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FIXED POINT THEORY

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Contents

Introduction

One of the most dynamic area of research of the last 60 years, with a lot of applications in various fields of pure and applied mathematics, as well as, in physical, economic or life sciences, is without doubt **the fixed point theory**. Not only solutions of several classes of equations or inclusions, but also equilibrium states of an economy, optimization processus solution, fractals, closed orbits in a system of mutually gravitating bodies, etc. are fixed points of an appropriate operator.

The dynamic of this topic is reflected, at least, by the following arguments:

★ **Over 120 books (monographs, lecture notes, proceedings) on fixed point theory and its applications:**

F.E. Browder (1948), M.A. Krasnoselskii (1962), F.F. Bonsal (1962), J. Cronin (1964), T. van der Walt (1963), J. Reinermann (1970), R.F. Brown (1971), I.A. Rus (1971), V.I. Istrăţescu (1973), I.A. Rus (1973), M. Hegedüs (1973), H. Amann (1974), S.P. Singh (1974), D.R. Smart (1974), K. Deimling (1974), M. Hegedüs (1974), M.A. Krasnoselskii and P. Zabrejko (1975), M. van de Vel (1975), D. Fromholzer et al. (1975), F.E. Browder (1976), B.C. Eaves (1976), L. Górniewicz (1976), A.A. Ivanov (1976), S. Swaminatham (1976), M.J. Todd (1976), J.W. de Bakker (1976), T. Riedrich (1976), R. Gaines and J. Mawhin (1977), M.L. Balinski and R.W. Cottle (1978), C. Eisenack and C. Fenske (1978), O. Hadžić (1978), M. Hegedüs (1978), N. Lloyd (1978), H.-O. Peitgen and H.-O. Walther (1979), I.A. Rus (1979), I.A. Rus (1979), J. Banas and K. Goebel (1980), S. Czerwik (1980), W. Forster (1980), A.J.J. Talman (1980), G. van der Laan (1980), St.M. Robinson (1980), E. Fadell and G. Fournier (1981), V.I. Istrăţescu (1981), J. Dugundji and A. Granas (1982),

R. Wegrzyk (1982), B.J. Jiang (1983), I.A. Rus (1983), R.C. Sine (1983), D. L. Goncalves and J.C. de Souza Kiihl (1983), K. Goebel and S. Reich, M. Lösch (1984), O. Hadžić (1984), K.C. Border (1985), J. Mawhin (1985), R.D. Nussbaum (1985), J. Bewersdorff (1985), E. Zeidler (1985), M.F. Iwano (1985), F.E. Browder (1986), F. Robert (1986), A. Dold (1986), K. Schilling (1986), M.R. Tasković (1986), R. Kuczumow (1987), B. Blümel (1987), R F. Brown (1988), H. Ulrich (1988), D. Guo and V. Lakshmikantham (1989), T.-H. Kiang (1989), B. J. Jiang (1989), Yu.A. Shashkin (1989), A.G. Aksoy and M.A. Khamsi (1990), K. Goebel and W.A. Kirk (1990), M.A. Théra and J.-B. Baillon (1991), G. Sommaruga-Rosolemos (1991), K.K. Tan (1992), R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A. E. Rodkina and B. N. Sadovskii (1992), L. Schwartz (1994), J. Jaworowski, W.A. Kirk and S. Park (1995), J. Oprea (1995), O. Hadžić (1995), W.V. Petryshyn (1995), J.J. Duistermaat (1996), V.F. Démyanov (1996), T. Dominguez Benavides (1996), V. Berinde (1997), S.P. Singh, B. Watson and P. Srivastava (1997), J.M. Ayerbe Toledano, T. Dominguez Benavides and G. López Acedo (1997), F.H. Clarke, Yu.S. Ledyaev and R.J. Stern (1997), V. Radu, C. Grecu, A. Pogan, L. Radu and T. Venţe, (1998), Z. Yang (1999), N. Negoescu (1999), L. Górniewicz (1999), R.P. Agarwal and D. O'Regan (2000), Y.J. Cho (2000), D. Butnariu and A.N. Iusem (2000), W. Takahashi (2000), M.A. Khamsi and W.A. Kirk (2001), R.P. Agarwal, M. Meehan and D. O'Regan (2001), D. O'Regan and R. Precup (2001), W.A. Kirk and B. Sims (2001), O. Hadžić and E. Pap (2001), A. Buică (2001), I.A. Rus (2001), K. Goebel (2002), A. Petruşel (2002), M.A. Şerban (2002), A. Muntean (2002), A. Bege (2002), A. Petruşel (2002), V. Berinde (2002), A. Petruşel, G. Petruşel and I.A. Rus (2002), J. Andres and L. Górniewicz (2003), A. Granas and J. Dugundji (2003), S.B. Nadler jr. (2003), Z. Denkowski, S. Migórski and N.S. Papageorgiou, A. Fryszkowski (2004), D. Guo, Y.J. Cho and J. Zhu (2004), D. Miklaszewski (2005), R.F. Brown, M. Furi, L. Górniewicz and B. Jiang (Eds.) (2005), S. Reich and D. Shoikhet (2005), M. Balaj (2006), T.A. Burton (2006), L. Górniewicz (2006), C. Vladimirescu and C. Avramescu (2006), L. Gasiński and N.S. Papageorgiou (2006), I.A. Rus (2006), G. Moţ, A. Petruşel and G. Petruşel (2007), R. Skiba (2007), V. Berinde (2007), T.A. Burton (2008), E.U. Tarafdar and M.S.R.

Chowdhury (2008), V.G. Angelov (2008).

★ **Over 12,000 papers on fixed point theory from 1940 until now.**

★ **Almost 4,000 papers on fixed point theory only between 2000-2008.**

★ Except these theoretical books and papers, there are **more than 2,000 books, monographs and proceedings and over 40,000 papers**, which use the abstract theory of fixed point for various problems of pure, applied and computational mathematics.

★ The field of the fixed point theory is today vast and open to lots of techniques and ideas. A large number of applications are also developed in various directions.

Let us present some topics of the fixed point theory:

A. Topics in terms of structured sets:

- ⧫ Fixed Point Theory in Sets
- ⧫ Fixed Point Theory in Ordered Sets
- ⧫ Fixed Point Theory in Groups
- ⧫ Fixed Point Theory in Rings
- ⧫ Fixed Point Theory in Algebras
- ⧫ Fixed Point Theory in Universal Algebras
- ⧫ Fixed Point Theory in Categories
- ⧫ Fixed Point Theory in Metric Spaces
- ⧫ Fixed Point Theory in Generalized Metric Spaces
- ⧫ Fixed Point Theory in Geodesic Spaces
- ⧫ Fixed Point Theory in Gauge Spaces
- ⧫ Fixed Point Theory in Hilbert Spaces
- ⧫ Fixed Point Theory in Banach Spaces
- ⧫ Fixed Point Theory in Banach Algebras
- ⧫ Fixed Point Theory in Locally Convex Spaces
- ⧫ Fixed Point Theory in Linear Topological Spaces
- ⧫ Fixed Point Theory in Topological Spaces
- ⧫ Fixed Point Theory in Algebraic Topology

- ⧫ Fixed Point Theory on Manifolds

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B. Topics in terms of some classes of operators:

- ⧫ Fixed Point Theory for Increasing Operators
- ⧫ Fixed Point Theory for Decreasing Operators
- ⧫ Fixed Point Theory for Progressive Operators
- ⧫ Fixed Point Theory for Continuous Operators
- ⧫ Fixed Point Theory for Operators with Closed Graph
- ⧫ Fixed Point Theory for Open Operators
- ⧫ Fixed Point Theory for Closed Operators
- ⧫ Fixed Point Theory for Differentiable Operators
- ⧫ Fixed Point Theory for Holomorphic Operators
- ⧫ Fixed Point Theory for Generalized Contractions
- ⧫ Fixed Point Theory for Nonexpansive Operators
- ⧫ Fixed Point Theory for Asymptotically Nonexpansive Operators
- ⧫ Fixed Point Theory for Rotative Operators
- ⧫ Fixed Point Theory for Isometries
- ⧫ Fixed Point Theory for Delating Operators
- ⧫ Fixed Point Theory for Accretive Operators
- ⧫ Fixed Point Theory for Pseudocontractive Operators
- ⧫ Fixed Point Theory for Monotone Operators
- ⧫ Fixed Point Theory for Acyclic Operators
- ⧫ Fixed Point Theory for Symplectic Operators

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C. Topics in deep connection to fixed point theory:

- ⧫ Coincidence Point Theory
- ⧫ Zero Point Theory
- ⧫ Surjectivity Theory
- ⧫ Spectral Theory
- ⧫ Bifurcation Theory
- ⧫ Topological Degree Theory
- ⧫ Dynamical System Theory

- ⧫ Invariant Subsets
- ⧫ Convexity Structures
- ⧫ Geometry of the Banach Space
- ⧫ Measure of Noncompactness
- ⧫ Measure of Nonconvexity
- ⧫ Complexity of Computation
- ⧫ Ramsey Theory
- ⧫ Extremal Element Theory

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D. Topics generated by some classical results:

- ⧫ Borsuk-Ulam Type Theorems
- ⧫ Tarski-Kantorovich Type Theorems
- ⧫ Schauder-Tychonoff Type Theorems
- ⧫ Darbo Type Theorems
- ⧫ Sadovskii Type Theorems
- ⧫ Caristi-Kirk Type Theorems
- ⧫ Caristi-Browder Type Theorems
- ⧫ Browder-Ghöde-Kirk Type Theorems
- ⧫ Browder Type Theorems
- ⧫ Frum-Ketkov Type Theorems
- ⧫ Krasnoselskii Type Theorems
- ⧫ Leray-Schauder Type Theorems
- ⧫ Granas Type Theorems
- ⧫ Knaster-Kuratowski-Mazurkiewicz Type Theorems
- ⧫ Ky Fan Type Lemmas
- ⧫ Markov-Kakutani Type Theorems
- ⧫ Lefschetz Type Theorems
- ⧫ Nielsen Type Theorems
- ⧫ Poincaré-Birkhoff Type Theorems
- ⧫ Rabinowitz-Nussbaum Type Theorems

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E. Other topics:

- ⧫ Periodic Point Theory

- ♦ Almost Fixed Point Theory
- ♦ Common Fixed Point Theory
- ♦ Fixed Point Algorithms
- ♦ Mathematics of Fractals

F. Applications of the fixed point theory to:

- ♦ Equations in $\mathbb{R}^n$
- ♦ Equations in $\mathbb{C}^n$
- ♦ Matrix Equations
- ♦ Functional Equations
- ♦ Ordinary Differential Equations
- ♦ Partial Differential Equations
- ♦ Integral Equations
- ♦ Functional-Differential Equations
- ♦ Functional-Partial Differential Equations
- ♦ Functional-Integral Equations
- ♦ Differential Inclusions
- ♦ Integral Inclusions
- ♦ Mathematical Economics
- ♦ Informatics

G. The topics of the **Handbook of Metric Fixed Point Theory (W.A. Kirk and B. Sims - Eds.)** R[1] are the following:

- ♦ Contraction Operators and Extensions
- ♦ Fixed Point Free Operators
- ♦ Nonexpansive Operators
- ♦ Geometric Theory of Banach Spaces and Fixed Points
- ♦ Fixed Point Theory in Terms of Measure of Noncompactness
- ♦ Fixed Point Theory in l^1 and c_0
- ♦ Fixed Point Theory of Nonself Nonexpansive Operators
- ♦ Fixed Point Theory of Rotative Operators
- ♦ Fixed Point Theory in Banach Function Lattices
- ♦ Fixed Point Theory in Hyperconvex Spaces
- ♦ Fixed Point Theory of Holomorphic Operators
- ♦ Fixed Points and Semigroups of Nonlinear Operators

♦ Generic Aspects of Metric Fixed Point Theory
♦ Minimal Displacement Problem
♦ Retractions and Fixed Points
♦ Order-Theoretic Aspects of Metric Fixed Point Theory
♦ Fixed Point Theory of Multivalued Operators

H. The topics of the **Handbook of Topological Fixed Point Theory (R.F. Brown, M. Furi, L. Górniewicz and B. Jiang - Eds.)** R[1] are the following:

♦ I. Homological Methods in Fixed Point Theory (coincidence theory, Lefschetz fixed point theorem, Nielsen classes, homotopy minimal periods, periodic points and braid theory, fixed point theory of multivalued weighted operators, fixed point theory for homogeneous spaces)

♦ II. Equivariant Fixed Point Theory (equivariant fixed point, equivariant degree theory, bifurcation of solutions of $SO(2)$-symmetric nonlinear problems with variational structure)

♦ III. Nielsen Theory (Nielsen theory, applications of Nielsen theory, algebraic and fibre techniques for calculating the Nielsen number, Wecken theorem, relative Nielsen theory)

♦ IV. Applications (applications to differential equations and inclusions, applications to multivalued dynamical systems, Poincaré translation operator on differentiable manifolds, Wazewski method)

I. The topics of the book **Principles and Applications of Fixed Point Theory (Ioan A. Rus)** B[73] are the following:

♦ I. Fixed Point Theory

- The fixed point set
- Tarski's fixed point theorem
- Bourbaki's fixed point theorem
- Contraction principle
- Perov's fixed point theorem
- Luxemburg-Jung's fixed point theorem
- Brouwer's fixed point theorem
- Schauder's fixed point theorem

• Tychonoff 's fixed point theorem

• Browder-Ghöde-Kirk's fixed point theorem

• Fixed point theorems for multivalued operators

• Problems and results in fixed point theory (the method of successive approximations, measures of noncompactness, topological degree, the fixed point set, sequences of operators and fixed points, data dependence of fixed points, operators on cartesian product, fixed point theorems in $\mathbb{R}^n$, fixed point theorems in $\mathbb{C}^n$, common fixed point theory, coincidence point theory, almost fixed points, fixed point theory in categories).

♦ II. Applications of the Fixed Point Theory

• Equations in $\mathbb{R}^n$

• Equations in $s(\mathbb{R})$

• Functional Equations

• Integral Equations

• Functional-Differential Equations

• Partial Differential Equations

• Equations in Applied Mathematics

J. There also exists a project of **M.S. Khamsi: Fixed Point Theory and its Applications on the Web.** The topics considered there are:

♦ The Contraction Principle

♦ Nonexpansive Mappings in Hilbert Spaces

♦ Nonexpansive Mappings in Banach Spaces

♦ Orbit, Omega-set

♦ Ergodic Theorems

♦ Approximation Techniques

♦ Non-classical Banach Spaces (Orlicz spaces, James' spaces, Tsirelson' spaces)

♦ Metric Spaces

♦ Measure of Non-compactness

♦ Caristi's Fixed Point Theorem

♦ Bifurcation Theory

♦ Multivalued Mappings

♦ Generalized Structures (Ordered Set, Generalized Metric Spaces, Modular Spaces)

⧫ Topological Fixed Point Theory (Brouwer's Theorem, Minimax Theorems, KKM-Maps, Degree Theory, Sperner's Lemma, Discrete Brouwer's Theorem, Leray-Schauder's Fixed Point Theorem, Degree Theory, ANR' Sets, Nielsen Theorems, Lefschetz Fixed Point Theorems, Bifurcation Theory, Complementarity Problems, Renorming Techniques)

★ **Fixed Point Theory-An International Journal on Fixed Point Theory, Computation and Applications** is the first journal entirely devoted to fixed point theory and its applications. Actually, the academic year 1999-2000 marked the 30-th anniversary of the Seminar on Fixed Point Theory Cluj-Napoca. This research seminar started in 1969 at the initiative and under the guidance of Professor Ioan A. Rus from Babeş-Bolyai University of Cluj-Napoca. The yearly publication of the Seminar was *Seminar on Fixed Point Theory, Preprint no. 3.* The journal *Seminar on Fixed Point Theory Cluj-Napoca* (between 2000 and 2002) and *Fixed Point Theory* (since 2003) are continuations of this publication. The Editorial Board of the journal *Fixed Point Theory* is the following: Ioan A. Rus (Editor-in-Chief), Adrian Petruşel (Managing Editor), George Isac, Radu Precup (Editors), Jan Andres, Vasil Angelov, Jürgen Appell, Vasile Berinde, Theodore A. Burton, Dan Butnariu, Constantin Corduneanu, Tomas Dominguez Benavides, Marlène Frigon, Vasile Glăvan, Kazimierz Goebel, Lech Górniewicz, Kiyoshi Iseki, Genaro López Acedo, Enrique Llorens Fuster, William Art Kirk, Valeri Obukhovskii, Donal O'Regan, Viorel Radu, Simeon Reich, Biagio Ricceri, S.P. Singh, Wataru Takahashi, Mihai Turinici, Hong-Kun Xu (Editorial Board). The journal *Fixed Point Theory* publishes important research and expository papers devoted to the theory, computation and applications of the fixed points.

