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Averaging methods for finding periodic orbits via Brouwer degree

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Abstract

We consider the problem of finding T-periodic solutions for a differential system whose vector field depend on a small parameter ε. An answer to this problem can be given using the averaging method. Our main results are in this direction, but our approach is new. We use topological methods based on Brouwer degree theory to solve operator equations equivalent to this problem. The regularity assumptions are weaker then in the known results (up to second order in ε). A result for third order averaging method is also given.

As an application we provide a way to study bifurcations of limit cycles from the period annulus of a planar system and notice relations with the displacement function. A concrete example is given.

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1. Introduction

In few words we can say that the averaging method [15,16] gives a quantitative relation between the solutions of some non autonomous differential system and the solutions of the averaged differential system, which is an autonomous one. The averaging is with respect to the independent variable and the right hand sides of these systems are sufficiently small, depending on a small parameter ε. Also, by using the Implicit Function Theorem, the

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averaging method leads to the existence of periodic solutions for periodic systems. Our main results are in the spirit of this last idea. But our approach is new. We use topological methods to solve operator equations equivalent to the problem of finding T-periodic solutions. These operator equations are either infinite dimensional (of coincidence type, as are called in [2,3,9]) or finite dimensional.

We start by presenting our main results Theorems 1.1, 3.1 and 3.2. The first step in their proof is to replace our problem to that of finding zeros of some finite dimensional function related directly to the given differential system. In fact, we have to study bifurcation of zeros of this finite dimensional function with respect to the parameter ε around $\varepsilon = 0$. Instead of the Implicit Function Theorem we use Brouwer degree theory.

We succeeded to weaken the hypothesis of analogous theorems in first order averaging, as Theorem 11.5, p. 158, [16] (see our Theorem 1.1) and in second order averaging, as Corollary 6, p. 6, [12] or Theorem 2.2 [10], (see Theorem 3.1). The result for third order in the case of 1-dimensional systems is also stated (see Theorem 3.2). As far as we know this is the first time that an explicit formulation of the third order averaging method has been written. Due to our new approach, which do not involve any change of variable in the given system, we consider that it could be more easy and transparent to obtain results corresponding to higher order averaging.

Here we state the main result on first order averaging method.

Theorem 1.1 (First order averaging method). *We consider the following differential system*

$$x'(t) = \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{1.1}$$

where $F_1 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *are continuous functions, T-periodic in the first variable and D is an open subset of $\mathbb{R}^n$. We define $f_1 : D \to \mathbb{R}^n$ as*

$$f_1(z) = \int_0^T F_1(s, z)\, \mathrm{d}s, \tag{1.2}$$

and assume that:

(i) *F_1 and R are locally Lipschitz with respect to x;*
(ii) *for $a \in D$ with $f_1(a) = 0$, there exists a neighborhood V of a such that $f_1(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$ and $d_B(f_1, V, 0) \neq 0$.*

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T-periodic solution $\varphi(\cdot, \varepsilon)$ of system (1.1) *such that $\varphi(\cdot, \varepsilon) \to a$ as $\varepsilon \to 0$.*

Theorem 1.1 has weaker hypothesis than the analogous result Theorem 11.5 of Verhulst [16], where instead of (i) is assumed that

(j) F_1, R, $D_x F_1$, $D_x^2 F_1$ and $D_x R$ are defined, continuous and bounded by a constant M (independent of ε) in $[0, \infty) \times D$, $-\varepsilon_f < \varepsilon < \varepsilon_f$, and instead of (ii) requires that

(jj) for $a \in D$ with $f_1(a) = 0$ we have $J_{f_1}(a) \neq 0$.

From now on we denote by $D_x F$ the Jacobian matrix of the derivatives of the components of F with respect to the components of x, and by $D_x^2 F$ some matrix of the second order derivatives. By $J_f(a)$ we denote the Jacobian determinant of f calculated in a.

Via coincidence degree theory [9] we obtain a first order averaging method for continuous systems (Theorem 1.2 below), without assuming for the vector field even the locally Lipschitz property. As in all our results, the conditions for the bifurcation functions (which are finite dimensional) are given in terms of the Brouwer degree instead of the Jacobian determinant.

Here is the result, whose proof can be found in Section 4.

Theorem 1.2. *The conclusion of Theorem* 1.1 *is valid also without assuming that* (i) *holds.*

We emphasize that our main contribution to the averaging theory is the dropping of regularity conditions. In fact, in Theorem 1.2 we can assume only integrability on $(0, T)$ instead of continuity of F_1 and R with respect to t. Although the results of Ellison, Sáenz and Dumas [8] do not state existence of periodic solutions, we can say that the smoothness hypotheses are comparable. They give an approximation theorem based on Nth order averaging and they claim that the regularity conditions are probably close to minimal.

Except in the case of Theorem 1.2, the proof of our main results is based on Lemma 2.1 stated and proved in Section 2. This lemma can be used to study bifurcation of zeros of a finite dimensional continuous function for which we know the expansion with respect to the bifurcation parameter up to some order k. The differences between this result and the Implicit Function Theorem or the Malgrange Preparation Theorem ([7], Theorem 1.10 p. 194), also used to study bifurcation of zeros, are the following. First, we notice the regularity conditions. Lemma 2.1 can be applied to functions which are only continuous. For $k = 0$, in weaker conditions than the Implicit Function Theorem, is assured only existence. By considering a higher order approximation, in some cases, existence of many branches of zeros can be obtained, as it can be seen in Example 1 of Section 2. For C^∞ functions, this complements the Malgrange Preparation Theorem.

Section 2 contains also concrete examples and remarks regarding the use of Lemma 2.1 as a tool in bifurcation theory. It is worth mentioning here that the Brouwer degree theory is rich in results which conclude the existence of zeros of some function. These can be used instead of Lemma 2.1 to get new conditions for existence of periodic solutions of differential systems.

