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Nonlinear Analysis 68 (2008) 1603–1610

Nonlinear Analysis

www.elsevier.com/locate/na

Contributions to coincidence degree theory of asymptotically homogeneous operators

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Received 4 May 2006; accepted 20 December 2006

Abstract

In this paper we give contributions to the coincidence degree theory of asymptotically homogeneous operators. Applications are given to the periodic problem for second-order functional differential equations.

MSC: 47H11; 54H25; 34C25; 34K15

Keywords: Asymptotically homogeneous; Coincidence degree; Functional differential equations; Periodic solutions

1. Introduction

Differential equations with asymptotically homogeneous nonlinearities are intensively studied in the literature since they appear as models of important physical phenomena; see [5,9,10,12,14,16,17]. Most of the results are dedicated to the existence of periodic solutions. Powerful tools for proving such results are the continuation method (see [9,8,15]) and, maybe the most popular but not essentially different, degree theory (see [11]), in particular the coincidence degree theory (see [7]). Typically, an equivalent operator equation is associated with the periodic problem and after it is proved that the degree of the operator involved is not zero. It is better, of course, if an abstract theoretical result is available in order to prove that this degree is not zero (see [2,4–7,13]). Another idea is that this result can be used also easily for other problems. In [2] we obtained results that allow the computation of the coincidence degree for homogeneous nonlinear operators and we applied them to the periodic problem for first-order functional differential equations. In this article we extend the results from [2] to the case of asymptotically homogeneous operators and apply them to the periodic problem for second-order functional differential equations. We consider that is important to emphasize that the abstract results can be applied also to the periodic problem for first-order functional differential equations. In this way, extensions of similar results for the case of ordinary differential equations from [9,5] could be obtained. But our results for second-order equations cannot be obtained as consequences of these ones. Moreover, they extend to the case of functional differential equations some similar results from [14], but not in their full generality. It remains an open problem whether abstract results inside the coincidence theory can be found as generalizations of the results from [14] for differential equations. At the end we point out that also the results from [10,13,16] can

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0362-546X/$ - see front matter
doi:10.1016/j.na.2006.12.047

be obtained as consequences of our results (in [10,13,16] the nonlinear operators are asymptotically homogeneous of order 1).

2. Preliminaries

The coincidence degree. Definition. Let X and Y be two Banach spaces and $L : X \to Y$ be a linear continuous operator. In the sequel, L will be a Fredholm operator of index 0, i.e. $\operatorname{Im} L$ is closed in Y and the linear spaces $\operatorname{Ker} L$ and $\operatorname{coIm} L$ have the same dimension which is finite. We define $X_1 = \operatorname{Ker} L$ and $Y_1 = \operatorname{coIm} L$ so we have the decompositions $X = X_1 \oplus \operatorname{coKer} L$ and $Y = Y_1 \oplus \operatorname{Im} L$. Now we consider the linear isomorphism $J : X_1 \to Y_1$ and the continuous linear projectors $P : X \to X_1$ and $Q : Y \to Y_1$ with $\operatorname{Ker} Q = \operatorname{Im} L$ and $\operatorname{Im} P = X_1$.

Let Ω be an open bounded subset of X and $N : \overline{\Omega} \to Y$ be a continuous and compact operator (i.e. the closure of $N\left(\overline{\Omega}\right)$ is compact in Y). In order to define the coincidence degree of (L, N) in Ω, as in [7], denoted by $d(L - N, \Omega)$, we assume that

$$Lx \neq N(x) \quad \text{for all } x \in \partial\Omega.$$

It is proved in [7] (see also [3]) that the operator $M : \overline{\Omega} \to X$, $M = (L + JP)^{-1}(N + JP)$ is well defined, continuous, compact and

$$Lx^* = N(x^*) \quad \text{if and only if } x^* = M(x^*).$$

Then, the Leray–Schauder degree of $I_X - M$ (where I_X is the identity map of X) is well defined in Ω and we will denote it by $d_{\mathrm{LS}}(I_X - M, \Omega)$. This number is independent of the choice of P, Q and J (up to a sign) and we can define

$$d(L - N, \Omega) := d_{\mathrm{LS}}(I_X - M, \Omega).$$

The periodic problem for second-order functional differential equations and the coincidence degree. We consider the ω-periodic problem ($\omega > 0$) for a second-order functional differential equation

$$x \text{ is } \omega\text{-periodic}, \quad x''(t) = F(x)(t), \quad t \in \mathbb{R}, \tag{eq1}$$

where $F : C^1_\omega \to L^1_\omega$ is a continuous operator, taking bounded sets into bounded sets. The Banach spaces C^1_ω and L_ω are defined below.

