Mediterr. j. math. 99 (9999), 1–13
1660-5446/99000-0, DOI 10.1007/s00009-003-0000

Mediterranean Journal of Mathematics

Numerical ranges for pairs of operators, duality mappings with gauge function, and spectra of nonlinear operators

Jürgen Appell and Adriana Buică

Abstract. We define and study numerical ranges for pairs of nonlinear operators F and J which act between some Banach space X and its dual X^*, with respect to some increasing gauge function φ. Connections with spectra for certain classes of nonlinear operators introduced recently in the literature are also established. As a sample example, we consider the case when F is the duality map of the Lebesgue space $L_p(\Omega)$, J is the duality map of the corresponding Sobolev space $W_0^{1,p}(\Omega)$, and $\varphi(t) = t^{p-1}$ $(1 < p < \infty)$. This leads to existence, uniqueness, and perturbation results for a homogeneous eigenvalue problem involving the p-Laplace operator.

Mathematics Subject Classification (2000). Primary 47A12, 47H05; Secondary 46B10, 46E30, 46E35, 35J65.

Keywords. Nonlinear operator, monotone operator, coercive operator, numerical range, nonlinear spectrum, gauge function, duality map, p-Laplace operator, eigenvalue problem.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with Lipschitz continuous boundary, and $1 < p < \infty$. It is well known that the *p-Laplace operator* on the domain Ω defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \tag{1.1}$$

acts from the Sobolev space $X = W_0^{1,p}(\Omega)$ to its dual, $X^* = W^{-1,p'}(\Omega)$, where $p' = p/(p-1)$. The *nonlinear eigenvalue problem* with Dirichlet boundary condition

$$\begin{cases} -\Delta_p u(x) = \lambda |u(x)|^{p-2}u(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

This work was carried out while A. Buică was visiting University of Würzburg on a DAAD Grant.

for this operator arises in many fields of applied mathematics and mechanics, see e.g. [17]. Of course, in case $p = 2$ this problem just reduces to the *linear* eigenvalue problem for the Laplace operator $-\Delta$ which has been studied over and over in the last 150 years.

If we denote by J the differential operator defined by $-\Delta_p$ in the weak form, i.e.,

$$\langle Ju, v\rangle = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,dx \qquad (u \in W_0^{1,p}(\Omega)), \tag{1.3}$$

and by F the Nemytskij operator generated by the nonlinearity on the right hand side of (1.2), also in weak form, i.e.,

$$\langle Fu, v\rangle = \int_\Omega |u(x)|^{p-2}u(x)v(x)\,dx \qquad (u \in W_0^{1,p}(\Omega)), \tag{1.4}$$

we obtain two operators acting from X to its dual X^*. Here the norm we consider on X is

$$||u|| = \left(\int_\Omega |\nabla u(x)|^p\,dx\right)^{1/p}$$

which is, by the classical Poincaré inequality, equivalent to the usual norm on X involving the L_p-norm of u as well.

In this way, the eigenvalue problem (1.2) may be rewritten, for $\lambda \neq 0$ and $\mu = 1/\lambda$, equivalently as operator equation

$$Fu = \mu Ju \tag{1.5}$$

which is a *nonlinear eigenvalue problem* for the operator pair (F, J). A survey of methods and results for such problems may be found in Chapter 10 of the recent monograph [2].

Now, since spectra are intimately related to *numerical ranges*, it seems reasonable to connect the study of equation (1.5) to some numerical range for operator pairs (F, J) acting from a Banach space X to its dual X^*, like those given in (1.3) and (1.4). This is the purpose of the present paper which is organized as follows. In the first section we study two numerical ranges for pairs of operators (F, J) in the spirit of the numerical range introduced by Věra Burýšková [3] in connection with adjoints of nonlinear maps. Some properties of these numerical ranges are discussed, with a particular emphasis on the case of monotone and coercive operators. Afterwards we consider some connections with certain spectra for nonlinear operators F introduced in the last years by various authors, mainly for the special choice $Ju = u$. An important example is considered in the next section, namely the case where J is the duality map of some Banach space X, and F is the duality map of some larger Banach space Y. In case $X = W_0^{1,p}(\Omega)$ and $Y = L_p(\Omega)$ one essentially gets the maps (1.3) and (1.4). Finally, we briefly sketch some examples, applications, and extensions.

2. Numerical ranges for pairs of operators

Throughout this section, X is a reflexive Banach space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and X^* denotes its dual. Let first $\mathbb{K} = \mathbb{R}$, and let $J : X \to X^*$ and $F : Y \to Y^*$ be two hemicontinuous operators such that $F(0) = J(0) = 0$ and J is *strictly monotone*, i.e.,

$$\langle Ju - Jv, u - v \rangle > 0 \qquad (u, v \in X,\ u \neq v), \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual pairing between X^* and X. We recall that an operator $A : X \to X^*$ is called *hemicontinuous* if

$$\lim_{t \to 0} \langle A(u + tv), w \rangle = \langle Au, w \rangle \qquad (u, v, w \in X).$$

The *point spectrum* $\sigma_\pi(F, J)$ is defined as set of all scalars $\mu \in \mathbb{K}$ such that equation (1.5) has a nontrivial solution $u \in X$. Of course, in case of linear F and $J = I$ (the identity operator), this definition reduces to the familiar notion of point spectrum for F.

