Abstract Generalized Quasilinearization Method for Coincidences

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Abstract An abstract unified theory of both monotone iterative and generalized quasilinearization methods is presented for operator equations of coincidence type in ordered Banach spaces. Applications are given for semilinear problems in $C\left(\overline{\Omega};\mathbb{R}^k\right)$ and $L^p\left(\Omega;\mathbb{R}^k\right)$.

1 Introduction

The monotone iterative method and the Newton's method are known to be two very available and efficient techniques for finding roots of nonlinear equations. The first one applies to equations involving monotone operators and produces a sequence which is monotone and convergent towards a solution. However, like as Banach's Contraction Mapping Principle, the convergence of the monotone iterations is slow. The Newton's method has the advantage of providing quadratically convergent sequences. Historically, the first general convergence theorem for Newton's method for solving iteratively the equation $F(u) = 0$ was given by L.V. Kantorovich (1948) assuming certain bounds for F'' and the inverse of F', and also that the first approximation is chosen close enough to the solution. Later, Kantorovich himself and many other authors have adapted this method for equations in ordered normed spaces, to produce monotone and quadratically convergent sequences. For this purpose, the main assumption was the convexity or a convexity-like property of F. For references and much information about the stage of art in the sixties we refer the reader to the paper of Vandergraft [22]. For further contributions see [2, 9, 18, 21].

Applied to differential equations Newton's method is known as the quasilinearization method. A remarkable contribution in this direction has been the monograph of Bellman and Kalaba [3]. This method applies to semilinear equations with convex nonlinearities and provides lower approximate solutions which converge quadratically to the solution of the given nonlinear equation. The lower approximate solutions are solutions to certain linear equations.

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The nineties brought new extensions of the quasilinearization method. The most interesting new idea is due to Lakshmikantham [12] (see also [13, 15]) and consists in an extension of the quasilinearization method for equations with nonlinearities which can be represented as a difference of two convex functions. This extension, now known as the generalized quasilinearization method, was possible by combining the method of upper and lower solutions and the monotone iterative technique together with differential inequalities and comparison results. This provides a very efficient tool to construct upper and lower approximate solutions that converge monotonically and quadratically to the solution of the problem under consideration. The method was used for solving several classes of equations; see [1, 8, 12, 13, 14, 15, 16, 17, 20]. Nevertheless, by our knowledge, a general theory to unify all these particular results have not been given until now.

The goal of this paper is to develop an abstract theory of the generalized quasilinearization method for semilinear operator equations of coincidence type in ordered Banach spaces. Our theory contains as a particular case, the monotone iterative method (see [4, 5, 11, 19]), which is applicable to a large class of problems. However, not all the results existed in the literature can be embedded in the theory given in the present paper. For example, our results do not cover the case of discontinuous problems, for which a specific theory was provided by Heikkilä-Lakshmikantham [10] and Carl-Heikkilä [7].

The general results are applied to semilinear equations in $C\left(\overline{\Omega};\mathbb{R}^k\right)$ and $L^p\left(\Omega;\mathbb{R}^k\right)$. Finally we present two results for the initial value problem for a scalar differential equation. The first one is given in terms of divided differences for nondifferentiable nonlinearities, and the second one in terms of derivatives.

Similar ideas are used in [6] for finding roots of nonlinear equations by Newton type iterations.

2 Abstract Theory

We start with a result in ordered linear spaces which provides two monotone sequences of upper and respectively lower estimations of any zero of a nonlinear mapping in a given order interval.

Lemma 2.1. *Let D, Z be two ordered linear spaces, $F : D \to Z$ be a mapping and let $\alpha_0, \beta_0 \in D$. Assume*

$$\alpha_0 \le \beta_0,\ F(\alpha_0) \le 0 \le F(\beta_0)$$

and that for every $u, v \in D$ satisfying $\alpha_0 \le u \le v \le \beta_0$, there exists a bijective linear operator $A(u,v) : D \to Z$ with positive inverse such that

$$F(v) \le F(u) + A(u,v)(v-u). \tag{2.1}$$

In addition assume that for every $\alpha, \beta, u, v \in D$ with $\alpha \le u \le v \le \beta$,

$$A(u,v)z \le A(\alpha,\beta)z \text{ for all } z \in D,\ z \ge 0. \tag{2.2}$$

Then the sequences (α_n), (β_n) given by the iterative schemes

$$F(\alpha_n) + A(\alpha_n,\beta_n)(\alpha_{n+1}-\alpha_n) = 0, \tag{2.3}$$

$$F(\beta_n) + A(\alpha_n, \beta_n)(\beta_{n+1} - \beta_n) = 0 \tag{2.4}$$

($n \in \mathbf{N}$) *are well and uniquely defined in* D *and*

$$\alpha_0 \leq \alpha_1 \leq ... \leq \alpha_n \leq ... \leq \beta_n \leq ... \leq \beta_1 \leq \beta_0. \tag{2.5}$$

Moreover, for each $u \in D$ *with* $\alpha_0 \leq u \leq \beta_0$ *and* $F(u) = 0$, *one has*

$$\alpha_n \leq u \leq \beta_n \text{ for all } n \in N. \tag{2.6}$$

Proof. We shall prove by induction that for each $n \in \mathbb{N}$,

$$\begin{cases} \alpha_{n+1}, \beta_{n+1} \text{ are well and uniquely defined,} \\ \alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n, \\ F(\alpha_{n+1}) \leq 0 \leq F(\beta_{n+1}). \end{cases} \tag{2.7}$$

First we prove (2.7) for $n = 0$. The bijectivity of $A(\alpha_0, \beta_0)$ guarantees that α_1, β_1 are well and uniquely defined by (2.3) and (2.4), respectively. From (2.3) using $F(\alpha_0) \leq 0$, we obtain

$$A(\alpha_0, \beta_0)\alpha_1 = -F(\alpha_0) + A(\alpha_0, \beta_0)\alpha_0 \geq A(\alpha_0, \beta_0)\alpha_0.$$