Since then, other three journals on fixed point theory appeared in the mathematics literature:

• **Fixed Point Theory and Applications** (since 2004). The Editorial Board of the journal is composed by: R.P. Agarwal (Editor-in-Chief), Mohamed Amine Khamsi, Thomas Bartsch, Hichem Ben-El-Mechaiekh, Jonathan

M. Borwein, Robert F. Brown, Tomas Dominguez Benavides, Patrick M. Fitzpatrick, Hélène Frankowska, Massimo Furi, Lech Górniewicz, Djairo Guedes de Figueiredo, Evelyn Hart, Jerzy Jezierski, William A. Kirk, V. Lakshmikantham, Anthony To-Ming Lau, Jean Mawhin, Huang Nanjing, Roger D. Nussbaum, Donal O'Regan, Simeon Reich, Billy E. Rhoades, Klaus Schmitt, Brailey Sims, Tomonari Suzuki, Andrzej Szulkin, Wataru Takahashi, J.R.L. Webb, Fabio Zanolin (Associate Editors). The aim of this journal is "to report new fixed point theorems and their applications where the essentiality of the fixed point theorems is highlighted. Fixed point theorems give the conditions under which maps (single or multivalued) have solutions. The theory itself is a beautiful mixture of analysis, topology, and geometry. Over the last 50 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. In particular fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, engineering, game theory, and physics."

• **Journal of Fixed Point Theory and Applications** (since 2007). The Editorial Board of this journal is the following: Andrzej Granas (Editor-in-Chief), Gilles Gauthier (Managing Editor); Section Editors: Michael Crabb (Algebraic and Geometric Topology), Octav Cornea (Symplectic Topology and Global Analysis), Krystyna Kuperberg (Dynamical Systems), Norman Dancer (Nonlinear Analysis), Simeon Reich (Classical Topics), Fon Che Liu (Games, Economics and Computation Theory), Richard S. Palais (Surveys and Research Expository Papers), Alberto Abbondandolo (Short Communications and Open Problems); Editorial Advisory Board: Haim Brezis, Felix Browder, Yvonne Choquet-Bruhat, Albrecht Dold Alexander Ioffe, Anatole Katok, Paul Malliavin, Victor Maslov Isaac Namioka, Paul Rabinowitz, Czeslaw Ryll-Nardzewski, Albert Schwarz Anatoli Skorokhod; Associate Editors: Hichem Ben-El-Mechaiekh, Vieri Benci, Robert Cauty, Kung-Ching Chang, Bernard Cornet, Edward Fadell, John Franks, Marlène Frigon, Kazimierz Geba, Peter Gilkey, Ronald B. Guenther, Charles Horvath, Jacek Jachymski, Jan Jaworowski, Boju Jiang, Sam B. Nadler jr., Roger Nussbaum, Kaoru Ono, Heinz-Otto Peitgen, Grzegorz Rosenberg, Yuli Rudyak, Slawomir Rybicki, Matthias Schwarz, Alexander N. Sharkovsky, Michael Shub, Evgenij

G. Sklyarenko, Gencho Skordev, Heinrich Steinlein, Andrzej Szulkin, Sergei Tabachnikov, Wataru Takahashi, John Toland, Aleksy Tralle, Gerard van der Laan, Victor Zvyagin.

A short description of this journal reads as follows: "This journal publishes high-quality, peer-reviewed research papers in all disciplines in which the use of tools of the fixed point theory plays an essential role. It details new developments in fixed point theory as well as in related topological methods and examines ramifications to symplectic topology, dynamical systems and global analysis. In addition, the Journal of Fixed Point Theory and Applications presents significant applications in nonlinear analysis, mathematical economics and computation theory. It also features contributions to important problems in geometry, fluid dynamics and mathematical physics."

The journal is organized into eight sections:

- Algebraic and Geometric Topology
- Dynamical Systems
- Symplectic Topology and Global Analysis
- Nonlinear Analysis
- Classical Topics
- Games, Economics and Computation Theory
- Surveys and Research Expository Papers
- Short Communications and Open Problems.

• **JP Journal of Fixed Point Theory and Applications** (since 2007). The Editorial Board of this journal is the following: K.K. Azad (Managing Editor), Bashir Ahmad, Tomas Dominguez Benavides, Antonio Carbone, Yeol Je Cho, Liang-Ju Chu, Sompong Dhompongsa, Marlène Frigon, Lech Gorniewicz, Lishan Liu, Jong Seo Park, Simeon Reich, B.E. Rhoades, Biagio Ricceri, Wataru Takahashi, Peter Wong, Hong-Kun Xu, L.C. Zeng (Editors). A short description of the aims of the journal is the following: "The JP Journal of Fixed Point Theory and Applications is a fully refereed international journal, which published original research papers and survey articles in all aspects of Fixed Point Theory and their Applications. Topics in detail to be covered are new developments in fixed point theory as well as in related topological methods: ramifications to symplectic topology, dynamical systems and global analysis, significant applications in nonlinear analysis, mathematical economics and computation theory, contributions to important problems in geometry, fluid dynamics and mathematical physics and other such areas of interest."

The purpose of the monograph is to present the most important results in the field of fixed point theory. Each chapter starts with precursors, guidelines and general references of the topic. Our book is based, to a certain extent, on the authors' former book *Fixed Point Theory 1950-2000: Romanian Contributions*, House of the Book of Science, Cluj-Napoca, 2002.

The References of the book are organized in two sections. First part consists of an exhaustive bibliography of the fixed point theory of Romanian authors, while the second part is a general references list containing:

• basic references of the fixed point theory

as well as,

• papers of Romanian authors which have applied the fixed point theory.

The list of symbols, the index of terms and the author's conclude the book.

Throughout the book, the symbol **B[...]** indicates titles from the **Romanian Bibliography of the Fixed Point Theory**, while **R[...]** refers to titles from the **General References** list.

Finally, we would like to point out that, by this book, our intention was not only **to provide a tool for further research**, but also, to give, in each chapter, a guideline of the field, **to have, at a glance, the entire history of the topic.**

Cluj-Napoca, September 2008 The Authors

Chapter 1

Set-theoretic aspects of the fixed point theory

Precursors: G. Cantor.
Guidelines: A. Abian (1968), S. Eilenberg (?).
General references: A. Abian R[3] and R[4], K. Wisniewski R[1], J. Dugundji and A. Granas R[1], D. Smart R[1], I.A. Rus B[23], B[28], B[29], B[73] and B[90], W. Grudzinski R[1], A. Bege B[1], A. Granas and J. Dugundji R[1]. See also 14.1, 15.1, 15.2, 18, 19 and 20.

1.0 Basic notions and results

Let X be a nonempty set, $f : X \to X$ be a singlevalued operator and $T : X \multimap X$ be a multivalued operator. Then we denote:

$$\mathcal{P}(X) := \{A \mid A \subset X\}$$

$$P(X) := \{A \subset X | A \neq \emptyset\}$$

$$\Delta(X) - \text{ the diagonal of } X \times X$$

$$CardX - \text{ the cardinal number of X}$$

$$1_X - \text{ the identity operator}$$

$$F_f := \{x \in X|\ f(x) = x\} - \text{ the fixed point set of f}$$

$$f^0 := 1_X,\ f^1 := f, ..., f^n := f \circ f^{n-1} - \text{ the iterates of f}$$

$$I(f) := \{A \in P(X)|f(A) \subset A\}$$

$$P_f := \bigcup_{n\in\mathbb{N}^*} F_{f^n} - \text{ the periodic point set of f}$$

$$T(Y) := \bigcup_{y\in Y} T(y)$$

$$T^1(Y) := T(Y), T^2(Y) := T(T(Y)), ..., T^n(Y) := T(T^{n-1}(Y)) - \text{ the iterates of T}$$

$$F_T := \{x \in X|\ x \in T(x)\} - \text{ the fixed point set of T}$$

$$(SF)_T := \{x \in X|\ \{x\} = T(x)\} - \text{ the strict fixed point set of T}$$

$$(SP)_T := \bigcup_{n\in\mathbb{N}^*} (SF)_{T^n} - \text{ the strict periodic point set of T}$$

Let X be a nonempty set and $Y \subseteq X$. By definition, a set retraction of X onto Y is an operator $\rho : X \to Y$ such that the restriction of ρ to Y is the identity operator. More general, if X is a structured set, then $\rho : X \to Y$ is a retraction with respect to that structure if ρ is a set retraction and ρ is a morphism with respect to that structure. If $\rho : X \to Y$ is a retraction, then Y is called a retract of X. An operator $f : Y \to X$ is retractible with respect to a retraction $\rho : X \to Y$ if $F_f = F_{\rho\circ f}$ (see 1.3).

The following theorems are the main results of the set-theoretical approach to the fixed point theory:

Abian's Theorem. (A. Abian R[4], K. Wisniewski R[1]) *Let X be a nonempty set and $f : X \to X$ be an operator. Then, the following statements are equivalent:*

(i) $F_f = \emptyset$;

(ii) there exist three mutually disjoint subset $X_1, X_2.X_3 \subset X$ such that:

(a) $X = X_1 \cup X_2 \cup X_3$,

(b) $X_i \cap f(X_i) = \emptyset$ for each $i \in \{1, 2, 3\}$.

Proof. It is obvious that $(ii) \Rightarrow (i)$. Let us prove now the reverse implication. We define the following equivalence relation:

$$x \overset{e}{\leftrightarrow} y \text{ if and only if there exist } n, m \in \mathbb{N} \text{ such that } f^n(x) = f^m(y).$$

This relation generates on X the following partition $X = \bigcup_{i\in I} X_i$, such that $f(X_i) \subset X_i$, for each $i \in I$. Thus, it is sufficient to prove the conclusion for the case $X = X_i$. Let $x_0 \in X_i$ and $x \in X_i$ be a generic element of X_i. Let $m(x) := \min\{m \in \mathbb{N} |\ \exists n \in \mathbb{N}\ : f^m(x_0) = f^n(x)\}$ and $n(x) := \min\{n \in \mathbb{N} |\ f^{m(x)}(x_0) = f^n(x)\}$. Notice that if $m(x) > 0$, then $m(f(x)) = m(x)$ and $n(f(x)) = n(x) - 1$. If there exists $x_1 \in X_i$ such that $x_1 = f^m(x_0)$ for some $m \in \mathbb{N}$, then a such x_1 is unique and we consider $X_{i_1} = \{x_1\}$. If such an element x_1 does not exist, then we consider $X_{i_1} = \emptyset$. Let

$$X_{i_2} := \{x \in X_i \setminus X_{i_1} |\ m(x) + n(x) \text{ is odd } \}$$

and

$$X_{i_3} := \{x \in X_i \setminus X_{i_1} |\ m(x) + n(x) \text{ is even } \}.$$

It is clear that $X_1 := {}_{i\in I} X_{i_1}$, $X_2 := {}_{i\in I} X_{i_2}$ and $X_3 := {}_{i\in I} X_{i_3}$ satisfy the following assertions:

$$X_1 \cup X_2 \cup X_3 = X \text{ and } f(X_i) \cap X_i = \emptyset,\ i \in \{1, 2, 3\}.$$

Eilenberg's Theorem. (see J. Dugundji, A. Granas [1], I.. Rus [29]) *Let X be a set, $R_n \subset X \times X$, $n \in \mathbb{N}$ be a sequence of equivalence relations and $f : X \to X$ be such that:*

(i) $X \times X = R_0 \supset R_1 \supset \cdots \supset R_n \supset \cdots$,

(ii) $\bigcap_{n\in\mathbb{N}} R_n = \Delta(X)$,

(iii) if $(x_n)_{n\in\mathbb{N}}$ is any sequence in X such that $(x_n, x_{n+1}) \in R_n$ for each $n \in \mathbb{N}$, then there exists a unique $x \in X$ such that $(x_n, x) \in R_n$, for each $n \in \mathbb{N}$,

(iv) if $(x, y) \in R_n$ then $(f(x), f(y)) \in R_{n+1}$ for each $n \in \mathbb{N}$.

Then:

(a) $F_f = \{x^*\}$;

(b) $(f^n(x_0), x^*) \in R_n$ for each $x_0 \in X$ and $n \in \mathbb{N}$.

Proof. (a) and (b). Let $x^*, y^* \in F_f$. From (i) and (iv) we have that if $(x^*, y^*) \in R_0$ implies $(x^*, y^*) = (f(x^*), f(y^*)) \in R_1$. By induction we get that $(x^*, y^*) \in R_n$ for each $n \in \mathbb{N}$. From (ii) it follows $x^* = y^*$. Thus, $card F_f \leq 1$.

Let $x_0 \in X$. Then $(x_0, f(x_0)) \in R_0$. From (iv) we have that $(f^n(x_0), f^{n+1}(x_0)) \in R_n$, for all $n \in \mathbb{N}$. From (iii) there exists a unique $x* \in X$ such that $(f^n(x_0), x^*) \in R_n$, for all $n \in \mathbb{N}$. On the other hand, from (iv), we get that $(f^n(x_0), f(x^*)) \in R_n$, for all $n \in \mathbb{N}$. Hence, $f(x^*) = x^*$. □

For other general set-theoretic aspects of fixed point theory, see I.A. Rus B[73] (pp. 9-16), B[85] and B[29].

The basic set-theoretic problems of the fixed point theory are the following: Let X be a nonempty set and $f : X \to X$ be an operator. Which are the sufficient conditions for:

Problem 1.0.1. $F_f \neq \emptyset$?

Problem 1.0.2. $F_f = \{x^*\}$?

Problem 1.0.3. $cardF_f \geq n$, with a given $n \in \mathbb{N}^*$?

Problem 1.0.4. f is a Bessaga operator ? (i.e., $F_f = F_{f^n} = \{x^*\}$, for each $n \in \mathbb{N}^*$);

Problem 1.0.5. $P_f \neq \emptyset$?

Problem 1.0.6. $F_{f^n} \neq \emptyset$ (with a given $n \in \mathbb{N}^*$) implies $F_f \neq \emptyset$?

Problem 1.0.7. $F_f = F_{f^n} \neq \emptyset$, for each $n \in \mathbb{N}^*$?

Problem 1.0.8. $\bigcap_{n\in\mathbb{N}} f^n(X) = \{x^*\}$? (i.e., f is a Janos operator);

Problem 1.0.9. $F_f = \bigcap_{n\in\mathbb{N}} f^n(X)$?

Let X be a nonempty set, $Y \subseteq X$ and $f : Y \to X$. Let $\rho : X \to Y$ be a set retraction.

Problem 1.0.10. Study the above mentioned problems for the operators f and $\rho \circ f$.

Let X be a nonempty set and $T : X \to P(X)$ be a multivalued operator. Consider the fractal operator generated by T, i.e., $\hat{T} : P(X) \to P(X)$ given by $T(Y) := \bigcup_{y\in Y} T(y)$.

In which conditions we have:

Problem 1.0.11. $F_T \neq \emptyset$?

Problem 1.0.12. $(SF)_T \neq \emptyset$?

Problem 1.0.13. $F_T = (SF)_T \neq \emptyset$?

Problem 1.0.14. $P_T = F_T$?
Problem 1.0.15. $(SP)_T = (SF)_T$?
Problem 1.0.16. $(SF)_T \neq \emptyset$ implies $F_T = (SF)_T = \{x^*\}$?
Problem 1.0.17. $T(F_T) = F_T$?
Problem 1.0.18. $F_{\hat{T}} \neq \emptyset$?
Problem 1.0.19. $F_{\hat{T}} = \{Y^*\}$?
Problem 1.0.20. $\hat{T}$ is a Bessaga operator ?
Problem 1.0.21. $\hat{T}$ is a Janos operator ?

Let X be a nonempty set, $Y \subseteq X$ and $T : Y \to P(X)$ be a multivalued operator. Let $\rho : X \to Y$ be a set retraction of X onto Y. By definition, T is retractible with respect to ρ if $F_T = F_{\rho \circ T}$.

In which conditions we have:
Problem 1.0.22. T is retractible with respect to ρ ?
Problem 1.0.23. $F_T \neq \emptyset$ and $(SF)_T \neq \emptyset$?
Problem 1.0.24. $F_T = (SF)_T \neq \emptyset$?
Problem 1.0.25. $(SF)_T \neq \emptyset$ implies $F_T = (SF)_T = \{x^*\}$?

One of the main aim of this book is to present some results for the above mentioned problems in terms of structured sets.

1.1 Total f-variant subsets and fixed points

Let X be a nonempty set and $f : X \to X$ an operator. By definition a subset $Y \subseteq X$ is called total f-variant if $Y \cap f(Y) = \emptyset$.