$$C^1_\omega := \left\{x : \mathbb{R} \to \mathbb{R}^n \text{ continuously differentiable and } \omega\text{-periodic}\right\},$$

with the norm $|x|_{C^1} = |x|_{\sup} + |x'|_{\sup}$, where $|x|_{\sup} = \sup_{t\in[0,\omega]}|x(t)|$.

$$L^1_\omega := \left\{x : \mathbb{R} \to \mathbb{R}^n \text{ integrable and } \omega\text{-periodic}\right\},$$

with the norm $|x|_{L^1} = \int_0^\omega |x(s)|\,\mathrm{d}s$.

A solution of the ω-periodic problem (eq1) is a function $x \in C^1_\omega$ such that x'' is absolutely continuous. In order to apply the coincidence degree theory to this problem we make use of the following notation and remarks.

$$Y := \left\{y : \mathbb{R} \to \mathbb{R}^n \text{ continuous and } y(t + \omega) = y(t) + y(\omega) \text{ for all } t \in \mathbb{R}\right\}.$$

With the norm $|\cdot|_{\sup}$ the linear space Y becomes a Banach space.

We define the operator

$$L : C^1_\omega \to Y, \qquad Lx(t) := x'(t) - x'(0),$$

which is linear and continuous, and its kernel and image are

$$X_1 := \operatorname{Ker} L = \left\{x : \mathbb{R} \to \mathbb{R}^n \text{ constant function}\right\},$$
$$Y_2 := \operatorname{Im} L = \{y \in Y \text{ with } y(\omega) = 0\} = \{y \in C_\omega \text{ with } y(0) = 0\} \subset C_\omega.$$

Now we consider the linear space

$$Y_1 := \left\{y \in Y \text{ such that } y(t) = ct \text{ for all } t \in \mathbb{R} \text{ and for some } c \in \mathbb{R}^n\right\},$$

and notice that $Y_1 = \text{co}\, Y_2$ (where co denotes the complementary space), Y_2 is closed in Y, and the linear spaces X_1 and Y_1 have dimension n. Hence, *L is a linear Fredholm operator of index* 0.

The map $J : X_1 \to Y_1$, $Ja(t) := (a/\omega)t$ for all $t \in \mathbb{R}$ and $a \in X_1$ is a linear isomorphism, while

$$P : C^1_\omega \to X_1, \qquad Px := x(0) \quad \text{and} \quad Q : Y \to Y_1, \quad Qy(t) := (1/\omega)y(\omega)t$$

are linear projectors as in the above definition of the coincidence degree.

It is useful also to note that $JPx(t) = (x(0)/\omega)t$, $J^{-1}Qy = y(\omega)$, and that $L + JP : C^1_\omega \to Y$ is a linear isomorphism with the inverse

$$(L + JP)^{-1}(y)(t) = \int_0^t y(s)\,\mathrm{d}s - \frac{t}{\omega}\int_0^\omega y(s)\,\mathrm{d}s + \left(1 + \frac{t}{2} - \frac{t^2}{2\omega}\right) y(\omega).$$

The nonlinear operator N defined by

$$N : C^1_\omega \to Y, \qquad N(x)(t) := \int_0^t F(x)(s)\,\mathrm{d}s \tag{1}$$

is continuous and compact. The operator $M := (L + JP)^{-1} \circ (N + JP) : C^1_\omega \to C^1_\omega$ is well defined and in this case is given by the formula

$$M(x)(t) = x(0) + \int_0^t N(x)(s)\,\mathrm{d}s - \frac{t}{\omega}\int_0^\omega N(x)(s)\,\mathrm{d}s + \left(1 + \frac{t}{2} - \frac{t^2}{2\omega}\right) N(x)(\omega).$$