The proof of Theorem 1.1, the statement and proof of the results on second and third order averaging method are contained in Section 3.

A concrete example as application of this theory is given in Section 5. First we propose a general way for applying the averaging method in order to study limit cycles of planar systems bifurcating from periodic trajectories of the period annulus (Theorem 5.1). This has been done before by Llibre in [12] for perturbations inside quadratic polynomial systems of

$$\dot{x} = -y(1 + \lambda_4 y),$$
$$\dot{y} = x(1 + \lambda_4 y),$$

and by Llibre, Pérez del Río and Rodríguez in [13] for perturbations inside polynomial systems of degree n of the above system. The way of applying the method is essentially the same as we propose in Theorem 5.1. But we prove in Theorem 5.2 that this is equivalent to study the displacement function of the given planar system. Thus, only for practical reasons someone have to choose between these two methods. Chicone and Jacobs found in [6] that, up to first order in the small parameter ε, at most two limit cycles bifurcate inside quadratic systems from the period annulus of

$$\dot{x} = -y + x^2,$$
$$\dot{y} = x + xy.$$

They studied the displacement function using some results of Bautin [1]. We will find the same, in a shorter way, by using the averaging method.

Even for planar systems, the averaging method and the use of the displacement function are not always equivalent. Other approaches can be found in [12].

2. Some remarks on the Brouwer degree

For bounded open subsets V of $\mathbb{R}^n$, such that $V \subset D$ and 0 does not lie in $f(\partial V, \varepsilon)$ for some ε, denote by $d_B(f(\cdot, \varepsilon), V, 0)$ the *Brouwer degree* of the function $f(\cdot, \varepsilon)$ with respect to the set V and the point 0, as is defined in [4].

One of the main properties of the topological degree is that, if $d_B(f(\cdot, \varepsilon), V, 0) \neq 0$, then the equation

$$f(z, \varepsilon) = 0 \tag{2.3}$$

has a solution in V (see again [4]).

The main result of this section is the following.

Lemma 2.1. *We consider the continuous functions $f_i : \overline{V} \to \mathbb{R}^n$, for $i = 0, \dots, k$, and $f, g, r : \overline{V} \times [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}^n$, given by*

$$g(\cdot, \varepsilon) = f_0(\cdot) + \varepsilon f_1(\cdot) + \varepsilon^2 f_2(\cdot) + \cdots + \varepsilon^k f_k(\cdot), \tag{2.4}$$
$$f(\cdot, \varepsilon) = g(\cdot, \varepsilon) + \varepsilon^{k+1} r(\cdot, \varepsilon). \tag{2.5}$$

Assume that

$$g(z, \varepsilon) \neq 0 \quad \text{for all } z \in \partial V,\ \varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}. \tag{2.6}$$

Then, for $|\varepsilon| > 0$ sufficiently small, $d_B(f(\cdot, \varepsilon), V, 0)$ is well defined and

$$d_B\big(f(\cdot, \varepsilon), V, 0\big) = d_B\big(g(\cdot, \varepsilon), V, 0\big).$$

Proof. We use the invariance under homotopy of the Brouwer degree. For each $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$ we consider the continuous homotopy

$$g_t(\cdot\,,\varepsilon) = g(\cdot\,,\varepsilon) + t\big(f(\cdot\,,\varepsilon) - g(\cdot\,,\varepsilon)\big), \quad \text{for } 0 \leqslant t \leqslant 1.$$

All we have to prove is that, when ε is sufficiently small, $0 \notin g_t(\partial V, \varepsilon)$ for all $0 < t \leqslant 1$. We assume by contradiction that, for some $t_0 \in (0,1]$ and some $x_0 \in \partial V$, $g_{t_0}(x_0, \varepsilon) = 0$. Let $M > 0$ be such that $|r(z,\varepsilon)| \leqslant M$ for all $z \in \overline{V}$ and every $\epsilon \in (0, \varepsilon_0]$. Then $|g(x_0,\varepsilon)| \leqslant M\varepsilon^{k+1}$, which is not true for ε sufficiently small, since $|g(x_0,\varepsilon)| = |f_0(x_0) + \varepsilon f_1(x_0) + \cdots + \varepsilon^k f_k(x_0)| \neq 0$. $\square$

Now we remind the definition of the Brouwer degree for C^1 functions (as it is given in [14]). Let $g \in C^1(D)$, $\overline{V} \subset D$ and $Z_g = \{x \in V\colon\ g(z) = 0\}$. We assume also that

$$J_g(z) \neq 0, \quad \text{for all } z \in Z_g,$$

where $J_g(z)$ is the Jacobian determinant of g at z. This assures that Z_g is finite (see Theorem 1.1.2 of [14]). Then

$$d_B(g, V, 0) = \sum_{z \in Z_g} \operatorname{sign}\big(J_g(z)\big).$$

In [14] there are some examples of computing the degree for functions which are not C^1 or, for which $J_g(z) = 0$ for some $z \in Z_g$ (for example, see p. 21).

Remark 1. Let $g : D \to \mathbb{R}^n$ be a C^1 function, with $g(a) = 0$, where D is an open subset of $\mathbb{R}^n$ and $a \in D$. Whenever $J_g(a) \neq 0$, there exists a neighborhood V of a such that $g(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$. Then $d_B(g, V, 0) \in \{-1, 1\}$.