Under the above hypotheses, we define a numerical range for to the pair (F, J) by

$$W(F, J) = \left\{ \frac{\langle Fu - Fv, u - v \rangle}{\langle Ju - Jv, u - v \rangle} : u, v \in X,\ \langle Ju - Jv, u - v \rangle \neq 0 \right\}. \tag{2.2}$$

In the special case when X is a Hilbert space, $Y = X$, $J = I$, and F being Lipschitz continuous, the numerical range (2.2) reduces to the numerical range in the sense of Zarantonello [23,24] defined by

$$W_Z(F) = \left\{ \frac{\langle Fu - Fv, u - v \rangle}{||u - v||^2} : u, v \in X,\ u \neq v \right\}.$$

It is well-known that the numerical range of a nonlinear operator may have a complicated structure if the underlying Banach space X is complex. Thus, it may be not convex, in contrast to the linear case, but it is still connected if X is a Hilbert space. In Banach spaces, however, the numerical range may even be disconnected; for some examples and counterexamples we refer to Chapter 11 of [2] and the references there.

However, in connection with monotone operators we always restrict ourselves to *real* Banach spaces, and in this case the set (2.2) has a simple structure: it is just a real interval. To see this, fix $\mu_0, \mu_1 \in W(F, J)$ and choose $u_0, v_0, u_1, v_1 \in X$ such that $u_0 \neq v_0$, $u_1 \neq v_1$, and

$$\mu_0 = \frac{\langle Fu_0 - Fv_0, u_0 - v_0 \rangle}{\langle Ju_0 - Jv_0, u_0 - v_0 \rangle}, \quad \mu_1 = \frac{\langle Fu_1 - Fv_1, u_1 - v_1 \rangle}{\langle Ju_1 - Jv_1, u_1 - v_1 \rangle}.$$

When $\dim X \geq 2$, there exist continuous functions $\phi, \psi : [0, 1] \to X$ with $\phi(0) = u_0$, $\psi(0) = v_0$, $\phi(1) = u_1$, $\psi(1) = v_1$, and $\phi(t) \neq \psi(t)$ for all $t \in [0, 1]$. Then the function $f : [0, 1] \to W(F, J)$ defined by

$$f(t) := \frac{\langle F\phi(t) - F\psi(t), \phi(t) - \psi(t) \rangle}{\langle J\phi(t) - J\psi(t), \phi(t) - \psi(t) \rangle}$$

satisfies $f(0) = \mu_0$ and $f(1) = \mu_1$. Moreover, f is continuous, since F and J are hemicontinuous. Consequently, its range contains the whole interval $[\mu_0, \mu_1]$ as claimed.

The following lemma shows that the operator $\mu J - F$ occurring in (1.5) has some nice "regularity property" if the scalar μ is "bounded away" from the numerical range (2.2).

Lemma 2.1. *Let $\mu \in \mathbb{R} \setminus \overline{W(F,J)}$ and $d_\mu := \operatorname{dist}(\mu, W(F,J))$. Then the relation*

$$\langle Au - Av, u - v\rangle \geq d_\mu \langle Ju - Jv, u - v\rangle \tag{2.3}$$

holds for all $u, v \in X$ with $u \neq v$, where either $A = \mu J - F$ or $A = F - \mu J$.

Proof. For $u, v \in X$ with $u \neq v$ we have

$$0 < d_\mu \leq \left| \mu - \frac{\langle Fu - Fv, u - v\rangle}{\langle Ju - Jv, u - v\rangle} \right| = \frac{|\langle (\mu J - F)u - (\mu J - F)v, u - v\rangle|}{|\langle Ju - Jv, u - v\rangle|}.$$

Multiplying by $\langle Ju - Jv, u - v\rangle$ gives the result. □

Observe that in the proof of Lemma 2.1 we used the fact that $W(F,J)$ is an interval.

