On Peetre's condition in the coincidence theory. II. Relations with other coincidence theorems and applications

A. Buică and F. Aldea
Babes-Bolyai University, Department of Applied Mathematics,
str. M. Kogalniceanu 1, 3400 Cluj-Napoca, Romania
email: abuica@math.ubbcluj.ro, aldea@math.ubbcluj.ro

Abstract

The aim of the second part of our work on Peetre's condition in the coincidence theory is to show how the abstract results from the first part can be extanded to operators defined on some set X and to illustrate the data dependence results with some applications to elliptic equations.

1 Introduction

The aim of the second part of our work on Peetre's condition in the coincidence theory is to show how the abstract results from the first part can be extanded to operators defined on some set X and to illustrate the data dependence results with some applications to elliptic equations.
We shall be able to prove the well-known coincidence theorem of Goebel [8] as a consequence of Theorem 2.1 [1] and the data dependence theorem related to the theorem of Goebel [2] as a consequence of Theorem 3.3. [1]
Corollary 3.3 is a new data dependence result related to a coincidence theorem from [3] involving operators which satisfy a generalized pseudo-contractivity condition.
In Section 4 we shall state and prove a general data dependence result for some fully nonlinear elliptic equations of the form

$$u \in H^2(\Omega) \cap H_0^1(\Omega), \quad a(x, D^2u) = f(x), \quad \text{a.a. } x \in \Omega,$$

where $f \in L^2(\Omega)$ and the function $a : \Omega \times \mathcal{M}_n \to \mathbb{R}$ satisfy the ellipticity condition (A4) stated in that section.
Let us mention that this kind of problems was considered in [6, 9, 4], where existence results were established. In [6, 9] a stronger ellipticity condition (A5) is used.
In Section 5, for the sake of simplicity, we shall consider the elliptic equation

$$\frac{\partial^2 u}{\partial x_1^2} + \lambda \frac{\partial^2 u}{\partial x_2^2} = f,$$

and prove that its solutions $u(\lambda) \in H^2(\Omega) \cap H_0^1(\Omega)$ depends continuously on $\lambda > 0$.

2 Preliminaries

In this section we collect some facts which allows us to extend the abstract results of the paper [1] from this proceedings to operators defined on some set X.

Let X be a set, (Y, ρ) be a metric space and $g : X \to Y$ be an operator.
Let us define $d_g : X \times X \to \mathbb{R}_+$ by the following formula.

$$d_g(x_1, x_2) = \rho(g(x_1), g(x_2)), \text{ for all } x_1, x_2 \in X.$$

Lemma 2.1 *([6]) If g is bijective then d_g is a metric on X.
Moreover, if (Y, ρ) is complete then (X, d_g) is complete.*

Let us consider $\tilde{X}$ the factorized set of X with the equivalence relation

$$x_1 \sim x_2 \text{ iff } g(x_1) = g(x_2).$$

For $h : X \to Y$ and some $\tilde{x} \in \tilde{X}$, let $\tilde{h}(\tilde{x}) = h(x)$, where $x \in \tilde{x}$. So, we obtain some operator $\tilde{h} : \tilde{X} \to Y$, which depends on the choice of x in $\tilde{x}$.
Let us notice that there exists a unique $\tilde{g}$ coresponding to g and $\tilde{g}$ is injective.
Let us consider also $f : X \to Y$ an operator. The following lemma shows that we can treat the coincidence problem for $\tilde{f}, \tilde{g}$ instead of f, g.

Lemma 2.2 *If $\tilde{x}^* \in C(\tilde{f}, \tilde{g})$ then there exists $x^* \in \tilde{x}^*$ such that $x^* \in C(f, g)$.*

Let us give now a new definition, which extend the notion of continuity in a metric space.

Definition 2.1 *The operator $f : X \to Y$ is said to be continuous w.r.t. g if for every sequence $(g(x_n))$ convergent in $g(X)$ to some y^*, there exists $x^* \in g^{-1}(y^*)$ such that the sequence $(f(x_n))$ converges to $f(x^*)$.*

The following lemmata will be useful in the next section. We shall state them without proof.