This yields $\alpha_1 \geq \alpha_0$ since $A(\alpha_0, \beta_0)$ has a positive inverse. Similarly, from (2.4) and $F(\beta_0) \geq 0$, we deduce

$$A(\alpha_0, \beta_0)\beta_1 = -F(\beta_0) + A(\alpha_0, \beta_0)\beta_0 \leq A(\alpha_0, \beta_0)\beta_0,$$

whence $\beta_1 \leq \beta_0$. Furthermore, using successively (2.3), (2.1) and (2.4), we obtain

$$\begin{aligned} A(\alpha_0, \beta_0)\alpha_1 &= -F(\alpha_0) + A(\alpha_0, \beta_0)\alpha_0 \\ &\leq -F(\beta_0) + A(\alpha_0, \beta_0)\beta_0 \\ &= A(\alpha_0, \beta_0)\beta_1. \end{aligned}$$

It follows that $\alpha_1 \leq \beta_1$. Now the inequalities $F(\alpha_1) \leq 0 \leq F(\beta_1)$ follow from the next relations by using (2.2):

$$\begin{aligned} F(\alpha_1) &\leq F(\alpha_0) + A(\alpha_0, \alpha_1)(\alpha_1 - \alpha_0) \\ &= (A(\alpha_0, \alpha_1) - A(\alpha_0, \beta_0))(\alpha_1 - \alpha_0), \end{aligned}$$

$$\begin{aligned} F(\beta_1) &\geq F(\beta_0) - A(\beta_1, \beta_0)(\beta_0 - \beta_1) \\ &= (A(\alpha_0, \beta_0) - A(\beta_1, \beta_0))(\beta_0 - \beta_1). \end{aligned}$$

Thus (2.7) holds for $n = 0$. Similar arguments can be used to prove (2.7) for $n+1$, when it is assumed (2.7) is true for n.

Now let $u \in D$ be any element satisfying $\alpha_0 \leq u \leq \beta_0$ and $F(u) = 0$. We shall prove (2.6) by induction. Let us assume that (2.6) holds for n. Then from

$$\begin{aligned} A(\alpha_n, \beta_n)\alpha_{n+1} &= -F(\alpha_n) + A(\alpha_n, \beta_n)\alpha_n \\ &\leq -F(u) + A(\alpha_n, u)(u - \alpha_n) + A(\alpha_n, \beta_n)\alpha_n \\ &= A(\alpha_n, \beta_n)u + (A(\alpha_n, u) - A(\alpha_n, \beta_n))(u - \alpha_n) \\ &\leq A(\alpha_n, \beta_n)u \end{aligned}$$

and

$$\begin{aligned} A(\alpha_n,\beta_n)\,\beta_{n+1} &= -F(\beta_n) + A(\alpha_n,\beta_n)\,\beta_n \\ &\geq -F(u) - A(u,\beta_n)(\beta_n - u) + A(\alpha_n,\beta_n)\,\beta_n \\ &= A(\alpha_n,\beta_n)\,u - (A(u,\beta_n) - A(\alpha_n,\beta_n))(\beta_n - u) \\ &\geq A(\alpha_n,\beta_n)\,u, \end{aligned}$$

since the inverse of $A(\alpha_n,\beta_n)$ is positive, we deduce that (2.6) also holds for $n+1$. □

In what follows we deal with the coincidence operator equation

$$Lu = N(u)\,,\ u \in D. \tag{2.8}$$

Our first result represents a generalization of the monotone iterative technique for coincidences.

Theorem 2.1. *Let X be an ordered Banach space, Z be an ordered topological linear space, D a linear subspace of X and $\alpha_0, \beta_0 \in D$. Let $L : D \to Z$ be a linear operator and $N : X \to Z$ be a mapping. Assume that the following conditions are satisfied:*

(i) $\alpha_0 \leq \beta_0,\ L\alpha_0 \leq N(\alpha_0)$ *and* $L\beta_0 \geq N(\beta_0)\,;$

(ii) *for every $u, v \in X$ with $\alpha_0 \leq u \leq v \leq \beta_0$, there is a linear operator $P(u,v) : X \to Z$ such that $L - P(u,v) : D \to Z$ is bijective with positive inverse,*

$$N(u) \leq N(v) - P(u,v)(v-u) \tag{2.9}$$

and

$$-P(u,v)\,z \leq -P(\alpha,\beta)\,z \tag{2.10}$$

for all $\alpha, \beta, u, v, z \in X$ with $\alpha_0 \leq \alpha \leq u \leq v \leq \beta \leq \beta_0$ and $z \geq 0$;

(iii) *either*
(a) the positive cone of X is regular and the operators

$$\begin{cases} (L - P(\alpha_0,\beta_0))^{-1} N,\ (L - P(\alpha_0,\beta_0))^{-1} P(\alpha_0,\beta_0)\,, \\ (L - P(\alpha_0,\beta_0))^{-1} P(u,u)\,,\ u \in X,\ \alpha_0 \leq u \leq \beta_0 \end{cases}$$

are continuous on $[\alpha_0,\beta_0]\,,$
or
(b) the positive cone of X is normal and the operators (2.11) are completely continuous on $[\alpha_0,\beta_0]\,.$

Then the sequences $(\alpha_n)\,,(\beta_n)$ given by the iterative schemes

$$L\alpha_{n+1} = N(\alpha_n) + P(\alpha_n,\beta_n)(\alpha_{n+1} - \alpha_n)\,, \tag{2.11}$$

$$L\beta_{n+1} = N(\beta_n) + P(\alpha_n,\beta_n)(\beta_{n+1} - \beta_n) \tag{2.12}$$

$(n \in \mathbf{N})$ are well and uniquely defined in D. In addition, they are monotonically convergent in X to the minimal and, respectively, to the maximal solution in $[\alpha_0,\beta_0]$ of (2.8).