Remark 2. The Brouwer degree of the function $f_0(z) = z^2$ is 0 in any neighborhood of the origin. The arguments follow. The function f_0 has a unique zero, $a = 0$ and we have that $f_0'(0) = 0$. In order to compute the degree, we consider an arbitrary $\lambda > 0$, the interval $V = (-2\lambda, 2\lambda)$ and the function $g(z) = z^2 - \lambda^2$. Then, g has two zeros in V: $-\lambda$ and λ. The Jacobian matrix is negative at $-\lambda$ and positive at λ. Therefore, $d_B(g, V, 0) = 0$. Since it is easy to see that $\sup_{z \in \overline{V}} |f_0(z) - g(z)| < \inf_{z \in \partial V} f_0(z)$, by Definition 1.4.1 of [14], we get that $d_B(f_0, V, 0) = 0$.

Remark 3. We intend to describe a method for using Lemma 2.1 in order to give some answers to our main problem of finding zeros of a convenient function $f : D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$. We will assume that D is an open subset of $\mathbb{R}^n$ and f is of the form (2.5) with g given by (2.4) and $r : D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ continuous. The first step for this is to find all zeros of f_0. Let $a \in D$ be such that $f_0(a) = 0$. If there exists a neighborhood V of a such that $d_B(f_0, V, 0) \neq 0$, then for $|\varepsilon|$ sufficiently small $f(\cdot\,,\varepsilon)$ has at least one zero in V.

If the Brouwer degree of f_0 is zero in small neighborhoods of a or it cannot be computed (this includes the possibility that f_0 is identically 0), we proceed to study $f_0 + \varepsilon f_1$ in some small neighborhood of a and for ε sufficiently small.

First we assume that there exists $a_{1\varepsilon}$ a zero of $f_0 + \varepsilon f_1$ and a bounded open subset V such that $a_{1\varepsilon} \in V$ for each $\varepsilon \neq 0$ sufficiently small and

$$d_B(f_0 + \varepsilon f_1, V, 0) \neq 0. \tag{2.7}$$

Thus, from Lemma 2.1, $f(\cdot, \varepsilon)$ has at least one zero in V. We notice that there is the possibility to exist other zeros of $f_0 + \varepsilon f_1$ in the same neighborhood of a, additionally to $a_{1\varepsilon}$.

In the case that (2.7) is not fulfilled, then we continue studying, analogously, the function $f_0 + \varepsilon f_1 + \varepsilon^2 f_2$, and so on.

Example 1. The previous remarks are illustrated here for the continuous function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(z, \varepsilon) = z^2 - \varepsilon^2 + \varepsilon^3 r(z, \varepsilon)$. Using the notations of Lemma 2.1, we have $f_0(z) = z^2$, $f_1(z) = 0$, $f_2(z) = -1$.

In any neighborhood of 0, the degree of f_0 is 0 (we have proved this in Remark 2). Thus, we continue with the study of $(f_0 + \varepsilon f_1 + \varepsilon^2 f_2)(z) = z^2 - \varepsilon^2$. This function has two zeros, $-\varepsilon$ and ε. We fix some $\varepsilon_0 > 0$ and consider the open intervals $V = (0, \varepsilon_0)$ and $U = (-\varepsilon_0, 0)$. Using Remark 1, we obtain that $d_B(f_0 + \varepsilon f_1 + \varepsilon^2 f_2, V, 0) \neq 0$ for $0 < \varepsilon < \varepsilon_0$, and the same relation holds for U instead of V. Then, by Lemma 2.1 and some previous remarks, for $\varepsilon > 0$ sufficiently small, $f(\cdot, \varepsilon)$ has at least two zeros, one in U and another one in V.

Whenever r is C^∞, since $f_0(z) = z^2$, by Malgrange Preparation Theorem [7], for $\varepsilon > 0$ sufficiently small, $f(\cdot, \varepsilon)$ has at most two zeros. Hence, it has exactly two zeros.

Remark 4. Assume that the hypotheses of Lemma 2.1 are fulfilled for $k = 0$ and, in addition, that

(i) for $a \in D$ with $f_0(a) = 0$, there exists a neighborhood V of a such that $f_0(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$ and $d_B(f_0, V, 0) \neq 0$.

First we notice that, since $f_0(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$, by the excision property of the degree (Theorem 2.2.1, p. 26 from [14]) and we deduce that $d_B(f_0, V_\mu, 0) \neq 0$ for every neighborhood $V_\mu \subset V$ of a. We choose V_μ such that $V_\mu \to \{a\}$ as $\mu \to 0$. Then it is easy to see that, for ε sufficiently small $f(\cdot, \varepsilon)$ has at least one zero $a_\varepsilon \in V_\mu$ and that we can choose a_ε such that $a_\varepsilon \to a$ as $\varepsilon \to 0$. In this case we say that *at least one branch of zeros bifurcates from a*. Moreover, if, in addition, $J_{f_0}(a) \neq 0$, by the Implicit Function Theorem, this branch is unique.

3. Averaging via the Brouwer degree

The main theoretical results on averaging are stated and proved in this section. We begin with the justification of the fact that *the problem of finding T-periodic solutions for some differential system is equivalent to that of finding zeros of some corresponding finite dimensional function.*

We consider the differential system,

$$x'(t) = F(t, x, \varepsilon), \tag{3.8}$$

where $F : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ is a continuous function, T-periodic in the first variable, locally Lipschitz in the second one and D is an open subset of $\mathbb{R}^n$. For each $z \in D$ we denote by $x(\cdot, z, \varepsilon) : [0, t_z) \to \mathbb{R}^n$ the solution of (3.8) with $x(0, z, \varepsilon) = z$. We assume that

$$t_z > T \quad \text{for all } z \in D. \tag{3.9}$$

We consider the function $f : D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$, given by

$$f(z, \varepsilon) = \int_0^T F\bigl(t, x(t, z, \varepsilon), \varepsilon\bigr)\,\mathrm{d}t. \tag{3.10}$$

Every solution of (3.8)

$$x : [0, T] \to \mathbb{R}^n \quad \text{with } x(0) = x(T) \tag{3.11}$$

can be extended by periodicity to $\mathbb{R}$ and we have the relation

$$x(T, z, \varepsilon) - x(0, z, \varepsilon) = f(z, \varepsilon).$$

Then, every $(z_\varepsilon, \varepsilon)$ such that

$$f(z_\varepsilon, \varepsilon) = 0 \tag{3.12}$$

provides the periodic solution $x(\cdot, z_\varepsilon, \varepsilon)$ of (3.8). The converse is also true, i.e. for every T-periodic solution of (3.8), if we denote by z_ε its value at $t = 0$ then (3.12) holds. *Hence, the problem of finding a T-periodic solution of* (3.8)*, can be replaced by the problem of finding zeros of the finite-dimensional function $f(\cdot, \varepsilon)$ given by* (3.10).