Lemma 2.3 *(i) If $g(x_1) = g(x_2)$ implies that $h(x_1) = h(x_2)$ then there exists a unique $\tilde{h}$ coresponding to h.*
(ii) If f is contraction w.r.t. g then $\tilde{f}$ is unique and is contraction w.r.t. $\tilde{g}$.
(iii) If f is continuous w.r.t. g then there exists some $\tilde{f}$ which is continuous w.r.t. $\tilde{g}$.

Lemma 2.4 *If g is bijective and f is continuous w.r.t. g then $f, g : (X, d_g) \to (Y, \rho)$ are continuous. If g is surjective and f is contraction w.r.t. g then $\tilde{f}, \tilde{g} : (\tilde{X}, d_g) \to (Y, \rho)$ are continuous.*

3 Coincidence theorems and data dependence results

Corollary 3.1 is the coincidence theorem of Goebel [8] and we shall prove it like a consequence of Theorem 2.1 [1].

Corollary 3.1 *[8] Let X be a set, Y be a complete metric space and $f, g : X \to Y$, be some operators such that the following conditions are fulfilled.*
(i) g is surjective;
(ii) f is a contraction w.r.t. g with the constant $\alpha \in (0, 1)$;
Then $C(f, g) \neq \emptyset$.
If, in addition, g is injective, then the coincidence point is unique.

Proof. By Lemma 2.3, $\tilde{f}$ and $\tilde{g}$ can be chosen in a unique way and $\tilde{f}$ is contraction w.r.t $\tilde{g}$. This implies that $\tilde{f}, \tilde{g} : (\tilde{X}, d_g) \to (Y, \rho)$ are continuous (from Lemma 2.4) and satisfy condition (P) (by Example 2., [1]). Using Theorem 2.1 [1] and Lemma 2.2, $C(f, g) \neq \emptyset$.
Let us suppose now that g is injective and denote by x_1^* and x_2^* two coincidence points of f and g. Because f is a contraction w.r.t. g we have $\rho(f(x_1^*), f(x_2^*)) \leq \alpha\rho(g(x_1^*), g(x_2^*))$, so $\rho(g(x_1^*), g(x_2^*)) \leq \alpha\rho(g(x_1^*), g(x_2^*))$. The constant α is less then 1 implies that $\rho(g(x_1^*), g(x_2^*)) = 0$, thus $g(x_1^*) = g(x_2^*)$. But g is injective, hence $x_1^* = x_2^*$.

In order to state the data dependence results, let us consider two pair of operators $f_i, g_i : X \to Y$, $i = \overline{1,2}$ such that there exist η_1, η_2 with

$$\rho(f_1(x), f_2(x)) \leq \eta_1, \text{ for all } x \in X,$$

and

$$\rho(g_1(x), g_2(x)) \leq \eta_2 \text{ for all } x \in X.$$

Also, we shall denote by x_i^* a generic element of $C(f_i, g_i)$, $i = \overline{1,2}$.

Corollary 3.2 *([2]) Let X be a set, Y be a complete metric space and $f_i, g_i : X \to Y$, $i = \overline{1,2}$ be some operators such that the following conditions are fulfilled.*
(i) g_1 is surjective;
(ii) f_1 is a contraction w.r.t. g_1 with the constant $\alpha \in (0, 1)$;
(iii) f_2, g_2 have at least one coincidence point.
Then the following estimation holds

$$\rho(g_1(x_1^*), g_1(x_2^*)) \leq \frac{1}{1-\alpha}(\eta_1 + \eta_2).$$

Proof. Let us consider $\tilde{f}_1, \tilde{g}_1 : (\tilde{X}, d_{g_1}) \to (Y, \rho)$ like in Corollary 3.1. which are continuous and satisfy condition (P).
Because $\tilde{g}_1$ is bijective, $C(\tilde{f}_1, \tilde{g}_1) = \{\tilde{x}_1^*\}$.
Let us choose $\tilde{f}_2$ and $\tilde{g}_2$ such that, for $\tilde{x}_2^* \in \tilde{X}$ with $x_2^* \in \tilde{x}_2^*$, $\tilde{f}_2(\tilde{x}_2^*) = f_2(x_2^*)$ and the same for g_2. Then $\tilde{x}_2^* \in C(\tilde{f}_2, \tilde{g}_2)$.
Thus, we can apply Theorem 3.2 [1].

