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Quasilinearization Method for Nonlinear Elliptic Boundary-Value Problems

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Abstract. The quasilinearization method is developed for strong solutions of semilinear and nonlinear elliptic boundary-value problems. We obtain two monotone, L^p-convergent sequences of approximate solutions. The order of convergence is two. The tools are some results on the abstract quasilinearization method and from weakly–near operators theory.

Key Words. Elliptic boundary-value problem, nonlinear boundary-value problems, strong solutions, lower and upper solutions, approximations, quasilinearization.

1. Introduction

The aim of our work is to develop the quasilinearization method for strong solutions of semilinear and fully nonlinear elliptic problems. We obtain two monotone, L^p-convergent sequences of approximate solutions and estimate that the order of convergence is two. The monotone iterative technique is presented as a consequence for the case of fully nonlinear elliptic problems.

Our results on semilinear elliptic problems complement in some sense and intersect, but do not include, the ones existing in the literature (mainly given by Lakshmikantham–Vatsala in Ref. 1 and by Lakshmikantham–Leela in Ref. 2). Anyway, by our knowledge, the method has not been initiated until now to fully nonlinear elliptic problems.

Recently, the ideas of the quasilinearization method have been refined, extended, and generalized to a variety of problems by Lakshmikantham et al (see Refs. 1–4).

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In Ref. 5, a joint paper by Buica and Precup, we gave an abstract version of this method. The results from Ref. 5 are for operator equations in ordered Banach spaces. They can be used easily in order to obtain new results for specific problems. One of the results is used also in the present paper. In the study of nonlinear elliptic problems, we need in addition the theory of weakly–near operators as developed in Ref. 6, a joint paper by Buica and Domokos.

2. Notations and Main Hypotheses

Let $1 < p < \infty$, let Ω be a C^2 bounded domain of $\mathbb{R}^n$. We denote by $\mathcal{M}_n$ the space of $n \times n$ real matrices; $|\cdot|_m$ is the Euclidean norm in $\mathbb{R}^m$; tr $N = \sum_{i=1}^n \xi_{ii}$ is the trace of the $n \times n$ matrix $N = (\xi_{ij})$. The Sobolev spaces $W^{2,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ are defined as in Ref. 7. We denote by B the following linear elliptic operator in nondivergence form:

$$Bu = \sum_{i,j=1}^{n} l_{ij}(x)[\partial^2 u/\partial x_i \partial x_j] + \sum_{i=1}^{n} l_i(x)[\partial u/\partial x_i], \tag{1}$$

where

$$L = (l_{ij}) \in C(\bar{\Omega}, \mathcal{M}_n), \quad l = (l_i) \in L^\infty(\Omega, \mathbb{R}^n),$$

and

$$\sum_{i,j=1}^{n} l_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n.$$

Let $a : \Omega \times \mathbb{R} \times \mathcal{M}_n \to \mathbb{R}$ be a function which fulfills the following conditions:

(A1) $a(x, 0, 0) = 0, a(\cdot, r, M)$ is measurable, and $a(x, \cdot, \cdot)$is continuous;

(A2) there exist $c_1, c_2 \geq 0$ such that

$$|a(x, r, M)| \leq c_1|r| + c_2|M|_{n^2},$$

or $n < 2p$ and there exist two continuous functions c_1, c_2 such that

$$|a(x, r, M)| \leq c_1(x, r)|M|_{n^2} + c_2(x, r).$$

We consider the semilinear elliptic problem

$$u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad -Bu = f(x,u), \quad \text{for a.e. } x \in \Omega, \tag{2}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and the nonlinear elliptic problem

$$u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad a(x,u,D^2u) = f(x), \quad \text{for a.e. } x \in \Omega, \tag{3}$$

where $f \in L^p(\Omega)$ is given and the following ellipticity condition is satisfied:

(A3) there exists $c > 0$ such that

$$\begin{aligned} &[a(x, r+s, d, N+M) - a(x, s, d, M)][tr(L(x)N)] \\ &\geq c|tr(L(x)N)|^2, \\ &\text{for almost all } x \in \Omega, \text{ for all } r, s \in \mathbb{R}, d \in \mathbb{R}^n, M, N \in \mathcal{M}_n. \end{aligned} \tag{4}$$

We will work in the presence of lower and upper solutions. We say that $\alpha_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is a lower solution of (2) if

$$-B\alpha_0 \leq f(x, \alpha_0),$$

or a lower solution of (3) if

$$a(x, \alpha_0, D^2\alpha_0) \geq f(x), \quad \text{for a.a. } x \in \Omega.$$

Whenever the reversed inequality holds for β_0, we say that β_0 is an upper solution.