In order to apply Lemma 2.1 we need the Mac-Laurin formula. Whenever $f : D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ is continuous and of class C^k in ε we write

$$f(z, \varepsilon) = g(z, \varepsilon) + \varepsilon^{k+1} r(z, \varepsilon), \tag{3.13}$$

with g given by

$$g(z, \varepsilon) = f(z, 0) + \varepsilon \frac{\partial f}{\partial \varepsilon}(z, 0) + \cdots + \varepsilon^k \frac{1}{k!} \frac{\partial^k f}{\partial \varepsilon^k}(z, 0). \tag{3.14}$$

Except in $\varepsilon = 0$, the function r is well-defined and continuous. If one can prove that r is bounded on some set of the form $K \times [-\varepsilon_0, \varepsilon_0]$ with K a compact subset of D, then we have that r is continuous on $D \times (-\varepsilon_f, \varepsilon_f)$. The continuity of r is needed in Lemma 2.1 and, in this case, from now on, instead of writing formula (3.13) with the function r given explicitly, we use the Landau's symbol (see, for example [15] p. 11) and write on $K \times [-\varepsilon_0, \varepsilon_0]$,

$$f(z, \varepsilon) = g(z, \varepsilon) + \varepsilon^{k+1} \mathrm{O}(1).$$

For example, if $\frac{\partial^k f}{\partial \varepsilon^k}$ is Lipschitz on $K \times [-\varepsilon_0, \varepsilon_0]$, then r is bounded on this set.

The statement of Theorem 1.1 on first order averaging method is in Section 1. Here is the proof.

Proof of Theorem 1.1. For all $z \in \overline{V}$, there exists $\varepsilon_0 > 0$ such that, whenever $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, $x(\cdot, z, \varepsilon)$ is defined on $[0, T]$, i.e. relation (3.9) is valid. Indeed, by the local existence and uniqueness theorem (see, for example, Theorem 1.2.2, p. 2 from [15]), $t_z > h_z$ and $h_z = \inf(T, \frac{b}{M(\varepsilon)})$ where $M(\varepsilon) \geqslant |\varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon)|$ for all $t \in [0, T]$, for each x with $|x - z| \leqslant b$ and for every $z \in \overline{V}$. When $|\varepsilon|$ is sufficiently small, $M(\varepsilon)$ can be arbitrarily large, such that $h_z = T$ for all $z \in \overline{V}$.

For all $t \in [0, T]$, $z \in \overline{V}$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ the following relation holds

$$x(t, z, \varepsilon) = z + \varepsilon \int_0^t F_1\bigl(s, x(s, z, \varepsilon)\bigr)\,\mathrm{d}s + \varepsilon^2 \int_0^t R\bigl(s, x(s, z, \varepsilon), \varepsilon\bigr)\,\mathrm{d}s, \tag{3.15}$$

and the function f given by (3.10) becomes for our system

$$f(z, \varepsilon) = \varepsilon \int_0^T F_1\bigl(s, x(s, z, \varepsilon)\bigr)\,\mathrm{d}s + \varepsilon^2 \int_0^T R\bigl(s, x(s, z, \varepsilon), \varepsilon\bigr)\,\mathrm{d}s.$$

We will prove now that

$$f(z, \varepsilon) = \varepsilon f_1(z) + \varepsilon^2 \mathrm{O}(1) \quad \text{on } \overline{V} \times [-\varepsilon_0, \varepsilon_0], \tag{3.16}$$

with f_1 given by (1.2). Let us first notice that there exists K a compact subset of D such that $x(t, z, \varepsilon) \in K$ for all $t \in [0, T]$, $z \in \overline{V}$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Then it is easy to see that

$$f(z, \varepsilon) - \varepsilon f_1(z) = \varepsilon \int_0^T \bigl[F_1(s, x(s, z, \varepsilon)) - F_1(s, z)\bigr]\,\mathrm{d}s + \varepsilon^2 \mathrm{O}(1). \tag{3.17}$$

Using that F_1 is Lipschitz with respect to x on $[0, T] \times K$ and formula (3.15), we obtain the following relations

$$\bigl|F_1\bigl(s, x(s, z, \varepsilon)\bigr) - F_1(s, z)\bigr| \leqslant L_K \bigl|x(s, z, \varepsilon) - z\bigr| = \varepsilon \mathrm{O}(1).$$

Thus, (3.16) holds. Using Remark 4, we obtain that the hypothesis (ii) assures the existence of z_ε such that $f(z_\varepsilon, \varepsilon) = 0$ and $z_\varepsilon \to a$ as $\varepsilon \to 0$. Then $\varphi(\cdot, \varepsilon) = x(\cdot, z_\varepsilon, \varepsilon)$ is a periodic solution of (1.1) and $\varphi(\cdot, \varepsilon) \to a$ as $\varepsilon \to 0$ (this is implied by the continuity property of the solutions of (1.1) with respect to a parameter and the initial data). □