The restriction of the map $J^{-1}QN$ to X_1 identified with $\mathbb{R}^n$ is the function $f : \mathbb{R}^n \to \mathbb{R}^n$ given by the formula

$$f(a) := \int_0^\omega F(a)(s)\,\mathrm{d}s. \tag{2}$$

Hence, we have defined a pair (L, N) and the associated spaces and operators that fit in the framework of the definition of the coincidence degree. Of course, the important thing is that *a function x is a solution of the ω-periodic problem for* (eq1) *if and only if $Lx = N(x)$*. For some Ω open and bounded subset of C^1_ω such that $Lx \neq N(x)$ for each $x \in \partial\Omega$, the coincidence degree $d(L - N, \Omega)$ is well defined. We will refer to this like the degree of the problem (eq1) and we use the notation

$$\text{cdeg}((\text{eq1}), \Omega) := d(L - N, \Omega).$$

The coincidence degree. Properties. From the properties of the Leray–Schauder degree (as given for example in [11]), one can obtain similar properties for the coincidence degree. We will present here some of them that will be used in the sequel.

Theorem 1. (i) *(Existence property). If $d((L, N), \Omega) \neq 0$ then there exists $x \in \Omega$ such that $Lx = N(x)$.*
(ii) *(The invariance under homotopies). If $H : \overline{\Omega} \times [0, 1] \to Y$ is continuous, compact and $Lx \neq H(x, \lambda)$ for all $x \in \partial\Omega$ and $\lambda \in [0, 1]$, then $d(L - H(., \lambda), \Omega)$ is independent of λ.*

The effect of small perturbations is negligible, as is proved in the next theorem ([7] Theorem III.3, page 24).

Theorem 2. *Assume that $Lx \neq N(x)$ for each $x \in \partial\Omega$. If N_p is such that $\sup_{x\in\partial\Omega} \|N_p(x)\|$ is sufficiently small, then $Lx \neq N(x) + N_p(x)$ for all $x \in \partial\Omega$ and $d(L - N - N_p, \Omega) = d(L - N, \Omega)$.*

The importance of the following result is that it gives sufficient conditions for being able to calculate the coincidence degree as the Brouwer degree (denoted with d_B) of a related finite dimensional mapping. It is known that the degree of finite dimensional mappings is easier to calculate (there are examples in [3,4,11]). The idea of the proof is the use of the homotopy of the problem $Lx = N(x)$ with the finite dimensional one $Lx = QN(x)$.

Theorem 3 *([7]). We assume that $Lx \neq \lambda N(x)$ for all $x \in \partial\Omega$ and $\lambda \in (0, 1]$. If $QN(x) \neq 0$ for all $x \in \partial\Omega \cap X_1$ then $d\,(L - N, \Omega) = d_B\left(J^{-1}QN, \Omega \cap X_1, 0\right)$.*

The first hypothesis of the above theorem is fulfilled in the case where the right hand side of the equation $Lx = N(x)$ is sufficiently small. This is precisely stated in the next theorem ([7] and Theorem IV.2, page 31).

Theorem 4. *Assume that for each sufficiently small $\varepsilon > 0$, the operator $N_s(\cdot, \varepsilon) : \overline{\Omega} \to Y$ is continuous and compact, $QN_s(x, \varepsilon) \neq 0$ for all $x \in \partial\Omega \cap X_1$ and $\sup_{x\in\partial\Omega} |N_s(x, \varepsilon) - N_s(x, 0)| \to 0$ as $\varepsilon \to 0$. Then for $\varepsilon > 0$ sufficiently small $Lx \neq \varepsilon N_s(x, \varepsilon)$ for all $x \in \partial\Omega$ and $d(L - \varepsilon N_s(\cdot, \varepsilon), \Omega) = d_B\left(J^{-1}QN_s(\cdot, \varepsilon), \Omega \cap X_1, 0\right)$.*

In the next section, the following statement will be used several times. Its proof is straightforward.