3. Theoretical Preliminaries

In this section, we will present some abstract results from the theory of weakly–near operators and the quasilinearization method. We will need them to prove the main results of this paper.

In Ref. 6, the notion of nearness in the sense of Campanato (see Refs. 8–11) is generalized using an accretivity–type condition, instead of a contraction–type one.

Let X be a nonempty set and let Z be a Banach space. Let $A, B : X \to Z$ be two operators. We denote by $J : Z \rightsquigarrow Z^*$ the normalized duality map of Z. We say that A is strongly accretive with respect to B, if there exists $c > 0$ and, for every $x, y \in X$, there exists $j(Bx - By) \in J(Bx - By)$ with

$$\langle Ax - Ay, j(Bx - By) \rangle \geq c\|Bx - By\|^2. \tag{5}$$

The map A is continuous with respect to B if $A \circ B^{-1} : B(X) \rightsquigarrow Z$ has a continuous selection.

The next definition introduces the notion of weak nearness. We notice that, in Ref. 6, it is given in a slightly more general form.

Definition 3.1. We say that A is weakly near B if A is strongly accretive with respect to B and continuous with respect to B.

This notion extends the property of the differential operator to be near (or to approximate) the map, as well as other approximation notions used in nonsmooth theory of inverse or implicit functions. For details in this direction, see also Refs. 12–14.

The proof of the following result can be found in Refs. 6 and 14–15.

Proposition 3.1. Let A be weakly near to B. If B is bijective, then A is bijective.

In what follows, we present a result from Ref. 5. This is an abstract version of the quasilinearization method for the coincidence operator equation

$$Lu = N(u), \quad u \in D. \tag{6}$$

Theorem 3.1. Let X be an ordered Banach space, let Z be an ordered topological linear space, let D be a linear subspace of X, and let $\alpha_0, \beta_0 \in D$ with $\alpha_0 \leq \beta_0$. Let $L: D \to Z$ be a linear operator and let

$$N: \{u \in D: \alpha_0 \leq u \leq \beta_0\} \to Z$$

be a continuous mapping. Assume that the following conditions are satisfied:

(i) $\alpha_0 \leq \beta_0$, $L\alpha_0 \leq N(\alpha_0)$, and $L\beta_0 \geq N(\beta_0)$;

(ii) for every $u, v \in D$ with $\alpha_0 \leq u \leq v \leq \beta_0$, there is a continuous linear operator $Q(u, v): D \to Z$ such that $L - Q(u, v): D \to Z$ is bijective with positive inverse and

$$N(u) \leq N(v) - Q(u, v)(v - u), \tag{7}$$

$$-Q(u, v)z \leq -Q(\alpha, \beta)z, \tag{8}$$

for all $\alpha, \beta, u, v, z \in D$ with $\alpha_0 \leq \alpha \leq u \leq v \leq \beta \leq \beta_0$ and $z \geq 0$;

(iii) the positive cone of X is regular and the operator $(L - Q(\alpha_0, \beta_0))^{-1}$ is continuous on every $u \in D$ with $\alpha_0 \leq u \leq \beta_0$.

Then, the sequences (α_n), (β_n) given by the iterative schemes

$$L\alpha_{n+1}=N(\alpha_n)+Q(\alpha_n,\beta_n)(\alpha_{n+1}-\alpha_n), \tag{9}$$
$$L\beta_{n+1}=N(\beta_n)+Q(\alpha_n,\beta_n)(\beta_{n+1}-\beta_n), \tag{10}$$

$n\in\mathbb{N}$, are well and uniquely defined in D and they are monotonically convergent in X to the minimal and to the maximal solution in $[\alpha_0,\beta_0]$ of (6), respectively.

4. Quasilinearization for Semilinear Elliptic Problems

In this section, we study approximate solutions for (2) given by the following iterative schemes:

$$-B\alpha_{n+1}=f(x,\alpha_n)+P(x,\alpha_n,\beta_n)(\alpha_{n+1}-\alpha_n), \tag{11}$$
$$-B\beta_{n+1}=f(x,\beta_n)+P(x,\alpha_n,\beta_n)(\beta_{n+1}-\beta_n). \tag{12}$$

In our main result, under some additional assumptions on f, we obtain that these schemes give monotone and quadratically convergent sequences of approximate solutions. As consequences, we obtain two results. One contains ideas similar to those used by Lakshmikantham et al. (see Refs. 1, 3–4) as regards the conditions for the nonlinear part and the form of the function P in the iterative schemes. The basic condition for f is some convexity and P is given in terms of the derivatives of f. The second consequence of our main result uses for P an expression in terms of divided differences and it can be used when f is not differentiable.