Theorem 3.1 (Second order averaging method). *We consider the following differential system*

$$x'(t) = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon), \tag{3.18}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. We assume that

(i) *$F_1(t,\cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, R and $D_x F_1$ are locally Lipschitz with respect to x, and R is differentiable with respect to ε.*

We define $f_1, f_2 : D \to \mathbb{R}^n$ as

$$f_1(z) = \int_0^T F_1(s,z)\,\mathrm{d}s,$$

$$f_2(z) = \int_0^T \left[D_z F_1(s,z) \cdot \int_0^s F_1(t,z)\,\mathrm{d}t + F_2(s,z) \right] \mathrm{d}s \tag{3.19}$$

and assume moreover that

(ii) *for $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists $a_\varepsilon \in V$ such that $f_1(a_\varepsilon) + \varepsilon f_2(a_\varepsilon) = 0$ and $d_B(f_1 + \varepsilon f_2, V, 0) \neq 0$.*

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T-periodic solution $\varphi(\cdot,\varepsilon)$ of system (3.18).

Proof. The idea of the proof is the same as for the previous theorem. We will write here only the main relations, and we will omit some details. For all relations which follow we will consider that they hold for $t \in [0,T]$, $z \in \overline{V}$, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Since the right-hand side of system (3.18) is differentiable with respect to ε, then the solution $x(t,z,\varepsilon)$ has the same quality. Hence, by an analogous with (3.15),

$$x(t,z,\varepsilon) = z + \varepsilon \int_0^t F_1(s,z)\,\mathrm{d}s + \varepsilon^2 \mathrm{O}(1),$$

and

$$\frac{\partial x}{\partial \varepsilon}(t,z,\varepsilon) = \int_0^t F_1(s,z)\,\mathrm{d}s + \varepsilon \mathrm{O}(1).$$

Using also that $D_x F_1$ is locally Lipschitz (then Lipschitz on $[0,T] \times \overline{V} \times [-\varepsilon_0, \varepsilon_0]$) we obtain the following relations:

$$F_1\big(t, x(t,z,\varepsilon)\big) = F_1(t,z) + \varepsilon D_z F_1(t,z) \cdot \frac{\partial x}{\partial \varepsilon}(t,z,0) + \varepsilon^2 \mathrm{O}(1),$$

$$F_2\big(t, x(t,z,\varepsilon)\big) = F_2(t,z) + \varepsilon \mathrm{O}(1).$$

Using the notation (3.19), the function f given by (3.10) can be written for our system $f(z,\varepsilon) = \varepsilon f_1(z) + \varepsilon^2 f_2(z) + \varepsilon^3 \mathrm{O}(1)$ on $\overline{V} \times [-\varepsilon_0, \varepsilon_0]$. The conclusion follows by applying Lemma 2.1. $\square$

Remark 5. Theorem 3.1 has weaker hypothesis than the analogous result Corollary 6 of Llibre [12], or Theorem 2.2 [10] where D is a bounded domain of $\mathbb{R}^n$, instead of (i) is assumed that

(j) $F_1, F_2, R, D_x F_1, D_x^2 F_1, D_x F_2, D_x R$ are defined, continuous and bounded in $[0,\infty) \times D \times (-\varepsilon_f, \varepsilon_f)$,

and instead of (ii) requires that

(jj) $f_1(z) = 0$ for all $z \in D$ and for $a \in D$ with $f_2(a) = 0$ we have $J_{f_2}(a) \neq 0$.

We will write the result of third order averaging for $n = 1$, although it holds for systems of arbitrary dimension, in order to avoid writing too much complicated formulas.

Theorem 3.2 (Third order averaging method in dimension 1). *We consider the following differential system*

$$x'(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 F_3(t,x) + \varepsilon^4 R(t,x,\varepsilon), \tag{3.20}$$

where $F_1, F_2, F_3 : \mathbb{R} \times D \to \mathbb{R}$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ are continuous functions, T-periodic in the first variable, D is an open interval of $\mathbb{R}$. We assume that

(i) *$F_1(t,\cdot) \in C^2(D)$, $F_2(t,\cdot) \in C^1(D)$ for all $t \in \mathbb{R}$, $F_1, F_2, F_3, R, D_x^2 F_1, D_x F_2$ are locally Lipschitz with respect to x, and R is twice differentiable with respect to ε.*

We take $f_1, f_2, f_3 : D \to \mathbb{R}$ given by (3.19) *and*

$$f_3(z) = \int_0^T \left[\frac{1}{2} \frac{\partial^2 F_1}{\partial z^2}(s,z)\big(y_1(s,z)\big)^2 + \frac{1}{2} \frac{\partial F_1}{\partial z}(s,z) y_2(s,z) + \frac{\partial F_2}{\partial z}(s,z) y_1(s,z) + F_3(s,z) \right] \mathrm{d}s,$$

where

$$y_1(s,z) = \int_0^s F_1(t,z)\,\mathrm{d}t, \qquad y_2(s,z) = \int_0^s \left[\frac{\partial F_1}{\partial z}(t,z) \int_0^t F_1(r,z)\,\mathrm{d}r + F_2(t,z) \right] \mathrm{d}t.$$

Moreover, assume that

(ii) *for $V \subset D$ an open and bounded interval and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$ there exists $a_\varepsilon \in V$ such that $f_1(a_\varepsilon) + \varepsilon f_2(a_\varepsilon) + \varepsilon^2 f_3(a_\varepsilon) = 0$ and $d_B(f_1 + \varepsilon f_2 + \varepsilon^2 f_3, V, 0) \neq 0$.*

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T-periodic solution $\varphi(\cdot,\varepsilon)$ of system (3.20).