Theorem 1.1.1. (M. Deaconescu B[3]). *Let $f : X \to X$ be an operator and $Y \subseteq X$ be a maximal total f-variant subset of X. Then:*
(i) $(X \setminus Y) \cap (X \setminus f(Y)) \cap (X \setminus f^{-1}(Y)) \subseteq F_f$
(ii) $X = F_f \cup Y \cup f(Y) \cup f^{-1}(Y)$
(iii) If f is injective, then

$$F_f = (X \setminus Y) \cap (X \setminus f(Y)) \cap (X \setminus f^{-1}(Y)).$$

The following result is a consequence of Theorem 1.1.1.

Theorem 1.1.2. (M. Deaconescu B[3]). *Let X be a Hausdorff topological space, $Y \subseteq X$ a connected and compact subset of X. Let $f : Y \to Y$ be an injective continuous operator such that exists a closed maximal total f-variant subset of X. Then $F_f = \emptyset$.*

Remark 1.1.1. Theorem 1.1.1. is in connection with the following result of A. Abian:

Theorem 1.1.3. (A. Abian R[3]). *An operator $f : X \to X$ has a fixed point if and only if there exists a subset $Y \subset X$ such that for every subset $A \subseteq Y$*

$$A \cap f(A) = \emptyset \text{ implies } Y \setminus (A \cup f(A) \cup f^{-1}(A)) \neq \emptyset.$$

1.2 Invariant subsets

Let X be a nonempty set, $f : X \to X$ be an operator and Y a nonempty subset of X. Then, by definition, Y is called:

(i) invariant for f if $f(Y) \subset Y$;

(ii) fixed (forward invariant) set for f if $f(Y) = Y$;

(iii) fixed for f^{-1} (backward invariant for f) if $f^{-1}(Y) = Y$;

(iv) completely invariant for f if $f(Y) = f^{-1}(Y) = Y$, i.e. Y is fixed for f and f^{-1}.

Let X be a nonempty set. An operator $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator if:

(i) $A \in \mathcal{P}(X)$ implies $A \subseteq \eta(A)$

(ii) $A \subset B$ implies $\eta(A) \subseteq \eta(B)$

(iii) $\eta \circ \eta = \eta$.

Theorem 1.2.1. (I.A. Rus B[35]). *Let X be a nonempty set, $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ a closure operator, $Y \in F_\eta$ and $f : Y \to Y$. Let A be a nonempty subset of Y. Then there exists $A_0 \subseteq Y$ such that:*

(i) $A_0 \supset A$

(ii) $A_0 \in F_\eta$

(iii) $A_0 \in I(f)$

(iv) $\eta(f(A_0) \cup A) = A_0$.

Theorem 1.2.2. (I.A. Rus, B[17]). *Let X be a nonempty set, $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ a closure operator, $Y \in F_\eta$ and $T : Y \to P(Y)$ a multivalued operator. Let A be a nonempty subset of Y. Then there exists $A_0 \subset Y$ such that:*

(i) $A_0 \supset A$

(ii) $A_0 \in F_\eta$

(iii) $A_0 \in I(T)$

(iv) $\eta(T(A_0) \cup A) = A_0$.

The above theorems generalize some known results, see M. Martelli R[1], R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina and B.N. Sadovskii R[1] J. Appell R[1] J. Banas and K. Goebel R[1], W.A. Kirk and B. Sims R[1].

For some applications in fixed point theory see Chapters 18 and 19.

1.3 R-contractions

Let X be a nonempty set and $R := (R_n)_{n\in\mathbb{N}}$, $R_n \subset X \times X$, a sequence of symmetric binary relations in X. Throughout this section we suppose that:

(c_1) $X \times X = R_0 \supset R_1 \supset \cdots \supset R_n \supset \ldots$

(c_2) $\bigcap\limits_{n\in\mathbb{N}} R_n = \Delta(X)$

(c_3) if $(x_n)_{n\in\mathbb{N}}$ is any sequence in X such that $(x_n, x_{n+p}) \in R_n$ for all n and $p \in \mathbb{N}$, then there is a unique $x^* \in X$ such that $(x_n, x^*) \in R_n$ for all $n \in \mathbb{N}$.

Definition 1.3.1. (R.F. Brown R[2]) Let X be a nonempty set and Y a nonempty subset of X. An operator $\rho : X \to Y$ is called a retraction of X onto Y if $\rho|_Y = 1_Y$.

Definition 1.3.2. (R.F. Brown R[2]) Let X be a nonempty set and Y a nonempty subset of X. An operator $f : Y \to X$ is retractible onto Y if there is a retraction $\rho : X \to Y$ such that $F_f = F_{\rho\circ f}$.

Definition 1.3.3. (I.A. Rus B[28]) Let X be a nonempty set. An operator $f : X \to X$ is called R-contraction if for all $n \in \mathbb{N}$, $(x, y) \in R_n$ implies $(f(x), f(y)) \in R_{n+1}$.

Definition 1.3.4. (I.A. Rus B[28]) Let X be a nonempty set. An operator $f : X \to X$ is R-nonexpansive if for all $n \in \mathbb{N}$, $(x,y) \in R_n$ implies $(f(x), f(y)) \in R_n$.

Definition 1.3.5. (I.A. Rus B[28]) Let X be a nonempty set. An operator $f : X \to X$ is R-continuous if $(x_n, x^*) \in R_n$, for all $n \in \mathbb{N}$ implies $(f(x_n), f(x^*)) \in R_n$.

Theorem 1.3.1. (I.A. Rus B[28]) *Let X be a nonempty set. If $f : X \to X$ is a R-contraction, then:*

(i) $F_f = \{x^*\}$;

(ii) $(f^n(x_0), x^*) \in R_n$, for all $x_0 \in X$, $n \in \mathbb{N}$.

Remark 1.3.1. Theorem 1.3.1. is a generalization of a result by S. Eilenberg (see J. Dugundji and A. Granas R[1] or A. Granas and J. Dugundji R[1]).

Theorem 1.3.2. (I.A. Rus B[28]) *Let X be a nonempty set, Y a nonempty subset of X, $\rho : X \to Y$ a retraction and $f : Y \to X$. We suppose that:*

(i) ρ is R-nonexpansive

(ii) f is R-contraction

(iii) f is retractible onto Y by means of ρ.

Then, $F_f = \{x^*\}$.

Theorem 1.3.3. (I.A. Rus B[28]). *Let X and Y be nonempty sets and $f, g : Y \to X$ two operators. We suppose that:*

(i) g is surjective

(ii) $(y_1, y_2) \in Y \times Y$ and $(g(y_1), g(y_2)) \in R_n$ imply $(f(y_1), f(y_2)) \in R_{n+1}$, for all $n \in \mathbb{N}$.

Then $C(f, g) \neq \emptyset$.

Remark 1.3.2. If f and g are R-continuous, then from Theorem 1.3.3. we obtain a result given by Holodovski (see I.A. Rus B[28] and B[73]).

Remark 1.3.3. The above results have some applications to nonlinear analysis (see I.A. Rus B[28] and A. Bege B[1]). For example:

Let $(X, +, R, \leq)$ be an ordered vector space. If X is a lattice and $x \in X$, then $|x| := x \vee (-x)$.

By definition a sequence (x_n) of elements in X is (0)-convergent to x^* (see R. Cristescu, R[1]) if there exist two sequences $(a_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$ in X such that:

(i) $(a_n)_{n\in\mathbb{N}}$ is increasing and $x^* = \bigvee\limits_{n\in N} a_n$

(ii) $(b_n)_{n\in\mathbb{N}}$ is decreasing and $x^* = \bigwedge\limits_{n\in N} b_n$

(iii) $a_n \leq x_n \leq b_n$, for all $n \in \mathbb{N}$.

Let $Y \subset X$ be a bounded and (0)-closed subset of X. Let $a \in (0,1)$, $M_0 \in X$ such that

$$|y_1 - y_2| \leq M_0, \text{ for all } y_1, y_2 \in Y.$$

Let

$$R_n := \left\{(x,y)|\ |x-y| \leq \frac{a^n}{1-a}M_0,\ x,y \in Y\right\}.$$

From Theorem 1.3.1. we have a result by F. Voicu, as follows:

Theorem 1.3.4. (F. Voicu B[3]). *Let X be a σ-complete vector lattice, Y a (0)-closed subset of X and $f : Y \to Y$. We suppose that:*

(i) there exists $M_0 \in X$ such that $|y_1 - y_2| \leq M_0$, for all $y_1, y_2 \in Y$

(ii) there exists $a \in (0,1)$ such that $|f(x) - f(y)| \leq a|x-y|$, for all $x, y \in Y$.

Then:

(a) $F_f = \{x^*\}$

(b) $f^n(x_0) \xrightarrow{(0)} x^*$ as $n \to \infty$, for all $x_0 \in Y$

(c) $|f^n(x_0) - x^*| \leq \dfrac{a^n}{1-a}M_0$.

Remark 1.3.3. For other results for R-contractions see A. Bege B[1].

1.4 Schröder's pairs

Let X be a nonempty set, $f : X \to X$ be an operator and $\psi : X \to \mathbb{R}_+$ be a functional. By definition, the pair (f, ψ) is a Schröder's pair if there exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that:

(i) φ is increasing;

(ii) $\varphi^n(t)$ converges to 0 as $n \to +\infty$, for all $t \in \mathbb{R}_+$;

(iii) the pair (f, ψ) is a solution of the Schröder's inequation:

$$\psi(f(x)) \leq \varphi(\psi(x)), \text{ for each } x \in X.$$

Let $\alpha \in]0,1[$. If we define $\varphi(t) := \alpha t$, $t \in \mathbb{R}_+$ and we suppose $\psi(f(x)) \leq \alpha\psi(x)$, for each $x \in X$, then (f, ψ) is a Schröder's pair.

For other examples and applications of Schröder's pairs, see Chapter 4 and Chapter 6.

We also have:

Lemma 1.4.1. *Let* (f, ψ) *be a Schröder's pair. Then* $F_f \subset Z_\psi$.

Proof. Let $x \in F_f$. Then
$0 \leq \psi(x) = \psi(f^n(x)) \leq \varphi^n(\psi(x)) \to 0$ as $n \to +\infty$. Thus $\psi(x) = 0$. □

Lemma 1.4.2. *If* $x \in Z_\psi$, *then* $f^n(x) \in Z_\psi$, *for all* $n \in \mathbb{N}^*$.

Proof. Since

$$\psi(f^n(x)) \leq \varphi^n(\psi(x))$$

we immediately get (using (i) and (ii)) that $\varphi(0) = 0$. □

From Lemma 1.4.2. we obtain:

Lemma 1.4.3. *If* (f, ψ) *is a Schröder's pair and* $cardZ_\psi = 1$, *then* $F_f = F_{f^n} = \{x^*\}$.

Remark 1.4.1. For the Schröder's pairs see I.A. Rus, A. Petruşel, M.A. Şerban B[1], I.A. Rus B[104] and the references therein. See also J. Jachymski R[1] and R[6], E. Akin R[1], Y.B. Rudyak and F. Schlenk R[1].

Chapter 2

Order-theoretic aspects of the fixed point theory

Precursors: E. Zermelo (1908), B. Knaster (1928).
Guidelines: L. Kantorovich (1939), N. Bourbaki (1949), A. Tarski (1955), A.C. Davis (1955), S. Abian and A.B. Brown (1961), A. Pelczar (1961), H. Brezis and F.E. Browder (1976), B. Fuchssteiner (1977), B. Baclawski and A. Björner (1981), B.S.W. Schröder (1996), J. Jachymski (2001).
General references: G. Birkhoff R[1], A. Tarski R[1], A.C. Davis R[1], S. Abian and A.B. Brown R[1], S.C. Kleene R[1], H. Amann R[2], R[3], J. Lambek R[1], R. Rival R[1], K. Baclawski and A. Björner R[1], D. Duffus and J. Rival R[1], M.R. Tasković R[2], S. Rudeanu B[1], I.A. Rus B[33] and B[90], M. Turinici B[1], G. Grätzer R[1], J. Dugundji and A. Granas R[1], W.A. Kirk and B. Sims (Eds.) R[1], V.I. Istrăţescu B[3] and B[5], E. Zeidler R[1], O. Stănăşilă R[1], A. Bege R[1], J. Jachymski R[9], R. Lemmert and P. Volkmann R[1], W. Oetlli and M. Théra R[1].

2.0 Basic notions and results

Let $(X, \leq)$ be a partially ordered set and $Y \subset X$. An element $x \in X$ is an upper bound for Y in X if $y \leq x$, for al $y \in Y$. If the set Y has an upper bound, then we say that Y is bounded above. An element $y \in Y$ is said to be a maximal element of Y if $(x \in Y, x \geq y) \Rightarrow (x = y)$. If $x \leq y$ for all $x \in Y$, then, by definition, y is the maximum element of Y. Dually, we can define the notions: lower bound, bounded below, minimal element, minimum element.

The minimum element of the set of all upper bounds of Y (if such an element exists) is called the supremum of Y and it is denoted by $supY$. Dually, one can define the infimum of Y, denoted by $infY$.

A partially ordered set X in which there exist $x \vee y := \sup\{x, y\}$ and $x \wedge y := \inf\{x, y\}$, for all $x, y \in X$ is called a lattice. If every subset of a partially ordered set X has a supremum and an infimum, then X is called a complete lattice. If X is a complete lattice, then we denote $0 := \inf X$ and $1 := \sup X$.

If $(X, \leq)$ is a partially ordered set, then we denote by $Max(X, \leq)$ - the set of all maximal elements of X and by $Min(X, \leq)$ - the set of all minimal elements of X.

A partially ordered set $(X, \leq)$ is said to be a chain (or totally ordered) if for every $x, y \in X$, either $x \leq y$ or $y \leq x$.

Let $(X, \leq)$ be a partially ordered set and $f : X \to X$ be an operator. By definition, the operator f is called:

a) increasing (or isotone) if $x_1, x_2 \in X$ $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$;

b) progressive if $x \leq f(x)$, for all $x \in X$.

Dually, we can define the concepts of decreasing operator and regressive operators.

We also denote:

$(UF)_f := \{x \in X |\ x \geq f(x)\}$.

$(LF)_f := \{x \in X |\ x \leq f(x)\}$.

Some of the main problems of fixed point theory in partially ordered sets are the following:

Problem 2.0.1. For which partially ordered sets $(X, \leq)$ we have that:

(a) $Max(X, \leq) \neq \emptyset$?

(b) $Min(X, \leq) \neq \emptyset$?

Problem 2.0.2. Let X be a nonempty set. Construct an ordered relation $\leq$ on X such that:

(a) $Max(X, \leq) \neq \emptyset$;

(b) $Min(X, \leq) \neq \emptyset$.

A basic result for Problem 2.0.1. is the following theorem (see Granas and Dugundji R[1], Zeidler R[1], etc.):

Zorn's Theorem. *Let* $(X, \leq)$ *be a partially ordered set in which every chain has an upper (lower) bound. Then* $Max(X, \leq) \neq \emptyset$ *(respectively* $Min(X, \leq) \neq \emptyset$*).*

From Zorn's theorem it follows:

Zermelo's Theorem. *Let* $(X, \leq)$ *be a partially ordered set and* $f : X \to X$ *be an operator. We suppose;*

(a) every chain in X *has an upper bound;*

(b) f *is a progressive operator.*

Then $F_f \neq \emptyset$.

Proof. Notice that the condition (b) implies that $Max(X, \leq) \subset F_f$. The proof follows now from Zorn's Theorem. □

The above proof gives rise to:

Problem 2.0.3. Let $(X, \leq)$ be a partially ordered set and $f : X \to X$ be a progressive operator. In which conditions we have that:

$$F_f \setminus Max(X, \leq) \neq \emptyset \ ?$$

Example 2.0.1. *Let* $([0, 3], \leq)$ *(where* $\leq$ *is the usual order relation on* $\mathbb{R}$*). Let* $f : [0, 3] \to [0, 3]$ *be defined by:*

$$f(x) := \begin{cases} x, & \text{if } x \in [0, 1] \\ 2, & \text{if } x \in]1, 2] \\ 3, & \text{if } x \in]2, 3] \end{cases}$$

Then, we have:

(i) f *is progressive;*

(ii) $F_f = [0, 1] \cup \{2\} \cup \{3\}$*;*

(iii) $Max([0, 3], \leq) = \{3\}$.

For the Problem 2.0.2. we have the following result (see A. Brondsted R[1], W.A. Kirk R[1], I. Ekeland R[1], J. Caristi R[1]).

Let (X, d) be a metric space and $\varphi : X \to \mathbb{R}_+$. We define the following partially ordered relation on X:

$$x \leq_\varphi y \text{ if and only if } d(x, y) \leq \varphi(x) - \varphi(y).$$

If the metric space (X, d) is complete and the functional φ is lower semicontinuous, then $Max(X, \leq) \neq \emptyset$. On the other hand, it is easy to see that an operator $f : (X, \leq) \to (X, \leq)$ is progressive if and only if

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \text{ for all } x \in X.$$

From the above considerations, we get:

Caristi-Kirk's Theorem. *Let (X, d) be a complete metric space and $\varphi : X \to \mathbb{R}_+$ be a lower semicontinuous functional. Let $f : X \to X$ be an operator such that:*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \textit{ for all } x \in X.$$

Then, $F_f \neq \emptyset$.