In what follows, for two functions $\alpha_0,\beta_0\in L^p(\Omega)$ with $\alpha_0\le\beta_0$, we consider the order interval $[\alpha_0,\beta_0]$ given by

$$[\alpha_0,\beta_0]=\{u\in L^p(\Omega):\alpha_0(x)\le u(x)\le\beta_0(x),\ \text{ for a.e. } x\in\Omega\}.$$

The next theorem is the main result of this section.

Theorem 4.1. Let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function. Assume that:

(i) there exist respectively α_0 and β_0, a lower and an upper solution of (2), with $\alpha_0\le\beta_0$ a.e. in Ω and $f(\cdot,\alpha_0(\cdot)), f(\cdot,\beta_0(\cdot))\in L^p(\Omega)$;

(ii) there exists a Carathéodory function $P:\Omega\times\mathbb{R}^2\to\mathbb{R}$ such that

$$f(x,u)\le f(x,v)-P(x,u,v)(v-u), \tag{13}$$

for $\alpha_0(x)\le u\le v\le\beta_0(x)$, a.e. in Ω; also, there exists a real number $M_1 \geq 0$ such that, for $\alpha_0(x) \leq \alpha \leq u \leq v \leq \beta \leq \beta_0(x)$, a.e. in Ω,

$$0 \leq -P(x, u, v) \leq -P(x, \alpha, \beta) \leq M_1. \tag{14}$$

Then, the sequences (α_n) and (β_n) given by the iterative schemes (11) and (12) are well and uniquely defined in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, are monotone, and converge in the L^p-norm to the minimal and respectively maximal solution of (2) in the order interval $[\alpha_0, \beta_0]$.

Problem (2) has a unique solution in the order interval $[\alpha_0, \beta_0]$ if, in addition, the following conditions are satisfied:

(iii) there exists a Carathéodory function $b : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x, u) \geq f(x, v) - b(x, v, u)(v - u), \tag{15}$$

for $\alpha_0(x) \leq u \leq v \leq \beta_0(x)$, a.e. in Ω;

(iv) $0 \leq -b(t, v, u) \leq M_2$, for all $\alpha_0(x) \leq u \leq v \leq \beta_0(x)$, a.e. in Ω.

Moreover, the condition

(v) there exist $c_1, c_2 \geq 0$ such that, for $\alpha_0(x) \leq \alpha \leq u \leq \beta \leq \beta_0(x)$,

$$b(x, u, \alpha) - P(x, \alpha, \beta) \leq c_1(u - \alpha) + c_2(\beta - \alpha) \tag{16}$$

assures that the convergence of (α_n) and (β_n) in $L^p(\Omega)$ is quadratic.

Proof. We divide the proof into several steps.

Step 1. All the hypotheses of Theorem 3.1 are fulfilled. Using the notations of Theorem 3.1, we consider

$$X = Z = L^p(\Omega), \quad D = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$
$$Lu = -Bu, \quad N(u) = f(\cdot, u(\cdot)), \quad Q(u, v)z = P(\cdot, u(\cdot), v(\cdot))z,$$

for $u, v \in D \cap [\alpha_0, \beta_0]$, with $u \leq v$ and $z \in D$.

The linear operator $Q(u, v)$ is well defined and continuous between D and $L^p(\Omega)$, since the function P is Carathéodory and satisfies condition (14), which assures that $P(\cdot, u(\cdot), v(\cdot)) \in L^\infty(\Omega)$. The fact that the nonlinear operator N is well defined and continuous between the set $\{u \in D : \alpha_0 \leq u \leq \beta_0\}$ and $L^p(\Omega)$ follows by the inequality (13) and the Lebesgue dominated convergence theorem.

It is easy to see that hypothesis (i) and relations (7)–(8) of Theorem 3.1 are valid. It is known that the positive cone of $L^p(\Omega)$ is regular. Also, for every $u, v, \in D$ with $\alpha_0 \leq u \leq v \leq \beta_0$, the mapping $L - Q(u, v)$ from D to $L^p(\Omega)$, in fact $w \longmapsto -Bw - l(\cdot)w$, where $l(x) = P(x, u(x), v(x)) \leq 0$, a.e. in Ω, is bijective, with positive and continuous inverse (Theorem 9.15 and Lemma 9.17 from Ref. 16). With these notations, we see that the relations (11)–(12) coincide with (9)–(10).