Sketch of the proof. This time we have to expand the function f given by (3.10) up to order 3. For this we will need again relations (3.20) for the functions F_2 and F_3 instead of F_1 and, respectively, F_2. Also, for F_1 we need the following relation:

$$
\begin{aligned}
F_1\big(t, x(t, z, \varepsilon)\big) = {} & F_1(t, z) + \varepsilon \frac{\partial F_1}{\partial z}(t, z) \cdot \frac{\partial x}{\partial \varepsilon}(t, z, 0) \\
& + \varepsilon^2 \frac{1}{2}\left[\frac{\partial^2 F_1}{\partial z^2}(t, z) \cdot \left(\frac{\partial x}{\partial \varepsilon}(t, z, 0)\right)^2 + \frac{\partial F_1}{\partial z}(t, z) \cdot \frac{\partial^2 x}{\partial \varepsilon^2}(t, z, 0)\right] \\
& + \varepsilon^3 \mathrm{O}(1).
\end{aligned}
$$

We notice that

$$
y_1(s, z) = \frac{\partial x}{\partial \varepsilon}(s, z, 0), \qquad y_2(s, z) = \frac{\partial^2 x}{\partial \varepsilon^2}(s, z, 0).
$$

Thus, $f(z, \varepsilon) = \varepsilon f_1(z) + \varepsilon^2 f_2(z) + \varepsilon^3 f_3(z) + \varepsilon^4 \mathrm{O}(1)$ on $\overline{V} \times [-\varepsilon_0, \varepsilon_0]$ and the conclusion follows by applying Lemma 2.1. □

4. Averaging method via the coincidence degree

The aim of this section is to give the idea of the proof of Theorem 1.2. For this we need some preliminaries from coincidence degree theory which can be found in more detail in [2,3,9].

We consider the differential system

$$
x'(t) = \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{4.21}
$$

where $F_1 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable and D is an open subset of $\mathbb{R}^n$. We define $f_1 : D \to \mathbb{R}^n$ as

$$
f_1(z) = \int_0^T F_1(s, z)\, \mathrm{d}s. \tag{4.22}
$$

We make the notation $C_T = \{x \in C[0, T]\colon\ x(0) = x(T)\}$ and notice that any solution of (4.21) which is in C_T can be extended to a T-periodic solution. Let V be a bounded and open set such that $\overline{V} \subset D$. We consider also the set

$$
\Omega = \big\{x \in C_T\colon\ x(t) \in V \ \text{ for all } t \in [0, T]\big\}
$$

which is open and bounded in the space C_T with respect to the sup-norm. We will also need the following objects: the space $C_0 = \{x \in C[0, T]\colon\ x(0) = 0\}$ with the sup-norm, the mapping $L : C_T \to C_0$ defined by $Lx(t) = x(t) - x(0)$ and the nonlinear operator $N(\cdot, \varepsilon) : \overline{\Omega} \to C_0$ defined by $N(x, \varepsilon)(t) = \int_0^t [\varepsilon F_1(s, x) + \varepsilon^2 R(s, x, \varepsilon)]\, \mathrm{d}s$. The linear continuous mapping L is a Fredholm operator of index 0, i.e. the image $\operatorname{Im} L$ is closed in C_0 and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = n < \infty$. The operator $N(\cdot, \varepsilon)$ is completely continuous, i.e. is continuous and $N(\overline{\Omega}, \varepsilon)$ is a relatively compact set.

We notice that the problem of finding a T-periodic solution of (4.21) can be written now as the abstract equation (called of *coincidence type*),

$$Lx = N(x, \varepsilon), \quad x \in \Omega.$$

Whenever $Lx \neq N(x)$ for every $x \in \partial\Omega$ the coincidence degree $d((L, N), \Omega)$ is defined in [9] (see also [3]) as the Leray–Schauder degree of some associated operator. From now on we shall refer to the number $d((L, N), \Omega)$ as being the *coincidence degree of system* (4.21). One of its main properties is that, if it is different than 0 then (4.21) has at least one solution in Ω, which is, in fact, a T-periodic solution.

Theorem IV.2 p. 31 of [9] is an abstract theorem on coincidence degree. A consequence of this theorem for our problem is the following statement.

(S) *For each ε sufficiently small, the coincidence degree for the system* (4.21) *in the set Ω is equal to the Brouwer degree $d_B(f_1, V, 0)$.*

In the hypotheses of Theorem 1.2, for each ε sufficiently small, the coincidence degree of (4.21) in Ω is not zero, hence the system (4.21) has a T-periodic solution, $\varphi(\cdot, \varepsilon) \in \Omega$. Like in the proof of Theorem 1.1, we notice that, instead of V we can consider a neighborhood $V_\mu \subset V$ of a such that $V_\mu \to a$ as $\mu \to 0$. This implies that the corresponding set Ω_μ is a neighborhood of the constant function a (in the space C_T with respect to the sup-norm) such that the diameter of Ω_μ is arbitrarily small when $\mu \to 0$. Hence, for ε sufficiently small, the system (4.21) has a T-periodic solution, $\varphi(\cdot, \varepsilon) \in \Omega_\mu$. We can choose solutions such that $\varphi(\cdot, \varepsilon) \to 0$ as $\varepsilon \to 0$.

5. Averaging for planar autonomous systems. Relation with the displacement function

We consider the planar system

$$\begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y), \end{aligned} \tag{5.23}$$

where $P, Q : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions under the assumption

(A1) System (5.23) has a period annulus around the singular point $(0, 0)$,

$$\Gamma_h\colon \left\{(x, y) \in \mathbb{R}^2\colon\ H(x, y) = h,\ h_c < h < h_s\right\}.$$

Here H is a first integral, h_c is the critical level of H corresponding to the center $(0, 0)$ and h_s denotes the value of H for which the period annulus terminates at a separatrix polycycle. Without loss of generality we can assume that $h_s > h_c \geqslant 0$. We denote by $\mu = \mu(x, y)$ the integrating factor of system (5.23) corresponding to the first integral H.