The numerical range (2.2) has some natural additivity and homogeneity properties which may be proved in exactly the same way as for Zarantonello's numerical range. For instance, it is rather straightforward to prove that

$$W(F+G, J) \subseteq W(F,J) + W(G,J), \quad W(\lambda F, J) = \lambda W(F,J),$$

$$W(F_z, J) = W(F,J) \qquad (F_z(x) := F(x) + z),$$

and

$$W(\lambda J - F, J) = \{\lambda\} - W(F,J).$$

As simple examples show, the set $W(F,J)$ is in general neither bounded nor closed. However, for special classes of operators one can say more. Recall that an operator $A : X \to X^*$ is *coercive* if

$$\lim_{||u|| \to \infty} \frac{\langle Au, u\rangle}{||u||} = \infty.$$

In view of the nonlinear eigenvalue problem, the following generalization of this notion is useful. Suppose that $\varphi : [0,\infty) \to [0,\infty)$ is a strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Such a function will be called a *gauge function* in what follows; typical examples are $\varphi(t) = t^{p-1}$ for $1 < p < \infty$, $\varphi(t) = e^t - 1$, or $\varphi(t) = \log(1+t)$. We say that an operator $A : X \to X^*$ is *φ-coercive* if

$$\lim_{||u|| \to \infty} \frac{\langle Au, u\rangle}{\varphi(||u||)} = \infty, \tag{2.4}$$

and *φ-monotone* if there is some $C_1 > 0$ such that

$$\langle Au - Av, u - v\rangle \geq C_1 \varphi(||u - v||)||u - v|| \quad (u, v \in X). \tag{2.5}$$

Moreover, we say that an operator $A : X \to X^*$ satisfies a *φ-Hölder condition* if there is some $C_2 > 0$ such that

$$||Au - Av|| \leq C_2\varphi(||u - v||) \quad (u, v \in X). \tag{2.6}$$

For example (see, e.g., [25, § 26.5]), the operator (1.3) satisfies for $p \geq 2$ in the space $X = W_0^{1,p}(\Omega)$ the estimate

$$\langle Ju - Jv, u - v\rangle \geq C||u - v||^p \quad (u, v \in X) \tag{2.7}$$

which means that J is φ-monotone for $\varphi(t) := t^{p-1}$. Putting $v = 0$ in (2.7) yields

$$\frac{\langle Ju, u\rangle}{||u||^{p-1}} \geq C||u|| \to \infty \qquad (||u|| \to \infty) \tag{2.8}$$

which shows that J is, again for $\varphi(t) = t^{p-1}$, also φ-coercive. Clearly, any φ-monotone operator is also strictly monotone in the sense of (2.1), and hence injective. In particular, the operator (1.3) is invertible on its range, and (2.7) immediately implies that its inverse satisfies the *global Hölder condition*

$$||J^{-1}f - J^{-1}g|| \leq \frac{1}{C^{1/(p-1)}}||f - g||^{1/(p-1)} \tag{2.9}$$

in case $p \geq 2$. For $1 < p < 2$, however, the situation is more complicated. In fact, it was shown in [5] that in this case (2.9) has to be replaced by

$$||J^{-1}f - J^{-1}g|| \leq \frac{2^{2-p}}{C}(||f|| + ||g||)^{(2-p)(p-1)}||f - g||, \tag{2.10}$$

i.e., a *local Lipschitz condition* for J^{-1}.

From standard estimates of scalar functions it follows that the operator (1.4) satisfies, again for $\varphi(t) = t^{p-1}$, a φ-Hölder condition of type

$$||Fu - Fv|| \leq \begin{cases} \varphi(||u - v||) & \text{if} \quad 1 < p \leq 2, \\ 2^{p-2}\varphi(||u - v||) & \text{if} \quad 2 \leq p < \infty. \end{cases} \tag{2.11}$$

The importance of the conditions (2.7), (2.8) and (2.11) is illustrated by the following

Proposition 2.2. *Suppose that J is φ-monotone, and F satisfies a φ-Hölder condition. Then the numerical range (2.2) is bounded.*

Proof. The proof is rather trivial: Given $\mu \in W(F, J)$, choose $u, v \in X$ such that

$$\mu = \frac{\langle Fu - Fv, u - v\rangle}{\langle Ju - Jv, u - v\rangle}.$$

Then

$$|\mu| \leq \frac{||u - v||\,||Fu - Fv||}{C_1\varphi(||u - v||)||u - v||} \leq \frac{C_2}{C_1}\frac{\varphi(||u - v||)||u - v||}{\varphi(||u - v||)||u - v||} = \frac{C_2}{C_1},$$

where C_1 and C_2 are the corresponding constants from (2.5) and (2.6), respectively. $\square$

Proposition 2.3. *Let* $\mu \in \mathbb{R} \setminus \overline{W(F,J)}$ *and* $d_\mu := \operatorname{dist}(\mu, W(F,J))$. *Then the following holds true.*

(a) *If* J *is* φ*-monotone then* $\mu J - F$ *is also* φ*-monotone.*

(b) *If* J *is* φ*-coercive then* $\mu J - F$ *is also* φ*-coercive.*

(c) *If* J *is both* φ*-monotone and* φ*-coercive, then* $\mu J - F$ *is a homeomorphism between* X *and* X^*.

