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SOME REMARKS ON THE MONOTONE ITERATIVE TECHNIQUE

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Abstract. We consider an abstract operator equation in coincidence form $Lu = N(u)$ and establish some comparison results and existence results via the monotone iterative technique. We use a generalized iteration method developed by Carl-Heikkila (1999). An application to a boundary value problem for a second-order functional differential equation is considered.

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1. INTRODUCTION

Let X be a nonempty set and Z be an ordered metric space. Let us consider the operator equation of the form

$$Lu = Nu, \tag{1.1}$$

and the iterative scheme

$$Lu_{n+1} = Nu_n, \tag{1.2}$$

where $L, N : X \to Z$.

In our work the operators L and N will satisfy some extended monotonicity conditions, which are described exactly in the following definition.

DEFINITION 1.1. *N is monotone increasing with respect to L if $u_1, u_2 \in X$ and $Lu_1 \le Lu_2$ imply that $Nu_1 \le Nu_2$.*

If in the last relation the reversed inequality holds, then N is monotone decreasing with respect to L.

Let X be an ordered set. If $Lu_1 \le Lu_2$ implies $u_1 \le u_2$ then L is said to be inverse-monotone (*see* [8]) *or of monotone-type* (*see* [9]).

The plan of our paper is as follows. In Section 2 we deal with operator inequalities corresponding with (1.1) and extend the abstract Gronwall lemma of Rus [6]. Let us mention that the result from [6] generalize some results from [9] and [11]. In Section 3 we generalize some known existence results for equation (1.1) ([5, 10, 4, 1, 9]) involving monotone increasing or monotone decreasing operators. We shall use a generalized

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iteration method developed in [2]. In Section 4 we shall apply some of our results to implicit second order functional-differential equations. For another treatment of this type of functional-differential equations it can be seen [7].

2. OPERATOR INEQUALITIES IN ORDERED METRIC SPACES

In this section we extend the notion of Picard operator [6], in Definition 2.1, and the abstract Gronwall lemma of Rus [6], in Theorem 2.3. The above mentioned notion and result correspond, in our setting, with the case when $X = Z$ and L is the identity mapping of Z. As a consequence of Theorem 2.3, we shall find a condition in Corollary 2.4, which assure the existence of ordered lower and upper solutions for equation (1.1).

Definition 2.1. *N is Picard with respect to L if there exists a unique $v^* \in Z$ with the following properties.*

(i) *there exists $u^* \in X$ such that $Lu^* = Nu^* = v^*$;*
(ii) *$N(X) \subset L(X)$;*
(iii) *for every $u_0 \in X$ a sequence defined by* (1.2) *is such that $(Lu_n)_{n\geq 0}$ is convergent to v^*.*

Example 2.1. If $X = Z$ and $N : Z \to Z$ is Picard [6] then N is Picard with respect to I, the identity mapping of Z.

Example 2.2. If L is inversable and $N \circ L^{-1} : Z \to Z$ is Picard, then N is Picard with respect to L.

Example 2.3. Let $L, N : (0,\infty) \to (-1,\infty)$ be given by $L(u) = u^2 - 1$ and $N(u) = \sqrt{u}$. Then N is Picard with respect to L. Let us mention that, also, N is monotone increasing with respect to L.

Example 2.4. If Z is also a complete metric space, L is surjective and N is contraction with respect to L then N is Picard with respect to L. Let us mention that N is contraction with respect to L if there exists $0 < a < 1$ such that for all $u_1, u_2 \in X$, $d(Nu_1, Nu_2) \leq a \cdot d(Lu_1, Lu_2)$.

For the proof of this result, also known as the *Coincidence Theorem of Goebel*, we refer to [3].

Lemma 2.2. *If N is monotone increasing* (*or monotone decreasing or contraction*) *with respect to L then*

$$Lu_1 = Lu_2 \text{ implies } Nu_1 = Nu_2.$$

If L is inverse-monotone then L is injective.

Proof. Let us consider only that N is monotone increasing with respect to L.

If $Lu_1 = Lu_2$ then $Lu_1 \leq Lu_2$ and $Lu_2 \leq Lu_1$. Thus, $Nu_1 \leq Nu_2$ and $Nu_2 \leq Nu_1$. This obviously implies the conclusion.

