On Peetre's condition in the coincidence theory. I. Abstract results

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1 Introduction

In this paper we give coincidence and data dependence results using Peetre's condition [10]. Let X and Y be two nonempty sets and $f, g : X \to Y$ be two mappings. Let us denote the coincidence points set for f and g by

$$C(f,g) = \{x \in X \mid f(x) = g(x)\}.$$

In the particular cases when g is a constant mapping, $g(x) = y,\ \forall x \in X$, where $y \in Y$ is given, we shall use the notation

$$S_y(f) = \{x \in X \mid f(x) = y\},$$

or g is the inclusion mapping $g(x) = x,\ \forall x \in X$, where $X \subset Y$, we shall use the well-known notation,

$$F(f) = \{x \in X \mid f(x) = x\}.$$

We shall work in the case when (X, d) and (Y, ρ) are two metric spaces. Let us also denote

$$\varphi(x) = \rho(f(x), g(x)),\ x \in X.$$

Now we can write the Peetre's condition below.
(P) *There exist* $0 < \alpha < 1$, $k > 0$ *and* $\psi : X \to X$ *such that:*

$$d(x, \psi(x)) \leq k \cdot \varphi(x) \text{ and } \varphi(\psi(x)) \leq \alpha \cdot \varphi(x),\ \forall\, x \in X.$$

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In the second paragraph we give existence result for coincidence points, using condition (P), reformulated in the sequences language (Lemma 2.1). Next section deals with data dependence results from same point of view. We mention that we apply theoretical results from section 2 and 3 for particular cases. In this way we find known fixed point result (when one of the operators is identity) and known surjectivity results (when one of the operators is constant).

2 Existence results

Lemma 2.1 *If condition (P) is satisfied, then for every $x_0 \in X$,*

(a) $\psi^n(x_0)$ is a Cauchy sequence;

(b) $\varphi(\psi^n(x_0)) \to 0$, as $n \to \infty$.

Proof. Let us fixed $x_0 \in X$. If $\varphi(x_0) = 0$ then $\psi(x_0) = x_0$, hence $\psi^n(x_0) = x_0$ for all $n \geq 0$. If $\varphi(x_0) > 0$ then we have the following estimations.

$$d(\psi^{n-1}(x_0), \psi^n(x_0)) \leq k \cdot \alpha^{n-1} \cdot \varphi(x_0) \text{ and } \varphi(\psi^n(x_0)) \leq \alpha^n \cdot \varphi(x_0).$$

Then $d(\psi^n(x_0), \psi^{n+p}(x_0)) \leq k(\alpha^n + \alpha^{n+1} + ... + \alpha^{n+p-1})\varphi(x_0) = k \cdot \alpha^n \frac{1-\alpha^p}{1-\alpha}\varphi(x_0) \leq k\frac{\alpha^n}{1-\alpha}\varphi(x_0) \to 0$, as $n, p \to \infty$. Thus, $(\psi^n(x_0))$ is a Cauchy sequence and $\varphi(\psi^n(x_0)) \to 0$.

Example 1. Let $X = Y$ and g be the identity mapping of X. *If f is a contraction on the orbit then Peetre's condition is fulfilled .*
Let us remember that f is a contraction on the orbit if there exists $0 < \alpha < 1$ such that

$$d(f(f(x)), f(x)) \leq \alpha \cdot d(x, f(x))$$

or, equivalently, $\varphi(f(x)) \leq \alpha \cdot \varphi(x)$.
Indeed, it is easy to see that (P) holds with $k = 1$ and $\psi(x) = f(x)$. The sequence $x_n = \psi^n(x_0)$ from Lemma 1.1 is given by the following relation,

$$x_{n+1} = f(x_n).$$

Example 2. Let g be bijective and such that there exists $k > 0$ with

$$d(x_1, x_2) \leq k \cdot \rho(g(x_1), g(x_2)) \text{ for all } x_1, x_2 \in X. \tag{2.1}$$

If f is contraction w.r.t. g then Peetre's condition is fulfilled.
Let us remember that f is contraction w.r.t. g if there exists $0 < \alpha < 1$ such that

$$\rho(f(x_1), f(x_2)) \leq \alpha \cdot \rho(g(x_1), g(x_2)), \quad \text{for all } x_1, x_2 \in X. \tag{2.2}$$