Proof. If J is φ-monotone, it follows from (2.3) and (2.5) that

$$\langle (\mu J - F)u - (\mu J - F)v \rangle \geq d_\mu C_1 \varphi(||u-v||)||u-v|| \qquad (u, v \in X)$$

which proves (a). Similarly, putting $v = 0$ in (2.3) yields

$$\frac{\langle (\mu J - F)u, u \rangle}{\varphi(||u||)} \geq d_\mu \frac{\langle Ju, u \rangle}{\varphi(||u||)} \to \infty \qquad (||u|| \to \infty)$$

which proves (b). Finally, under the hypotheses of (c) the operator $\mu J - F$ is hemicontinuous, strictly monotone, and φ-coercive, and the assertion follows from Minty's fundamental theorem [20] on monotone operators. □

Apart from (2.2), we consider now the numerical range

$$W_0(F,J) = \left\{ \frac{\langle Fu, u \rangle}{\langle Ju, u \rangle} : u \in X, \langle Ju, u \rangle \neq 0 \right\}. \tag{2.12}$$

By our assumption $F(0) = J(0) = 0$, this is a subset of the numerical range (2.2). In the special case when X is a Hilbert space, $Y = X$, $J = I$, and F is continuous, the numerical range (2.12) reduces to the numerical range in the sense of Feng [11]. Moreover, Věra Burýšková [3] considered the numerical range (2.12) for continuous positively homogeneous operators (of the same degree) F and J, which is of course motivated by the eigenvalue problem (1.2).

3. Connections with nonlinear spectra

The following proposition gives a connection between the numerical range (2.12) and the point spectrum of (F, J) introduced above.

Proposition 3.1. *The inclusions*

$$\sigma_\pi(F,J) \subseteq W_0(F,J) \subseteq W(F,J) \tag{3.1}$$

holds. Moreover, if J *is* φ*-coercive,* F *is compact, both* J *and* F *are odd, and* $\mu \in \mathbb{R} \setminus (\overline{W_0(F,J)} \cup \{0\})$, *then the operator* $\mu J - F$ *is surjective.*

Proof. The inclusions (3.1) are obvious, while the second assertion is a direct consequence of Theorem 1 from [7]. □

We point out that the operators (1.3) and (1.4) satisfy all hypotheses of Proposition 3.1 in the reflexive Sobolev space $X = W_0^{1,p}(\Omega)$. In fact, they are obviously odd, and J is φ-coercive for $\varphi(t) = t^{p-1}$, as observed before. By Krasnosel'skij's theorem [16], the Nemytskij operator (1.4) is continuous and bounded from $L_p(\Omega)$ into $L_{p'}(\Omega)$, and so it is *compact* from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$, by classical Sobolev imbedding theorems.

The numerical range is related to another nonlinear spectrum which was introduced by Weber [22] in connection with surjectivity results of Fredholm type. Following [22], we denote by $\sigma_\varphi(F, J)$ the set of all scalars μ for which $[\mu J - F]_\varphi = 0$, where

$$[A]_\varphi := \liminf_{||u||\to\infty} \frac{||Au||}{\varphi(||u||)}.$$

Observe that any φ-coercive operator satisfies $[J]_\psi > 0$ if $\psi(t) := \varphi(t)/t$ is a gauge function, but not vice versa. In case $\varphi(t) = t$ (and $J = I$), the spectrum $\sigma_\varphi(F, J)$ reduces to the spectrum $\Sigma(F)$ introduced by Furi and Vignoli in [12], i.e.,

$$\Sigma(F) = \{\lambda \in \mathbb{K} : \liminf_{||u||\to\infty} \frac{||Fu - \lambda u||}{||u||} = 0\}. \tag{3.2}$$

The spectrum $\sigma_\varphi(F, J)$ shares some natural properties with the familiar spectrum of bounded linear operators; for example, it is always closed, and even compact in case $[J]_\varphi > 0$. An important special case is of course $J = I$, where this condition is fulfilled for $\varphi(t) = t$; so the Furi-Vignoli spectrum (3.2) is always compact.

We remark that neither the spectrum $\sigma_\varphi(F, J)$ nor the point spectrum $\sigma_\pi(F, J)$ is included in the other, even in case $J = I$ and $\varphi(t) = t$. For example, in the scalar example $X = \mathbb{R}$ and $Fu = \sqrt{|u|}$ we have $\sigma_\pi(F, I) = \mathbb{R} \setminus \{0\}$, but $\sigma_\varphi(F, I) = \Sigma(F) = \{0\}$.

Proposition 3.2. *Assume that $[J]_\varphi > 0$, and suppose that $\psi(t) := \varphi(t)/t$ is a gauge function. Then the inclusion*

$$\sigma_\psi(F, J) \subseteq \overline{W_0(F, J)}$$

holds true.