Lemma 1. *Let $\varepsilon > 0$ and $x \in \overline{\Omega}$. We have that $Lx = \varepsilon N(x, \varepsilon)$ if and only if $Lx = QN(x, \varepsilon) + \varepsilon N(x, \varepsilon)$.*

3. Main results on coincidence degree theory

In this section we will work in the framework of the above section. Here we will consider two continuous and compact operators $N, N_p : X \to Y$ (N is compact if the closure of $N(\Omega)$ is compact in Y for any Ω, a bounded subset of X). We denote by $B_r(0)$ the ball of center 0 and radius $r > 0$ from the Banach space X. Usually, here Ω will be $B_r(0)$ for some suitably chosen radius r. Let $\alpha > 0$ be a given real number. We assume that N is positively homogeneous of order α, i.e.

$$N(rx) = r^\alpha N(x) \quad \text{for all } r > 0 \text{ and } x \in X. \tag{N1}$$

As regards the operator N_p we assume that it satisfies either the asymptotic condition

$$\lim_{r\to\infty} \frac{1}{r^\alpha} \sup_{|y|=1} |N_p(ry)| = 0, \tag{Np1}$$

or the following condition in the neighborhood of the origin:

$$\lim_{r\searrow 0} \frac{1}{r^\alpha} \sup_{|y|=1} |N_p(ry)| = 0. \tag{Np2}$$

Our aim is to obtain one of the following two conclusions.

$$d(L - N, B_r(0)) = d_B(J^{-1}QN, B_r(0) \cap X_1, 0) \tag{C1}$$

$$d(L - N - N_p, B_r(0)) = d(L - N, B_r(0)). \tag{C2}$$

We will consider three cases to discuss: $\alpha > 1$, $\alpha < 1$ and $\alpha = 1$.

Theorem 5. *Assume that the operator N satisfies (N1) with $\alpha \neq 1$. If the only solution in X of the equation $Lx = N(x)$, and the only solution in X_1 of $QN(x) = 0$ is $x = 0$, then (C1) holds for every $r > 0$.*

Proof. Let $r > 0$ be fixed. We apply Theorem 3 with $\Omega = B_r(0)$. Since we do indeed have $QN(x) \neq 0$ for all $x \in X_1$ with $|x| = r$, it remains to prove that

$$Lx \neq \lambda N(x) \quad \text{for all } x \text{ with } |x| = r \text{ and } \lambda \in (0, 1]. \tag{3}$$

We assume by contradiction that there exists some $x^* \in X$ with $|x^*| = r$ and some $\lambda^* \in (0, 1]$ such that $Lx^* = \lambda^* N(x^*)$. Let $y^* = (\lambda^*)^{1/(\alpha-1)}x^*$ and notice that $y^* \neq 0$ satisfies $Ly^* = N(y^*)$. This contradicts one of the hypotheses. Hence, (3) is valid and the conclusion follows. $\square$

Theorem 6. *Assume that the operator N satisfies (N1) with $\alpha > 1$.*

(i) *If $QN(y) \neq 0$ for all $y \in X_1$ with $|y| = 1$ then (C1) holds for r sufficiently small.*
(ii) *If the operator N_p satisfies (Np2) and $QN(y) \neq 0$ for all $y \in X_1$ with $|y| = 1$ then (C2) holds for r sufficiently small.*
(iii) *If the operator N_p satisfies (Np1) and there exists $\mu > 0$ such that $\mu \leq |N(y)|$ for all y with $|y| = 1$ then (C2) holds for r sufficiently large.*

Proof. (i) First we prove that for each r sufficiently small,

$$Lx \neq \lambda N(x) \quad \text{for all } x \text{ with } |x| = r \text{ and } \lambda \in (0, 1]. \tag{4}$$

Through the change $y = \frac{1}{r}x$, using that L is linear and N is α-homogeneous, (4) is equivalent to $Ly \neq \lambda r^{\alpha-1}N(y)$ for all y with $|y| = 1$. This last relation follows by Theorem 4 noticing that $\lambda r^{\alpha-1}$ is sufficiently small. So, (4) is proved. Our hypotheses assure that $QN(x) \neq 0$ for all $x \in X_1$ with $|x| = r$. This can be easily seen by considering again the change $y = \frac{1}{r}x$. Now we apply Theorem 3 with $\Omega = B_r(0)$ and obtain the conclusion.