For the last statement we have to prove that, if L is of monotone type, then $Lu_1 = Lu_2$ implies $u_1 = u_2$. This can be done like above. □

THEOREM 2.3. *If N is monotone increasing with respect to L and N is Picard with respect to L then*

(i) $Lu_0 \leq Nu_0$ *implies* $Lu_0 \leq v^*$,
(ii) $Lu_0 \geq Nu_0$ *implies* $Lu_0 \geq v^*$.

If, in addition, L is of monotone type then

(j) $Lu_0 \leq Nu_0$ *implies* $u_0 \leq u^*$,
(jj) $Lu_0 \geq Nu_0$ *implies* $u_0 \geq u^*$.

Proof. Let us consider $u_0 \in X$ such that $Lu_0 \leq Nu_0$ and the sequence defined by (1.2) starting from u_0. The following relations hold,

$$Lu_0 \leq Nu_0 = Lu_1 \leq Nu_1 = Lu_2 \leq Nu_2 \leq \ldots .$$

Thus, for all $n \geq 0$,

$$Lu_0 \leq Nu_n,$$

and, passing to the limit when $n \to \infty$,

$$Lu_0 \leq v^*.$$

The next relation can be proved similarly.

If, in addition, L is inverse-monotone, then, by Lemma 2.2, u^* given by Definition 2.1 is unique and, of course, $Lu_0 \leq Lu^*$ implies $u_0 \leq u^*$. □

We say that $\underline{u} \in X$ is a *lower solution* of (1.1) if $L\underline{u} \leq N\underline{u}$. Similarly, $\bar{u} \in X$ is an *upper-solution* of (1.1) if $L\bar{u} \geq N\bar{u}$.

COROLLARY 2.4. *Let us consider two operators $\underline{N}, \bar{N} : X \to Z$ such that they are monotone increasing with respect to L and Picard with respect to L. If*

$$\underline{N}u \leq Nu \leq \bar{N}u, \quad \text{for all } u \in X, \tag{2.1}$$

then there exist $\underline{u}$ a lower solution and $\bar{u}$ an upper-solution of (1), *such that*

$$L\underline{u} \leq L\bar{u}.$$

If, in addition, L is of monotone type then,

$$\underline{u} \leq \bar{u}.$$

Proof. $\underline{N}$ and $\bar{N}$ being Picard with respect to L, there exist $\underline{u}$ such that

$$L\underline{u} = \underline{N}\underline{u},$$

and $\bar{u}$ such that

$$L\bar{u} = \bar{N}\bar{u}.$$

Then, by (3), $L\underline{u} \le N\underline{u}$ and $L\bar{u} \ge N\bar{u}$, which mean that $\underline{u}$ is a lower solution and $\bar{u}$ is a super-solution of (1.1).
Also by (2.1), the following inequality holds

$$L\underline{u} \le \bar{N}\underline{u}.$$

We apply now Theorem 2.3 for $\bar{N}$ and deduce that

$$L\underline{u} \le L\bar{u}.$$

The last part of the conclusion follows in an obvious way. □

3. OPERATOR EQUATIONS IN ORDERED BANACH SPACES

In this section we shall establish two existence results for equation (1.1), involving an operator N which is increasing with respect to L, in Theorem 3.2, respectively monotone decreasing with respect to L, in Theorem 3.3. We shall use a generalized iteration method developed in [2]. As it is mentioned in [2], this method enlarges the range of applications since neither L nor N need be continuous. In this spirit, Theorem 3.2 generalizes Theorem 3.1 in [4], and Theorem 3.3 generalizes Theorem 3 in [10] and Theorem 2 in [5] (these are given in the case $X = Z$ and $L = I$).

The following result is Proposition 3.4 from [2] and we shall use it to derive Theorem 3.2.

Proposition 3.1. *Assume that the following conditions hold.*

(i) *There exists $\underline{u}$ a lower solution of* (1.1), $\underline{u} \in W \subset X$;
(ii) N *is monotone increasing with respect to* L;
(iii) $L(W)$ *is an ordered metric space and if* (u_n) *is a sequence in* W *such that the sequences* (Lu_n) *and* (Nu_n) *are increasing, then* (Nu_n) *converges in* $L(W)$.