Corollary 3.3 *Let X be a set, Y be a Banach space and $f_i, g_i : X \to Y$, $i = \overline{1,2}$ be some operators such that the following conditions are fulfilled.*
(i) f_1 is continuous w.r.t. g_1; g_1 is bijective;
(ii) f_1 is a strong pseudo-contraction w.r.t. g_1 with the constant $\alpha \in (0,1)$;
(iii) f_2, g_2 have at least one coincidence point.
Then the following estimation holds

$$||g_1(x_1^*) - g_1(x_2^*)|| \leq \frac{1}{1-\alpha}(\eta_1 + \eta_2).$$

Proof. The operator f is a strong pseudocontraction w.r.t. g [3] with the constant α, if for all $x_0, x_1^* \in X$ the following relation hold.

$$\langle f_1(x_0) - f_1(x_1^*), j(g_1(x_0) - g_1(x_1^*))\rangle \leq \alpha ||g_1(x_0) - g_1(x_1^*)||^2.$$

The existence of a coincidence point for f_1 and g_1 is assured by Theorem 3.1 [3]. For $x_1^* \in C(f_1, g_1)$ the following implications hold.

$$||g_1(x_0) - g_1(x_1^*)||^2 \leq \frac{1}{1-\alpha}\langle g_1(x_0) - g_1(x_1^*) - (f_1(x_0) - f_1(x_1^*)), j(g_1(x_0) - g_1(x_1^*))\rangle$$

$$\implies ||g_1(x_0) - g_1(x_1^*)||^2 \leq \frac{1}{1-\alpha}\langle g_1(x_0) - f_1(x_0), j(g_1(x_0) - g_1(x_1^*))\rangle$$

$$\implies d_{g_1}(x_0, x_1^*) = ||g_1(x_0) - g_1(x_1^*)|| \leq \frac{1}{1-\alpha}||g_1(x_0) - f_1(x_0)|| = \frac{1}{1-\alpha}\varphi_1(x_0).$$

If in the last inequality we put $x_0 \in C(f_1, g_1)$ then $\varphi(x_0) = 0$ and, hence, $d_{g_1}(x_0, x_1^*) = 0$. Then $x_0 = x_1^*$ or $C(f_1, g_1) = \{x_1^*\}$. Also, using Remark 1 [1], this last inequality assures that $f_1, g_1 : (X, d_{g_1}) \to (Y, d_{||\cdot||})$ fulfill condition (P). Since all the hypotesis of Theorem 3.2 [1] are fulfilled, the conclusion holds with constants α and $k = 1$.

4 A general data dependence result for some fully nonlinear elliptic equations

Let Ω be a C^2 bounded domain from $\mathbb{R}^n$. We denote by $\mathcal{M}_n$ the space of $n \times n$ real matrix; $|\cdot|_m$ is the euclidian norm from $\mathbb{R}^m$ and $trN = \sum_{n=1}^{n} \xi_{ii}$ is the trace of the $n \times n$ matrix $N = (\xi_{ij})$.
For a function $a : \Omega \times \mathcal{M}_n \to \mathbb{R}$ let us list the following hypothesis.
(A1) $a(\cdot, M)$ is measurable for all $M \in \mathcal{M}_n$.
(A2) $a(x, \cdot)$ continuous for almost every $x \in \Omega$ and there exists $\alpha > 0$ such that

$$|a(x,N)| \leq \alpha\left(|N|_{n^2} + |trN|\right),$$

for almost every $x \in \Omega$ and for all $N \in \mathcal{M}_n$.
(A3) $a(\cdot, 0) = 0$.
(A4) there exist δ and $\gamma > 0$ with $\gamma + \delta < 1$ such that

$$[trN - (a(x, M+N) - a(x,M))]\, trN \leq \gamma |N|_{n^2}^2 + \delta |trN|^2, \tag{4.1}$$

for almost every $x \in \Omega$ and for all $M, N \in \mathcal{M}_n$.