(ii) The conclusion follows by the invariance under homotopies of the coincidence degree for $\Omega = B_r(0)$ and $H(x, \lambda) = N(x) + \lambda N_p(x)$. All we have to do is to check that for all $\lambda \in [0, 1]$,

$$Lx \neq N(x) + \lambda N_p(x) \quad \text{for all } x \text{ with } |x| = r. \tag{5}$$

Through the change $y = \frac{1}{r}x$, this is equivalent to

$$Ly \neq r^{\alpha-1}\left[N(y) + \lambda \frac{1}{r^\alpha} N_p(ry)\right] \quad \text{for all } y \text{ with } |y| = 1. \tag{6}$$

We define $N_s(y, r) = N(y) + \lambda \frac{1}{r^\alpha} N_p(ry)$ and, using Lemma 1, we deduce that (6) is equivalent to

$$Ly \neq QN_s(y, r) + r^{\alpha-1}N_s(y, r) \quad \text{for all } y \text{ with } |y| = 1. \tag{7}$$

The linearity of Q implies that $QN_s(y, r) = QN(y) + \lambda \frac{1}{r^\alpha} QN_p(ry)$ and, using the notation $\tilde{N}_p(y) = \lambda \frac{1}{r^\alpha} QN_p(ry) + r^{\alpha-1}N_s(y, r)$, relation (7) becomes

$$Ly \neq QN(y) + \tilde{N}_p(y) \quad \text{for all } y \text{ with } |y| = 1. \tag{8}$$

We claim that the hypotheses of Theorem 2 are fulfilled for the operators L, QN and $\tilde{N}_p$. Hence, relation (8) holds true, and, by the chain of the above equivalences, (5) is also valid.

In the sequel, we will prove the claim. If, for some y^* with $|y^*| = 1$, $Ly^* = QN(y^*)$ then $y^* \in X_1$ and $QN(y^*) = 0$. This contradicts one of the hypotheses. It remains to prove that $\sup_{|y|=1}|\tilde{N}_p(y)|$ can be arbitrarily small as $r \to 0$. Let r be sufficiently small. The relation (Np2) and the continuity of Q assures that the first term of $\tilde{N}_p$ can be arbitrarily small. The compactness of N and N_p, and relation (Np2) imply that $\sup_{|y|=1}|N_s(y, r)|$ is bounded by a constant that does not depend on r. Hence, the second term of $\tilde{N}_p$ can also be arbitrarily small. The proof is done.

(iii) We use also the homotopy $H(x, \lambda) = N(x) + \lambda N_p(x)$ in order to apply the invariance under homotopies of the coincidence degree. All we have to prove is relation (5) that, through the change $x = ry$, is equivalent with (6). Moreover, this becomes

$$N(y) \neq \frac{1}{r^{\alpha-1}} Ly - \lambda \frac{1}{r^\alpha} N_p(ry) \quad \text{for all } y \text{ with } |y| = 1. \tag{9}$$

The right hand side of (9) can be arbitrarily small as $r \to \infty$, while the left hand side is above $\mu > 0$. Then, for r sufficiently large, (9) is true. □

In the case $\alpha < 1$ an analogous result holds. Its statement, without the proof, follows.

Theorem 7. *Assume that the operator N satisfies (N1) with $\alpha < 1$.*

(i) *If $QN(y) \neq 0$ for all $y \in X_1$ with $|y| = 1$ then (C1) holds for r sufficiently large.*
(ii) *If the operator N_p satisfies (Np1) and $QN(y) \neq 0$ for all $y \in X_1$ with $|y| = 1$ then (C2) holds for r sufficiently large.*
(iii) *If the operator N_p satisfies (Np2) and there exists $\mu > 0$ such that $\mu \leq |N(y)|$ for all y with $|y| = 1$ then (C2) holds for r sufficiently small.*

Remark 1. Let N be such that it satisfies (N1) with $\alpha \neq 1$ and there exists $\mu > 0$ such that $\mu \leq |N(y)|$ for all y with $|y| = 1$ (i.e. N satisfies the hypotheses of Theorem 6(iii) when $\alpha > 1$ and, respectively, of Theorem 7(iii) when $\alpha < 1$). As a consequence of Theorem 10.2 from [1] we obtain that there exists $x \neq 0$ such that $x = N(x)$. Hence Theorem 5 cannot be used in order to calculate the degree $d(I - N, B_r(0))$.