Then (1.1) *has a solution* u_* *with the property*

$$Lu_* = \min\{Lw \in L(W) \mid L\underline{u} \le Lw \text{ and } Lw \ge Nw \}.$$

If, in addition, W *is an ordered space and* L *is of monotone type, then* u_* *is the minimal solution of* (1.1) *in* $W_0 = \{u \in W \mid L\underline{u} \le Lu\}$.

We notice that the dual result is valid.

In the following results, i.e. Theorem 3.2 and Theorem 3.3, Z will be an ordered Banach space (OBS) with a normal cone K.

Let us remember, (see [5, 1, 10]) that the cone $K = \{v \in Z \mid v \ge 0\}$ is said to be *normal* if there exists $\delta > 0$ such that $0 \le v \le w$ implies $||v|| \le \delta ||w||$.

For $v \le w$ the order interval $[v, w]$ is the set of all $u \in Z$ such that $v \le u \le w$. Every order interval for an OBS is bounded if and only if the cone K is normal. In an OBS with a normal cone, every monotone increasing sequence which has a convergent subsequence, is convergent.

A cone K is said to be *regular* if every monotone increasing sequence contained in some order interval, is convergent.

Theorem 3.2. *Assume that the following conditions hold.*

(i) $\underline{u}$ *is a lower solution and* $\bar{u}$ *is a super-solution of* (1.1) *with* $L\underline{u} \leq L\bar{u}$*;*
(ii) N *is monotone increasing with respect to* L*;*
(iii) $[L\underline{u}, L\bar{u}] \subset L(X)$*;*
(iv) K *is regular or* $[N\underline{u}, N\bar{u}] \cap N(X)$ *is a compact subset of* Z*.*

Then (1.1) *has a solution* u_* *with the property*

$$Lu_* = \min\{Lw \in [L\underline{u}, L\bar{u}] \mid Lw \geq Nw\ \}$$

and a solution u^* *with the property*

$$Lu^* = \max\{Lw \in [L\underline{u}, L\bar{u}] \mid Lw \leq Nw\ \}.$$

If, in addition, L *is of monotone type, then* u_* *is the minimal solution, and* u^* *the maximal solution of* (1) *in* $[\underline{u}, \bar{u}]$*.*

Proof. Let us consider $W = \{u \in X \mid L\underline{u} \leq Lu \leq L\bar{u}\}$. Then, using also (iii), $L(W) = [L\underline{u}, L\bar{u}]$, which is a closed subset of Z, thus is an ordered metric space.
Let (u_n) be a sequence in W such that (Lu_n) and (Nu_n) are increasing. Using (i) and (ii), $L\underline{u} \leq Lu_n \leq L\bar{u}$ imply that $L\underline{u} \leq N\underline{u} \leq Nu_n \leq N\bar{u} \leq L\bar{u}$. Then (Nu_n) is an increasing sequence in the bounded (because K is normal) interval $L(W)$.
If K is regular, then (Nu_n) converges.
If $[N\underline{u}, N\bar{u}] \cap N(X)$ is compact, then (Nu_n) has a convergent subsequence. By the monotonicity of the sequence (Nu_n), it converges.
All the hypotheses of Proposition 3.1 are fulfilled. Hence, the conclusion follows. □

Remark 3.1. If, in addition to the hypotheses of Theorem 3.2, N is continuous with respect to L then u^* can be obtained by (1.2) starting from $\underline{u}$, in the sense that a sequence defined by (1.2) with $u_0 = \underline{u}$ is such that (Lu_n) converges to Lu^*.
Let us mention that N is said to be continuous with respect to L if for every sequence (Lu_n) from $L(X)$ convergent to $Lu^* \in L(X)$, the sequence (Nu_n) converges to Nu^*.