Proof. We apply Lemma 2.1. Here $F = L - N$ and $A(u,v) = L - P(u,v)$. Clearly, (ii) guarantees (2.1) and (2.2). Also (2.11)-(2.12) coincide with (2.3)-(2.4). Hence the sequences $(\alpha_n), (\beta_n)$ are well and uniquely defined in D, and satisfy (2.5) and (2.6). For the convergence we discuss two cases which correspond to the alternatives in (iii).
Case 1. Assume (iii)(a) holds. Then the regularity of the positive cone of X implies that the monotone sequences $(\alpha_n), (\beta_n)$ are convergent, say to $u_* \in X$ and $u^* \in X$, respectively. Also

$$\alpha_n \leq u_* \leq u^* \leq \beta_n, \; n \in \mathbf{N}$$

and

$$u_* \leq u \leq u^*$$

for any $u \in D$ with $\alpha_0 \leq u \leq \beta_0$ and $Lu = N(u)$. It remains to prove that $u_*, u^* \in D$ and $Lu_* = N(u_*)$, $Lu^* = N(u^*)$. From (2.11) we have

$$\begin{aligned} \alpha_{n+1} &= (L - P(\alpha_0, \beta_0))^{-1} [N(\alpha_n) \\ &\quad + P(\alpha_n, \beta_n)(\alpha_{n+1} - \alpha_n) - P(\alpha_0, \beta_0)\alpha_{n+1}]. \end{aligned} \tag{2.13}$$

Also (2.10) guarantees

$$\begin{aligned} -P(u_*, u_*)(\alpha_{n+1} - \alpha_n) &\leq -P(\alpha_n, \beta_n)(\alpha_{n+1} - \alpha_n) \\ &\leq -P(\alpha_0, \beta_0)(\alpha_{n+1} - \alpha_n). \end{aligned} \tag{2.14}$$

Then (2.13) yields

$$(L - P(\alpha_0, \beta_0))^{-1} [N(\alpha_n) - P(\alpha_0, \beta_0)\alpha_n] \leq \alpha_{n+1}$$

$$\leq (L - P(\alpha_0, \beta_0))^{-1} [N(\alpha_n) + P(u_*, u_*)(\alpha_{n+1} - \alpha_n) - P(\alpha_0, \beta_0)\alpha_{n+1}].$$

Letting $n \to \infty$ and using the continuity of the operators (2.11) we obtain

$$u_* = (L - P(\alpha_0, \beta_0))^{-1} [N(u_*) - P(\alpha_0, \beta_0) u_*].$$

This shows that $u_* \in D$ and yields $Lu_* = N(u_*)$.
Case 2. Assume (iii)(b) holds. From (2.11) we also have

$$\begin{aligned} \alpha_{n+1} &= (L - P(\alpha_0, \beta_0))^{-1} [N(\alpha_n) + \\ &\quad (P(\alpha_n, \beta_n) - P(\alpha_0, \beta_0))(\alpha_{n+1} - \alpha_n) - P(\alpha_0, \beta_0)\alpha_n]. \end{aligned} \tag{2.15}$$

From (2.14),

$$\begin{aligned} 0 &\leq (P(\alpha_n, \beta_n) - P(\alpha_0, \beta_0))(\alpha_{n+1} - \alpha_n) \\ &\leq (P(u_*, u_*) - P(\alpha_0, \beta_0))(\alpha_{n+1} - \alpha_n). \end{aligned}$$

In particular, this implies that the sequence (γ_n) given by

$$\gamma_n = (L - P(\alpha_0, \beta_0))^{-1} (P(u_*, u_*) - P(\alpha_0, \beta_0))\alpha_n$$

is increasing. On the other hand, since the operators

$$(L - P(\alpha_0, \beta_0))^{-1} P(u_*, u_*), \ (L - P(\alpha_0, \beta_0))^{-1} P(\alpha_0, \beta_0)$$

are completely continuous, (γ_n) has a convergent subsequence. Now its monotonicity together with the semimonotonicity of the norm of X (a consequence of the normality of the cone) guarantees that the whole sequence (γ_n) is convergent. Consequently,

$$(L - P(\alpha_0, \beta_0))^{-1} (P(\alpha_n, \beta_n) - P(\alpha_0, \beta_0))(\alpha_{n+1} - \alpha_n) \to 0.$$

Then from (2.15), since

$$(L - P(\alpha_0, \beta_0))^{-1} N, \ (L - P(\alpha_0, \beta_0))^{-1} P(\alpha_0, \beta_0)$$

are completely continuous, we see that (α_n) contains a convergent subsequence. Thus the whole sequence (α_n) is convergent. Further we continue as in *Case 1.*

Similar arguments yield the analogue conclusion for (β_n). □

Remark 2.1. *If N and $P(u,v)$ are continuous then the assumption on operators (2.11) in (iii) is satisfied if $(L - P(\alpha_0, \beta_0))^{-1}$ is continuous, in case (a), and if N is bounded and $(L - P(\alpha_0, \beta_0))^{-1}$ is completely continuous, in case (b).*

Remark 2.2. *In particular, if $P(u,v) = 0$ for every u, v, Theorem 2.1 reduces to the monotone iterative method for the operator equation $Lu = N(u)$ with an increasing mapping N. The reader can see that in this case, (iii) (b) requires that $L^{-1}N$ is completely continuous.*

The next result gives conditions so that $(\alpha_n), (\beta_n)$ converge quadratically to the unique solution in $[\alpha_0, \beta_0]$ of (2.8).

Theorem 2.2. *Assume all the assumptions of Theorem 2.1 hold. If*

(iv) *for every $u, v \in D$ with $\alpha_0 \le u \le v \le \beta_0$, there exists a mapping $R(v,u) : D \to Z$ such that*

$$N(u) \ge N(v) - R(v,u)(v - u); \tag{2.16}$$

(v) *$L - R(v,u)$ is inverse positive, i.e. $(L - R(v,u))z \ge 0$ implies $z \ge 0$,*

then (2.8) has a unique solution u^* in $[\alpha_0, \beta_0]$.