The following result was given by J. Jachymski in R[6].

Jachymski's Theorem. *Let X be a nonempty set and $f : X \to X$ be an operator. The following statements are equivalent:*

(i) $F_f = P_f \neq \emptyset$;

(ii) there exists a partial ordering $\leq$ on X such that every chain in $(X, \leq)$ has a supremum and f is progressive with respect to$\leq$;

(iii) there exists a complete metric d on X and a lower semicontinuous functional $\varphi : X \to \mathbb{R}_+$ such that:

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \textit{ for all } x \in X.$$

By definition, a partially ordered set $(X, \leq)$ has the fixed point property (briefly f.p.p.) if and only if:

$$f \text{ is increasing } \Rightarrow F_f \neq \emptyset.$$

One of the most important problem of fixed point theory in partially ordered sets is:

Problem 2.0.4. Which partially ordered sets have f.p.p. ?

Some partial answers are:

Lemma 2.0.1. *Let* $(X, \leq)$ *be a partially ordered set and* $Y \subset X$*. We suppose:*

(a) $(X, \leq)$ *has the f.p.p.;*

(b) there exists an ordered set retraction $\varphi : X \to Y$ *(i.e.,* φ *is a set retraction and it is increasing).*

Then, $(Y, \leq)$ *has the f.p.p.*

Lemma 2.0.2. *Let* $(X, \leq)$ *and* $(Y, \leq)$ *be two partially ordered sets. We suppose:*

(a) $(X, \leq)$ *has the f.p.p.;*

(b) there exists an ordered set isomorphism $\varphi : X \to Y$ *(i.e.,* φ *is an increasing bijection).*

Then, $(Y, \leq)$ *has the f.p.p.*

The main result on this topic is:

Tarski's Theorem. *Let* $(X, \leq)$ *be a complete lattice and* $f : X \to X$ *be an increasing operator. Then:*

(a) $F_f \neq \emptyset$*;*

(b) $(F_f, \leq)$ *is a complete lattice.*

Proof. Since f is increasing we have that $(LF)_f, (UF)_f \in I(f)$.

Let $x^* := \sup(LF)_f$. then we have $x \leq x^*$, for all $x \in (LF)_f$ and $x \leq f(x) \leq f(x^*)$, for all $x \in (LF)_f$.

Hence,$x^* \leq f(x^*)$, i.e, $f(x^*) \in (LF)_f$. Since $(LF)_f$ is invariant with respect to f, we get that $x^* \in F_f$ and $x^* = \sup F_f$. By a similar approach, we obtain that $x_* := \inf(UF)_f \in F_f$ and $x_* = \inf F_f$. □

From the above proof, we have (see Abian-Brown (1961), Pelczar (1961), Amann (1977), $\cdots$) the following result:

Abian-Brown-Pelczar-Amann's Theorem. *Let* $(X, \leq)$ *be a partially ordered set and* $f : X \to X$ *be an operator. We suppose that:*

(i) every chain in X *has a supremum;*

(ii) f *is increasing;*

(iii) $(LF)_f \neq \emptyset$.

Then, $F_f \neq \emptyset$.

A converse of Tarski's theorem is:

Davis's Theorem. *Let* $(X, \leq)$ *be a lattice. If for all increasing operators* $f : X \to X$ *we have that* $F_f \neq \emptyset$*, then the lattice* $(X, \leq)$ *is complete.*

Moreover, a special case of Tarski's theorem is the following result:

Corollary 2.0.1. *Every finite lattice has the fixed point property.*

Let $(X, \leq)$ be a finite lattice. By definition, an element $y \in X$ is said to be a complement of the element $x \in X$ if and only if $x \wedge y = 0$ and $x \vee y = 1$. A lattice $(X, \leq)$ is called complemented if every $x \in X$ has at least a complement. For noncomplemented lattices we have:

Baclawski-Björner's Theorem. *Let* $(X, \leq)$ *be a finite lattice. If* X *is noncomplemented, then* $X \setminus \{0, 1\}$ *has the f.p.p.*

For the case of multivalued operators, we have the follwing result given by Brézis and Browder in 1976.

Brézis-Browder's Theorem. *Let* $(X, \leq)$ *be a partially ordered set,* $\varphi : X \to \mathbb{R}_+$ *be a functional and* $T : X \multimap X$ *be a multivalued operator given by* $T(x) := \{y \in X | \ x \leq y\}$*. We suppose that:*

(i) $x \leq y, \ x \neq y$ *implies* $\varphi(x) < \varphi(y)$*;*

(ii) for any increasing sequences $(x_n)_{n \in \mathbb{N}}$ *in* X*, such that* $(\varphi(x_n))_{n \in \mathbb{N}}$ *is bounded, there exists* $y \in X$ *with the property* $x_n \leq y$*, for all* $n \in \mathbb{N}$*.*

(iii) for each $x \in X$ *the set* $\varphi(T(x))$ *is bounded from above.*

Then, $(SF)_T \cap T(x) \neq \emptyset$*, for each* $x \in X$*.*

For other basic results of fixed point theory for multivalued operators on ordered sets see R.E. Smithson R[3]. See also A. Száz R[1].

Remark 2.0.1 Let X be a Banach space, $z \in X$, $r > 0$ and $x \in X \setminus \overline{B}(z, r)$. By definition, the set $co(\{x\} \cup \overline{B}(z, r))$ is called a drop and it is denoted by $DP(x, \overline{B}(z, r))$.

As an application of the Caristi-Kirk Theorem we have:

Danes' Theorem. *Let* X *be a Banach space,* Y *be a closed subset of* X*,* $z \in X \setminus Y$ *and* $0 < r < DP(z, Y)$*. Let* $f : Y \to Y$ *be an operator such that:*

$$f(y) \in Y \cap DP(y, \overline{B}(z, r)), \ \text{for each } y \in Y.$$

Then, for each $y \in Y$ *we have*

$$F_f \cap DP(y, \overline{B}(z,r)) \neq \emptyset.$$

For other details on this result, as well as, for the drop theory see J. Danes R[2] and R[3], J.-P. Penot R[2], I. Monterde and V. Montesinos R[1].

2.1 Other fixed point theorems in ordered sets

We recall first some useful concepts.

Definition 2.1.1. Let $(X, \leq)$ be a partially ordered set. Then, X is well ordered if each nonempty subset has a minimal element.

Definition 2.1.2. (M. Turinici B[1]) Let $(X, \leq)$ be a partially ordered set and $f : X \to X$. The operator f is almost-increasing if:

$$x, y \in X,\ x \leq f(x) \leq \cdots \leq f^n(x) \leq y \ \Rightarrow\ x \in f(y).$$

Theorem 2.1.1. (M. Turinici B[1]). *Let* $(X, \leq)$ *be a partially ordered set and* $f : X \to X$*. We suppose that:*

(i) f *is progressive*

(ii) each f*-invariant well ordered subset of* X *is bounded from above.*

Then $F_f \neq \emptyset$ *and is cofinal in* X*.*

Theorem 2.1.2. (M. Turinici B[1]). *Let* $(X, \leq)$ *be a partially ordered set and* $f : X \to X$ *an operator. We suppose that:*

(i) $(LF)_f \neq \emptyset$*;*

(ii) f *is almost-increasing;*

(iii) $(LF)_f$ *is* f*-invariant;*

(iv) $\{x, f(x)\}$ *has an infimum, for all* $x \in X$*;*

(v) each f*-invariant well ordered subset* Y *of* X *has a minimal upper bound in* $ubd(Y)$*.*

Then:

(a) $F_f \neq \emptyset$ *and is cofinal in* $(LF)_f$*;*

(b) $\max(F_f)$ *is nonempty and cofinal in* $(LF)_f$*.*

Remark 2.1.1. These results are in connection with Zermelo-Bourbaki fixed point principle and with Manka's fixed point theorem R[1] (see M. Turinici B[1]).

Remark 2.1.2. For some applications of the fixed point theory in ordered sets to the metrical fixed point theory, see A. Baranga B[2], W.A. Kirk and B. Sims (Eds.) R[1] (J. Jachymski, pp. 613-641).

2.2 Fixed point theorems for Boolean type operators

The following result is given by S. Rudeanu.

Theorem 2.2.1. (S. Rudeanu B[1]). *Let B be a Boolean algebra and $f : B^n \to B^n$ be an increasing Boolean operator. Then $F_f \neq \emptyset$.*

Remark 2.2.1. For $n = 1$ the previous theorem is a result given by G. Scognamiglio. (see S. Rudeanu B[1] for more details).

Remark 2.2.2. For the general theory of Boolean equations see S. Rudeanu R[1].

2.3 Fixed point theorems for non self-operators

Let X be a nonempty set. For $A, B \in P(X)$ we denote

$$\mathbb{M}(A, B) := \{f : A \to B |\ f \text{ is an operator}\},$$

$$\mathbb{M}(A) := \mathbb{M}(A, A).$$

Definition 2.3.1. (I.A. Rus B[33]). A triple $(X, S(X), M)$ is a large fixed point structure if:

(i) $S(X) \subset P(X),\ S(X) \neq \emptyset$;

(ii) M is an operator which attaches to each pair $(A, B) \in P(X) \times P(X)$, a nonempty subset of $\mathbb{M}(A, B)$;

(iii) every $Y \in S(X)$ has the fixed point property with respect to $M(Y)$, i.e. $Y \in S(X),\ f \in M(Y)\ \Rightarrow\ F_f \neq \emptyset$.

For some examples of large fixed point structures see Chapter 18, Section 18.1.

Lemma 2.3.1. (I.A. Rus B[33]). *Let $(X, S(X), M)$ be a large fixed point structure. Let $Y \in S(X)$ and $\rho : X \to Y$ a retraction. Let $f : Y \to X$ be such that:*

(i) $\rho \circ f \in M(Y)$;

(ii) f is retractible onto Y by ρ.

Then, $F_f \neq \emptyset$.

From Lemma 2.3.1. we have:

Theorem 2.3.1. (I.A. Rus B[33]). *Let $(X, \leq)$ be a partially ordered set with the least element 0_X. Let $Y \in P(X)$ and $f : Y \to X$ be such that:*

(i) $0_X \in Y$;

(ii) $(Y, \leq)$ is a complete lattice;

(iii) f is increasing operator;

(iv) $f(x) \in X \setminus Y$ implies $\sup_Y([0_X, f(x)] \cap Y) \neq x$.

Then $F_f \neq \emptyset$.

Theorem 2.3.2. (I.A. Rus B[33]). *Let $(X, \leq)$ be a partially ordered set, $(Y, \leq)$ a complete maximal chain of X and α a well ordering of Y. Let $f : Y \to X$ be such that:*

(i) f is an increasing operator;

(ii) If $f(x) \in X \setminus Y$, then x is not the least element of the set $\{y \in Y \mid f(x)$ is not comparable with $y\}$ with respect to α.

Then, $F_f \neq \emptyset$.

Remark 2.3.1. For some example of ordered set retractions see D. Duffus and J. Rival R[1], I.A. Rus B[33].

Remark 2.3.2. For other aspects of the fixed point theory in ordered set see R. Cristescu R[1], A. Bege B[2] and B[7], M. Deaconescu B[2], I.A. Rus B[90], F. Voicu B[5] and B[7], D. Kurepa R[1].

Remark 2.3.3. For the theory of ordered sets, see G. Birkhoff R[1], G. Grätzer R[1], N. Bourbaki R[2], M.A. Khamsi and W.A. Kirk R[1].

Remark 2.3.4. For the Ekeland variational principle see also D.G. De Figueiredo R[1].

Chapter 3

Generalized contractions on metric spaces

Precursors: E. Picard (1890).
Guidelines: S. Banach (1922), R. Caccioppoli (1930), V.V. Niemytzki (1936), E. Rakotch (1962), M. Edelstein (1962), R. Kannan (1968), M.G. Maia (1968), A. Meir and E. Keeler (1969), M.A. Krasnoselskii (1972), J. Caristi (1976), B.E. Rhoades (1977), I.A. Rus (1979).
General references: M. Angrisani and M. Clavelli R[1], L.B. Ćirić R[2], S. Czerwik R[1], M. Edelstein R[1], K. Goebel and W.A. Kirk R[1], O. Hadžić R[3], W.A. Kirk and B. Sims (Eds.) R[1], M.A. Krasnoselskii and P. Zabrejko R[1], A.A. Ivanov R[1], V.I. Opoitsev R[1], P.L. Papini R[1], S. Reich R[1] and R[2], B.E. Rhoades R[1], M.R. Tasković R[1], V. Berinde B[7], V.I. Istrăţescu B[3], B[5] and B[1], A.S. Mureşan B[4], I.A. Rus B[26], B[4], B[70], B[73] and B[81], M. Turinici B[22], T. Zamfirescu B[9], B[11].

3.0 Preliminaries

3.0.1 Topological spaces

Let X be a nonempty set. By definition, a topology on X is a family $\tau \subset \mathcal{P}(X)$ of subsets of X, with the following properties:

(i) X and $\emptyset$ are elements of τ;

(ii) $O_1, O_2, \cdots, O_n \in \tau$ $(n \in \mathbb{N})$ imply $\bigcap_{k=1}^{n} O_k \in \tau$;

(iii) $O_i \in \tau$ $(i \in I)$ imply $\cup_{i \in I} O_i \in \tau$.

The pair (X, τ) is called a topological space.

An element of τ is called an open set in X. A subset Y of X is called closed if $X \setminus Y$ is open. A base for a topology τ on X is a subset τ_1 of τ, such that each open subset of X is a union of some elements of τ_1. A subbase for the topology τ on X is a subset τ_2 of τ, such that, the collection of all finite intersection of elements in τ_2 is a base for τ.

Let (X, τ) be a topological space and Y a subset of X. By definition, the closure $\overline{Y}$ of Y is the smallest closed subset of X that contains Y. The interior $int(Y)$ of Y is the largest open subset of X that is contained in Y. The boundary ∂Y of Y is defined as $\partial Y := \overline{Y} \setminus int(Y)$.

Let x be an element of (X, τ). By definition, a subset Y of X is a neighborhood of x if there exists an open subset Z of X such that $x \in Z \subset Y$.

For other aspects of the theory of topological spaces (nets and sequences (convergent, cluster point, etc.) subsets (compact, connected, dense, accumulation point, adherent point, etc.) and operators on topological spaces (continuous, open, closed, isomorphism, etc.) see N. Bourbaki R[3], N.M. Bliznyakov, Ya.A. Izrailevich and T.N. Fomenko R[1], J. Dugundji R[2], J.L. Kelly R[1], K. Kuratowski R[1], R. Engelking R[1], A. Brown and C. Pearcy R[1], L. Schwartz R[1], G. Beer R[1], etc.

3.0.2 Metric spaces

By a metric space we understand a pair (X, d), where X is a nonempty set and $d : X \times X \to \mathbb{R}_+$ is a functional such that:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(iii) $d(x, y) \le d(x, z) + d(z, y)$, for all $x, y, z \in X$.

The topology which has as open sets the elements of:

$$\tau_d := \{Y \subset X \ \text{ for each } x \in Y \text{ there is } r > 0 \text{ such that } B(x; r) \subset Y\},$$

(where $B(x; r) := \{y \in X | \ d(x, y) < r\}$) is called the topology on X generated by the metric d.

Two metrics d_1 and d_2 on X are said to be topological equivalent if $\tau_{d_1} = \tau_{d_2}$.

Two metrics d_1 and d_2 on X are said to be metric equivalent if there exist $c_1, c_2 > 0$ such that $c_1 d_1(x, y) \le d_2(x, y) \le c_2 d_1(x, y)$, for all $x, y \in X$.

The following problems are, in close connection, to the theory of operatorial equations (i.e., fixed point theory, coincidence point theory, surjectivity theory, etc.):

Problem 3.0.1. *Let d be a metric on a set X and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a function. Which are the assumptions on φ such that the functional $\varphi \circ d : X \times X \to \mathbb{R}_+$ is:*
(a) a metric on X;
(b) an equivalent metric with d on X ?

Problem 3.0.2. *Given a set X, construct a metric d on X with a given property.*

For the above problems see M.A. Şerban B[9] and the references therein (T.K. Sreenivasan (1947), P. Corraza (1999)), C. Bessaga R[1], L. Janos R[1], P.R. Meyers R[1], V. I. Opoitsev R[1], I.A. Rus B[16], J. Jachymski R[1], etc.

Let (X, d) be a metric space, $x_n \in X$ for $n \in \mathbb{N}$ and $x \in X$. Then, by definition:

(a) the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x if $d(x_n, x)$ converges to 0, as $n \to +\infty$;

(b) the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy if $d(x_n, x_m)$ converges to 0, as $n, m \to +\infty$;

Always, a convergent sequence is Cauchy, but not reversely.