Theorem 3.3. *Assume that the following conditions hold.*

(i) $Lu \geq 0$ *implies* $Nu \geq 0$*;*
(ii) N *is monotone decreasing with respect to* L*;*
(iii) *if* u_0 *and* u_1 *are such that* $Lu_0 = 0$, $Nu_0 = Lu_1$ *then* $Nu_1 > 0$ *and* $[0, Lu_1] \subset L(X)$*;*
(iv) *there exists* $\alpha \in (-1, 0)$ *such that* $Nu_\mu \leq \mu^\alpha Nu$ *for all* $u, u_\mu \in X$ *with (*$0 \leq Lu \leq Lu_1$ *and* $Lu_\mu = \mu Lu$*) and for all* $\mu \in (0,1)$*;*
(v) *for every* v, w *with* $0 < v \leq w \leq Lu_1$ *there is* $\mu \in (0,1)$ *such that* $\mu w \leq v$*,*
(vi) *the cone* K *is regular or* $[Nu_1, Nu_0] \cap N(X)$ *is a compact subset of* Z*.*

Then (1.1) *has a solution,* u^* *with* $Lu^* > 0$*.*

Proof. For every $u \in X$, if $\tilde{u}$ is such that $Nu = L\tilde{u}$, let us define $\tilde{N}u = N\tilde{u}$. By Lemma 2.2, $\tilde{N}u$ does not depend on the choice of $\tilde{u}$, thus the operator $\tilde{N} : X \to Z$ is well-defined.

$\tilde{N}$ is monotone increasing with respect to L. Indeed, $Lu_1 \leq Lu_2 \Rightarrow$, $L\tilde{u_1} = Nu_1 \geq Nu_2 = L\tilde{u_2} \Rightarrow \tilde{N}u_1 = N\tilde{u_1} \leq N\tilde{u_2} = \tilde{N}u_2$.

Let us consider also u_2, u_3 such that $Nu_1 = Lu_2$ and $Nu_2 = Lu_3$. By (i) and (iii), $Lu_1 \geq 0 = Lu_0$, which implies, by (ii), that $Nu_1 \leq Nu_0$. Using the definitions of u_2 and u_1, the following relation holds.

$$Lu_2 \leq Lu_1. \tag{3.1}$$

We shall focus our attention to the equation

$$Lu = \tilde{N}u. \tag{3.2}$$

We shall prove that u_2 is a lower solution and u_1 is an upper solution of (3.2). This follows by the following implications.

$$Nu_1 \geq 0 = Lu_0 \Rightarrow Lu_2 \geq Lu_0 \Rightarrow Nu_2 \leq Nu_0 \Rightarrow \tilde{N}u_1 \leq Lu_1,$$

and

$$Nu_2 \leq Nu_0 \Rightarrow Lu_3 \leq Lu_1 \Rightarrow Nu_3 \geq Nu_1 \Rightarrow \tilde{N}u_2 \geq Lu_2.$$

We use Theorem 3.2 and deduce that equation (3.2) has a solution u^*, i.e.

$$Lu^* = \tilde{N}u^*,$$

with the property

$$Lu^* = \min\{Lw \in [Lu_2, Lu_1] \mid Lw \geq Nw\}.$$

Let us consider $\tilde{u}^*$ such that

$$L\tilde{u}^* = Nu^*.$$

By the definition of $\tilde{N}$, $Lu^* = N\tilde{u}^*$. And now, using also again the definition of $\tilde{N}$, we obtain

$$L\tilde{u}^* = \tilde{N}\tilde{u}^*.$$

If $Lu^* = L\tilde{u}^*$ then, the existence of a solution for (1.1) is proved. Using (iii), (iv) and (v) we shall prove that this always holds. First, let us notice that $0 < Lu^* \leq L\tilde{u}^* \leq Lu_1$. According to (v), let $\mu_0 = \sup\{\mu \in (0,1] \mid \mu L\tilde{u}^* \leq Lu^*\}$. Clearly, $\mu_0 L\tilde{u}^* \leq Lu^*$. We have to prove that $\mu_0 = 1$. Then, $L\tilde{u}^* = Nu^* \leq N\tilde{u}^*_{\mu_0} \leq \mu_0^{\alpha} N\tilde{u}^* = \mu_0^{\alpha} Lu^*$. Here, $\tilde{u}^*_{\mu_0}$ is such that $L\tilde{u}^*_{\mu_0} = \mu_0 L\tilde{u}^*$. Consequently, $\mu_0^{-\alpha} \leq \mu_0$, that is $-\alpha \geq 1$, a contradiction. Thus, $Lu^* = L\tilde{u}^*$. □

4. APPLICATION

In this section we shall establish a weak maximum principle for the functional-differential operator

$$Lu = -u'' - \lambda u(g(x))$$

and an existence result for the following boundary value problem for a second order implicit functional-differential equation.

$$\begin{cases} -u''(x) = f(x, u(g(x)), u(x), -u''(x)), \text{ a.a. } x \in (0,1) \\ u \in H^2(0,1) \cap H^1_0(0,1). \end{cases} \tag{4.1}$$

Let us list the following hypotheses.