In addition assume that the following conditions are satisfied:

(vi) *$(L - P(u,u))^{-1} : Z \to X$ is continuous for every $u \in D$, $\alpha_0 \le u \le \beta_0$;*

(vii) *there exist two constants $c_1, c_2 > 0$ such that*

$$|(R(w,\alpha) - P(\alpha,\beta))z|_Z \le c_1 |w - \alpha|_X |z|_X + c_2 |\alpha - \beta|_X |z|_X \tag{2.17}$$

for all $\alpha, \beta, w, z \in D$, $\alpha_0 \le \alpha \le w \le \beta \le \beta_0$, $z \ge 0$.

Then the convergence of $(\alpha_n), (\beta_n)$ to u^* is quadratic.

Proof. First we prove that $u_* = u^*$, where u_* is the minimal and u^* is, respectively, the maximal solution in $[\alpha_0, \beta_0]$ of (2.8). We write (2.16) for $u_* \leq u^*$, replace $N(u_*)$ by Lu_* and $N(u^*)$ by Lu^*, and obtain

$$Lu_* - R(u^*, u_*)u_* \geq Lu^* - R(u^*, u_*)u^*.$$

The hypothesis (v) assures that $u_* \geq u^*$. Then $u_* = u^*$, i.e. equation (2.8) has a unique solution in $[\alpha_0, \beta_0]$.

Let

$$p_n = u^* - \alpha_n, \quad q_n = \beta_n - u^*.$$

Using (2.9), (2.10) and (2.16) we obtain the following two inequalities:

$$\begin{aligned} Lp_{n+1} - P(u^*, u^*) p_{n+1} &\leq Lp_{n+1} - P(\alpha_n, \beta_n) p_{n+1} \\ &= -P(\alpha_n, \beta_n) p_n - N(\alpha_n) + N(u^*) \\ &\leq (R(u^*, \alpha_n) - P(\alpha_n, \beta_n)) p_n. \end{aligned}$$

Let $\Gamma = (L - P(u^*, u^*))^{-1} : Z \to X$. Since Γ is positive, we deduce

$$0 \leq p_{n+1} \leq \Gamma (R(u^*, \alpha_n) - P(\alpha_n, \beta_n)) p_n.$$

Furthermore, the norm of X being semimonotone and Γ being continuous, we obtain

$$\begin{aligned} |p_{n+1}|_X &\leq c|\Gamma| \, |(R(u^*, \alpha_n) - P(\alpha_n, \beta_n)) p_n|_Z \\ &\leq c|\Gamma| \left(c_1 |p_n|_X^2 + c_2 |\alpha_n - \beta_n|_X |p_n|_X \right) \\ &\leq c|\Gamma| \left(c_1 |p_n|_X^2 + c_2 (|p_n|_X + |q_n|_X) \cdot |p_n|_X \right) \\ &\leq a|p_n|_X^2 + b|q_n|_X^2 . \end{aligned}$$

Here $c > 0$ comes from the semimonotonicity of $|\cdot|_X$, and $|\Gamma|$ is the norm of the operator Γ.

A similar inequality can be established for $|q_{n+1}|_X$. □

Remark 2.3. *All the above results are valid if the operator N is defined only on $[\alpha_0, \beta_0] \cap D$, instead on the whole space X.*

As a consequence of Theorem 2.2, we obtain the following abstract version of Lakshmikantham's generalized quasilinearization method for the semilinear operator equation (2.8).

Theorem 2.3. *Let X be an ordered Banach space, Z be another ordered Banach space, D a linear subspace of X and $\alpha_0, \beta_0 \in D$. Let $L : D \to Z$ be a linear operator and $N : X \to Z$ be a mapping. Assume that the following conditions are satisfied:*

(a) $\alpha_0 \leq \beta_0,\ L\alpha_0 \leq N(\alpha_0)$ *and* $L\beta_0 \geq N(\beta_0)$;

(b) *$N = N_1 - N_2$, where $N_1, N_2 : X \to Z$ are C^1-Gâteaux differentiable mappings which are convex on $[\alpha_0, \beta_0]$, and for every $u, v, z \in X$ with $\alpha_0 \le u \le v \le \beta_0$ and $z \ge 0$,*

$$N_i'(u)z \le N_i'(v)z, \quad i = 1, 2;$$

(c) *$L - N_1'(u) + N_2'(v) : D \to Z$ are bijective with positive inverse for every $u, v \in [\alpha_0, \beta_0]$ with $u \le v$ or $v \le u$;*

(d) *either*
(1) the positive cone of X is regular and the operator

$$\left(L - N_1'(\alpha_0) + N_2'(\beta_0)\right)^{-1} \tag{2.18}$$

is continuous on $[\alpha_0, \beta_0]$,
or
(2) the positive cone of X is normal, the mapping N is bounded and the operator (2.18) is completely continuous on $[\alpha_0, \beta_0]$.

Then (2.8) has a unique solution u^* in $[\alpha_0, \beta_0]$ and the sequences (α_n), (β_n) given by the iterative schemes

$$L\alpha_{n+1} = N(\alpha_n) + \left(N_1'(\alpha_n) - N_2'(\beta_n)\right)(\alpha_{n+1} - \alpha_n), \tag{2.19}$$

$$L\beta_{n+1} = N(\beta_n) + \left(N_1'(\alpha_n) - N_2'(\beta_n)\right)(\beta_{n+1} - \beta_n) \tag{2.20}$$

$(n \in \mathbf{N})$ are well and uniquely defined in D and they are monotonically convergent in X to u^*.

If in addition $(L - N'(u))^{-1} : Z \to X$ is continuous for every $u \in D$, $\alpha_0 \le u \le \beta_0$, and N_1', N_2' are Lipschitz on $[\alpha_0, \beta_0]$, then the convergence of $(\alpha_n), (\beta_n)$ in X is quadratic.