We consider perturbations of (5.23) of the form

$$\begin{aligned} \dot{x} &= P(x, y) + \varepsilon p(x, y, \varepsilon), \\ \dot{y} &= Q(x, y) + \varepsilon q(x, y, \varepsilon), \end{aligned} \tag{5.24}$$

where $p, q : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

We will propose a way for applying the averaging method in order to study limit cycles of (5.24) for ε sufficiently small, which bifurcate at $\varepsilon = 0$ from periodic trajectories of the period annulus of (5.23). The first aim is to write system (5.24) in the standard form for applying the averaging method, i.e. of the form (1.1). The differential system in this standard form describes the dependence between the square root of energy, $R = \sqrt{h}$ and the angle φ of the polar coordinates. The vector field of this equation will be 2π-periodic and its 2π-periodic solutions will be periodic trajectories of (5.24).

Theorem 5.1. *Assume* (A1) *holds for system* (5.23) *and that*

$$xQ(x, y) - yP(x, y) \neq 0 \text{ for all } (x, y) \text{ in the period annulus.} \tag{5.25}$$

Let $\rho : (\sqrt{h_c}, \sqrt{h_s}) \times [0, 2\pi) \to [0, \infty)$ *be a continuous function such that*

$$H\big(\rho(R, \varphi)\cos\varphi, \rho(R, \varphi)\sin\varphi\big) = R^2, \tag{5.26}$$

for all $R \in (\sqrt{h_c}, \sqrt{h_s})$ *and all* $\varphi \in [0, 2\pi)$. *Then the differential equation which describes the dependence between the square root of energy,* $R = \sqrt{h}$ *and the angle* φ *for system* (5.24) *is*

$$\frac{dR}{d\varphi} = \varepsilon \frac{\mu(x^2 + y^2)(Qp - Pq)}{2R(Qx - Py) + 2R\varepsilon(qx - py)}, \tag{5.27}$$

where $x = \rho(R, \varphi)\cos\varphi$ *and* $y = \rho(R, \varphi)\sin\varphi$.

We take $\varepsilon_f > 0$ *sufficiently small and* $D = \bigcup_{h_{c*} < h < h_{s*}} \Gamma_h$, *where* $h_c < h_{c*} < h_{s*} < h_s$ *are fixed but arbitrarily closed to* h_c *and* h_s, *respectively. The vector field of Eq.* (5.26) *is well defined and continuous on* $D \times (-\varepsilon_f, \varepsilon_f)$ *and is* 2π*-periodic with respect to* φ.

Proof. We need the relations,

$$\frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q = 0, \qquad \frac{\partial H}{\partial y} = -\mu P, \qquad \frac{\partial H}{\partial x} = \mu Q$$

which are valid on the period annulus since H is a first integral and μ is an integrating factor of (5.23). We define the function

$$G(r, R, \varphi) = H(r\cos\varphi, r\sin\varphi) - R^2,$$

at every point (r, ϕ) from the period annulus (which is an open set) and for each $R \in (\sqrt{h_c}, \sqrt{h_s})$. Here (r, ϕ) denote the polar coordinates. We have that

$$\frac{\partial G}{\partial r} = \frac{\partial H}{\partial x}\cos\varphi + \frac{\partial H}{\partial y}\sin\varphi = \frac{\mu(x, y)}{r}\big(Q(x, y)x - P(x, y)y\big),$$

where $x = r\cos\varphi$ and $y = r\sin\phi$. For every (r_0, φ_0) in the period annulus there is a R_0 such that $G(r_0, R_0, \varphi_0) = 0$. Assumption (5.25) assures that $\frac{\partial G}{\partial r}(r_0, R_0, \varphi_0) \neq 0$. By the Implicit Function Theorem, around every such point (R_0, φ_0) there is a unique continuous function $\rho = \rho(R, \varphi)$ such that relation (5.26) holds. Hence this function is well defined on the whole domain $(\sqrt{h_c}, \sqrt{h_s}) \times [0, 2\pi)$ and satisfies (5.26).

The dependence between the square root of energy and the time is given by $R(t) = \sqrt{H(x(t), y(t))}$, and between the angle φ and the time is $\varphi(t) = \arctan\frac{y(t)}{x(t)}$, whenever $(x(t), y(t)) \in \Gamma_h$, $t \in \mathbb{R}$. Then we get

$$\dot{R} = \varepsilon\frac{\mu(Qp - Pq)}{2R}, \qquad \dot{\varphi} = \frac{(Qx - yP) + \varepsilon(qx - py)}{x^2 + y^2}.$$

Eliminating the time we obtain equation (5.27). Condition (5.25) implies that the vector field of (5.27) is well defined in $D \times (-\varepsilon_f, \varepsilon_f)$ for ε_f sufficiently small. Also, it is easy to see that it is cotinuous and 2π-periodic in φ. □

An important result is the following, which states that the application of the averaging method for planar systems in the conditions of this section is equivalent to the study of the displacement function. In particular, the first order averaging method is equivalent to the study of first order Melnikov function. For more details in this direction we refer to [1,6,7,11,17].

The proof of this theorem is a direct consequence of Theorem 5.1 and the definition of the displacement and Melnikov functions.