In order to prove this, let $x \in X$ and $\psi(x) = g^{-1}(f(x))$. Then we can write

$$\varphi(x) = \rho(g(\psi(x)), g(x)), \quad \varphi(\psi(x)) = \rho(f(\psi(x)), f(x)).$$

Now, using also relations (2.1) and (2.2), we obtain $d(x, \psi(x)) \leq k \cdot \varphi(x)$, $\varphi(\psi(x)) \leq \alpha\varphi(x)$.
The sequence from Lemma 1.1, $x_n = \psi^n(x_0)$ is given by the following relation

$$g(x_{n+1}) = f(x_n).$$

Example 3. Let Y be a normed space, g be like in Example 2 and $\lambda > 0$. *If $h_\lambda = \lambda f + (1-\lambda)g$ is contraction w.r.t. g, then condition (P) is fulfilled.*
The sequence is given by the following relation for every $x_0 \in X$,

$$g(x_{n+1}) = \lambda f(x_n) + (1-\lambda)g(x_n).$$

Example 4. Let X, Y be Banach spaces, y an arbitrary element from Y and $g : X \to Y$, $g(x) = y$ for all x in X. *If f is as in Lemma 2.2 and g as above, then condition (P) is fulfilled.*

Lemma 2.2 *[1] Let X, Y be Banach spaces and $f : X \to Y$ be Gateaux differentiable operator on X. let $\delta : [0, \infty) \to [0, \infty)$ be a continuous, bounded away from zero and suppose that for each x in X we have*

$$df_x(B(0,1)) \supset B(0, \delta(||x||))$$

Then for each $y \in Y$, there are operator $\psi(.;y) : X \to X$ and $\alpha \in (0, 1/2]$ such that

$$\begin{aligned} ||f(\psi(.;y)(x)) - y|| &\leq \alpha ||f(x) - y|| \text{ and} \\ ||\psi(.;y)(x) - x|| &\leq \frac{1}{\delta(||x||)} ||f(x) - y|| \end{aligned}$$

The main results of this section are Theorem 2.1 and Theorem 2.2.

Theorem 2.1 *Let (X, d) be a complete metric space and f and g be continuous. If f, g satisfy Peetre's condition then*

(i) $C(f,g) \neq \emptyset$;
(ii) for every $x_0 \in X$ *there exists* $x^* \in C(f,g)$ *such that* $\psi^n(x_0) \to x^*(x_0)$
(iii) for every $x_0 \in X$ *the following estimation holds*

$$d(x_0, x^*(x_0)) \leq \frac{k}{1-\alpha} \cdot \varphi(x_0). \tag{2.3}$$

Proof. By Lemma 2.1, for every $x_0 \in X$ the sequence $x_n = \psi^n(x_0)$ is a Cauchy sequence in X and such that $\varphi(x_n) \to 0$ The metric space (X,d) is complete, thus the sequence (x_n) is convergent to some $x^* \in X$. Using the continuity of f and g, we deduce that the functional $\varphi : X \to \mathbb{R}_+$ is also continuous. Hence, $\varphi(x_n) \to \varphi(x^*)$. From the unicity of the limit in a metric space, $\varphi(x^*)$ must be 0. But this means that $\rho(f(x^*), g(x^*)) = 0$, which implies that $f(x^*) = g(x^*)$, i.e. $C(f,g) \neq \emptyset$.
From the proof of Lemma 2.1 we use the relation $d(x_n, x_{n+p}) \leq k \cdot \alpha^n \frac{1-\alpha^p}{1-\alpha} \varphi(x_0)$ to deduce that $d(x_0, x_n) \leq k \frac{1-\alpha^n}{1-\alpha} \varphi(x_0)$. Taking by the limit for $n \to \infty$ we obtain exactly (2.3).

Remark 1. In fact, in the hypothesis of Theorem 1.1, condition (P) is equivalent to (i)+(ii). The reversed implication coresponds with the trivial case, when we can choose ψ from condition (P) the constant operator $\psi(x) = x^*,\ x \in X$, where x^* is given by (ii).