Proof. Suppose that $d_\mu := \mathrm{dist}\,(\mu, W_0(F, J)) > 0$. Choose $\eta \in (0, [J]_\varphi)$ and $\rho > 0$ such that $||u|| \geq \rho$ implies $\langle Ju, u\rangle \geq \eta\varphi(||u||)$. Then

$$0 < d_\mu \leq \left|\mu - \frac{\langle Fu, u\rangle}{\langle Ju, u\rangle}\right| = \frac{|\langle(\mu J - F)u, u\rangle|}{|\langle Ju, u\rangle|} \leq \frac{||(\mu J - F)u||\,||u||}{\eta\varphi(||u||)} = \frac{||(\mu J - F)u||}{\eta\psi(||u||)},$$

hence

$$\frac{||(\mu J - F)u||}{\psi(||u||)} \geq d_\mu \eta \qquad (||u|| \geq \rho).$$

This shows that $[\mu J - F]_\psi \geq d_\mu\eta > 0$, hence $\mu \notin \sigma_\psi(F, J)$, which proves the assertion. □

There are some other more sophisticated spectra for nonlinear operators which may be connected to the numerical ranges (2.5) or (2.12). For example, Zarantonello's numerical range for Lipschitz continuous operators mentioned above is intimately related to a spectrum introduced by Maddox and Wickstead [19] in 1978. We consider here another class of operators which is defined by means of compactness properties and has been considered first by Tarafdar and Thompson [21].

Let A be an operator between two Banach spaces X and Y such that $Au \neq 0$ on the sphere $S_r(X) = \{u \in X : ||u|| = r\}$. We say that A is *k-epi* on the ball $B_r(X) = \{u \in X : ||u|| \leq r\}$ $(k \geq 0)$ if, given any continuous operator $C : X \to Y$ which has measure of noncompactness $\leq k$ (see, e.g. [1]) and vanishes on the sphere $S_r(X)$, we can find a solution of the coincidence equation $Au = Cu$ in the interior of the ball $B_r(X)$. Obviously, if A is k-epi on some ball, then A is also k'-epi on the same ball for $k' \leq k$; so it seems reasonable to introduce the characteristic

$$\nu(A) := \inf_{r>0} \sup \{k \geq 0 : A \text{ is } k\text{-epi on } B_r(X)\}.$$

In finite dimensional spaces X, one can have only the alternative $\nu(A) = 0$ or $\nu(A) = \infty$, as a consequence of Brouwer's fixed point theorem, and this is of course not interesting. In infinite dimensional spaces, however, it is important to distinguish the cases $\nu(A) = 0$ and $\nu(A) > 0$ which describe, loosely speaking, a certain *measure of solvability* of the coincidence equation $Au = Cu$. For example, from Darbo's well-known fixed point theorem [6] it follows that the identity operator I satisfies $\nu(I) = 1$ in infinite dimensions. Therefore, the set

$$\sigma_\nu(F, J) = \{\mu \in \mathbb{K} : \nu(\mu J - F) = 0\}$$

plays an important role in nonlinear spectral theory (see Chapter 8 of [2]).

Proposition 3.3. *Assume that $\nu(J) > 0$. Then the inclusion*

$$\sigma_\nu(F, J) \subseteq \operatorname{co} W_0(F, J)$$

holds true.

Proof. The assumption $\nu(J) > 0$ means that the operator μJ is k-epi, for $k > 0$ small enough, on any ball $B_r(X)$. Fix $\mu \in \mathbb{R} \setminus \operatorname{co} W_0(F, J)$ and consider the set

$$S := \{u \in X : \mu Ju = tFu \text{ for some } t \in [0, 1]\}.$$

Clearly, $0 \in S$; we claim that $S = \{0\}$. In fact, suppose that $u \in S$, $u \neq 0$. Then we find $t \in (0, 1]$ such that $tFu = \mu Ju$, hence

$$\mu = t \frac{\langle Fu, u\rangle}{\langle Ju, u\rangle} \in \operatorname{co} W_0(F, J),$$

contradicting our choice of μ. From the homotopy invariance of k-epi operators [21] it follows that $H(u, t) := \mu Ju - tFu$ is an admissible homotopy joining $H(\cdot, 0) = \mu J$ and $H(\cdot, 1) = \mu J - F$, and so $\nu(\mu J - F) > 0$. But this means exactly that $\mu \notin \sigma_\nu(F, J)$ as claimed. □

4. A special case: duality maps

Recall that the *duality map* $\mathcal{D} : X \to X^*$ of a Banach space is defined by

$$\mathcal{D}(u) = \{\ell_u \in X^* : \langle u, \ell_u \rangle = ||u||^2, \, ||\ell_u|| = ||u||\}. \tag{4.1}$$

More generally, given a gauge function φ as in the preceding section, one may define a *duality map with gauge function* $\mathcal{D}_\varphi : X \to X^*$ by

$$\mathcal{D}_\varphi(u) = \{\ell_u^\varphi \in X^* : \langle u, \ell_u^\varphi \rangle = \varphi(||u||)||u||, \, ||\ell_u^\varphi|| = \varphi(||u||)\}. \tag{4.2}$$

Obviously, (4.1) is a special case of (4.2) for $\varphi(t) = t$. As far as we know, the generalized duality map (4.2) was considered first by J. L. Lions [18] and has useful applications in the theory of partial differential equations.