We apply now Theorem 3.1 and deduce that the sequences (α_n) and (β_n), given by the iterative schemes (11) and (12), are well and uniquely defined in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, are monotone, and converge in the L^p-norm to the minimal and respectively maximal solution of (2) in the order interval $[\alpha_0, \beta_0]$.

Step 2. The solution is unique in $[\alpha_0, \beta_0]$. Let us denote by u_* the minimal solution, by u^* the maximal solution, and also put

$$l_*(x) = b(x, u^*(x), u_*(x)).$$

Using (15) and the above notations, we obtain that

$$-Bu_* \geq -Bu^* - l_*(x)(u^* - u_*), \qquad \text{a.e. in } \Omega.$$

Then,

$$-Bu_* - l_*(x)u_* \geq -Bu^* - l_*(x)u^*, \quad \text{a.e. in } \Omega,$$

where $l_* \in L^\infty(\Omega)$ and $l_*(x) \leq 0$. The weak maximum principle implies that $u_* \geq u^*$. But $u_* \leq u^*$. Hence, $u_* = u^*$ and the solution is unique in $[\alpha_0, \beta_0]$.

Step 3. The convergence is quadratic. We denote

$$p_n = u^* - \alpha_n \quad \text{and} \quad q_n = \beta_n - u^*.$$

Using (11) and (13)–(16), we obtain

$$\begin{aligned}
&-Bp_{n+1} - P(x, u^*, u^*)p_{n+1} \\
&\leq -Bp_{n+1} - P(x, \alpha_n, \beta_n)p_{n+1} \\
&= -P(\alpha_n, \beta_n)p_n - f(x, \alpha_n) + f(x, u^*) \\
&\leq (b(x, u^*, \alpha_n) - P(x, \alpha_n, \beta_n))p_n \\
&\leq c_1 p_n^2 + c_2(\beta_n - \alpha_n)p_n \\
&= c_1 p_n^2 + c_2(q_n + p_n)p_n \\
&\leq c_3 p_n^2 + c_4 q_n^2.
\end{aligned}$$

Whenever $p_n^2, q_n^2 \in L^p(\Omega)$, using the fact that the linear operator $-B - P(\cdot, u^*, u^*)I$ has a bounded inverse, we obtain that

$$\|p_{n+1}\|_{L^p} \leq C_1 \|p_n^2\|_{L^p} + C_2 \|q_n^2\|_{L^p}.$$

This ends the proof. □

The next result is a consequence of the above theorem.

Theorem 4.2. Let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function and let α_0,β_0 respectively be a lower and an upper solution of (2), such that $\alpha_0\le\beta_0$ a.e. in Ω and $f(\cdot,\alpha_0(\cdot)), f(\cdot,\beta_0(\cdot))\in L^p(\Omega)$. Assume that $f=f_1-f_2$, where $f_1,f_2:\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory, $f_1(x,\cdot)$ and $f_2(x,\cdot)$ are C^1 on $\mathbb{R}$ and convex on $[\alpha_0(x),\beta_0(x)]$ for a.a. $x\in\Omega$. In addition, assume that $\frac{\partial f_1}{\partial u}(x,\cdot)$ and $\frac{\partial f_2}{\partial u}(x,\cdot)$ are Lipschitz on $[\alpha_0(x),\beta_0(x)]$, with Lipschitz constants not depending on x, and that

$$-M\le\frac{\partial f_1}{\partial u}(t,u)-\frac{\partial f_2}{\partial u}(t,v)\le 0,$$

for all $u,v\in[\alpha_0(x),\beta_0(x)]$ and for a.a. $x\in\Omega$. Then, the sequences (α_n) and (β_n), given by the iterative schemes

$$-B\alpha_{n+1}=f(x,\alpha_n)+\left(\frac{\partial f_1}{\partial u}(x,\alpha_n)-\frac{\partial f_2}{\partial u}(x,\beta_n)\right)(\alpha_{n+1}-\alpha_n),$$
$$-B\beta_{n+1}=f(x,\beta_n)+\left(\frac{\partial f_1}{\partial u}(x,\alpha_n)-\frac{\partial f_2}{\partial u}(x,\beta_n)\right)(\beta_{n+1}-\beta_n),$$

are well and uniquely defined in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ and converge monotonically and quadratically in $L^p(\Omega)$ to the unique solution of (2) in $[\alpha_0,\beta_0]$.