Proof. We shall apply Theorems 2.1-2.2. In order to prove relations (2.9) and (2.16) we use Lemma 4.1 from [22] which assures that $N_1'(z)h \le N_1(z+h) - N_1(z)$ for all z, h with $z, z+h \in [\alpha_0, \beta_0]$. A similar relation holds for N_2. Then, for every u, v satisfying $\alpha_0 \le u \le v \le \beta_0$, we have

$$N_1'(u)(v-u) \le N_1(v) - N_1(u) \le N_1'(v)(v-u),$$

$$-N_2'(v)(v-u) \le -N_2(v) + N_2(u) \le -N_2'(u)(v-u).$$

By summing up, we obtain that (2.9) and (2.16) hold with

$$P(u,v) = N_1'(u) - N_2'(v) \text{ and } R(v,u) = N_1'(v) - N_2'(u).$$

It is easy to see that hypotheses (b) guarantees (2.10). Also, the hypothesis (d) and Remark 2.1 imply (iii) in Theorem 2.1. As regards condition (vi) from Theorem 2.2, it is satisfied since $P(u,u) = N'(u)$. Finally, the Lipschitz property of N_1' and N_2' guarantee (vii) in Theorem 2.2. □

Remark 2.4. *The hypothesis (b) can be replaced by the assumption that N_1 and N_2 are twice uniformly differentiable on every segment of X, the positive cone of Z is normal, and $N_i''(u) \geq 0$ for every $u \in X$ and $i = 1, 2$ (see [22]).*

3 Applications to Semilinear Problems

3.1 Systems of Semilinear Equations

In this section we discuss the semilinear equation in $\mathbb{R}^k$

$$Lu = N_f(u), \;\; u \in D, \tag{3.1}$$

where D is a linear subspace of $L^p(\Omega; \mathbb{R}^k)$ ($\Omega \subset \mathbb{R}^m$ open, $1 < p \leq \infty$), $L : D \subset L^p(\Omega; \mathbb{R}^k) \to L^q(\Omega; \mathbb{R}^k)$ is a general linear operator, $q \in [1, \infty)$ and N_f is the superposition operator associated to a given (p, q)-Carathéodory function $f : \Omega \times \mathbb{R}^k \to \mathbb{R}^k$.

If $p, q \in [1, \infty)$, we say that a function $g : \Omega \times \mathbb{R}^k \to \mathbb{R}^n$ is (p, q)-Carathéodory if g satisfies the Carathéodory conditions (i.e. $g(\cdot, z) : \Omega \to \mathbb{R}^n$ is measurable for each $z \in \mathbb{R}^k$, $g(x, \cdot) : \mathbb{R}^k \to \mathbb{R}^n$ is continuous for a.e. $x \in \Omega$) and there exist a function $h \in L^q(\Omega; \mathbb{R}_+)$ and a constant $c \in \mathbb{R}_+$ with

$$|g(x, z)| \leq h(x) + c|z|^{p/q} \text{ a.e. } x \in \Omega, \text{ for all } z \in \mathbb{R}^k.$$

Here $|\cdot|$ stands for the euclidean norm in $\mathbb{R}^k$ or $\mathbb{R}^n$. Also, we say that g is (∞, q)-Carathéodory if g satisfies the Carathéodory conditions, and for each $r > 0$ there is a function $h_r \in L^q(\Omega; \mathbb{R}_+)$ such that

$$|g(x, z)| \leq h_r(x) \text{ a.e. } x \in \Omega, \text{ for all } z \in \mathbb{R}^k \text{ with } |z| \leq r.$$

Similar definitions are given for matrix-valued functions $g : \Omega \times \mathbb{R}^k \to \mathcal{M}_{k \times k}(\mathbb{R})$, by identification of $\mathcal{M}_{k \times k}(\mathbb{R})$ and $\mathbb{R}^{k^2}$.

Recall if g is a (p, q)-Carathéodory function ($1 \leq p, q < \infty$), then the superposition operator $N_g : L^p(\Omega; \mathbb{R}^k) \to L^q(\Omega; \mathbb{R}^n)$, given by

$$N_g(v)(x) = g(x, v(x))$$

is well defined, bounded and continuous. Also, if g is an (∞, q)-Carathéodory function ($1 \leq q < \infty$), and Ω is bounded, then N_g maps $C(\overline{\Omega}; \mathbb{R}^k)$ into $L^q(\Omega; \mathbb{R}^n)$, is bounded and continuous.

In what follows by $|\cdot|_p$ we denote the norm of $L^p(\Omega; \mathbb{R}^k)$,

$$|u|_p = \left(\int_\Omega |u(x)|^p dx \right)^{1/p}, \quad \text{if } 1 \leq p < \infty,$$

$$|u|_\infty = \inf \{ c \in \mathbb{R}_+ \; : \; |u(x)| \leq c \;\text{ a.e. } x \in \Omega \}, \quad \text{if } p = \infty.$$

Obviously $|\cdot|_\infty$ also stands for the sup-norm of the space $C(\overline{\Omega}; \mathbb{R}^k)$ when Ω is bounded.

Also recall that the positive cone of $C\left(\overline{\Omega};\mathbb{R}^k\right)$,

$$K_{C\left(\overline{\Omega};\mathbb{R}^k\right)}=\left\{u\in C\left(\overline{\Omega};\mathbb{R}^k\right)\ :\ u(x)\geq 0 \text{ on } \overline{\Omega}\right\}$$

is normal, and the positive cone of $L^p\left(\Omega;\mathbb{R}^k\right)$ $(1\leq p<\infty)$,

$$K_{L^p(\Omega;\mathbb{R}^k)}=\left\{u\in L^p\left(\Omega;\mathbb{R}^k\right)\ :\ u(x)\geq 0 \text{ a.e. on } \Omega\right\}$$

is regular. Here, as well as in what follows, the relation $\leq$ in $\mathbb{R}^k$ means the usual partial ordering by components.

We will discuss simultaneously the cases of continuous solutions and, respectively, L^p-solutions $(1<p<\infty)$ to (3.1), by considering $1<p\leq\infty$.