In general, the duality maps (4.1) and (4.2) may be multivalued; in special spaces, however, they are singlevalued. For example, the map (4.1) is singlevalued if and only if the underlying space X is smooth (i.e., its norm is Gâteaux differentiable on $X \setminus \{0\}$).

In this section we make the following general hypotheses. We suppose that X and Y are smooth reflexive Banach spaces, so their duality maps (4.2) are singlevalued. Moreover, we assume that X is strictly convex and compactly imbedded in Y with imbedding constant $C_{X,Y}$, i.e.,

$$||u||_Y \leq C_{X,Y}||u||_X. \tag{4.3}$$

To emphasize the difference between the norms of an element u in these spaces, we will use norm indices in the sequel. In order to apply the results of the preceding section, we let J be the duality map with gauge function φ on X, i.e.,

$$Ju = \{\ell_u^\varphi \in X^* : \langle u, \ell_u^\varphi \rangle = \varphi(||u||_X)||u||_X, \, ||\ell_u^\varphi||_{X^*} = \varphi(||u||_X)\},$$

and F be the duality map with the same gauge function φ on the larger space Y, i.e.,

$$Fu = \{\ell_u^\varphi \in Y^* : \langle u, \ell_u^\varphi \rangle = \varphi(||u||_Y)||u||_Y, \, ||\ell_u^\varphi||_{Y^*} = \varphi(||u||_Y)\}.$$

Then $F : Y \to Y^*$ is monotone and, since X is strictly convex, $J : X \to X^*$ is even strictly monotone [9,15]. The fact that X and Y are reflexive and their duality maps are singlevalued ensures that both J and F are hemicontinuous [9, Proposition 3]. From its definition and the properties of φ we see that J is coercive, and both J and F are clearly odd operators. Consequently, the results of the preceding section work in this case. In particular, the operator $\mu J - F : X \to X^*$ is onto for $\mu \notin \overline{W_0(F, J)}$, and even a homeomorphism for $\mu \notin \overline{W(F, J)}$.

So the problem arises to describe the numerical ranges (2.5) and (2.12) more explicitly in case of duality maps J and F. It is clear that $W(F, J) \subseteq [0, \infty)$. Moreover,

$$W_0(F, J) = \left\{ \frac{\varphi(||u||_Y)||u||_Y}{\varphi(||u||_X)||u||_X} : u \neq 0 \right\} \tag{4.4}$$

in this case, hence $W_0(F,J) \subseteq [0, \tilde{C}_{X,Y}]$, where

$$\tilde{C}_{X,Y} := C_{X,Y} \sup_{t>0} \frac{\varphi(C_{X,Y}t)}{\varphi(t)}, \tag{4.5}$$

with $C_{X,Y}$ from (4.3).

The most important example seems to be $X = W_0^{1,p}(\Omega)$, hence $X^* = W^{-1,p'}(\Omega)$, and $Y = L_p(\Omega)$, hence $Y^* = L_{p'}(\Omega)$ $(1 < p < \infty,\ p' = p/(p-1))$. Put $\varphi(t) := t^{p-1}$. In this case, the duality map J of X with gauge function φ is precisely (1.3), and the duality map of Y with the same gauge function is precisely (1.4). Moreover, it is well-known that the lowest eigenvalue λ_1 of the problem (1.2) for the p-Laplace operator (1.1) is positive, isolated, and simple (in the sense that all eigenfunctions u corresponding to λ_1 are scalar multiples of a fixed normalized eigenfunction u_1), and may be "calculated" as variational Rayleigh quotient

$$\lambda_1 = \inf_{u \in X \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u(x)|^p \, dx}{\displaystyle\int_\Omega |u(x)|^p \, dx},$$

precisely as in the linear case $p = 2$. Equivalently, the positive number $\lambda_1^{1/p}$ may be considered as best constant $C_{X,Y}$ in the Poincaré inequality (4.3). So (4.5) yields here

$$\tilde{C}_{X,Y} := \lambda_1^{1/p} \sup_{t>0} \frac{\lambda_1^{(p-1)/p} t^{p-1}}{t^{p-1}} = \lambda_1$$

in this case, hence $W_0(F,J) \subseteq [0, \lambda_1]$. In fact, it was shown in [8] that even $W_0(F,J) = [0, \lambda_1]$, so $W_0(F,J)$ is compact in this case. From our discussion above it follows that the operator $J - \lambda F : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is surjective for $\lambda > \lambda_1$, and even bijective for $\lambda < 0$.