Proof. Apply Theorem 4.1 for

$$P(t,u,v)=b(t,u,v)=\frac{\partial f_1}{\partial x}(t,u)-\frac{\partial f_2}{\partial x}(t,v).$$

The differentiability of $f_1(x,\cdot)$ and $f_2(x,\cdot)$ and their convexity on $[\alpha_0(x),\beta_0(x)]$ imply that the following relations hold:

$$f_1(x,v)-\frac{\partial f_1}{\partial u}(x,v)(v-u)\le f_1(t,u)\le f_1(x,v)-\frac{\partial f_1}{\partial u}(x,u)(v-u),$$
$$-f_2(x,v)+\frac{\partial f_2}{\partial u}(x,u)(v-u)\le f_2(x,u)\le -f_2(x,v)+\frac{\partial f_2}{\partial u}(x,v)(v-u),$$

for all $\alpha_0(x)\le u\le v\le\beta_0(x)$. By summing up these inequalities, we obtain relations (13) and (15). Relation (14) is also valid, since the derivative of a convex function is monotone increasing. Now, it is clear that hypotheses (i)–(iv) of Theorem 4.1 are fulfilled. It remains to prove (v). This is valid, indeed as follows by the next inequalities. We use the fact that f_1 and f_2 are convex and have Lipschitz derivatives on $[\alpha_0(x), \beta_0(x)]$,

$$\begin{aligned} b(x,u,\alpha) - P(x,\alpha,\beta) &= \frac{\partial f_1}{\partial u}(x,u) - \frac{\partial f_1}{\partial u}(x,\alpha) - \frac{\partial f_2}{\partial u}(x,\alpha) + \frac{\partial f_2}{\partial u}(x,\beta) \\ &\le c_1(u-\alpha) + c_2(\beta-\alpha), \end{aligned}$$

for all $\alpha_0(x) \le \alpha \le u \le \beta \le \beta_0(t)$. This completes the proof. □

It is important to notice that the main result of this section is valid also when the function f is not differentiable in the second variable. Instead of the derivative, we can use for example the divided difference. By our knowledge, such kind of results have not been given until now for elliptic problems and respectively for any other problem until the paper (Ref. 5) was published.

For a function $g:[c,d] \to \mathbb{R}$ and two given points $u, v \in [c,d], u \ne v$, the divided difference of g on the points u, v is defined by

$$[g; u, v] = [g(u) - g(v)]/(u - v).$$

Recall that, if the function g is convex, then by the Jensen inequality,

$$[g; u, v] \le [g; u, w] \le [g; v, w], \tag{17}$$

whenever $c \le u \le v \le w \le d$.

Theorem 4.3. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function and let $\alpha_0, \beta_0 \in C(\Omega)$ be respectively a lower and an upper solution of (2), such that $\alpha_0 \le \beta_0$ a.e. in Ω and $f(\cdot, \alpha_0(\cdot)), f(\cdot, \beta_0(\cdot)) \in L^p(\Omega)$. Let $\alpha_{-1}, \beta_{-1} \in C(\Omega)$ be such that $\alpha_{-1}(x) < \alpha_0(x)$ and $\beta_0(x) < \beta_{-1}(x)$ for each $x \in \Omega$. Assume that $f = f_1 - f_2$ where $f_1, f_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory, $f_1(t,\cdot)$ and $f_2(t,\cdot)$ are convex on $[\alpha_{-1}(x), \beta_0(x)]$ and respectively on $[\alpha_0(x), \beta_{-1}(x)]$ for a.a. $x \in \Omega$. In addition, assume that

$$\begin{aligned} -M &\le [f_1(x,\cdot); \alpha_{-1}(x), u] - [f_2(x,\cdot); v, \beta_{-1}(x)] \le 0, \\ -M &\le [f_1(x,\cdot); v, \beta_{-1}(x)] - [f_2(x,\cdot); \alpha_{-1}(x), u] \le 0, \end{aligned}$$

for all $\alpha_0(x) \le u \le v \le \beta_0(x)$ and for a.a. $x \in \Omega$. Then, the sequences (α_n) and (β_n), given by the iterative schemes

$$\begin{aligned} -B\alpha_{n+1} &= f(x,\alpha_n) + \big([f_1; \alpha_{-1}, \alpha_n] - [f_2; \beta_{-1}, \beta_n]\big)(\alpha_{n+1} - \alpha_n), \\ -B\beta_{n+1} &= f(x,\beta_n) + \big([f_1; \alpha_{-1}, \alpha_n] - [f_2; \beta_{-1}, \beta_n]\big)(\beta_{n+1} - \beta_n), \end{aligned}$$

are well and uniquely defined in $W^{2,p}(\Omega) \bigcap W_0^{1,p}(\Omega)$ and converge monotonically in $L^p(\Omega)$ to the unique solution of (2) in $[\alpha_0, \beta_0]$.