Remark 2. If in previous theorem we take the constant operator g as in Example 4., then we obtain the Kasahara surjectivity theorem in formulation given in [1]. In other words the hypotesis from Kasahara's surjectivity theorem imply condition (P) for operators f and g define as above.

Remark 3. The contraction mapping principle can be proved like a consequence of Theorem 2.1.

The following result is a Maia-type theorem.

Theorem 2.2 *Let* d' *be another metric on* X *and* ρ' *on* Y*. Let us suppose that the following conditions are fulfilled.*
(i) Peetre's condition holds for $f, g : (X,d) \to (Y,\rho)$;
(ii) there exist $a, b > 0$ *such that*

$$\begin{aligned} d'(x_1,x_2) &\leq a \cdot d(x_1,x_2) \text{ for all } x_1,x_2 \in X, \text{ and} \\ \rho'(y_1,y_2) &\leq b \cdot \rho(y_1,y_2) \text{ for all } y_1,y_2 \in Y; \end{aligned}$$

(iii) (X, d') *is a complete metric space;*
(iv) $f, g : (X, d') \to (Y, \rho')$ *are continuous.*
Then
(i) $C(f, g) \neq \emptyset$
(ii) for every $x_0 \in X$ *there exists* $x^* \in C(f, g)$ *such that* $\psi^n(x_0) \to x^*(x_0)$
(iii) for every $x_0 \in X$ *the following estimation holds*

$$d'(x_0, x^*(x_0)) \leq \frac{a \cdot k}{1 - \alpha} \cdot \varphi(x_0). \tag{2.4}$$

Proof. Let us denote $\varphi(x) = \rho(f(x), g(x))$ and $\varphi'(x) = \rho'(f(x), g(x))$. By Lemma 2.1, the Cauchy sequence (x_n), $x_n = \psi^n(x_0)$ in (X, d) such that $\varphi(x_n) \to 0$. Using (ii) we have that $0 \leq \varphi'(x_n) \leq b \cdot \varphi(x_n)$. So, $\varphi'(x_n) \to 0$. Also, by (ii) we have that $d'(x_n, x_{n+p}) \leq c \cdot d(x_n, x_{n+p})$. But the right hand member of this inequality tends to 0 when $n, p \to \infty$, which implies that $d'(x_n, x_{n+p}) \to 0$ when $n, p \to \infty$. This means that (x_n) is a Cauchy sequence in (X, d'), too. Using (iii), there exists $x^*(x_0) \in X$ the limit of $(\psi^n(x_0))$ in (X, d'). The condition (iv) assures that $\varphi' : (X, d') \to \mathbb{R}_+$ is continuous. As in the proof of Theorem 2.1, we deduce that $C(f, g) \neq \emptyset$ and $d(x_0, x_n) \leq k\dfrac{1 - \alpha^n}{1 - \alpha}\varphi(x_0)$ which imples that $d'(x_0, x_n) \leq k \cdot a\dfrac{1 - \alpha^n}{1 - \alpha}\varphi(x_0)$. When $n \to \infty$, estimation (2.4) holds.

3 Data dependence

Throughout this section we shall consider two pair of operators $f_i, g_i : X \to Y$, $i = \overline{1, 2}$ such that there exist η_1, η_2 with

$$\rho(f_1(x), f_2(x)) \leq \eta_1, \text{ for all } x \in X,$$

and

$$\rho(g_1(x), g_2(x)) \leq \eta_2 \text{ for all } x \in X.$$

Also, we shall denote by x_i^* a generic element of $C(f_i, g_i)$, $i = \overline{1, 2}$.

Theorem 3.1 *Let* X *be a complete metric space and suppose that* f_i, g_i *are continuous and satisfy condition* **(P)** *with the constants* $k_i > 0$ *and* $\alpha_i \in (0, 1)$, $i = \overline{1, 2}$. *Then* $C(f_1, g_1)$ *and* $C(f_2, g_2)$ *are closed subsets of* X *and the following estimation holds*

$$H(C(f_1, g_1), C(f_2, g_2)) \leq \frac{\max\{k_1, k_2\}}{1 - \max\{\alpha_1, \alpha_2\}}(\eta_1 + \eta_2).$$