An explicit description of the larger numerical range (2.5) is more complicated. In the special case $N = 1$ and $p \neq 2$, it follows from a theorem of Drábek and Takác [10] that $J - \lambda F$ is not injective whenever $\lambda > 0$, and so $(0, \infty) \subseteq \overline{W(F,J)}$. Together with the obvious inclusion $W(F,J) \subseteq [0, \infty)$ this implies that $\overline{W(F,J)} = [0, \infty)$ in this case, so $W(F,J)$ need not be compact. In particular, the operators (1.3) and (1.4), together with the gauge function $\varphi(t) = t^{p-1}$, may serve as an example for strict inequality in the second inclusion in (3.1). We summarize our discussion with the following

Theorem 4.1. *Let $X = W_0^{1,p}(\Omega)$, hence $X^* = W^{-1,p'}(\Omega)$, and $Y = L_p(\Omega)$, hence $Y^* = L_{p'}(\Omega)$ $(1 < p < \infty, p' = p/(p-1))$. Denote by $C_{X,Y}$ the best constant in the Poincaré inequality (4.3). Put $\varphi(t) = t^{p-1}$, and let J and F be defined by (1.3) and (1.4), respectively. Then the following is true.*

(a) *The operator $J : X \to X^*$ is a φ-coercive and φ-monotone homeomorphism, and its inverse satisfies the Hölder condition*

$$||J^{-1}f - J^{-1}g|| \leq C_{X,Y}^{-1/(p-1)} ||f-g||^{1/(p-1)} \qquad (f, g \in W^{-1,q}(\Omega)).$$

(b) *The operator* $F : Y \to Y^*$ *satisfies the* φ*-Hölder condition (2.11).*

(c) *The operator* $J - \lambda F : X \to X^*$ *is bijective for* $\lambda < 0$ *and surjective for* $\lambda > C^p_{X,Y}$.

(d) *The numerical range (2.12) is contained in the interval* $[0, C^p_{X,Y}]$.

(e) *For* $\mu > C^p_{X,Y}$ *one has both* $\mu \notin \sigma_\pi(F, J)$ *and* $\mu \notin \sigma_\psi(F, J)$*, where* $\psi(t) = t^{p-2}$ *for* $p > 2$.

5. Concluding remarks

We briefly sketch how to apply our abstract results to nonlinear problems. Consider, for instance, the inhomogeneous nonlinear elliptic problem

$$\begin{cases} -\Delta_p u(x) = \lambda |u(x)|^{p-2} u(x) + g(x, u(x)) + h(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

for $h \in W^{-1,q}(\Omega), h(x) \not\equiv 0$, which may be considered as a *perturbation* of (1.2). We assume that the derivative of the nonlinearity g in (5.1) satisfies the growth condition

$$\left| \frac{\partial g}{\partial u}(x, u) \right| \leq \alpha |u|^q + \beta |u|^{p^*-2} \qquad (x \in \Omega, u \in \mathbb{R}),$$

where $p^* = Np/(N-p)$ denotes the critical Sobolev exponent. A typical example is the nonlinearity $g(u) = \alpha |u|^q u + \beta |u|^{p^*-2} u$ which has been studied, e.g., in [4]. Since p^* is the critical Sobolev exponent and the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L_{p^*}(\Omega)$ is only continuous, not compact, the lower semicontinuity properties of the corresponding functional is in question, if one tries to look at the problem (5.1) from the viewpoint of variational methods.

However, topological methods still work. For instance, building on the local Lipschitz condition (2.10) the author of [5] studied problem (5.1) for $\lambda = 0$ and proved existence of solutions (for sufficiently small α and β) by means of Banach's contraction mapping principle. On the other hand, using the spectrum $\sigma_\nu(F, J)$ from Proposition 3.3, together with the numerical range (2.12), one may deal with this problem by imposing topological conditions, rather than metric conditions. Details will be given in a subsequent paper.

We close with some remarks on possible extensions. Our main example has been the Lebesgue and Sobolev spaces treated in Theorem 4.1. Of course, one may formulate a parallel result in the reflexive sequence space $X = l_p$ for $1 < p < \infty$. Here the usual (singlevalued) duality map (4.1) in X is given by $\mathcal{D}(u) = \{\ell_u\}$ with

$$\langle v, \ell_u \rangle = \frac{1}{||u||^{p-2}} \sum_{n=1}^{\infty} |u_n|^{p-2} u_n v_n \qquad (v = (v_n)_n \in l_{p/(p-1)}). \tag{5.2}$$

Consequently, choosing again $\varphi(t) := t^{p-1}$, we obtain $\mathcal{D}(u) = \{\ell_u^\varphi\}$ with $\ell_u^\varphi(v)$ given by the series in (5.2).

The special choice $\varphi(t) = t^{p-1}$ for the gauge function in the above theorem is of course suggested by the special homogeneity behaviour of both sides in the eigenvalue problem (1.2). However, more complicated choices of φ are also possible. For example, given an arbitrary gauge function φ, as in [14] one may consider the so-called *φ-Laplace operator* defined by

$$\Delta_\varphi u = \operatorname{div}\left(\varphi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right).$$

Clearly, for $\varphi(t) = t^{p-1}$ one gets the p-Laplace operator (1.1). However, if one is interested in eigenvalue problems involving nonlinearities with non-polynomial (e.g., exponential) growth, it is a useful device to replace Sobolev spaces by Sobolev-Orlicz spaces, see e.g. [13]. To this end, a typical choice for a rapidly increasing gauge function would be, for example, $\varphi(t) = e^t - 1$.