Proof. Apply Theorem 4.1 for

$$P(x,u,v)=[f_1(x,\cdot);\alpha_{-1}(x),u]-[f_2(x,\cdot);v,\beta_{-1}(x)],$$
$$b(x,u,v)=[f_1(x,\cdot);v,\beta_{-1}(x)]-[f_2(x,\cdot);\alpha_{-1}(x),u].$$

Using inequalities (17), we have

$$[f_1(x,\cdot);\alpha_{-1}(x),u]\le[f_1(x,\cdot);u,v],$$
$$[f_2(x,\cdot);v,\beta_{-1}(x)]\ge[f_2(x,\cdot);u,v],$$
$$[f_1(x,\cdot);\alpha_{-1}(x),u]\ge[f_1(x,\cdot);\alpha_{-1}(x),\alpha],$$
$$[f_2(x,\cdot);v,\beta_{-1}(x)]\le[f_2(x,\cdot);\beta,\beta_{-1}(x)],$$

whenever

$$\alpha_{-1}(x)<\alpha_0(x)\le\alpha\le u\le v\le\beta\le\beta_0(x)<\beta_{-1}(x);$$

hence, by summing up the first two inequalities and the last two ones, we obtain (13) and (14) respectively. Using again (17), we obtain

$$[f_1(x,\cdot);u,v]\le[f_1(x,\cdot);v,\beta_{-1}],\quad [f_2(x,\cdot);\alpha_{-1}(x),u]\le[f_2(x,\cdot);u,v],$$

for $\alpha_0(t)\le u\le v\le\beta_0(t)$; hence, by summing up, we obtain (15). □

5. Quasilinearization for Nonlinear Elliptic Problems

We establish a quasilinearization method for fully nonlinear elliptic problem (3). As consequence, we obtain a monotone iterative method for this problem. The theory of weakly-near operators is combined with the abstract quasilinearization result, Theorem 3.1. Both have been presented in Section 3. We consider the iterative schemes

$$-B\alpha_{n+1}=-Bu_{\alpha_n}+(1/c)P(x,\alpha_n,\beta_n)(\alpha_{n+1}-\alpha_n), \tag{18}$$
$$a(x,\alpha_n,u_{\alpha_n})=f(x), \tag{19}$$
$$-B\beta_{n+1}=-Bu_{\beta_n}+(1/c)P(x,\alpha_n,\beta_n)(\beta_{n+1}-\beta_n), \tag{20}$$
$$a(x,\beta_n,u_{\beta_n})=f(x). \tag{21}$$

The following theorem is the main result of this section.

Theorem 5.1. Assume that conditions (A1)–(A2) are fulfilled and that:

(i) there exist a lower solution α_0 and an upper solution β_0 of (3) with

$$\alpha_0(x) \le \beta_0(x), \quad \text{for a.a. } x \in \Omega;$$

(ii) there exists a Carathéodory function $P : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ such that, for every $r, s \in \mathbb{R}$ and for a.a. $x \in \Omega$ with $\alpha_0(x) \le r \le s \le \beta_0(x)$, we have

$$0 \le a(x, r, M) - a(x, s, M) \le -P(x, r, s)(s - r);$$

(iii) for all $\alpha_0(x) \le \alpha \le r \le s \le \beta_0(x)$, a.e. in Ω,

$$0 \le -P(x, r, s) \le -P(x, \alpha, \beta) \le M.$$

Then, (α_n) and (β_n) given by (18)–(21) are well and uniquely defined in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and are two monotone sequences which converge in $L^p(\Omega)$ to the unique solution u^* of (3) with $\alpha_0(x) \le u^*(x) \le \beta_0(x)$.

Proof. Let $w \in L^p(\Omega)$. We consider the mapping A_w defined by

$$A_w : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega), \quad A_w(u)(x) = a(x, w, D^2u).$$

We consider also the equation

$$A_w(u) = f \tag{22}$$

and the linear elliptic operator $B : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$ given by (1).

We divide the proof into several steps.