(g1) the function $g : [0,1] \to [0,1]$ is continuous.

(f1) the function $f : (0,1) \times R^3 \to R$ is Caratheodory and there exists a continuous function $\varphi : [0,1] \times R^2 \to R$ such that

$$|f(x,u,v,w)| \leq \varphi(x,u,v), \text{ a.a. } x \in (0,1), u,v,w \in R.$$

(f2) f is monotone increasing with respect to the last three variables.

Let us denote

$$Nu = f(x, u(x), u(g(x)), -u''),$$
$$Z = L^2(0,1),\ X = H^2(0,1) \cap H_0^1(0,1).$$

Then, we obtain two operators $L, N : X \to Z$ and the BVP can be written in the following form (with $\lambda = 0$).

$$Lu = Nu,\ u \in X. \tag{4.2}$$

Let us notice that (f1) and the inclusion $X \subset C[0,1]$ imply that N is well-defined. Also, for our existence result, we shall not need another growth condition for the function f.

Next we shall prove that, when $0 \leq \lambda < 8$, the weak maximum principle holds for the functional-differential operator L.

THEOREM 4.1. *If* $0 \leq \lambda < 8$ *then* $L : X \to Z$ *is surjective and it is inverse-monotone.*

Proof. In order to prove that L is surjective we study the solvability of the following equation for an arbitrary $w \in Z$.

$$Lu = w,\ u \in X. \tag{4.3}$$

Let us consider the following integral operator.

$$A_w : C[0,1] \to C[0,1], \quad A_w u = \int_0^1 G(x,s)[\lambda u(g(s)) + w(s)]\mathrm{d}s.$$

The Green function $G : [0,1] \times [0,1] \to R$ is given by

$$G(x,s) = \begin{cases} s(1-x), & \text{if } s \leq x \\ x(1-s), & \text{if } s \geq x. \end{cases}$$

Then equation (4.3) is equivalent to

$$A_w u = u,\ u \in C[0,1].$$

By o straightforward calculation, the following relation can be proved

$$||A_w u_1 - A_w u_2||_C \leq \lambda \cdot \tfrac{1}{8} ||u_1 - u_2||_C.$$

Thus, A_w is a contraction on the Banach space $C[0,1]$, so it has a unique fixed point. Hence, L is surjective.

In order to prove that L is inverse-monotone, because L is linear it is sufficient to prove that $Lu \leq 0$ implies $u \leq 0$.

Let $u^* \in X$ be such that $Lu^* \leq 0$. Let us denote by $w^*(x) = Lu^*(x)$. Then $w^*(x) \leq 0$ and $A_{w^*}u^* = u^*$.

The operator A_{w^*} is Picard and monotone increasing and, in this case, it is easy to see that $A(0) \leq 0$. Then, by Theorem 2.3 (or Theorem 4.1 in [6]) $u^* \leq 0$. □

The following theorem is an existence result for the BVP considered at the beginning of this section.

THEOREM 4.2. *If conditions* (g1), (f1) *and* (f2) *hold and there exists a subsolution* $\underline{u}$ *and an upper solution* $\bar{u}$ *for problem* (6) *with*

$$-\underline{u}'' \leq -\bar{u}''$$

then (4.1) *has a solution.*

Proof. This follows easily by Theorem 3.2. Let us omit the details and notice only some useful facts.

$Z = L^2(0,1)$ is an ordered Banach space with a regular cone (see [1]).

$[L\underline{u}, L\bar{u}] \subset L(X)$ because L is surjective.

The condition (f2) and that L is inverse-monotone imply that N is monotone increasing with respect to L. □

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