A metric space is said to be complete if each Cauchy sequence is convergent.

A metric space is said to be compact if each sequence in X has a convergent subsequence. A subset Y of X is compact if each sequence in Y has a convergent subsequence in Y.

A subset Y of X is called:

(a) bounded if there exist $x \in X$ and $r > 0$ such that $Y \subset B(x; r)$;

(b) totally bounded if for every $\epsilon > 0$ there exists a finite ϵ-net for Y;

(c) precompact if the closure $\bar{Y}$ is compact.

We have (see A. Brown and C. Pearcy R[1]):

Theorem 3.0.1. *Let (X, d) be a metric space and $Y \subset X$. The following statements are equivalent:*

(i) Y is totally bounded;

(ii) for every $\epsilon > 0$ there exists a finite covering of Y consisting of sets of diameter less than ϵ;

(iii) for every $\epsilon > 0$ there exists a finite partition of Y into sets of diameter less than ϵ;

(iv) for every $\epsilon > 0$ there exists a finite ϵ-net in Y.

Theorem 3.0.2. (Cantor) *Let (X, d) be a complete metric space and Y_n, $n \in \mathbb{N}$ be nonempty closed subsets of X such that $Y_{n+1} \subset Y_n$, $n \in \mathbb{N}$ and $\delta(Y_n) \to 0$ as $n \to \infty$.*

Then, $\bigcap\limits_{n\in\mathbb{N}} Y_n = \{x^*\}$.

Theorem 3.0.3. (Hausdorff) *A metric space (X, d) is compact if and only if it is complete and totally bounded.*

Let (X, d) be a metric space. A functional defined by

$$\alpha_K : P_b(X) \to \mathbb{R}_+,\ \alpha_K(Y) := \inf\left\{\varepsilon > 0 |\, Y = \bigcup_{i=1}^{n} Y_i,\ \delta(Y_i) \leq \varepsilon,\ n \in \mathbb{N}\right\},$$

is called the Kuratowski measure of noncompactness.

Some basic properties of the functional α_K are given by:

Theorem 3.0.4. *Let (X, d) be a metric space and α_K be the Kuratowski measure of noncompactness of X. Then:*

(i) $0 \leq \alpha_K(Y) \leq \delta(Y)$, for all $Y \in P_b(X)$;

(ii) $Y_1, Y_2 \in P_b(X)$, $Y_1 \subset Y_2 \Rightarrow \alpha_K(Y_1) \leq \alpha_K(Y_2)$;

(iii) $\alpha_K(Y_1 \cup Y_2) = \max(\alpha_K(Y_1), \alpha_K(Y_2))$, $Y_1, Y_2 \in P_b(X)$;

(iv) $\alpha_K(V_r(Y)) \leq \alpha_K(Y) + 2r$, for all $Y \in P_b(X)$, for all $r > 0$ (where $V_r(Y) := \{x \in X |\ D(x, Y) < r\}$);

(v) $\alpha_K(\overline{Y}) = \alpha_K(Y)$, for all $Y \in P_b(X)$;

(vi) if $Y_n \in P_{b,cl}(X)$, $Y_{n+1} \subset Y_n$, $n \in \mathbb{N}$, are such that $\alpha_K(Y_n) \to 0$ as

$n \to \infty$, then

$$Y_\infty := \bigcap_{n\in\mathbb{N}} Y_n \neq \emptyset \text{ and } \alpha_K(Y_\infty) = 0,$$

i.e., Y_∞ is a compact set.

For other considerations on measures of noncompactness see I.A. Rus B[95], J.M. Ayerbe Toledano, T. Dominguez Benavides and G. López Acedo R[1], R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina and B.N. Sadovskii R[1], J. Appell R[1], J. Banas and K. Goebel R[1], etc.

3.0.3 Comparison functions

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. Let us consider, with respect to φ, the following assumptions:

(i_φ) φ is increasing.

(ii_φ) $\varphi(t) < t$, for all $t > 0$.

(iii_φ) $\varphi(0) = 0$.

(iv_φ) $\varphi^n(t) \to 0$ as $n \to \infty$, for all $t \in \mathbb{R}_+$.

(v_φ) $t - \varphi(t) \to \infty$ as $t \to \infty$.

(vi_φ) $\sum_{n\in\mathbb{N}} \varphi^n(t) < +\infty$.

By definition, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison function if φ satisfies the conditions (i_φ) and (iv_φ).

A comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be:

(a) a strict comparison function if it satisfies (v_φ).

(b) a strong comparison function if it satisfies (vi_φ).

It is easy to check that:

(i_φ) and (ii_φ) imply (iii_φ);

(i_φ) and (iv_φ) imply (ii_φ);

(i_φ), (iv_φ) and (vi_φ) imply (ii_φ) and (iii_φ), as well as, the fact that the functions $t \mapsto \sum_{n\in\mathbb{N}} \varphi^n(t) < +\infty$ and φ are continuous in 0.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a strict comparison function. We denote $\varphi_\eta := \sup\{t \in \mathbb{R}_+ |\ t - \varphi(t) \leq \eta\}$.

Example 3.0.1.

(1) Let $\lambda \in [0,1[$*. Then* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$*,* $\varphi(t) := \lambda t$*, is a strict and strong comparison function. Notice that in this case* $\varphi_\eta = \frac{\eta}{1-\lambda}$*;*

(2) $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$*,* $\varphi(t) := \dfrac{t}{1+t}$ *is a strict comparison function, but it isn't a strong comparison function. Notice that in this case* $\varphi_\eta = \frac{1}{2}(\eta + \sqrt{\eta^2+4})$*.*

(3) The function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$*,* $\varphi(t) := \dfrac{1}{2}t$ *for* $t \in [0,1]$ *and* $\varphi(t) := t - \dfrac{1}{2}$ *for* $t > 1$*, is a comparison function.*

Notice that if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, then each iterate φ^k, $k \in \mathbb{N}^*$ is a comparison function.

For more considerations on comparison functions see I.A. Rus B[4] (pp. 41-42), V. Berinde B[7], M.A. Şerban B[2] (pp. 33-36), J. Jachymski and J. Jóźwik R[1] and the references therein.

We will present now some notions with respect to functions $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$, where $k \in \{2,3,\cdots\}$. Denote first $\phi_\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\phi(t) := \varphi(t,t,\cdots,t)$.

By definition:

(a) φ is a comparison function if φ is increasing and ϕ_φ satisfies (iv_φ);

(b) φ is a strict comparison function if φ is a comparison function and ϕ_φ satisfies (v_φ);

(b) φ is a strong comparison function if φ is a comparison function and ϕ_φ satisfies (vi_φ).

For other details on the above concepts, see I.A. Rus B[4] (pp. 47-48).

3.1 Operators on metric spaces

3.1.1 Basic concepts

Let (X,d) and (Y,ρ) be two metric spaces and $f : X \to Y$ an operator. By definition, the operator f is:

(1) continuous if $x_n \in X$, $n \in \mathbb{N}$ with $x_n \to x$ as $n \to +\infty$ implies $f(x_n) \to f(x)$ as $n \to +\infty$;

(2) with closed graph if $x_n \in X$, $n \in \mathbb{N}$ with $x_n \to x$ and $f(x_n) \to y$ as $n \to +\infty$ implies $y = f(x)$, i.e., $G(f) \subset X \times Y$ is a closed subset;

(3) asymptotically regular if $d(f^n(x), f^{n+1}(x)) \to 0$ as as $n \to +\infty$, for

each $x \in X$;

(4) bounded if $A \in P_b(X)$ implies $f(A) \in P_b(Y)$;

(5) compact if $A \in P_b(X)$ implies $\overline{f(A)} \in P_{cp}(X)$;

(6) completely continuous if it is compact and continuous;

(7) Lipschitz if there exists $l \in \mathbb{R}_+$ such that

$$d(f(x), f(y)) \leq l d(x, y), \quad \text{for all } x, y \in X;$$

(8) contraction if there exists $l \in [0, 1[$ such that f is l-Lipschitz (i.e. Lipschitz with the constant l);

(9) contractive if

$$d(f(x), f(y)) < d(x, y), \quad \text{for all } x, y \in X, \; x \neq y;$$

(10) nonexpansive if it is 1-Lipschitz;

(11) noncontractive if

$$d(f(x), f(y)) \geq d(x, y), \quad \text{for all } x, y \in X;$$

(12) expansive if

$$d(f(x), f(y)) > d(x, y), \quad \text{for all } x, y \in X, \; x \neq y;$$

(13) dilatation if there exists $l > 1$ such that

$$d(f(x), f(y)) \geq l d(x, y), \quad \text{for all } x, y \in X;$$

(14) isometry if

$$d(f(x), f(y)) = d(x, y), \quad \text{for all } x, y \in X;$$

(15) similarity if there exists $l > 0$ such that

$$d(f(x), f(y)) = l d(x, y), \quad \text{for all } x, y \in X.$$

3.1.2 Generalized contractions

Let (X, d) be a metric space and $f : X \to X$ an operator. The contraction principle states that if (X, d) is a complete metric space and f is a contraction, then f has a unique fixed point x^* and $f^n(x)$ converges to x^* as $n \to \infty$, for all $x \in X$.

In the last fifty years many papers established various metrical fixed point theorems. In these theorems f satisfies at various contraction-type conditions. Here are some of them:

(i) (Niemytzki (1936); Edelstein (1962)): (X, d) is compact and f is contractive.

(ii) (Rakotch (1962)): there exists a decreasing function $\alpha : R_+ \to R_+$, such that $\alpha(t) < 1$, for $t > 0$ and

$$d(f(x), f(y)) \leq \alpha(d(x,y))d(x,y), \text{ for all } x, y \in X.$$

(iii) (Browder (1968)): there exists a right continuous function $\varphi : R_+ \to R_+$ satisfying (i_φ) and (ii_φ) and

$$d(f(x), f(y)) \leq \varphi(d(x,y)), \text{ for all } x, y \in X.$$

(iv) (Boyd and Wong (1969)): there exists a right upper continuous function $\varphi : R_+ \to R_+$ satisfying (i_φ) and (ii_φ) and

$$d(f(x), f(y)) \leq \varphi(d(x,y)), \text{ for all } x, y \in X.$$

(v) (J. Matkowski (1975), I. A. Rus (1982)): there exists a comparison function $\varphi : R_+ \to R_+$ such that

$$d(f(x), f(y)) \leq \varphi(d(x,y)), \text{ for all } x, y \in X.$$

(vi) (Kannan (1968)): there exists $a \in \left[0, \frac{1}{2}\right[$ such that:

$$d(f(x), f(y)) \leq a[d(x, f(x)) + d(y, f(y))], \text{ for all } x, y \in X.$$

(vii) (Ćirić (1971), Reich (1971), I. A. Rus (1971)): there exist $a, b \in R_+$, with $a + 2b < 1$, such that:

$$d(f(x), f(y)) \leq ad(x,y) + b[d(x, f(x)) + d(y, f(y))], \text{ for all } x, y \in X.$$

(viii) (Ćirić (1974)): there exists $a \in [0, 1[$ such that:

$$d(f(x), f(y)) \leq$$

$\leq a\max\{d(x,y), d(x,f(x)), d(y,f(y)), d(x,f(y)), d(y,f(x))\}$, for all $x,y \in X$.

(ix) (T. Zamfirescu (1972)): there exist $a,b,c \in R_+$, with $a < 1$, $b < \frac{1}{2}$ and $c < \frac{1}{2}$ such that for each $x,y \in X$ at least one of the following conditions is true:

(1) $d(f(x),f(y)) \leq ad(x,y)$,
(2) $d(f(x),f(y)) \leq b[d(x,f(x)) + d(y,f(y))]$,
(3) $d(f(x),f(y)) \leq c[d(x,f(y)) + d(y,f(x))]$.

(x) (V. I. Istrăţescu (1981)): there exist $a,b \in \mathbb{R}_+$, with $a+b<1$ such that:

$$d(f^2(x), f^2(y)) \leq ad(f(x),f(y)) + bd(x,y), \text{ for all } x,y \in X.$$

(xi) (Meir and Keeler (1969)): for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq d(x,y) \leq \varepsilon + \delta \ \Rightarrow \ d(f(x),f(y)) < \varepsilon.$$

(xii) (Krasnoselskii and Zabrejko (1975)): for each $0 < a < b$ there exists $l(a,b) \in]0,1[$ such that:

$$a \leq d(x,y) \leq b \ \Rightarrow \ d(f(x),f(y)) \leq l(a,b)d(x,y).$$

(xiii) (Burton (1996)): for each $a > 0$ there exists $l(a) \in [0,1[$ such that:

$$d(x,y) \geq a \text{ implies } d(f(x),f(y)) \leq l(a)d(x,y).$$

(xvi) (I. A. Rus (1972), S. Kasahara (1975), Hicks and Rhoades (1979)): f is a graphic a-contraction if it has closed graph and there exists $a \in [0,1[$ such that:

$$d(f^2(x), f(x)) \leq ad(x,f(x)), \text{ for all } x \in X.$$

For other generalizations of the contraction condition, as well as, comparison results and applications, see B.E. Rhoades R[1], V. Berinde B[1], I.A. Rus B[4], M. Hegedüs and T. Szilágyi R[1], W.A. Kirk and B. Sims (Eds.) R[1] (pp. 1-34), J. Jachymski and I. Jóźwik R[1] and the references therein, A. Branciari R[1], W. Walter R[1], etc.

The following conditions appear in some fixed point theorems and they imply the triviality for the operator.

Degenerate condition no. 1. *Let (X, d) be a metric space and $f : X \to X$ be an operator, for which there exists $\alpha > 0$ such that:*

$$d(f(x), f(y)) \leq \alpha[d(x, f(x)) \cdot d(y, f(y))]^{\frac{1}{2}}, \text{ for all } x, y \in X.$$

If $x_0 \in F_f$, then $f(x) = x_0$, for all $x \in X$.

Degenerate condition no. 2. *Let (X, d) be a metric space and $f : X \to X$ be an operator, for which there exists $a_i \in \mathbb{R}_+$ with $a_2 a_3 > 0$ such that:*

$$d(f(x), f(y)) \geq a_1 d(x, y) + a_2 d(x, f(x)) + a_3 d(y, f(y)), \text{ for all } x, y \in X.$$

Then $f = 1_X$.

Degenerate condition no. 3. *Let (X, d) be a metric space and $, g : X \to X$ be two operators. Suppose:*

(i) f and g are surjective;

(ii) there exist $a, b, c \in \mathbb{R}_+^*$ such that:

$$d(f(x), g(y)) \geq ad(x, y) + bd(x, f(x)) + cd(y, g(y)), \text{ for all } x, y \in X.$$

Then $f = g = 1_X$.

For other examples of such metric conditions, see B. Fisher R[1] and R[2], I.A. Rus B[19], D. Trif B[2] and the references therein.

3.2 Basic fixed point principles

The aim of this section is to to present some basic fixed point principles on a metric space.

Contraction Principle. (Banach (1922) and Caccioppoli (1930)) *Let (X, d) be a complete metric space and $f : X \to X$ be an α-contraction. Then we have:*

(i) $F_f = F_{f^n} = \{x^*\}$, for each $n \in \mathbb{N}^*$;

(ii) for each $x \in X$ the sequence of successive approximations $f^n(x)$ of f starting from x converges to x^*;

(iii) $d(x, x^*) \leq \frac{1}{1-\alpha} \cdot d(x, f(x))$, for each $n \in \mathbb{N}$.

Proof. (i) and (ii) By the contraction condition, we get that $Card F_f \leq 1$. Let $x \in X$ be arbitrary chosen. Then $d(f^n(x), f^{n+p}(x)) \leq d(f^n(x), f^{n+1}(x)) +$

$d(f^{n+1}(x), f^{n+2}(x)) + \cdots + d(f^{n+p-1}(x), f^{n+p}(x)) \leq \frac{\alpha^n}{1-\alpha} d(x, f(x)) \to 0$ as $n \to +\infty$. Hence the sequence $(f^n(x))_{n\in\mathbb{N}}$ is Cauchy. Since (X,d) is complete, we get that $f^n(x) \to x^*$ as $n \to +\infty$. From the continuity of f we get that $x^* \in F_f$. Thus $F_f = \{x^*\}$. Notice now that from (ii) we get that $F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$.

(iii) The conclusion follows from: $d(x, x^*) \leq d(x, f(x)) + d(f(x), x^*) \leq d(x, f(x)) + \alpha d(x, x^*)$. □

Extensions and generalizations of the above result are:

Matkowski's Theorem. (1975) *Let (X,d) be a complete metric space and $f : X \to X$ be an φ-contraction, i.e., φ is a comparison function and*

$$d(f(x), f(y)) \leq \varphi(d(x,y)), \text{ for all } x, y \in X.$$

Then we have:

(i) $F_f = F_{f^n} = \{x^*\}$, for each $n \in \mathbb{N}^*$;

(ii) for each $x \in X$ the sequence of successive approximations $f^n(x)$ of f starting from x converges to x^*;

(iii) if, additionally, φ is a strict comparison function, then $d(x, x^*) \leq \varphi_{d(x,f(x))}$.

