrix from the lower right corner of the $n \times n$ matrix $Y^{-1}(0) - Y^{-1}(T)$.*

Then there exists $\beta_0 : \mathbb{R}^k \to \mathbb{R}^{n-k}$ such that, for all $\alpha \in \mathbb{R}^k$, $z_\alpha = (\alpha, \beta_0(\alpha))$ satisfies

$$[Y^{-1}(T) - Y^{-1}(0)]z = \int_0^T Y^{-1}(s)q(s)ds. \tag{13}$$

Moreover, for all $\alpha \in \mathbb{R}^k$, the unique solution of

$$x' = P(t)x + q(t), \tag{14}$$

with $x(0) = z_\alpha$, is T–periodic.

Proof. Since the matrix $Y^{-1}(T) - Y^{-1}(0)$ has the first k lines identically 0 and we have (i), the first k equations in the system (13) are the trivial ones, i.e $0 = 0$. Using (ii) we obtain the solution of this system as $z_\alpha = (\alpha, \beta_0(\alpha))$ for all $\alpha \in \mathbb{R}^k$.

Denoting by $x(\cdot, z)$ the solution of (14) with $x(0) = z$ and $f_0(z) = x(T, z) - z$, we have that

$$Y^{-1}(T)f_0(z) = -[Y^{-1}(T) - Y^{-1}(0)]z + \int_0^T Y^{-1}(s)q(s)ds.$$

Then, every zero of f_0 is a solution of the linear algebraic system (13). The last part of the conclusion follows now from the correspondence between the zeros of f_0 and the T–periodic solutions of (14). □

As a consequence of Theorem 3.1 it is easy to obtain the following Corollary. This result is known as the Theorem of Malkin (see [4]).

Corollary 3. *Consider the system (1) with $F_0(t, x) = P(t)x + q(t)$ and assume that all the hypotheses of Lemma 5.2 are satisfied. Let the function $f_1 : \mathbb{R}^k \to \mathbb{R}^k$ be given by*

$$f_1(\alpha) = \pi \int_0^T Y^{-1}(t)F_1(t, x_\alpha(t))dt.$$

If there exists $a \in V$ with $f_1(a) = 0$ and $\det((df_1/d\alpha)(a)) \neq 0$, then there exists a T–periodic solution $\varphi(\cdot, \varepsilon)$ of system (1) such that $\varphi(0, \varepsilon) \to z_a$ as $\varepsilon \to 0$.

6. **Case (iii).** In this section we consider the system

$$\begin{aligned} u'(t) &= F_0^1(t, u) + \varepsilon F_1^1(t, u, v) + \varepsilon^2 R^1(t, u, v, \varepsilon), \\ v'(t) &= F_0^2(t, u, v) + \varepsilon F_1^2(t, u, v) + \varepsilon^2 R^2(t, u, v, \varepsilon), \end{aligned} \tag{15}$$

where $F_0 = (F_0^1, F_0^2)$, $F_1 = (F_1^1, F_1^2)$ and $R = (R^1, R^2)$ satisfy the hypotheses stated in the Introduction, and the splitting is with respect to the projectors $(\pi, \pi^\perp)$. We assume that there exists an open set V with $\overline{V} \subset \pi\Omega$ such that, for each $\alpha \in \overline{V}$, the unique solution u_α of $u'(t) = F_0^1(t,u)$ satisfying $u(0) = \alpha$ is T–periodic, and the system $v' = F_0^2(t,u_\alpha(t),v)$ has a unique T–periodic solution. Before stating the main results, we give the following lemma.

Lemma 6.1. *Let $P : \mathbb{R} \to \mathcal{M}_n$ be a continuous and T–periodic function such that, for all $t \in \mathbb{R}$, the matrix $P(t)$ has in the upper right corner the null $k \times (n-k)$ matrix and it has the block form*

$$P(t) = \begin{pmatrix} A(t) & 0 \\ B(t) & C(t) \end{pmatrix}.$$

Then there exists $Y(t)$ a fundamental matrix of solutions of the system

$$y' = P(t)y, \tag{16}$$

such that $Y^{-1}(t)$ has in the upper right corner the null $k \times (n-k)$ matrix. Moreover,

$$Y^{-1}(t) = \begin{pmatrix} U^{-1}(t) & 0 \\ W(t) & V^{-1}(t) \end{pmatrix},$$

where $U(t)$ and $V(t)$, respectively, are fundamental matrices solutions of $u' = A(t)u$ and $v' = C(t)v$.

Proof. For $y \in \mathbb{R}^n$ we define $u = \pi y \in \mathbb{R}^k$ and $v = \pi^\perp y \in \mathbb{R}^{n-k}$. Then, the adjoint system, $y' = -P^*(t)y$, can be written as

$$u' = -A^*(t)u - B^*(t)v, \quad v' = -C^*(t)v. \tag{17}$$

Denoting $U_a(t)$ and $V_a(t)$, respectively, some fundamental matrix solutions for $u' = -A^*(t)u$ and $v' = -C^*(t)v$, we see that

$$Y_a(t) = \begin{pmatrix} U_a(t) & W_a(t) \\ 0_{(n-k)\times k} & V_a(t) \end{pmatrix}$$

is a fundamental matrix solution for (17). Hence, the fundamental matrix of solutions of (16), $Y(t)$, satisfying $Y^{-1}(t) = Y_a^*(t)$, has the required property. □

Let $U_\alpha(t)$ and $V_\alpha(t)$, respectively, be fundamental matrix solutions for the systems $u' = A_\alpha(t)u$ and $v' = C_\alpha(t)v$, where $A_\alpha(t) = D_u F_0^1(t,u_\alpha(t))$ and $C_\alpha(t) = D_v F_0^2(t,u_\alpha(t),v_\alpha(t))$. The following corollary of Theorem 3.1 is the main result of this section.