References

[1] R. R. Akhmerov, M. I. Kamenskij, A. S. Potapov, A. E. Rodkina, B. N. Sadovskij, *Measures of Noncompactness and Condensing Operators* [in Russian]. Nauka, Novosibirsk 1986; Engl. transl.: Birkhäuser, Basel 1992.

[2] J. Appell, E. De Pascale, A. Vignoli, *Nonlinear Spectral Theory.* DeGruyter, Berlin 2004.

[3] V. Burýšková, *Some properties of nonlinear adjoint operators.* Rocky Mount. J. Math. **28** (1998), 41–59.

[4] J. Chabrowski, *On multiple solutions for the non-homogeneous p-Laplacian with a critical Sobolev exponent.* Diff. Integral Equ. **8, 4** (1995), 705–716.

[5] Y. J. Cheng, *Hölder continuity of the inverse of the p-Laplacian.* J. Math. Anal. Appl. **221** (1998), 734–748.

[6] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto*. Rend. Sem. Mat. Univ. Padova **24** (1955), 84–92.

[7] G. Dincă, *A Fredholm-type result for a couple of nonlinear operators*. C. R. Acad. Sci. Paris **333** (2001), 415–419.

[8] G. Dincă, P. Jebelean, *Some existence results for a class of nonlinear equations involving a duality mapping.* Nonlin. Anal. TMA **46** (2001), 347–363.

[9] G. Dincă, P. Jebelean, J. Mawhin, *Variational and topological methods for Dirichlet problems with p-Laplacian.* Portugaliae Math. **58, 3** (2001), 339–378.

[10] P. Drábek, P. Takác, *A counterexample to the Fredholm alternative for the p-Laplace operator*. Proc. Amer. Math. Soc. **127** (1999), 1079–1087.

[11] W. Feng, *A new spectral theory for nonlinear operators and its applications*. Abstr. Appl. Anal. **2** (1997), 163–183.

[12] M. Furi, A. Vignoli, *A nonlinear spectral approach to surjectivity in Banach spaces.* J. Funct. Anal. **20** (1975), 304–318.

[13] J.-P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly increasing coefficients.* Trans. Amer. Math. Soc. **190** (1974), 163–205.

[14] J.-P. Gossez, R. Manásievich, *On a nonlinear eigenvalue problem in Orlicz-Sobolev spaces.* Proc. Royal Soc. Edinburgh Sect. A **132** (2002), 891–909.

[15] P. Jebelean, *Metode de analiză neliniară cu aplicatii in probleme la limită cu p-Laplacian.* Editura Universităţii de Vest, Timişoara 2001.

[16] M. A. Krasnosel'skij, *On the continuity of the operator* $Fu(x) = f(x, u(x))$ [in Russian]. Doklady Akad. Nauk SSSR **77, 2** (1951), 185–188.

[17] P. Lindqvist, *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$. Proc. Amer. Math. Soc. **109** (1990), 157–163.

[18] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires.* Gauthier Villars, Paris 1969.

[19] I. J. Maddox, A. W. Wickstead, *The spectrum of uniformly Lipschitz mappings.* Proc. Royal Irish Acad. Sect. A **89** (1989), 101–114.

[20] G. Minty, *Monotone nonlinear operators in Hilbert space.* Duke Math. J. **29** (1962), 341–346.

[21] E. U. Tarafdar, H. B. Thompson, *On the solvability of nonlinear noncompact operator equations.* J. Austral. Math. Soc. **43** (1987), 103–114.

[22] H. Weber, *Ein φ-asymptotisches Spektrum und Surjektivitätssätze vom Fredholm-Typ für nichtlineare Operatoren mit Anwendungen.* Math. Nachr. **117** (1984), 7–35.

[23] E. H. Zarantonello, *The closure of the numerical range contains the spectrum.* Pacific J. Math. **22** (1967), 575–595.

[24] E. H. Zarantonello, *Proyecciones sobre conjuntos convexos en el espacio de Hilbert y teoría espectral.* Revista Unión Mat. Argentina **26** (1972), 187–201.

[25] E. Zeidler, *Nonlinear Functional Analysis and its Applications II/B: Nonlinear Monotone Operators.* Springer, Berlin 1990.

Acknowledgment

A. Buică wishes to thank to the Department of Mathematics of University of Würzburg for hospitality and to the DAAD (Deutscher Akademischer Austausch Dienst) for the financial support.

Jürgen Appell
University of Würzburg
Department of Mathematics
Am Hubland, D-97074
Würzburg, Germany
e-mail: `appell@mathematik.uni-wuerzburg.de`

Adriana Buică
Babeş-Bolyai University
Department of Applied Mathematics
1 Kogălniceanu st.
RO-400084 Cluj-Napoca, Romania
e-mail: `abuica@math.ubbcluj.ro`