Proof. (i) and (ii) By the φ-contraction condition, we get that $CardF_f \leq 1$. Indeed, if $x^*, y^* \in F_f$, then $d(f^n(x^*), f^n(y^*)) \leq \varphi^n(d(x^*, y^*)) \to 0$ as $n \to +\infty$.

Next, we will prove that $I(f) \cap P_{b,cl}(X) \neq \emptyset$. Let $x \in X$ be arbitrary chosen. We have $d(f^n(x), f^{n+1}(x)) \leq \varphi^n(d(x, f(x))) \to 0$ as $n \to +\infty$. Let $\epsilon > 0$. From the above relation and using the fact that $\varphi(t) < t$ for all $t > 0$, it follows there exists $x_0 \in X$ such that $d(x_0, f(x_0)) \leq \epsilon - \varphi(\epsilon)$. We obtain now that $\bar{B}(x_0, \epsilon) \in I(f)$. Indeed, if $y \in \bar{B}(x_0, \epsilon)$, then $d(f(y), x_0) \leq d(f(y), f(x_0)) + d(f(x_0), x_0) \leq \epsilon$.

On the other hand, a φ-contraction is a (δ, φ)- contraction, i.e.,

$$\delta(f(Y)) \leq \varphi(\delta(Y)), \text{ for all } Y \in I(f) \cap P_b(X).$$

We can prove now that $F_f \neq \emptyset$. For this purpose, let $Y \in I(f) \cap P_{b,cl}(X)$. Then $\delta(Y_n) = \delta(f(\bar{Y}_{n-1}) = \delta(f(Y_{n-1}) \leq \varphi(\delta(Y_{n-1})) \leq \cdots \leq \varphi^n(\delta(Y)) \to 0$ as $n \to +\infty$. Hence, we have $\bigcap\limits_{n\in\mathbb{N}} Y_n = \{x^*\} \in I(f)$. This $F_f = \{x^*\}$.

In order to prove (ii), notice that:

$$d(f^n(x), x^*) \leq \varphi^n(d(x, x^*)) \to 0 \text{ as } n \to +\infty.$$

From (ii) we get now that $F_f = F_{f^n}$, for each $n \in \mathbb{N}^*$. $\Box$

Ćirić-Reich-Rus's Theorem. (1971) *Let (X, d) be a complete metric space and $f : X \to X$ be an operator. Suppose there exist $\alpha, \beta \in \mathbb{R}_+$ with $\alpha + 2\beta < 1$ such that*

$$d(f(x), f(y)) \leq \alpha d(x, y) + \beta[d(x, f(x)) + d(y, f(y))] \text{ for all } x, y \in X.$$

Then we have:

(i) $F_f = F_{f^n} = \{x^*\}$, for each $n \in \mathbb{N}^*$;

(ii) for each $x \in X$ the sequence of successive approximations $f^n(x)$ of f starting from x converges to x^*;

(iii) $d(x, x^*) \leq \frac{1-\beta}{1-\alpha-2\beta} \cdot d(x, f(x))$, for each $x \in X$.

Proof. Let $x \in X$ and $y := f(x)$. Then we have:

$$d(f(x), f^2(x)) \leq \frac{\alpha + \beta}{1 - \beta} d(x, f(x)), \text{ for all } x \in X.$$

Since $\frac{\alpha+\beta}{1-\beta} < 1$ we get that the sequence $(f^n(x))_{n\in\mathbb{N}}$ is Cauchy and hence convergent to a certain $x^* \in X$. Let us prove that $x^* \in F_f$. We have: $d(x^*, f(x^*)) \leq d(x^*, f^n(x)) + d(f^n(x), f(x^*)) \leq d(x^*, f^n(x)) + \alpha d(f^{n-1}(x), x^*) + \beta[d(f^{n-1}(x), f^n(x) + d(x^*, f(x^*))] \to 0$ as $n \to +\infty$. Hence $x^* \in F_f$. Thus $F_f = \{x^*\}$. From (ii) we have that $F_f = F_{f^n} = \{x^*\}$, for each $n \in \mathbb{N}^*$.

(iii) $d(x, x^*) \leq d(x, f(x)) + d(f(x), f^2(x)) + \cdots + d(f^{n-1}(x), f^n(x)) + d(f^n(x), x^*) \leq [1 + \frac{\alpha+\beta}{1-\beta} + (\frac{\alpha+\beta}{1-\beta})^2 + \cdots + (\frac{\alpha+\beta}{1-\beta})^{n-1}] \cdot d(x, f(x)) + d(f^n(x), x^*) \to \frac{1-\beta}{1-\alpha-2\beta} \cdot d(x, f(x))$ as $n \to +\infty$. $\Box$

Notice that the case $\alpha = 0$, in the previous theorem, is Kannan's fixed point theorem.

Meir-Keeler's Theorem. (1969) *Let (X, d) be a complete metric space and $f : X \to X$ be a Meir-Keeler type operator, i.e., for each $\epsilon > 0$ there exists $\eta > 0$ such that for $x, y \in X$ with $\epsilon \leq d(x, y) < \epsilon + \eta$ we have $d(f(x), f(y)) < \epsilon$. Then we have:*

(i) $F_f = \{x^*\}$;

(ii) the sequence $(f^n(x))_{n\in\mathbb{N}}$ *converges to* x^*, *for each* $x \in X$.

Proof. Denote $x_n := f^n(x_0)$, $n \in \mathbb{N}$.

The proof of the theorem can be organized in four steps.

Step 1. We prove that

$$d(f(x), f(y)) < d(x, y), \text{ for each } x, y \in X \text{ with } x \neq y.$$

Let $x, y \in X$ be such that $x \neq y$. Then by letting $\epsilon := d(x, y)$ in the definition of Meir-Keeler operator we get $d(f(x), f(y)) < d(x, y)$.

Step 2. We prove that the sequence $a_n := d(x_n, x_{n+1}) \searrow 0$ as $n \to +\infty$.

If there is $n_0 \in \mathbb{N}$ such that $a_{n_0} = 0$ then $x_{n_0} \in F_f$.

If $a_n \neq 0$, for each $n \in \mathbb{N}$, then $a_n = d(f(x_{n-1}), f(x_n)) < d(x_{n-1}, x_n) = a_{n-1}$. Hence the sequence $(a_n)_{n\in\mathbb{N}}$ converges to a certain $a \geq 0$. Suppose that $a > 0$. Then, for each $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $\epsilon \leq a_n < \epsilon + \eta$, for all $n \geq n_\epsilon$. Then, by the Meir-Keeler condition we obtain $a_{n+1} < \epsilon$, which is a contradiction with the above relation.

Step 3. We will prove that the sequence (x_n) is Cauchy.

Suppose, by contradiction, that (x_n) is not a Cauchy sequence. Then, there exists $\epsilon > 0$ such that $\limsup d(x_m, x_n) > 2\epsilon$. For this ϵ there exists $\eta := \eta(\epsilon) > 0$ such that for $x, y \in X$ with $\epsilon \leq d(x, y) < \epsilon + \eta$ we have $d(f(x), f(y)) < \epsilon$. Choose $\delta := \min\{\epsilon, \eta\}$. Since $a_n \searrow 0$ as $n \to +\infty$ it follows that there is $p \in \mathbb{N}$ such that $a_p < \frac{\delta}{3}$. Let $m, n \in \mathbb{N}^*$ with $n > m > p$ such that $d(x_n, x_m) > 2\epsilon$. For $j \in [m, n]$ we have $|d(x_m, x_j) - d(x_m, x_{j+1}| \leq a_j < \frac{\delta}{3}$. Also, $d(x_m, x_{m+1} < \epsilon$ and $d(x_m, x_n) > \epsilon + \delta$ we obtain that there exists $k \in [m, n]$ such that $\epsilon < \epsilon + \frac{2\delta}{3} < d(x_m, x_k) < \epsilon + \delta$.

On the other hand, for any $m, l \in \mathbb{N}$ we have: $d(x_m, x_l) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{l+1}) + d(x_{l+1}, x_l) = a_m + d(f(x_m), f(x_l)) + a_l < \frac{\delta}{3} + \epsilon + \frac{\delta}{3}$. The contradiction proves that (x_n) is Cauchy.

Step 4. We prove that $x^* := \lim\limits_{n\to+\infty} x_n$ is a fixed point of f.

Since f is continuous and $x_{n+1} = f(x_n)$, we get by passing to the limit that $x^* = f(x^*)$.

If $x^*, y \in F_f$ are two distinct fixed points of f then, by the contractive condition, we get the following contradiction: $d(x^*, y) = d(f(x^*), f(y)) < d(x^*, y)$.

This completes the proof. □

Krasnoselskii's Theorem. (1972) *Let (X, d) be a complete metric space and $f : X \to X$ be an operator. Suppose that for each $0 < a \leq b < +\infty$ there is $l(a, b) \in [0, 1[$ such that*

$$x, y \in X,\ a \leq d(x, y) \leq b \text{ implies } d(f(x), f(y)) \leq l(a, b)d(x, y).$$

Then we have:

(i) $F_f = F_{f^n} = \{x^*\}$, for each $n \in \mathbb{N}^*$;

(ii) the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to x^*, for each $x \in X$.

Proof. Notice first that:

a) $d(f(x), f(y)) \leq l(a, b)d(x, y)$ for each $x, y \in X$ with $x \neq y$;

b) $CardF_f \leq 1$;

c) f is continuous.

On the other hand, for $x \in X$ with $f^{n+1}(x) \neq f^n(x)$ for each $n \in \mathbb{N}$, we have: $d(x, f(x)) > d(f(x), f^2(x)) > \cdots > d(f^n(x), f^{n+1}(x)) > \cdots > 0$. If $\lim\limits_{n\to+\infty} d(f^n(x), f^{n+1}(x)) > 0$, we get a contradiction. Thus, $\lim\limits_{n\to+\infty} d(f^n(x), f^{n+1}(x)) = 0$.

Let $r > 0$ and $\epsilon > 0$ such that $2\epsilon < r$ and $l(\frac{r}{2}, r)r + \epsilon \leq r$. Then there exists $x_0 \in X$ such that $d(x_0, f(x_0)) < \epsilon$ and $\bar{B}(x_0; r) \in I(f)$.

Indeed, let $y \in \bar{B}(x_0; r)$ be arbitrary chosen.

If $d(y, x_0) \leq \frac{r}{2}$, then $d(f(y), x_0) \leq d(f(y), f(x_0)) + d(f(x_0), x_0) \leq l(d(y, x_0), d(y, x_0))d(, x_0) + \epsilon \leq r$.

If $d(y, x_0) \geq \frac{r}{2}$, then $d(f(y), x_0) \leq l(\frac{r}{2}, r) + \epsilon \leq r$.

Now we will apply the above conclusion to $f : \bar{B}(x_0; r) \to \bar{B}(x_0; r)$ for the case $\frac{r}{2}$. Hence there exists $x_1 \in B_1 := \bar{B}(x_0; r)$ such that $B_2 := \bar{B}(x_1; \frac{r}{2}) \cap \bar{B}(x_0; r) \in I(f)$. By induction, we obtain $B_n \in I(f) \cap P_{b,cl}(X)$ such that $B_n \subset B_{n+1}$ for each $n \in \mathbb{N}^*$ and $\delta(B_n) \to)$ as $n \to +\infty$. By Cantor's theorem we get that $\bigcap\limits_{n\in\mathbb{N}} B_n = \{x^*\} \in I(f)$. Thus $F_f = \{x^*\}$.

Let $x \in X$ with $x \neq x^*$. Then we have $d(f^n(x), x^*) \to 0$ as $n \to +\infty$. Indeed, since the sequence $(d(f^n(x), x^*))_{n\in\mathbb{N}}$ is decreasing, it is convergent too. If, by contradiction $d(f^n(x), x^*) \to u > 0$ as $n \to +\infty$, then $d(f^n(x), x^*) \leq l(u, d(x, x^*))^n d(x, x^*) \to 0$ as $n \to +\infty$. Thus $d(f^n(x), x^*) \to 0$ as $n \to +\infty$.

Finally, notice that from (ii) we obtain $F_{f^n} = F_f = \{x^*\}$. The proof is now complete. □

Graphic Contraction Principle. (I.A. Rus (1972), S. Kasahara (1975), T.L. Hicks and B.E. Rhoades (1979)) *Let (X, d) be a complete metric space, $f : X \to X$ and $\alpha \in [0, 1[$. We suppose that:*

(a) $d(f^2(x), f(x)) \leq \alpha d(x, f(x))$, for all $x \in X$;

(b) the operator f has closed graph.

Then:

(i) $F_f = F_{f^n} \neq \emptyset$, for each $n \in \mathbb{N}^*$;

(ii) $f^n(x) \to f^\infty(x)$ as $n \to \infty$, and $f^\infty(x) \in F_f$, for all $x \in X$;

(iii) $d(x, f^\infty(x)) \leq \dfrac{1}{1-\alpha} d(x, f(x))$, for all $x \in X$.

Proof. (i)+(ii). From (a) we have that $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is a complete metric space it follows that $(f^n(x))_{n\in\mathbb{N}}$ is convergent we denote by $f^\infty(x)$ its limit. From (b) we have that $f^\infty(x) \in F_f$, i.e., $F_f \neq \emptyset$. From (ii) it follows that $F_{f^n} = F_f$.

(iii) $d(x, f^{(n+1)}(x)) \leq d(x, f(x)) + d(f(x), f^2(x)) + \cdots + d(f^n(x), f^{n+1}(x))$

$$\leq (1 + \alpha + \alpha^2 + \cdots + \alpha^n) d(x, f(x)).$$

Letting $n \to \infty$, we have

$$d(x, f^\infty(x)) \leq \frac{1}{1-\alpha} d(x, f(x)), \quad \text{for all } x \in X.$$

□

Caristi-Browder's Theorem. (J. Caristi (1976), F.E. Browder (1976)) *Let (X, d) be a complete metric space, $f : X \to X$ an operator and $\varphi : X \to \mathbb{R}_+$ a functional. We suppose that:*

(a) $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$, for all $x \in X$;

(b) the operator f has closed graph.

Then:

(i) $F_f = F_{f^n} \neq \emptyset$;

(ii) $f^n(x) \to f^\infty(x)$ as $n \to \infty$, and $f^\infty(x) \in F_f$, for all $x \in X$;

(iii) if there is $\alpha \in \mathbb{R}_+^*$ such that $\varphi(x) \leq \alpha d(x, f(x))$, then

$$d(x, f^\infty(x)) \leq \alpha d(x, f(x)), \quad \textit{for all } x \in X.$$

Proof. (i)+(ii). Let $x \in X$. From (a) it follows

$$\sum_{k=0}^{n} d(f^k(x), f^{k+1}(x)) \le \varphi(x) - \varphi(f^{n+1}(x)) \le \varphi(x).$$

This implies that $(f^n(x))_{n\in\mathbb{N}}$ is a convergent sequence. Let us denote by $f^\infty(x)$ is limit. From (b) we have that $f^\infty(x) \in F_f$.

(iii) $d(x, f^{n+1}(x)) \le \sum_{k=0}^{n} d(f^k(x), f^{k+1}(x)) \le \varphi(x) \le \alpha d(x, f(x))$.

So, $d(x, f^\infty(x)) \le \alpha d(x, f(x))$, for all $x \in X$. □

It is well-known that Caristi-Browder's fixed point theorem is equivalent to the variational principle of Ekeland.

Another interesting concept was introduced by Clarke R[1]. Recall that, if (X, d) is a metric space, then for $x, y \in X$, we denote by the symbol

$$[x, y] := \{z \in X \mid d(x, z) + d(z, y) = d(x, y)\},$$

the metric segment between x and y.

If (X, d) is a metric space, then an operator $f : X \to X$ is said to be a directional contraction provided that:

(i) f is continuous;

(ii) there exists $k \in]0, 1[$ such that, for any $x \in X$ with $f(x) \neq x$ there exists $z \in [x, f(x)] \setminus \{x\}$ such that $d(f(x), f(z)) \le kd(x, z)$.

Remark 3.2.1. Any contraction is a directional contraction, but the reverse implication isn't true. For example, if $X := (\mathbb{R}^2, \|\cdot\|_M)$ (where $\|\cdot\|_M$ denotes the Minkowski norm on X) and $f : X \to X$ given by

$$f(x_1, x_2) := (\frac{3x_1}{2} - \frac{x_2}{3}, x_1 + \frac{x_2}{3})$$

is a directional contraction, but it isn't a contraction. Moreover, $F_f = \{(x, \frac{3x}{2}) \mid x \in \mathbb{R}\}$. See also J.M. Borwein and Q.J. Zhu R[1].