Our main result in the case $\alpha = 1$ is the following.

Theorem 8. *Assume that the operator N satisfies (N1) with $\alpha = 1$ and that the only solution of $Lx = N(x)$ is $x = 0$.*
(i) *If the operator N_p satisfies (Np1) then (C2) holds for r sufficiently large.*
(ii) *If the operator N_p satisfies (Np2) then (C2) holds for r sufficiently small.*

Proof. We treat the two cases (i) and (ii) simultaneously. Like in the proof of Theorem 6, the conclusion follows by the invariance under homotopies of the coincidence degree for $\Omega = B_r(0)$ and $H(x, \lambda) = N(x) + \lambda N_p(x)$. So, all we have to prove is that relation (6) holds true, i.e.

$$Ly \neq N(y) + \lambda \frac{1}{r} N_p(ry) \quad \text{for all } y \in X \text{ with } |y| = 1. \tag{10}$$

Our hypothesis (both variants) assure that $\sup_{|y|=1} \lambda \frac{1}{r} |N_p(ry)|$ can be arbitrarily small on choosing r either sufficiently large in case (i), or sufficiently small in case (ii). Also, $Ly \neq N(y)$ for all $y \in X$ with $|y| = 1$. Theorem 2 confirms then that (10) holds true. □

Remark 2. If N_p is positively homogeneous of order β, then (Np1) is satisfied in the case where $\beta < \alpha$, while (Np2) is satisfied in the case where $\beta > \alpha$.

Remark 3. All the results remain valid if the limits in (Np1) and (Np2), respectively, are sufficiently small positive numbers.

Remark 4. Of course, existence results can be obtained as consequences of our theorems. For example, as a consequence of Theorem 8(i), we obtain:

Theorem 9. *Assume that the operator N satisfies (N1) with $\alpha = 1$, that the only solution of $Lx = N(x)$ is $x = 0$ and that $d(L - N,\ B_r(0)) \neq 0$ for some $r > 0$. If the operator N_p satisfies (Np1) then the equation $Lx = N(x) + N_p(x)$ has at least one solution in X.*

This result is an extension from the Leray–Schauder degree theory to the coincidence degree theory of Theorem 3.2 from [17]. Moreover, in [17] the condition that replaces (Np1) is a global one, while in our case it is only an asymptotic condition.

4. Periodic solutions for second-order functional differential equations with asymptotically homogeneous nonlinearities

We consider the problems

$$x \text{ is } \omega\text{-periodic}, \quad x''(t) = F(x)(t), \quad t \in \mathbb{R}, \tag{eq1}$$

$$x \text{ is } \omega\text{-periodic}, \quad x''(t) = F(x)(t) + F_p(x)(t), \quad t \in \mathbb{R}, \tag{eq2}$$

where $F, F_p : C^1_\omega \to L^1_\omega$ are continuous operators, taking bounded sets into bounded sets. We shall use all the notation and results given in Section 2 and, in addition, we define

$$N_p : C^1_\omega \to Y, \qquad N_p(x) := \int_0^t F_p(x)(s)\,\mathrm{d}s. \tag{11}$$

Like in Section 2 we talk about the degree of the problems (eq1) and (eq2) and we use the notation

$$\mathrm{cdeg}((\text{eq1}), \Omega) := d(L - N, \Omega), \qquad \mathrm{cdeg}((\text{eq2}), \Omega) := d(L - N - N_p, \Omega).$$

We assume that F is positively homogeneous of order $\alpha > 0$ and that F_p satisfies one of the following two conditions:

$$\lim_{r\to\infty} \frac{1}{r^\alpha} \sup_{|y|_{C^1}=1} |F_p(ry)|_{L^1} = 0, \tag{Fp1}$$

$$\lim_{r\searrow 0} \frac{1}{r^\alpha} \sup_{|y|_{C^1}=1} |F_p(ry)|_{L^1} = 0. \tag{Fp2}$$

The proof of the next result is straightforward and we omit it.