The main result in [6] states that, if a satisfies (A1)-(A3) and the ellipticity condition
(A5) there exist α', δ' and $\gamma' > 0$ with $\gamma' + \delta' < 1$ such that

$$|trN - \alpha'\left(a(x, M+N) - a(x,M)\right)| \leq \gamma' |N|_{n^2} + \delta' |trN|,$$

for almost every $x \in \Omega$ and for all $M, N \in \mathcal{M}_n$;
then, for every $f \in L^2(\Omega)$ the problem

$$u \in H^2(\Omega) \cap H_0^1(\Omega), \quad a(x, D^2 u) = f(x), \quad \text{a.a. } x \in \Omega$$

has a unique solution. Let us notice that (A5) implies (A2) and (A4).
Using the weaker ellipticity condition (A4) instead of (A5) we shall give an existence and data dependence result for this problem. We shall use Corollary 3.3.
In order to state the main result of this section, let us consider the equations.

$$u \in H^2(\Omega) \cap H_0^1(\Omega), \quad a_1(x, D^2 u) = f_1(x), \tag{4.2}$$

$$u \in H^2(\Omega) \cap H_0^1(\Omega), \quad a_2(x, D^2 u) = f_2(x). \tag{4.3}$$

Theorem 4.1 *Let $f_1, f_2 \in L^2(\Omega)$, a_1, a_2 satisfy assumptions (A1)-(A4) with the same constants δ, γ and*

$$|f_1(x) - f_2(x)| \le \eta\ , \ \text{a.a. } x \in \Omega, \tag{4.4}$$

$$|a_1(x, M) - a_2(x, M)| \le \xi\ , \ \text{a.a. } x \in \Omega, \text{ and for all } M \in \mathcal{M}_n. \tag{4.5}$$

Then the following estimation holds

$$||u_1^* - u_2^*||_{H^2} \le C \frac{\sqrt{mes(\Omega)}}{1 - (\gamma + \delta)} (\eta + \xi), \tag{4.6}$$

where u_1^*, u_2^* are unique solutions of (4.2) and (4.3) and $C > 0$ is taken such that $||u||_{H^2} \le C||\Delta u||_{L^2}$ for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Proof. Let us denote $X = H^2(\Omega) \cap H_0^1(\Omega)$, $Y = L^2(\Omega)$ and, for $i = 1$ and $i = 2$,

$$F_i u = \Delta u - a_i(x, D^2 u), \qquad G_i u = \Delta u - f_i.$$

Then, using (A1)-(A3), we obtain the operators $F_i : X \to Y$ and $G_i : X \to Y$ such that we can write the problem (4.2) as the coincidence equation

$$F_1 u = G_1 u, \quad u \in X,$$

and problem (4.3) as the coincidence equation

$$F_2 u = G_2 u, \quad u \in X.$$

We intent to apply Corollary 3.3.
Operators G_1 and G_2 are bijective (see [7]).
Also, G_1^{-1} is continuous ([7]) and, by (A1)-(A2), F_1 is continuous, too. This implies that F_1 is continuous w.r.t. G_1.
In order to prove that F_1 is a strong pseudo-contraction w.r.t. G_1 let us consider $u, v \in X$. Using (A4) we obtain the following estimations.

$$\begin{aligned}(F_1 u(x) - F_1 v(x))\,(G_1 u(x) - G_1 v(x)) &= \left[\Delta(u - v) - \left(a_1(x, D^2 u) - a_1(x, D^2 v)\right)\right] \Delta(u - v) \le \\ &\le \gamma |D^2(u - v)(x)|_{n^2}^2 + \delta |\Delta(u - v)(x)|^2.\end{aligned}$$

Then, integrating on Ω and using that $||D^2 u||_{L^2} = ||\Delta u||_{L^2}$ for all $u \in X$ (Corollary 9.10 page 235, [7]), we deduce that

$$\langle F_1 u - F_1 v, G_1 u - G_1 v \rangle_{L^2} \le \gamma ||D^2(u - v)||_{L^2}^2 + \delta ||\Delta(u - v)||_{L^2}^2 = (\gamma + \delta) ||\Delta(u - v)||_{L^2}^2.$$