The main result for directional contraction was established by Clarke R[1] in 1978. For the multivalued version of the next theorem see H.K. Xu R[5].

Clarke's Theorem. *Let (X, d) be a complete metric space and $f : X \to X$ be a directional contraction with constant k. Then $F_f \neq \emptyset$.*

Proof. Define $\varphi(x) := d(x, f(x))$, for each $x \in X$. Then φ is continuous and bounded from below. By Ekeland variational principle, applied to φ with $\epsilon \in]0, 1-k[$, we get that there exists $y \in X$ such that

$$\varphi(y) \le \varphi(x) + \epsilon d(x, y), \text{ for all } x \in X.$$

If $f(y) = y$ we are done. Otherwise, by the directional contraction assumption there exists $z \in X$ such that $z \in [y, f(y)] \setminus \{y\}$ such that $d(f(z), f(y)) \le kd(z, y)$.

By $\varphi(y) \le \varphi(z) + \epsilon d(z, y)$ and taking into account that $d(y, z) + d(z, f(y)) = d(y, f(y)) = \varphi(y)$ we get that:

$$d(y, z) \le d(z, f(z)) - d(z, f(y)) + \epsilon d(x, y).$$

Then: $d(z, f(z)) - d(z, f(y)) \le d(f(y), f(z)) \le kd(y, z)$. By combining the last two relations we conclude that $d(y, z) \le (k+\epsilon)d(y, z)$, which is a contradiction. Hence $y \in F_f$. □

The above results give rise to the following definitions:

Definition 3.2.1. Let (X, d) a metric space. An operator $f : X \to X$ is weakly Picard operator (briefly WPO) if the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit, denoted by $f^\infty(x)$, is a fixed point f.

Definition 3.2.2. If f is a WPO and $F_f = \{x^*\}$, then by definition f is a Picard operator (briefly PO).

If f is a PO then f is a Bessaga operator, i.e.,

$$F_f = F_{f^n} = \{x^*\}, \text{ for all } n \in \mathbb{N}^*.$$

If f is a WPO, then

$$F_{f^n} = F_f \neq 0, \quad \text{for all } n \in \mathbb{N}^*.$$

Definition 3.2.3. If f is a WPO, then we define the operator f^∞ by

$$f^\infty : X \to X, \quad f^\infty(x) := \lim_{n\to\infty} f^n(x).$$

Definition 3.2.4. Let f be a WPO and $c > 0$. Then f is said to be c-WPO if

$$d(x, f^\infty(x)) \le cd(x, f(x)), \text{ for all } x \in X.$$

Remark 3.2.2. For the above definitions and the theory of WPOs see I.A. Rus B[4], B[14], B[16], B[30], B[34], B[41] and B[49]. See also Chapter 10.

Now we continue with the basic metrical fixed point principles in the case of compact metric spaces. We have:

Niemytzki-Edelstein's Theorem. (V. Niemytzki (1936), M. Edelstein (1962)) *Let (X, d) be a compact metric space and $f : X \to X$ be a contractive operator. Then:*

(i) $F_f = F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$, i.e., f is Bessaga operator;

(ii) $f^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$, i.e., f is Picard operator (PO).

Proof. (i)+(ii). f contractive implies that f is continuous and $cardF_f \leq 1$. So, for to have $cardF_f = 1$, we prove that $F_f \neq \emptyset$.

Let $x \in X$. We consider the sequence of successive approximations, $(f^n(x))_{n\in\mathbb{N}}$. Since (X, d) is compact there exists a subsequence $(f^{n_k}(x))_{k\in\mathbb{N}}$ which converges to an element $x^* \in X$. This implies that

$$d(f^{n_k}(x), f(f^{n_k}(x))) \to d(x^*, f(x^*)) \text{ as } k \to \infty.$$

But the sequence $(d(f^n(x), f^{n+1}(x)))_{n\in\mathbb{N}}$ is decreasing. Hence it is convergent. So, we have

$$d(f^n(x), f^{n+1}(x)) \to d(x^*, f(x^*)) \text{ as } n \to \infty.$$

From the continuity of f, this implies that

$$d(x^*, f(x^*)) = d(f(x^*), f^2(x^*)).$$

The contractive condition on f implies $x^* \in F_f$. Thus, $F_f = \{x^*\}$ and $f^n(x) \to x^*$ as $n \to \infty$. The fact that $F_{f^n} = F_f$, for all $n \in \mathbb{N}^*$ follow from (ii).

Remark 3.2.3. If f is a contraction and (X, d) is compact, then

$$\bigcap_{n\in\mathbb{N}} f^n(X) = \{x^*\},$$

i.e. f is Janos operator.

Remark 3.2.4. In the Niemytzki-Edelstein's Theorem, we can put "$\overline{f(X)}$ is compact" instead "(X,d) is a compact".

More general we have:

Theorem 3.2.1. *Let (X,d) be a bounded and complete metric space, α_K be the Kuratowski measure of noncompactness of X and $f : X \to X$ be an operator. We suppose that:*

(i) there is a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\alpha_K(f(Y)) \leq \varphi(\alpha_K(Y)), \quad \textit{for all} \;\; Y \in I(f) \cap P_b(X);$$

(ii) f is a contractive operator.

Then:

(i) $F_f = F_{f^n} = \{x^*\}$

(ii) $f^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$.

Proof. (i)+(ii). Let $Y_1 := \overline{f(X)}$, $Y_2 := \overline{f(Y_1)}, \ldots, Y_{n+1} := \overline{f(Y_n)}, \ldots$

We have $Y_n \in I(f)$,

$$\alpha_K(Y_n) = \alpha_K(\overline{f(Y_{n-1})}) \leq \varphi(\alpha_K(Y_{n-1})) \leq \cdots \leq \varphi^n(\alpha_K(X)) \to 0 \text{ as } n \to \infty.$$

From this we have that

$$Y_\infty := \bigcap_{n\in\mathbb{N}} Y_n \neq \emptyset, \quad Y_\infty \in I(f) \text{ and } \alpha_K(Y_\infty) = 0.$$

Now the proof follows from the Niemytzki-Edelstein's theorem for the operator $f|_{Y_\infty} : Y_\infty \to Y_\infty$. □

Remark 3.2.5. For the above results and for other metrical fixed point theory see W.A. Kirk and B. Sims R[1], A.A. Ivanov R[1], I.A. Rus B[4], B[49], B[70], M. Kikkawa, T. Suzuki R[1], O. Hadžić R[2], V. Berinde B[7], B[37], D. Blebea and G. Dincă B[1], V.I. Istrăţescu B[3], V. Popa B[7], M. Turinici B[22], B[24], T. Zamfirescu B[4], B[5] and B[11], Z. Kominek R[1], D. Downing and W.A. Kirk R[1], T. Shibata R[1], etc.

3.3 Fixed point theorems on sets with two metrics

The following result was given by M. G. Maia in 1968:

Theorem 3.3.1. (Maia) *Let X be a nonempty set, d and ρ two metrics on X and $f : X \to X$ an operator. We suppose that:*

(i) $d(x,y) \leq \rho(x,y)$, for all $x, y \in X$;

(ii) (X,d) is a complete metric space;

(iii) $f : (X,d) \to (X,d)$ is continuous;

(iv) $f : (X,\rho) \to (X,\rho)$ is an l-contraction.

Then:

(a) $F_f = \{x^*\}$;

(b) $f^n(x) \xrightarrow{d} x^*$ as $n \to \infty$, for all $x \in X$;

(c) $f^n(x) \xrightarrow{\rho} x^*$ as $n \to \infty$, for all $x \in X$;

(d) $\rho(x,x^*) \leq \frac{1}{1-l}\rho(x,f(x))$, for each $x \in X$.

Proof. (a) and (b) Let $x \in X$ and $(f^n(x))_{n\in\mathbb{N}}$ be the corresponding sequence of successive approximations. From (iv) it follows that this sequence is Cauchy in (X,ρ). From (i) we get that it is Cauchy in (X,d) too. From (ii) we have that $(f^n(x))_{n\in\mathbb{N}}$ is convergent in (X,d) to some $x^* \in X$. From (iii) we obtain that $x^* \in F_f$. Notice that (iv) implies $cardF_f \leq 1$. Thus $F_f = \{x^*\}$. Moreover, since f is a Picard operator in (X,d) we have that $F_{f^n} = F_f$.

(c) and (d) Take $y = x^*$ in (iv) and follow the proof of the Contraction principle. □

I.A. Rus B[76] noticed that Maia's theorem remains true if the condition (i) is replaced by:

(i') there exists a number $c > 0$ such that $d(f(x),f(y)) \leq c\rho(x,y)$, for all $x, y \in X$,

or if condition (iv) is replaced by:

(iv') $f : (X,\rho) \to (X,\rho)$ some generalized contraction condition (such as Kannan, Ćirić-Reich-Rus, etc.)

For other generalizations of the Maia's fixed point theorem see M. Albu B[1], V. Berinde B[18], N. Gheorghiu B[1], A. S. Mureşan B[3] and B[4], A. S. Mureşan and V. Mureşan B[1], V. Mureşan B[1], R. Precup B[26], R. Precup and D. O'Regan B[1], I. A. Rus B[49], B[57], B[75] and B[76], D. Trif B[3]. See also V. Berinde B[7], R.P. Agarwal, M. Meehan and D. O'Regan R[1], B. Rzepecki R[2].

3.4 Basic problems of the metric fixed point theory

We formulate now several open problems of the metric fixed point theory.

Problem 3.4.1. Let (X, d) be a metric space and $f : X \to X$ be an operator. Which are the metric conditions on f which imply that every periodic point of f is a fixed point, i.e.,

$$F_f = F_{f^n}, \quad \text{for all } n \in \mathbb{N}?$$

Problem 3.4.2. Give metric conditions on f implying that:
(i) $F_f = \{x^*\}$;
(ii) $f^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$.

Problem 3.4.3. Give metric conditions on f implying that:
(i) $F_f \neq \emptyset$;
(ii) $f^n(x) \to x^*(x)$ as $n \to \infty$, for all $x \in X$.

Problem 3.4.4.

Problem 3.4.4a. Let (X, d) be a complete metric space, (Y, τ) be a topological space and $f : X \times Y \to X$ a continuous operator. Give metric conditions on $f(\cdot, y) : X \to X$ implying:
(i) $F_{f(\cdot,y)} = \{x^*_y\}$,
(ii) the operator $P : Y \to X$, $y \mapsto x^*_y$ is continuous.

Problem 3.4.4b. Let (X, d) be a (complete, bounded, compact,etc.) metric space and $f, g : X \to X$ be such that:
(i) $F_g \neq \emptyset$
(ii) there exists $\eta > 0$ such that:

$$d(f(x), g(x)) \leq \eta, \quad \text{for all } x \in X.$$

Let $x^*_g \in F_g$ and $F_f = \{x^*_f\}$. For which generalized contractions f can we estimate $d(x^*_f, x^*_g)$?

Problem 3.4.4c. Let (X, d) be a (complete, bounded, compact,etc.) metric space and $f, f_n : X \to X$, $n \in \mathbb{N}$ be such that:
(i) f_n converges uniformly to f;

(ii) $F_f = \{x^*\}$;

(iii) $F_{f_n} \neq \emptyset$.

Let $x_n^* \in F_{f_n}$. For which generalized contractions f we have $x_n^* \to x^*$ as $n \to \infty$?

Problem 3.4.5. Let $(X, \|\cdot\|)$ be a Banach space. For which generalized contractions $f : X \to X$, we have that:

(a) $1_X - f$ is a surjection ?

(b) $1_X - f$ is a bijection ?

(c) $1_X - f$ is a topological isomorphism ?

Problem 3.4.6. Let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ $(x_n \in X)$ a bounded sequence. For which Picard operators $f : X \to X$ we have that $f^n(x_n) \to x^*$ as $n \to \infty$?

Let (X, d) a metric space and $f : X \to X$ an operator such that $F_f = \{x^*\}$. By definition, (see F.S. De Blasi and J. Myjak R[2]) the fixed point problem for the operator f is well-posed if

$$x_n \in X,\ n \in \mathbb{N},\ d(x_n, f(x_n)) \to 0 \text{ as } n \to \infty \Rightarrow x_n \to x^* \text{ as } n \to \infty.$$

Problem 3.4.7.

Problem 3.4.7a. For which generalized contractions the fixed point problem is well-posed ?

Problem 3.4.7b. For which Picard operators the fixed point problem is well posed ?

Let (X, d) be a metric space. An operator $f : X \to X$ has, by definition, the limit shadowing property (see A.M. Ostrowski R[1], J. Jachymski R[4], T. Eirola, O. Nevanlina and S.Yu. Pilyugin R[1]) if

$$x_n \in X,\ n \in \mathbb{N} \text{ and } d(x_{n+1}, f(x_n)) \to 0 \text{ as } n \to \infty \Rightarrow$$

$$\text{there exists } x \in X \text{ such that } d(x_n, f^n(x)) \to 0 \text{ as } n \to \infty.$$

Problem 3.4.8.

Problem 3.4.8a. Which generalized contractions have the limit shadowing property ?

Problem 3.4.8b. Which Picard operators do have the limit shadowing property ?

For example in the case of contractions we have:

Theorem 3.4.1. *Let (X,d) be a complete metric space and $f : X \to X$ an α-contraction. Then:*

(i) f is Bessaga operator;

(ii) f is Picard operator ($F_f = \{x^*\}$);

(iii) f is $\dfrac{1}{1-\alpha}$-Picard operator;

(iv) the fixed point problem for the operator f is well posed;

(v) the operator f has the limit shadowing property;

(vi) if $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in X, then $f^n(x_n) \to x^*$ as $n \to \infty$.

(vii) if $g : X \to X$ is such that there exists $\eta > 0$ with

$$d(f(x), g(x)) \le \eta, \quad \text{for all} \ \ x \in X,$$

then:

$$x_g^* \in F_g \Rightarrow d(x^*, x_g^*) \le \frac{\eta}{1-\alpha};$$

(viii) if $f_n : X \to X$, $f_n \overset{unif.}{\to} f$, $x_n^* \in F_{f^n}$, $n \in \mathbb{N}$, then $x_n^* \to x^*$ as $n \to \infty$;

(ix) if (X,d) is a bounded metric space, then f is Janos operator;

(x) if X is a Banach space, then $1_X - f : X \to X$ is a topological isomorphism.

For the basic problems of the metrical fixed point theory, see I.A. Rus B[70], B[49], B[26], B[4], B[108], W.A. Kirk and B. Sims R[1], M.A. Krasnoselskii and P. Zabrejko R[1], V. Berinde B[7], B[2], B[37], K. Deimling R[3], A. Granas and J. Dugundji R[1], D.R. Smart R[1], E. Zeidler R[1], T.H. Kim and K.M. Park R[1]. See also, F. Aldea B[3], V. Berinde B[13], B[20], B[21], V.I. Istrăţescu B[3], A.S. Mureşan B[1], B[3], B[7], V. Mureşan B[1], B[3], I.A. Rus B[30], B[34], B[51], B[54], B[57], B[108], etc.

For the well-posed of the fixed point problem see F.S. De Blasi and J. Myjak R[2], E. Matouskova, S. Reich and A.J. Zaslavski R[1], S. Reich and A.J. Zaslavski R[5], I.A. Rus B[106], B[108], etc.

For the limit shadowing property see A.M. Ostrowski R[1], J. Jachymski R[4], T. Eirola, O. Nevanlinna and S.Yu. Pilyugin R[1], I.A. Rus B[102], T. Žáčik R[1], etc.

3.5 Equivalent statements

In what follow we shall present three types of equivalent statements which appear in the metrical fixed point theory.

Theorem 3.5.1. *Let X be a nonempty set and $f : X \to X$ be an operator. Then the following statement are equivalent:*

(P_1) *There exists a metric d on X such that $f : (X, d) \to (X, d)$ is a Picard operator.*

(P_2) *f is a Bessaga operator.*

(P_3) *There exist $\alpha \in]0, 1[$ and $\chi : X \to \mathbb{R}_+$ such that*

(i) $card(Z_\chi) = 1$;

(ii) $\chi(f(x) \leq \alpha\chi(x)$, for all $x \in X$, i.e., (χ, α) is a Schröder pair.