Step 1. A_w is well defined and continuous. Using condition (A2), we obtain that, for every $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$,

$$|a(x, u, D^2u)| \le c_1|u(x)| + c_2|D^2u|_{n^2},$$

or

$$n < 2p \quad \text{and} \quad |a(x, u, D^2u)| \le c_1(x, u(x))(|D^2u|_{n^2}) + c_2(x, u(x)).$$

When $n < 2p$, the Sobolev imbedding theorem (Ref. 7) assures that $W^{2,p}(\Omega) \subset C(\bar{\Omega})$. Then, in both cases, the right side of this inequality is an L^p-function and we can deduce that A_w is well-defined and continuous.

Step 2. A_w is weakly near B. We shall prove first that A_w is strongly accretive with respect to B. The normalized duality map of the Banach space $L^p(\Omega)$ is (here, $1/p+1/q=1$)

$$J: L^p(\Omega) \to L^q(\Omega), \quad Ju(x) = u(x)|u(x)|^{p-2}\|u\|^{2-p}.$$

For $u, v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, we obtain the following estimations:

$$\begin{aligned}
&\langle A_w(u) - A_w(v), J(Bu - Bv)\rangle \\
&= \|B(u-v)\|_{L^p}^{2-p} \int_\Omega [a(x, w, D^2u) - a(x, w, D^2v)]B(u-v)|B(u-v)|^{p-2}dx \\
&\geq c\,\|B(u-v)\|_{L^p}^{2-p} \int_\Omega |B(u-v)(x)|^p dx = c\,\|B(u-v)\|_{L^p}^2 .
\end{aligned}$$

Thus,

$$\langle A_w(u) - A_w(v), J(Bu - Bv)\rangle \geq c\,\|Bu - Bv\|_{L^p}^2 . \tag{23}$$

A_w is continuous with respect to B because it is continuous and B is bijective with a continuous inverse.

Step 3. Choice of the objects of Theorem 3.1. Because A_w is weakly near B and B is bijective, A is bijective too. Then, Equation (22) has a unique solution; we denote it by u_w.

We consider now another operator, related to Equation (22),

$$\mathcal{U}: L^p(\Omega) \to L^p(\Omega), \quad \mathcal{U}(w) = -Bu_w.$$

We notice that w is a coincidence point of $\mathcal{U}$ and $-B$, i.e.,

$$-Bw = \mathcal{U}(w), \quad \text{if and only if } w = u_w,$$

which means that u_w is a solution of problem (3). We apply Theorem 3.1 with

$$L = -B, \quad X = Z = L^p(\Omega), \quad D = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad N = \mathcal{U}.$$

Step 4. $\mathcal{U}$ is continuous in every $v \in L^p(\Omega)$. Let $u_v \in D$ be the unique solution of $A_v(u) = f$. The hypotheses (A1)–(A2) assure that the mapping $w \longmapsto a(\cdot, w, D^2u_v)$ is continuous from $L^P(\Omega)$ to itself; in particular, it is continuous in v. Then, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that, whenever $w \in L^p(\Omega)$ with $\|w - v\|_{L^p} \leq \delta$, we have

$$\|a(\cdot, w, D^2u_v) - a(\cdot, v, D^2u_v)\|_{L^p} \leq \varepsilon.$$

Then,

$$\|A_w(u_v) - A_v(u_v)\|_{L^p} \leq \varepsilon.$$

We replace in this relation $A_v(u_v)$ by $A_w(u_w)$, since both are equal to f, and obtain

$$\|A_w(u_v) - A_w(u_w)\|_{L^p} \leq \varepsilon.$$

Now, we write the relation (23) for u_v and u_w, use the inequality

$$|\langle x, J(y)\rangle \leq \|x\| \cdot \|y\|,$$

which holds in every Banach space, and obtain that

$$\|Bu_w - Bu_v\|_{L^p} \leq \varepsilon/c.$$

We write again this relation using the definition of $\mathcal{U}$ and obtain

$$\|\mathcal{U}(w) - \mathcal{U}(v)\|_{L^p} \leq \varepsilon/c.$$

Step 5. We prove now that

$$-B\alpha_0 \leq \mathcal{U}(\alpha_0) \quad \text{and} \quad \mathcal{U}(\beta_0) \leq -B\beta_0.$$

Here, $\alpha_0 \in D$ is such that

$$a(x, \alpha_0, D^2\alpha_0) \geq f(x), \quad \text{for a.a.}\, x \in \Omega.$$

We notice that the following implication is valid for all $v \in L^p(\Omega)$ and all $u_1, u_2 \in D$:

$$a(x, v, D^2u_1) \geq a(x, v, D^2u_2), \ \text{on } \Omega \Rightarrow Bu_1 \geq Bu_2, \ \text{on } \Omega. \tag{24}$$