In the next theorems the notation $pq/(p-q)$ stands for q when $p=\infty$.

Theorem 3.1. *Let $\Omega\subset\mathbf{R}^m$ be open, $1\leq q<p\leq\infty$, $f:\overline{\Omega}\times\mathbb{R}^k\to\mathbb{R}^k$ a (p,q)-Carathéodory function, $L:D\subset L^p\left(\Omega;\mathbb{R}^k\right)\to L^q\left(\Omega;\mathbb{R}^k\right)$ a linear operator, and $\alpha_0,\beta_0\in D$. Assume*

(i) $\alpha_0\leq\beta_0,\ L\alpha_0\leq N_f\left(\alpha_0\right),\ N_f\left(\beta_0\right)\leq L\beta_0;$

(ii) *there exists a $(p,pq/\left(p-q\right))$-Carathéodory function $a:\Omega\times\mathbb{R}^{2k}\to\mathcal{M}_{k\times k}\left(\mathbb{R}\right)$ such that*

$$f\left(x,u\right)\leq f\left(x,v\right)-a\left(x,u,v\right)\left(v-u\right) \tag{3.2}$$

for $\alpha_0\left(x\right)\leq u\leq v\leq\beta_0\left(x\right)$, a.e. $x\in\Omega$, and

$$-a\left(x,u,v\right)\leq -a\left(x,\alpha,\beta\right) \tag{3.3}$$

for $\alpha_0\left(x\right)\leq\alpha\leq u\leq v\leq\beta\leq\beta_0\left(x\right)$, a.e. $x\in\Omega$;

(iii) *if*

$$\mathcal{L}_a=\left\{a(\cdot,u(\cdot),v(\cdot)):\ \alpha_0\leq u\leq v\leq\beta_0 \text{ and } u,v\in L^p\left(\Omega;\mathbb{R}^k\right)\right.$$
$$\left.\text{for } p<\infty,\ \text{resp. } u,v\in C\left(\overline{\Omega};\mathbb{R}^k\right) \text{ for } p=\infty\right\}$$

then for each $l\in\mathcal{L}_a$, the linear mapping from D to $L^q\left(\Omega;\mathbb{R}^k\right)$,

$$w\longmapsto Lw-l(\cdot)w \tag{3.4}$$

is bijective and has a positive inverse which is continuous from $L^q\left(\Omega;\mathbb{R}^k\right)$ to $L^p\left(\Omega;\mathbb{R}^k\right)$, resp. completely continuous when $p=\infty$.

Then the sequences (α_n) and (β_n) given by the iterative schemes

$$L\alpha_{n+1}=f\left(\cdot,\alpha_n\right)+a\left(\cdot,\alpha_n,\beta_n\right)\left(\alpha_{n+1}-\alpha_n\right),$$
$$L\beta_{n+1}=f\left(\cdot,\beta_n\right)+a\left(\cdot,\alpha_n,\beta_n\right)\left(\beta_{n+1}-\beta_n\right)$$

are well and uniquely defined in D and converge monotonically in $L^p\left(\Omega;\mathbb{R}^k\right)$, to the minimal and respectively maximal solution of (3.1) in $\left[\alpha_0,\beta_0\right]$.

If in addition the following conditions are satisfied

(iv) *there exists a* $(p, pq/(p-q))$*-Carathéodory function* $b : \Omega \times \mathbb{R}^{2k} \to \mathcal{M}_{k\times k}(\mathbb{R})$ *such that*

$$f(x,u) \geq f(x,v) - b(x,v,u)(v-u) \tag{3.5}$$

for $\alpha_0(x) \leq u \leq v \leq \beta_0(x)$, *a.e.* $x \in \Omega$;

(v) *for every* $l \in \mathcal{L}_b$, *the linear mapping (3.4) is inverse positive,*

then (3.1) has a unique solution in $[\alpha_0, \beta_0]$.

Moreover, the next condition

(vii) *there exist two constants* $c_1, c_2 \geq 0$ *such that*

$$|b(x,u,\alpha)z - a(x,\alpha,\beta)z| \leq c_1|u-\alpha|\cdot|z| + c_2|\beta-\alpha|\cdot|z|$$

for $\alpha_0(x) \leq \alpha \leq u \leq \beta \leq \beta_0(x)$, *a.e.* $x \in \Omega$, $z \in \mathbb{R}^k$,

assures that the convergence of (α_n), (β_n) *in* $L^p(\Omega;\mathbb{R}^k)$ *is quadratic.*

Proof. Apply Theorems 2.1-2.2 for $X = L^p(\Omega;\mathbb{R}^k)$ if $p < \infty$ and $X = C(\overline{\Omega};\mathbb{R}^k)$ if $p = \infty$, $Z = L^q(\Omega;\mathbb{R}^k)$, $N = N_f$,

$$P(u,v)w = a(\cdot,u(\cdot),v(\cdot))w,$$

$$R(v,u)w = b(\cdot,v(\cdot),u(\cdot))w.$$

In order to follow easily the correspondence between the hypotheses of Theorem 3.1 on one hand, and those of Theorems 2.1-2.2 on the other hand, we kept the notations of their numbering items. For example, (i), (ii) in Theorem 3.1 imply (i), (ii) in Theorem 2.1. Notice N is bounded and continuous, as a superposition operator. Also, $P(u,v)$ is continuous from X to $L^q(\Omega;\mathbb{R}^k)$. Indeed, if $p < \infty$, since $p > q$ and $q/p + (p-q)/p = 1$, by Hölder's inequality we have for every $w \in L^p(\Omega;\mathbb{R}^k)$,

$$\int_\Omega |P(u,v)w(x)|^q\,dx \leq \int_\Omega |a(x,u(x),v(x))|^q\,|w(x)|^q\,dx$$

$$\leq \left(\int_\Omega (|w(x)|^q)^{p/q}\,dx\right)^{q/p}\left(\int_\Omega (|a(x,u(x),v(x))|^q)^{p/(p-q)}\,dx\right)^{(p-q)/p}$$

$$= |w|_p^q\,|a(\cdot,u(\cdot),v(\cdot))|_{pq/(p-q)}^q < \infty.$$

Here we have used the fact that a is $(p, pq/(p-q))$ -Carathéodory. Hence $P(u,v)w \in L^q(\Omega;\mathbb{R}^k)$ and

$$|P(u,v)w|_q \leq |a(\cdot,u(\cdot),v(\cdot))|_{pq/(p-q)}\,|w|_p\,,$$

which shows that $P(u,v)$ is continuous.