Corollary 4. *Assume that there exists an open set V with $\overline{V} \subset \pi\Omega$ such that, for each $\alpha \in \overline{V}$, $u_\alpha(\cdot)$ is T–periodic, and the system $v' = F_0^2(t,\alpha,v)$ has a unique T–periodic solution, denoted v_α. Moreover, assume that the matrix $\Delta_\alpha = V_\alpha^{-1}(0) - V_\alpha^{-1}(T)$ has $\det(\Delta_\alpha) \neq 0$ and consider the function $f_1 : \overline{V} \to \mathbb{R}^k$ given by*

$$f_1(\alpha) = \int_0^T U_\alpha^{-1}(t) F_1(t,u_\alpha(t),v_\alpha(t))dt. \tag{18}$$

If there exists $a \in V$ with $f_1(a) = 0$ and $\det\left((df_1/d\alpha)(a)\right) \neq 0$, then there exists a T–periodic solution $\varphi(\cdot,\varepsilon)$ of system (15) such that $\varphi(0,\varepsilon) \to (a, v_a(0))$ as $\varepsilon \to 0$.

Proof. We consider the function $\beta_0 : \overline{V} \to \mathbb{R}^{n-k}$ given by $\beta_0(\alpha) = v_\alpha(0)$. Then the set $\mathcal{Z} = \left\{ z_\alpha = (\alpha, \beta_0(\alpha)),\ \alpha \in \overline{V} \right\}$ satisfies hypothesis (i) of Theorem 3.1.

The matrix $P(t,z)$ given by (6) has in the upper right corner the null $k \times (n-k)$ matrix because F_0^1 does not depend on v. Then, by Lemma 6.1, there exists $Y(t)$ a fundamental matrix of solutions of the system (5) such that $Y^{-1}(t,z)$ has in the upper right corner the null $k \times (n-k)$ matrix. In particular, this is true for the matrix $Y_\alpha^{-1}(0) - Y_\alpha^{-1}(T)$. Since, also by Lemma 6.1, this matrix has in the lower right corner the matrix $V_\alpha^{-1}(0) - V_\alpha^{-1}(T)$, we see that also hypothesis (ii) is fulfilled. The form of the function f_1 follows from the specific form of $Y_\alpha^{-1}(t)$. □

For the particular case when F_0^1 is identically zero, the result is given in the following corollary.

Corollary 5. *Assume that $F_0^1(t,u)$ is identically zero and that the system $v' = F_0^2(t,\alpha,v)$ has a unique T–periodic solution, denoted v_α. Moreover, assume that the matrix $\Delta_\alpha = V_\alpha^{-1}(0) - V_\alpha^{-1}(T)$ has $\det(\Delta_\alpha) \neq 0$, and consider the function $f_1 : \overline{V} \to \mathbb{R}^k$ given by*

$$f_1(\alpha) = \int_0^T F_1(t,\alpha,v_\alpha(t))dt. \tag{19}$$

If there exists $a \in V$ with $f_1(a) = 0$ and $\det\left(\left(df_1/d\alpha\right)(a)\right) \neq 0$, then there exists a T–periodic solution $\varphi(\cdot,\varepsilon)$ of system (15) such that $\varphi(0,\varepsilon) \to (a, v_a(0))$ as $\varepsilon \to 0$.

7. **Remarks.** 1- Weaker versions of the theorem presented here have been used in applications. Let us, for instance, mention the repeated use of Malkin's theorem to establish synchronization of weakly coupled oscillators. We mention references linked with mathematical physiology. Synchronization of the electrical activity of cardiac cells in the sinusal node explains the formation of the cardiac rythm (see for instance [7], p. 427). Also, it is now believed that synchronization of electrical neurons plays a key role in explaining brain activity in neurosciences (see for instance [6]). There are many other applications to mechanics and physics, which are, in some sense, more classical.

2- There are possible applications to frequency locking, as it appears, for instance in the periodically forced Van der Pol oscillator.

Consider the perturbed equation

$$\frac{dx}{dt} = F_0(x) + \varepsilon F_1(x,t,\varepsilon),$$

where the unperturbed part displays a periodic solution of period T. Assume that the perturbation is periodic of period $T' = pT(1+\varepsilon\delta(\varepsilon))/q$. Perform the change of variables $t = \tau(1+\varepsilon\delta(\varepsilon))$, which transforms the equation into

$$\frac{dx}{d\tau} = F_0(x) + \varepsilon G(x,\tau,\varepsilon),$$

with G periodic of period $T'' = \frac{p}{q}T$ relatively to the time τ. The preceding theorem shows, under some conditions, the existence of periodic solutions for the perturbed system of period pT and hence of period qT'. This "adaptation" of the oscillation on a multiple of the period of the forcing term was observed for the first time by van der Pol.

3- Finally, consider the special case of Hamiltonian dynamics in dimension $n = 2m$:

$$H(p,q,\varepsilon) = H_0(p,q) + \varepsilon H_1(p,q) + O(\varepsilon^2).$$

It is interesting to note that in the case where the unperturbed dynamics is isochronous (all orbits of the associated Hamiltonian system $H_0(p,q)$ are periodic of same period T), the bifurcation function takes the special form

$$f_1(p,q) = \left(\frac{\partial \overline{H}_1}{\partial q}(p,q), \frac{\partial \overline{H}_1}{\partial p}(p,q)\right),$$

where $\overline{H}_1(p,q)$ is the Hamiltonian $H_1(p,q)$ averaged along the periodic orbits of H_0. Our theorem extends in this case a well–known theorem of J. Moser (see [8]).

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