Proof. Using the continuity of f_1 and g_1 it is easy to see that $C(f_1, g_1)$ is a closed subset of X. Analogously for $C(f_2, g_2)$.
By Theorem 1.1, for every $x_0 \in X$ there exists $x^* \in C(f_1, g_1)$ such that $d(x_0, x^*) \leq \frac{k_1}{1-\alpha_1}\varphi_1(x_0)$.
Now let $x_0 \in C(f_2, g_2)$. The following estimation hold $\varphi_1(x_0) = \rho(f_1(x_0), g_1(x_0)) \leq \rho(f_1(x_0), f_2(x_0)) + \rho(f_2(x_0), g_2(x_0)) + \rho(g_2(x_0), g_1(x_0)) \leq \eta_1 + \eta_2$. Thus,

$$d(x_0, x^*) \leq \frac{k_1}{1-\alpha_1}(\eta_1 + \eta_2). \tag{3.5}$$

By a similar way, we have that for all $y_0 \in C(f_1, g_1)$ there exists $y^* \in C(f_2, g_2)$ such that

$$d(y_0, y^*) \leq \frac{k_2}{1-\alpha_2}(\eta_1 + \eta_2). \tag{3.6}$$

Relations (3.5) and (3.6) assure that the conclusion holds.

Theorem 3.2 *Let X be a complete metric space, $f_n, f, g_n, g : X \to Y$, $n \geq 1$ be continuous operators such that (f_n, g_n), $n \geq 1$ and (f, g) satisfy condition* **(P)** *with the same constants k and α. Let us also suppose that there exists $\eta_n \to 0$ and $\beta_n \to 0$, as $n \to \infty$ such that*

$$\rho(f_n(x), f(x)) \leq \eta_n, \text{ for all } x \in X,$$

and

$$\rho(g_n(x), g(x)) \leq \beta_n \text{ for all } x \in X.$$

Then $H(C(f_n, g_n), C(f, g)) \to 0$, as $n \to \infty$.

Corollary 3.1 *Let (X, d) be a complete metric space, (Y, ρ) be a metric space and operators $f_i : X \to Y$, $g_i : X \to Y$, $g_i(x) = y_i$, $i = 1, 2$ which respect the Kasahara's surjectivity theorem [8]. Then $S_{y_i}(f_i)$ are closed subsets and*

$$H(S_{y_1}(f_1), S_{y_2}(f_2)) \leq \frac{\max\{k_1, k_2\}}{1 - \max\{\alpha_1, \alpha_2\}}(\eta_1 + \eta_2).$$

Proof. From Remark 2 we have that pair (f_i, g_i) satisfy condition (P) with constant α_i and k_i and from Theorem 3.1 the conclusions of corrollary hold.

Theorem 3.3 *Let X be a complete metric space and suppose that the following conditions are fulfilled.*
(i) f_1, g_1 are continuous and satisfy condition (P) with the constants $k > 0$ and

$\alpha \in (0,1)$;
(ii) f_1, g_1 have unique coincidence point;
(iii) f_2, g_2 have at least one coincidence point.
Then the following estimation holds

$$d(x_1^*, x_2^*) \leq \frac{k}{1-\alpha}(\eta_1 + \eta_2). \tag{3.7}$$

Proof. $C(f_1, g_1) = \{x_1^*\}$. Using (i), by Theorem 2.1, for $x_2^* \in X$, $d(x_2^*, x_1^*) \leq \frac{k}{1-\alpha}\varphi_1(x_2^*)$.
$\varphi_1(x_2^*) = \rho(f_1(x_2^*), g_1(x_2^*)) \leq \rho(f_1(x_2^*), f_2(x_2^*)) + \rho(f_2(x_2^*), g_2(x_2^*)) + \rho(g_2(x_2^*), g_1(x_2^*)) \leq \eta_1 + \eta_2$. Thus, (3.7) holds.

Corollary 3.2 *[13] Let $X = Y$ be a complete metric space and suppose that the following conditions are fulfilled.*
(i) f_1 is a contraction with the constant $\alpha \in (0,1)$;
(ii) f_2 has at least one fixed point.
Then the following estimation holds

$$\rho(x_1^*, x_2^*) \leq \frac{1}{1-\alpha}\eta_1.$$

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