The case $p = \infty$ is left to the reader.

Now (iii) together with Remark 2.1 guarantees (iii) ((a) for $p < \infty$, resp. (b) for $p = \infty$) in Theorem 2.1. It is easily seen that conditions (iv)-(vii) in Theorem 2.2 are also satisfied. □

Before stating the consequence of Theorem 3.1, let us precise some notations. For a function $g : \Omega \times \mathbb{R}^k \to \mathbb{R}^k$, and a given point $x \in \Omega$, by $g'(x, \cdot)$ we mean the matrix

$$g'(x,z) = \left[\frac{\partial g_i(x,z)}{\partial z_j}\right]_{1\leq i,j\leq k}.$$

Thus, differentiability refers only to the z argument of $g(x, z)$.

Corollary 3.1. *Let $\Omega \subset \mathbb{R}^m$ be open, $1 \leq q < p \leq \infty$, $f : \Omega \times \mathbb{R}^k \to \mathbb{R}^k$ a (p, q)-Carathéodory function, $L : D \subset L^p\left(\Omega; \mathbb{R}^k\right) \to L^q\left(\Omega; \mathbb{R}^k\right)$ a linear operator, and $\alpha_0, \beta_0 \in D$. Assume*

(i) $\alpha_0 \leq \beta_0,\ L\alpha_0 \leq N_f(\alpha_0),\ N_f(\beta_0) \leq L\beta_0;$

(ii) *there exist functions $f_1, f_2 : \Omega \times \mathbb{R}^k \to \mathbb{R}^k$ with*

$$f = f_1 - f_2,$$

such that for a fixed x, $f_1(x, \cdot)$, $f_2(x, \cdot)$ are differentiable in $\mathbb{R}^k$ and convex on $[\alpha_0(x), \beta_0(x)]$, a.e. $x \in \Omega$, and $f_1', f_2' : \Omega \times \mathbb{R}^k \to \mathcal{M}_{k\times k}(\mathbb{R})$ are $(p, pq/(p-q))$-Carathéodory functions;

(iii) *for every $l \in \mathcal{L} = \{f_1'(\cdot, u(\cdot)) - f_2'(\cdot, v(\cdot))\ :\ \alpha_0 \leq u, v \leq \beta_0,$ $u, v \in L^p\left(\Omega; \mathbb{R}^k\right)$ for $p < \infty$ resp. , $u, v \in C\left(\overline{\Omega}; \mathbb{R}^k\right)$ for $p = \infty\}$, the linear mapping $w \longmapsto Lw - l(\cdot)w$ from D to $L^q\left(\Omega; \mathbb{R}^k\right)$ is bijective and has a positive inverse which is continuous, resp. completely continuous if $p = \infty$, from $L^q\left(\Omega; \mathbb{R}^k\right)$ to $L^p\left(\Omega; \mathbb{R}^k\right)$.*

Then the sequences (α_n) and (β_n) given by the iterative schemes

$$\begin{aligned} L\alpha_{n+1} &= f(., \alpha_n) + (f_1'(., \alpha_n) - f_2'(., \beta_n))(\alpha_{n+1} - \alpha_n), \\ L\beta_{n+1} &= f(., \beta_n) + (f_1'(., \alpha_n) - f_2'(., \beta_n))(\beta_{n+1} - \beta_n) \end{aligned}$$

are well and uniquely defined in D and converge monotonically in $L^p\left(\Omega; \mathbb{R}^k\right)$ to the unique solution of (3.1) in $[\alpha_0, \beta_0]$.

If in addition $f_1'(x, \cdot)$ and $f_2'(x, \cdot)$ are Lipschitz on $[\alpha_0(x), \beta_0(x)]$ for a.e. $x \in \Omega$ with Lipschitz constants independent on x, then the convergence of (α_n) and (β_n) in $L^p\left(\Omega; \mathbb{R}^k\right)$ is quadratic.

Proof. Apply Theorem 3.1 for

$$\begin{aligned} a(x, u, v) &= f_1'(x, u) - f_2'(x, v), \\ b(x, v, u) &= f_1'(x, v) - f_2'(x, u). \end{aligned} \tag{3.6}$$

□

3.2 The Initial Value Problem

Consider the initial value problem

$$\begin{cases} u' = f(x,u), \ x \in [0,T], \\ u(0) = 0. \end{cases} \tag{3.7}$$

For any number $1 \le q < \infty$, we let $W^{1,q}[0,T]$ be the space of all absolutely continuous functions $u : [0,T] \to \mathbb{R}$ with $u' \in L^q[0,T]$. We seak solutions u to (3.7) in $W^{1,q}[0,T]$ with $u(0) = 0$.

We discuss (3.7) under the assumption that f can be represented in the form $f = f_1 - f_2$ with f_1, f_2 convex in the second variable. Without any differentiability assumptions on f, we can give an iterative procedure for monotone approximation of the extremal solutions of (3.7) in a given interval of functions, in terms of divided differences.

For a function $g : [c,d] \to \mathbb{R}$ and two given points $u, v \in [c,d]$, $u \ne v$, we let the divided difference of g on points u, v be defined by

$$[g; u, v] = \frac{g(u) - g(v)}{u - v}.$$

Recall if the function g is convex, then (by Jensen's inequality),

$$[g; u, v] \le [g; u, w] \le [g; v, w] \tag{3.8}$$

whenever $c \le u \le v \le w \le d$.