Theorem 5.2. *The function* $f : (\sqrt{h_{c*}}, \sqrt{h_{s*}}) \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ *described by* (3.10) *for Eq.* (5.27) *is given by*

$$f(R, \varepsilon) = \varepsilon \int_0^{2\pi} \frac{\mu(x^2 + y^2)(Qp - Pq)}{2R(Qx - Py) + 2R\varepsilon(qx - py)}\, \mathrm{d}\varphi, \tag{5.28}$$

and the function $f_1 : (\sqrt{h_{c*}}, \sqrt{h_{s*}}) \to \mathbb{R}$ *described by* (1.2) *for Eq.* (5.26) *is*

$$f_1(R) = \int_0^{2\pi} \frac{\mu(x^2 + y^2)(Qp - Pq)}{2R(Qx - Py)}\, \mathrm{d}\varphi, \tag{5.29}$$

where $\mu = \mu(x, y)$ *is the integrating factor of system* (5.23) *corresponding to the first integral* H, *and* $x = \rho(R, \varphi)\cos\varphi$ *and* $y = \rho(R, \varphi)\sin\varphi$.

Moreover, the function (5.28) *is a displacement function and* (5.29) *is the corresponding first order Melnikov function of the system* (5.24).

Example 1 (*Bifurcation of limit cycles from an isochronous center via averaging*). Following the notations used in [6] or [5] the quadratic differential system

$$\begin{aligned} \dot{x} &= -y + x^2, \\ \dot{y} &= x + xy, \end{aligned} \tag{5.30}$$

with an isochronous center at the origin belongs to the class $\mathcal{S}_2$ and a first integral in the period annulus has the expression $H(x, y) = \frac{x^2+y^2}{(1+y)^2}$. For this system we note that $h_c = 0$, $h_s = 1$, and that the function ρ which satisfies the hypotheses of Theorem 5.1 is given by $\rho(R, \varphi) = \frac{R}{1-R\sin\varphi}$ for all $0 < R < 1$ and $\varphi \in [0, 2\pi)$.

We consider perturbations in the normal Bautin form

$$\begin{aligned} \dot{x} &= -y + x^2 + \varepsilon p(x, y), \\ \dot{y} &= x + xy + \varepsilon q(x, y), \end{aligned} \tag{5.31}$$

where $p(x, y) = a_1 x - a_3 x^2 + (2a_2 + a_5)xy + a_6 y^2$ and $q(x, y) = a_1 y + a_2 x^2 + a_4 xy - a_2 y^2$. The corresponding 1-dimensional Eq. (5.27) is

$$\frac{dR}{d\varphi} = \varepsilon \frac{a_1 R + a(\varphi)R^2 + b(\varphi)R^3}{1 - R\sin\varphi + \varepsilon c(\varphi)R}, \tag{5.32}$$

where

$$\begin{aligned} a(\varphi) &= (-2a_1 + 3a_2 + a_5)\sin\varphi + (a_4 + a_6)\cos\varphi \\ &\quad + (-4a_2 - a_5)\sin^3\varphi + (-a_3 - a_4 - a_6)\cos^3\varphi, \\ b(\varphi) &= a_1 + a_2 + (-a_1 - 2a_2)\cos^2\varphi - a_4\cos\varphi\sin\varphi, \\ c(\varphi) &= (a_3 + a_4)\sin\varphi + (-3a_2 - a_5)\cos\varphi + (-a_3 - a_4 - a_6)\sin^3\varphi \\ &\quad + (4a_2 + a_5)\cos^3\varphi. \end{aligned}$$

We denote

$$\begin{aligned} F_1(\varphi, R) &= \frac{a_1 R + a(\varphi)R^2 + b(\varphi)R^3}{1 - R\sin\varphi}, \\ G(\varphi, R, \varepsilon) &= -\frac{[a_1 R + a(\varphi)R^2 + b(\varphi)R^3]c(\varphi)R}{(1 - R\sin\varphi)(1 - R\sin\varphi + \varepsilon c(\varphi)R)}, \end{aligned}$$

such that (5.32) becomes

$$\frac{dR}{d\varphi} = \varepsilon F_1(\varphi, \varepsilon) + \varepsilon^2 G(\varphi, R, \varepsilon),$$

which is of the form (1.1), i.e. the standard form for first order averaging. In order to apply Theorem 1.1, we need function (1.2) which is for our problem $f_1 : (0, 1) \to \mathbb{R}$,

$$f_1(z) = \int_0^{2\pi} \frac{a_1 z + a(\varphi)z^2 + b(\varphi)z^3}{1 - z\sin\varphi}\, \mathrm{d}\varphi.$$

We compute this integral using *Maple* and obtain

$$\begin{aligned} f_1(z) = &-\frac{1}{2(z\sqrt{1 - z^2})}\big[2a_2 z^4 + (6a_2 + a_5 - 2a_1)z^2\sqrt{1 - z^2} - (10a_2 + 2a_5)z^2 \\ &- (2a_5 + 8a_2)\sqrt{1 - z^2} + 8a_2 + 2a_5\big]. \end{aligned}$$

When we take the new variable $\xi \in (0, 1)$ defined by $z = \sqrt{1 - \xi^2}$, we get

$$\begin{aligned} &f_1\big(\sqrt{1 - \xi^2}\big) \\ &\quad = \frac{1}{2(\sqrt{1 - \xi^2})}(1 - \xi)\big(2a_2\xi^2 + (2a_1 - 4a_2 - a_5)\xi + 2a_1 + 2a_2 + a_5\big). \end{aligned}$$

We notice that $z \in (0,1)$ is a zero of f_1 if and only if $\xi \in (0,1)$ is a zero of the polynomial function $g(\xi) = 2a_2\xi^2 + c_1\xi + c_2$, where $c_1 = 2a_1 - 4a_2 - a_5$ and $c_2 = 2a_1 + 2a_2 + a_5$. It is easy to see that, in our discussion about the zeros of g we can consider its coefficients as arbitrary real numbers. So, we can conclude that the number of zeros of g in the interval $(0,1)$ is at most 2. This means that the number of zeros of f_1 is at most 2. Hence at most two limit cycles bifurcates from the period annulus of system (5.30).

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