Lemma 2. *$F : C^1_\omega \to L^1_\omega$ is positively homogeneous of order $\alpha > 0$; then N defined by (1) has the same property. Also, if $F_p : C^1_\omega \to L^1_\omega$ satisfies (Fp1) or (Fp2), then N_p defined by (11) satisfies (Np1) or (Np2), respectively.*

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$f(a) = \int_0^\omega F(a)(s)\,\mathrm{d}s. \tag{12}$$

The following results are direct consequences of Theorems 5–8.

Theorem 10. *Let $F : C^1_\omega \to L^1_\omega$ be a positively homogeneous operator of order $\alpha \neq 1$. We assume that the function f given by (12) satisfies $f(a) \neq 0$ for all $a \neq 0$ and the only ω-periodic solution of $x''(t) = F(x)(t)$ is the trivial one. Then* $\mathrm{cdeg}((\text{eq1}), \Omega) = d_B(f, \Omega \cap \mathbb{R}^n, 0)$.

Theorem 11. *Let $F : C^1_\omega \to L^1_\omega$ be a positively homogeneous operator of order $\alpha > 1$ and $F_p : C^1_\omega \to L^1_\omega$ be an operator that satisfies (Fp2). We assume that the function f given by (12) satisfies $f(a) \neq 0$ for all $a \neq 0$. Then* $\mathrm{cdeg}((\text{eq2}), \Omega) = \mathrm{cdeg}((\text{eq1}), \Omega) = d_B(f, \Omega \cap \mathbb{R}^n, 0)$ *for Ω any ball in C^1_ω with a sufficiently small radius.*

Theorem 12. *Let $F : C^1_\omega \to L^1_\omega$ be a positively homogeneous operator of order $\alpha < 1$ and $F_p : C^1_\omega \to L^1_\omega$ be an operator that satisfies (Fp1). We assume that the function f given by (12) satisfies $f(a) \neq 0$ for all $a \neq 0$. Then* $\mathrm{cdeg}((\text{eq2}), \Omega) = \mathrm{cdeg}((\text{eq1}), \Omega) = d_B(f, \Omega \cap \mathbb{R}^n, 0)$ *for Ω any ball in C^1_ω with a sufficiently large radius.*

Theorem 13. *Let $F : C^1_\omega \to L^1_\omega$ be a positively homogeneous operator of order $\alpha = 1$. We assume that the only ω-periodic solution of $x''(t) = F(x)(t)$ is the trivial one.*

(i) *If the operator F_p satisfies (Fp1) then* $\mathrm{cdeg}((\text{eq2}), \Omega) = \mathrm{cdeg}((\text{eq1}), \Omega)$ *for Ω any ball in C^1_ω with a sufficiently large radius.*
(ii) *If the operator F_p satisfies (Fp2) then* $\mathrm{cdeg}((\text{eq2}), \Omega) = \mathrm{cdeg}((\text{eq1}), \Omega)$ *for Ω any ball in C^1_ω with a sufficiently small radius.*

Again, like we noticed in Section 3, existence results for the problem (eq2) can be obtained as consequences of the above theorems. For example, we obtain the following result as a consequence of Theorem 12 for (eq2) with $F(x)(t) = g(x(t))$ where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function.

Theorem 14. *Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, positively homogeneous of order $\alpha < 1$ such that $g(a) \neq 0$ for all $a \neq 0$ and $d_B(g, V, 0) \neq 0$ for some neighborhood of the origin V. Let $F_p : C^1_\omega \to L^1_\omega$ be an operator that satisfies (Fp1). Then the equation $x''(t) = g(x(t)) + F_p(x)(t)$ has at least one ω-periodic solution.*

This result is analogous to Corollary 10 from [5] that is given for first-order ordinary differential equations.

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