Theorem 3.2. *Let $1 \le q < \infty$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, $\alpha_0, \beta_0 \in W^{1,q}[0,T]$ and $\alpha_{-1}, \beta_{-1} \in C[0,T]$ such that $\alpha_0(0) = \beta_0(0) = 0$, $\alpha_{-1} < \alpha_0 \le \beta_0 < \beta_{-1}$ on $[0,T]$ and*

$$\alpha_0'(x) \le f(x, \alpha_0(x)), \quad \beta_0'(x) \ge f(x, \beta_0(x) \quad a.e. \ on \ [0,T].$$

In addition assume that $f = f_1 - f_2$ where $f_1, f_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$ are (∞, q)-Carathéodory and for each fixed $x \in [0,T]$, $f_1(x,\cdot)$, $f_2(x,\cdot)$ are convex on $[\alpha_{-1}(x), \beta_0(x)]$ and respectively on $[\alpha_0(x), \beta_{-1}(x)]$, for a.e. $x \in [0,T]$. Then the sequences (α_n), (β_n) given by the iterative procedures

$$\begin{aligned} \alpha_{n+1}' &= f(\cdot, \alpha_n) + ([f_1; \alpha_{-1}, \alpha_n] - [f_2; \beta_{-1}, \beta_n]) (\alpha_{n+1} - \alpha_n), \\ \beta_{n+1}' &= f(\cdot, \beta_n) + ([f_1; \alpha_{-1}, \alpha_n] - [f_2; \beta_{-1}, \beta_n]) (\beta_{n+1} - \beta_n) \end{aligned}$$

are well and uniquely defined in $D = \{u \in W^{1,q}[0,T] \ : \ u(0) = 0\}$ and converge monotonically in $C[0,T]$ to the unique solution of (3.7) in $[\alpha_0, \beta_0]$.

Proof. Apply Theorem 3.1. Here $p = \infty$, $k = 1$, $Lu = u'$,

$$a(x,u,v) = [f_1(x,\cdot); \alpha_{-1}(x), u] - [f_2(x,\cdot); v, \beta_{-1}(x)].$$

Using inequalities (3.8) we have

$$[f_1(x,\cdot); \alpha_{-1}(x), u] \le [f_1(x,\cdot); u, v], \quad [f_2(x,\cdot); v, \beta_{-1}(x)] \ge [f_2(x,\cdot); u, v],$$

$$[f_1(x,\cdot); \alpha_{-1}(x), u] \ge [f_1(x,\cdot); \alpha_{-1}(x), \alpha],$$

$$[f_2(x,\cdot);v,\beta_{-1}(x)] \le [f_2(x,\cdot);\beta,\beta_{-1}(x)]\,,$$

whenever $\alpha_{-1}(x) < \alpha_0(x) \le \alpha \le u \le v \le \beta \le \beta_0(x) < \beta_{-1}(x)$. Whence, by summing up the first two inequalities and the last two ones, we obtain (3.2) and (3.3), respectively.

Notice the mapping (3.4) is bijective, since the initial value problem for a linear equation has a unique global solution. Its inverse is the Volterra integral operator V, given by

$$V(f)(x) = \int_0^x e^{l(s)(x-s)} f(s)ds, \quad \text{for } f \in L^q[0,T],$$

which is trivially positive, and completely continuous as follows by the Ascoli-Arzèla Theorem.

In order to prove the uniqueness of the solution, we choose

$$b(x,v,u) = [f_1(x,\cdot);v,\beta_{-1}(x)] - [f_2(x,\cdot);\alpha_{-1}(x);u]\,.$$

Now, we use again (3.8), and obtain that

$$[f_1(x,\cdot);u,v] \le [f_1(x,\cdot);\alpha_{-1}(x),v]\,, \quad [f_2(x,\cdot);\alpha_{-1}(x),u] \le [f_2(x,\cdot);u,v]$$

whenever $\alpha_0(x) \le u \le v \le \beta_0(x)$. Whence, by summing up we get (3.5). □

Corollary 3.2 in particular yields the following theorem which extends a similar result from [15].

Theorem 3.3. *Let $1 \le q < \infty$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\alpha_0, \beta_0 \in C^1[0,T]$ such that $\alpha_0(0) = \beta_0(0) = 0$, $\alpha_0 \le \beta_0$ and*

$$\alpha_0' \le f(x,\alpha_0)\,,\ \beta_0' \ge f(x,\beta_0) \ \ on\ \ [0,T]\,.$$

In addition assume that $f = f_1 - f_2$, where $f_1, f_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$, $f_1(x,.)$, $f_2(x,.)$ are (∞,q)-Carathéodory and for each fixed $x \in [0,T]$, $f_1(x,\cdot)$, $f_2(x,\cdot)$ are differentiable on $\mathbb{R}$ and convex on $[\alpha_0(x), \beta_0(x)]$ for a.e. $x \in [0,T]$, and their derivatives are (∞,q)-Carathéodory. Also assume that $f_1'(x,\cdot)$, $f_2'(x,\cdot)$ are Lipschitz on $[\alpha_0(x), \beta_0(x)]$ with Lipschitz constants not depending on x. Then the sequences (α_n) and (β_n) given by the iterative schemes

$$\begin{aligned}\alpha_{n+1}' &= f(\cdot,\alpha_n) + (f_1'(\alpha_n) - f_2'(\beta_n))(\alpha_{n+1} - \alpha_n)\,,\\ \beta_{n+1}' &= f(\cdot,\beta_n) + (f_1'(\alpha_n) - f_2'(\beta_n))(\beta_{n+1} - \beta_n)\end{aligned}$$

are well and uniquely defined in $D = \left\{u \in C^1[0,T] :\ u(0) = 0\right\}$ and converge monotonically and quadratically in $C[0,T]$ to the unique solution of (3.7) in $[\alpha_0, \beta_0]$.

Proof. The result is a direct consequence of Corollary 3.2 with $p = \infty$ and $k = 1$. □

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