Hence,

$$\langle F_1 u - F_1 v, G_1 u - G_1 v \rangle_{L^2} \le (\gamma + \delta) ||G_1 u - G_1 v||_{L^2}^2, \quad \text{for all } u, v \in X,$$

i.e. F_1 is strong pseudo-contraction w.r.t. G_1 with the constant $(\gamma + \delta)$.
Also, from (4.4) and (4.5) we obtain that

$$||G_1 u - G_2 u||_{L^2} \le \eta \sqrt{mes(\Omega)}, \tag{4.7}$$

$$||F_1 u - F_2 u||_{L^2} \le \xi \sqrt{mes(\Omega)}. \tag{4.8}$$

By Theorem 3.1 [3] (like in the proof of Corollary 3.3), let u_1^* be the unique coincidence point of F_1, G_1 and u_2^* be the unique coincidence point of F_2, G_2 (it can be proved similarly that F_2 is a strong pseudo-contraction w.r.t. G_2 and is continuous w.r.t. G_2).
All the hypothesis of Corollary 3.3 are fulfilled, then we have the estimation

$$||G_1(u_1^*) - G_1(u_2^*)||_{L^2} \le \frac{\sqrt{mes(\Omega)}}{1 - (\gamma + \delta)} (\eta + \xi).$$

Because $G_1(u_1^*) - G_1(u_2^*) = \Delta(u_1^* - u_2^*)$ it is easy to see that (4.6) holds.

5 Continuity with respect to the coefficients for a linear elliptic equation

Let us consider the linear elliptic equation

$$\frac{\partial^2 u}{\partial x_1^2} + \lambda \frac{\partial^2 u}{\partial x_2^2} = f, \quad u \in H^2(\Omega) \cap H_0^1(\Omega),$$

where $\lambda > 0$, $f \in L^2(\Omega)$ and denote by $u(\lambda)$ its solution (see [7]). Here Ω is a C^2 bounded domain of $\mathbb{R}^n$. It is easy to see that $u(1)$ is the solution of

$$\Delta u = f, \quad u \in H^2(\Omega) \cap H_0^1(\Omega).$$

The main result of this section is the following theorem.

Theorem 5.1 *$u(\lambda) \to u(1)$ in $H^2(\Omega)$ as $\lambda \to 1$.*

Proof. Let us denote $X = H^2(\Omega) \cap H_0^1(\Omega)$ and

$$F_\lambda u = (\lambda - 1)\frac{\partial^2 u}{\partial x_2^2}, \quad G(u) = f - \Delta u.$$

Then we obtain the operators $F_\lambda, G : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$, such that $u(\lambda)$ is the coincidence point of F_λ, G.
Let $r > 0$, $X = \{u \in H^2(\Omega) \cap H_0^1(\Omega) \mid ||Gu|| \leq r\}$ and $Y = G(X)$.
Let us denote $t = \dfrac{r}{r + ||f||} \in (0,1)$ and, from now on, consider λ such that $|\lambda - 1| \leq t$.
Let $u, v \in X$ and $w = u - v$. The following estimations hold.

$$(5.1) \qquad ||F_\lambda u - F_\lambda v||_{L^2}^2 = ||F_\lambda w||_{L^2}^2 = (\lambda - 1)^2 \int_\Omega \left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 dx \leq$$
$$\leq \quad (\lambda - 1)^2 \int_\Omega |D^2 w|_4^2 dx = (\lambda - 1)^2 \int_\Omega |\Delta w|^2 dx = |\lambda - 1|^2 \cdot ||Gu - Gv||_{L^2}^2.$$

Then $||F_\lambda u - F_\lambda v|| \leq t||Gu - Gv||$.
This last relation also implies that $||F_\lambda u|| \leq t||Gu - f|| \leq t(r + ||f||) = r$. Because G is bijective, $Y = \{y \in L^2(\Omega) \mid ||y|| \leq r\}$. Then we have proved that $F_\lambda u \in Y$ for all $u \in X$.
For applying Corollary 3.2 we consider $F_\lambda, G : X \to Y$ and notice that we have proved that F_λ is contraction w.r.t. G with the constant t.
From (5.2) with $v = 0$ we obtain

$$||F_\lambda u - F_1 u|| = ||F_\lambda u|| \leq |\lambda - 1| ||Gu - f|| \leq |\lambda - 1|(r + ||f||).$$

We apply now Corollary 3.3 and obtain the estimation

$$||Gu(\lambda) - Gu(1)|| \leq \frac{1}{1 - t} \cdot |\lambda - 1|(r + ||f||).$$

Thus, $||u(\lambda) - u(1)||_{H^2} \leq C||\Delta u(\lambda) - \Delta u(1)||_{L^2} = ||Gu(\lambda) - Gu(1)||_{L^2} \to 0$ when $\lambda \to 1$. Hence the conclusion holds.

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