Indeed, it is easy to see that this is true from the inequality

$$[a(x, v, D^2u_1) - a(x, v, D^2u_2)][B(u_1 - u_2)] \geq c|B(u_1 - u_2)|^2, \tag{25}$$

which follows by the ellipticity condition (A3). By taking $v = u_1 = \alpha_0$ and $u_2 = u_{\alpha_0}$ in (24), we obtain that

$$-B\alpha_0 \leq \mathcal{U}(\alpha_0).$$

Similarly, we can prove that

$$\mathcal{U}(\beta_0) \leq -B\beta_0.$$

Step 6. We prove now that (7) and (8) are valid with

$$Q(v,w):D\to L^p(\Omega), \quad Q(v,w)z=(1/c)P(\cdot,v(\cdot),w(\cdot))z,$$

where $v,w\in D$ are such that

$$\alpha_0(x)\le v\le w\le \beta_0(x).$$

The linear operator $Q(v,w)$ is well defined and continuous, like in the proof of Theorem 4.1. It is clear that (8) holds. In order to obtain (7), we use first (ii) and write

$$0\le a(x,v,D^2u_w)-a(x,w,D^2u_w)\le -P(x,v,w)(w-v).$$

By the definition of u_w and u_v, respectively, we have that

$$a(x,w,D^2u_w)=f(x)=a(x,v,D^2u_v);$$

hence,

$$0\le a(x,v,D^2u_w)-a(x,v,D^2u_v)\le -P(x,v,w)(w-v). \tag{26}$$

Implication (24) assures that

$$B(u_w-u_v)\ge 0.$$

Then, relation (25) implies

$$a(x,v,D^2u_w)-a(x,v,D^2u_v)\ge c[B(u_w-u_v)].$$

Using (26) and the definition of $\mathcal{U}$, we obtain the inequality

$$\mathcal{U}(v)\le\mathcal{U}(w)-(1/c)P(\cdot,v,w)(w-v).$$

Step 7. For every $u,v\in D$ with $\alpha_0\le u\le v\le\beta_0$, the mapping $L-Q(u,v)$ from D to $L^p(\Omega)$, in fact,

$$w\longmapsto -Bw-l(\cdot)w, \quad \text{where } l(x)=(1/c)\ P(x,u(x),v(x))\le 0, \quad \text{on } \Omega,$$

is bijective with positive and continuous inverse (Theorem 9.15 and Lemma 9.17 from Ref. 16). Hence, we have proved that all the hypotheses of Theorem 3.1 hold. Let u_* be the minimal solution and let u^* be the maximal solution of (3) in the order interval $[\alpha_0,\beta_0]$. Then, $u_*\le u^*$ and, using (ii),

$$0\le a(x,u_*,D^2u^*)-a(x,u^*,D^2u^*)=a(x,u_*,D^2u^*)-a(x,u_*,D^2u_*);$$

hence, by (24),

$$Bu^* \geq Bu_*.$$

It follows that $u^* \leq u_*$ and that $u^* = u_*$; i.e., the solution is unique. Then, the sequences (α_n) and (β_n) given by (18)–(21) are monotonically convergent to the unique solution $u^* \in [\alpha_0, \beta_0]$ of (3). □

The following is a monotone iterative scheme for (3):

$$-B\alpha_{n+1} = -Bu_{\alpha_n} + P/c(\alpha_{n+1} - \alpha_n), \quad a(x, \alpha_n, u_{\alpha_n} = f(x), \tag{27}$$

$$-B\beta_{n+1} = -Bu_{\beta_n} + P/c(\beta_{n+1} - \beta_n), \quad a(x, \beta_n, u_{\beta_n} = f(x). \tag{28}$$

We prove this like a consequence of the previous theorem.

Corollary 5.1. Assume that:

(i) there exist a lower solution α_0 and an upper solution β_0 of (3) with

$$\alpha_0(x) \leq \beta_0(x), \quad \text{for all } x \in \Omega;$$

(ii) there exists a real number $P < 0$ such that, for every $r, s \in \mathbb{R}$ and for a.a. $x \in \Omega$ with $\alpha_0(x) \leq r \leq s \leq \beta_0(x)$, we have

$$0 \leq a(x, r, M) - a(x, s, M) \leq -P(s - r).$$

Then, (α_n) and (β_n) given by (27)–(28) are two monotone sequences which converge in $W^{2,p}(\Omega)$ to the unique solution u^* of (3) with $\alpha_0(x) \leq u^*(x) \leq \beta_0(x)$.

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