(P_4) *There exist $\alpha \in]0, 1[$ and a complete metric d on X such that $f : (X, d) \to (X, d)$ is an α-contraction.*

(P_5) *There exist a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and a complete metric d on X such that $f : (X, d) \to (X, d)$ is a φ-contraction.*

(P_6) *There exist $x^* \in F_f$, $\alpha \in]0, 1[$ and a metric d on X such that*

$$d(f(x), x^*) \leq \alpha d(x, x^*), \quad \text{for all } x \in X$$

(P_7) *There exist $x^* \in F_f$ and a metric d on X such that:*

$$Y \in I_{cl}(X) \Rightarrow x^* \in Y.$$

(P_8) *There exists $x^* \in F_f$ and a metric d on X such that:*

$$x_n \in X,\ (x_n) \text{ is a bounded sequence } \Rightarrow f^n(x_n) \to x^* \text{ as } n \to \infty.$$

(P_9) *There exists a metric d on X such that the fixed point problem is well-posed for f with respect to d.*

Proof. $(P_1) \Rightarrow (P_2)$. Let $F_f = \{x^*\}$ and $y^* \in F_{f^m}$. Then $f^n(y^*) \to x^*$ as $n \to \infty$. Since $f^{km}(y^*) = y^*$, for $k \in \mathbb{N}$, we have $x^* = y^*$.

$(P_2) \Rightarrow (P_3)$. This is a theorem by J. Jachymski (see Jachymski R[1]).

$(P_3) \Rightarrow (P_4)$. The functional $d : X \times X \to \mathbb{R}_+$ defined by $d(x, y) := \chi(x) + \chi(y)$ is the desired metric.

$(P_4) \Rightarrow (P_5)$. We take $\varphi(t) = \alpha t$.

$(P_5) \Rightarrow (P_6)$. We remark that $(P_5) \Rightarrow (P_2) \Rightarrow (P_4) \Rightarrow (P_6)$.

$(P_6) \Rightarrow (P_7)$. Let $y \in I_{cl}(X)$ and $F_f = \{x^*\}$. If $x \in Y$, then $f^n(x) \in Y$ and $d(f^n(x), x^*) \leq \alpha^n d(x, x^*)$, for all $n \in \mathbb{N}$. Hence $f^n(x) \to x^*$ as $n \to \infty$ and $x^* \in Y$.

$(P_7) \Rightarrow (P_8)$. We observe that $(P_7) \Rightarrow (P_2) \Rightarrow (P_4)$. Now we prove that $(P_4) \Rightarrow (P_7)$. Let $x_n \in X$, $n \in \mathbb{N}$ such that the sequence (x_n) is bounded. Since $(x_n)_{n\in\mathbb{N}}$ is bounded, there is $M > 0$ such that $d(x_n, x^*) \leq M$, for all $n \in \mathbb{N}$. We have

$$d(f^n(x_n), x^*) \leq \alpha^n d(x_n, x^*) \leq \alpha^n M \to 0 \text{ as } n \to \infty.$$

$(P_8) \Rightarrow (P_9)$. First we prove that $(P_8) \Rightarrow (P_2)$. Let $x^* \in F_f$. If $y^* \in F_f$, then we consider the bounded sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_{2n} = x^*$, $x_{2n+1} = y^*$. From $f^n(x_n) \to x^*$ as $n \to \infty$, it follows that $y^* = x^*$. In a similar way we prove that $F_{f^n} = \{x^*\}$, $n \in \mathbb{N}^*$. Thus, f is a Bessaga operator. This implies that there exists a metric ρ on X such that $f : (X, \rho) \to (X, \rho)$ is an α-contraction $((P_2) \Rightarrow (P_4))$.

Let $y_n \in X$, $n \in \mathbb{N}$, such that $\rho(y_n, f(y_n)) \to 0$ as $n \to \infty$. Then we have, from the Contraction Principle (c)

$$\rho(y_n, x^*) \leq \frac{1}{1-\alpha}\rho(y_n, f(y_n)) \to 0 \text{ as } n \to \infty.$$

$(P_9) \Rightarrow (P_2)$. Let d a metric on X such that the fixed point problem is well-posed for f with respect to d. Let $F_f = \{x^*\}$. Let $y^* \in F_{f^m}$. If we take $y_n = y^*$, we have that $y^* = x^*$. So, $F_f = F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}$.

$(P_2) \Rightarrow (P_1)$. This is Bessaga's Theorem (see C. Bessaga R[1]). □

For the above and other equivalent statements see C. Bessaga R[1], P.R. Meyers R[1], V.I. Opoitsev R[1], I.A. Rus B[4], B[108], K. Deimling R[3], J. Jachymski R[1], I.A. Rus, A. Petruşel and M.A. Şerban B[1], etc.

Theorem 3.5.2. *Let X be a nonempty set and $f : X \to X$ be n operator. Then the following statements are equivalent:*

(WP_1) *There exists a metric d on X such that $f : (X, d) \to (X, d)$ is a weakly Picard operator.*

(WP_2) $F_f = F_{f^n} \neq \emptyset$, *for all* $n \in \mathbb{N}^*$.

(WP_3) *There exists a partial ordering,* $\leq$, *such that the set of all maximal elements of* X *is nonempty and* $f : (X, \leq) \to (X, \leq)$ *is progressive.*

(WP_4) *There exist a complete metric* d *on* X *and a number* $\alpha \in]0, 1[$ *such that*

(i) $f : (X, d) \to (X, d)$ *has closed graph;*

(ii) $d(f^2(x), f(x)) \leq \alpha d(x, f(x)),$ *for all* $x \in X$.

(WP_5) *There exist a complete metric* d *on* X *and a lower semicontinuous functional* $\varphi : X \to \mathbb{R}_+$ *such that*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \quad \text{for all } \ x \in X.$$

(WP_6). *There exist a complete metric* d *on* X *and a functional* $\varphi : X \to \mathbb{R}_+$ *such that:*

(i) f *has closed graph;*

(ii) $d(x, f(x)) \leq \varphi(x) - \varphi(f(x)),$ *for all* $x \in X$.

(WP_7) *There exists a partition,* $X = \bigcup_{i \in I} X_i$, *of* X *such that* $f(X_i) \subset X_i$ *and* $f|_{X_i} : X_i \to X_i$ *is a Bessaga operator for all* $i \in I$.

(WP_8) *There exists a partition,* $X = \bigcup_{i \in I} X_i$, *of* X *such that* $f(X_i) \subset X_i$ *and* $f|_{X_i} : X_i \to X_i$ *is a Picard operator.*

Proof. $(WP_1) \Rightarrow (WP_2)$. Let d be a metric on X such that $f : (X, d) \to (X, d)$ is a weakly Picard operator. From the definition of a WPO it follows that $F_f \neq \emptyset$ and $F_{f^n} = F_f$, for all $n \in \mathbb{N}$.

$(WP_2) \Rightarrow (WP_7)$. Since $F_f = F_{f^n}$, for all $n \in \mathbb{N}$, there exists a partition of X, i.e. $X := \bigcup_{i=1}^{n} = \cup X_i$ such that $X_i \in I(f)$, $card(F_f \cap X_i) = 1$ and $f|_{X_i} : X_i \to X_i$ is a Bessaga operator (see I.A. Rus B[16] and J. Jachymski R[6]). From a theorem of Bessaga there exists a complete metric d_i on X_i such that $f|_{X_i}$ is an α-contraction for all $i \in I$. Now we define a complete metric d on X. Let $x_i^* \in X_i \cap F_f$, $i \in I$, we take

$$d(x, y) := \begin{cases} d_i(x, y), & \text{if } \ x, y \in X_i, \\ d_i(x, x_i^*) + d_j(y, x_j^*) + 1, & \text{if } \ x \in X_i, \ y \in X_j, \ i \neq j. \end{cases}$$

It is clear that $d(x, y) < 1 \Rightarrow \exists \ i \in I$ such that $x, y \in X_i$. So, d is a complete metric.

On the other hand if $x \in X$, then there exists a unique $i \in I$ such that $x \in X_i$, and

$$d(f^2(x), f(x)) = d_i(f^2(x), f(x)) \le \alpha d(f(x), x) = \alpha d(f(x), x).$$

$(WP_4) \Rightarrow (WP_6)$. We take $\varphi : X \to \mathbb{R}_+$ defined by

$$\varphi(x) := \frac{1}{1-\alpha} d(x, f(x)).$$

$(WP_6) \Rightarrow (WP_3)$. See J. Jachymski R[6].

$(WP_3) \Rightarrow (WP_2)$. See J. Jachymski R[6].

$(WP_4) \Rightarrow (WP_1)$. This is the Graphic Contraction Principle.

$(WP_4) \Rightarrow (WP_5)$.

We take $\varphi(x) := \dfrac{1}{1-\alpha} d(x, f(x))$.

$(WP_5) \Rightarrow (WP_2)$. From Caristi-Kirk's theorem it follows that $F_f \neq \emptyset$. Since $f : (X, \le_\varphi) \to (X, \le_\varphi)$ is progressive, we have that $F_{f^n} = F_f$, for each $n \in \mathbb{N}^*$.

$(WP_1) \Rightarrow (WP_7)$. Let $x \in F_f$. If $X_x := \{y \in X \mid f^n(y) \to x \text{ as} n \to +\infty\}$, then $X = \bigcup_{x \in F_f} X_x$ is the solution of our problem.

$(WP_7) \Rightarrow (WP_8)$. See I.A. Rus B[16].

$(WP_8) \Rightarrow (WP_1)$. It is obvious. □

In the case of compact metric space we have

Theorem 3.5.3. (J. Janos (1967), S. Leader (1982), I.A. Rus (1983)) *Let (X, d) be a compact metric space and $f : X \to X$ be a continuous operator. Then the following statements are equivalent:*

(i) The operator f is Janos operator, i.e.,

$$\bigcap_{n \in \mathbb{N}} f^n(X) = \{x^*\};$$

(ii) The operator f is Picard operator and $f^n(x) \overset{unif.}{\to} x^*$ as $n \to \infty$, for each $x \in X$;

(iii) There exist a metric ρ topological equivalent with d and $\alpha \in]0,1[$ such that $f : (X, \rho) \to (X, \rho)$ is an α-contraction;

(iv) f is contractive with respect to some metric topological equivalent with d.

For a continuous operator we have:

Theorem 3.5.4. (E. Wattel R[1]) *Let (X, d) be a metric space and $f : X \to X$ be an operator. We suppose that:*

(i) f is continuous;

(ii) there exists $x_0 \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ contains a convergent subsequence;

$\lim\limits_{n\to+\infty} d(f^n(x), f^n(y)) = 0$*, for all $x, y \in X$.*

Then f is a Picard operator.

For other results see J. Jachymski R[1].

For some references for Theorem 3.5.2. and 3.5.3., see I.A. Rus B[108].

For other fixed point theorems in metric spaces see W.A. Kirk and B. Sims (Eds.) R[1], V. Berinde B[7], I.A. Rus B[4], B[70], O. Hadžić R[2], A.A. Ivanov R[1], B.E. Rhoades R[1], R[4], M.R. Tasković R[1], V.I. Istrăţescu B[3], T. Araki R[1], etc.

3.6 Generalized contractions and quasibounded operators

Let $(X, +, \mathbb{R}, \|\cdot\|)$ be a linear normed space. An operator $f : X \to X$ is called quasibounded if there are $m, M \geq 0$ such that

$$\|f(x)\| \leq m\|x\| + M, \quad \text{for all } x \in X.$$

The quasinorm of f is by definition

$$\|f\| = \inf\{m \in \mathbb{R}_+ |\ \|f(x)\| \leq m\|x\| + M, \quad \text{for all } x \in X\}.$$

A quasibounded operator is called norm contraction if $\|f\| < 1$.

The following problem was proposed by I.A. Rus.

Which generalized contractions are norm contraction ?

An answer to this question is:

Lemma 3.6.1. (M.C. Anisiu B[7]). *The following generalized contractions are norm contractions:*

(1) $$\|f(x)-f(y)\| \leq a\|x-y\| + b\|x-f(x)\| + c\|y-f(y)\| +$$

$$+d\|x-f(y)\| + c\|y-f(x)\|, \quad \textit{for all } x,y \in X$$

where $a,b,c,d,e \in \mathbb{R}_+$, $a+b+c+d<1$

(2) for any $x,y \in X$, *at least one of the following conditions is satisfied:*

(i) $\|f(x)-f(y)\| \leq a\|x-y\|$

(ii) $\|f(x)-f(y)\| \leq b(\|x-f(x)\| + \|y-f(y)\|)$

(iii) $\|f(x)-f(y)\| \leq c(\|x-f(y)\| + \|y-f(x)\|)$

where $0 \leq a < 1,\ 0 \leq b < \frac{1}{2},\ 0 \leq c < \frac{1}{2}$*;*

(3) $$\|f(x)-f(y)\| \leq a \max\{\|x-f(x)\|, \|y-f(y)\|\}, \quad \textit{for all } x,y \in X,$$

where $0 \leq a < \frac{1}{2}$.

For other results of this type see A. Granas R[4], J. Mawhin R[6], M. C. Anisiu B[7] and F. Aldea B[1].

Chapter 4

Generalized contractions on g.m.s. $(d(x, y) \in \mathbb{R}_+)$

Guidelines: T.A. Brown and W.W. Comfort (1960), I. Colojoară (1961), A.F. Monna (1961), W.J. Kammerer and R.H. Kasriel (1964), A.F. Monna (1964), J. Dugundji (1966), S. Kasahara (1968), F.W. Schäfke (1970), L. Janos (1971), K.K. Tan (1972), T.L. Hicks (1988), I.A. Bakhtin (1989), S.G. Matthews (1992).
General references: W.A. Kirk and B. Sims (Eds.) R[1], M.M. Bonsangue, F. von Breugel and J.J.M.M. Rutten R[1], M. Frigon R[2], I.A. Rus B[104], S. Oltra and O. Valero R[1], J. Reinermann R[1], R. P. Agarwal, D. O'Regan and N. Shahzad R[1], J. Jachymski, J. Matkowski and T. Swiatkowski R[1].

4.0 Generalized metric spaces $(d(x, y) \in \mathbb{R}_+)$

In this section we consider a generalized metric on a given set X as a functional, $d : X \times X \to \mathbb{R}_+$, which satisfies some axioms. The following axioms appear in the definitions of several types of generalized metrics:
(i) $d(x, y) = 0$ if and only if $x = y$;
(i_1) $d(x, x) = 0$, for all $x \in X$;
(i_2) $d(x, y) = 0$ implies $x = y$;
(i_3) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;

(i_4) $d(x,y) = d(y,x) = 0$ imply $x = y$;
(i_5) $d(x,x) = d(y,y) = d(x,y)$ if and only if $x = y$;
(i_6) $d(x,x) \le d(x,y)$, for all $x, y \in X$;
(i_7) $d(y,y) \le d(x,y)$, for all $x, y \in X$;
(ii) $d(x,y) = d(y,x)$, for all $x, y \in X$;
(iii) $d(x,y) \le d(x,z) + d(y,z)$, for all $x, y, z \in X$;
(iii_1) $d(x,y) \le \max(d(x,z), d(z,y))$, for all $x, y, z \in X$;
(iii_2) $d(x,y) \le s[d(x,z) + d(z,y)]$, for all $x, y, z \in X$, with $s > 1$;
(iii_3) $d(x,y) \le d(x,z) + d(z,y) - d(z,z)$, for all $x, y, z \in X$.
By definition d is a:

- pseudometric if satisfies: $(i_1) + (ii) + (iii)$;
- quasimetric if satisfies: $(i_3) + (iii)$;
- premetric ($\equiv$ quasi-pseudometric) if satisfies: $(i_1) + (iii)$;
- semimetric if satisfies: $(i) + (ii)$;
- symmetric if satisfies: $(i_2) + (ii)$;
- ultrametric if satisfies: $(i) + (ii) + (iii) + (iii_1)$;
- b-metric if satisfies: $(i) + (ii) + (iii_2)$;
- partial metric if satisfies: $(i_5) + (i_6) + (ii) + (iii)$.

For the above definitions and for the mathematics on a generalized metric space see: M. Fréchet R[1], F. Hausdorff R[1], L.M. Blumenthal R[1], K. Kunen and J.F. Vaugham (Eds.) R[1], J. Dugundji R[2], J. Kelley R[1], C.E. Aull and R. Lowen R[1], R. Engelking R[1], M.A. Khamsi and W.A. Kirk R[1], R. Kopperman R[1], J.L. Reilly R[1], M.M. Bensangue, F. van Breugel and J.J.M.M. Rutten R[1] and I.A. Rus B[104].

4.1 Fixed point theory in b-metric spaces

We start this section by presenting the concept of b-metric space.

Definition 4.1.1. (Bakhtin R[1], see also Czerwik R[1]) Let X be a set and let $s > 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric on X if and only if the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(iii_2) $d(x, z) \leq s[d(x, y) + d(y, z)]$, for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space if and only if X is a nonempty space and d is a b-metric on X.

We give next some examples of b-metric spaces.

Example 4.1.1. (V. Berinde B[13]) The space $l_p (0 < p < 1)$, $l_p = \{(x_n) \subset \mathbb{R} | \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, together with the function $d : l_p \times l_p \to \mathbb{R}$,

$$d(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{1/p},$$

where $x = (x_n), y = (y_n) \in l_p$ is a b-metric space.

By an elementary calculation we obtain: $d(x, z) \leq 2^{1/p}[d(x, y) + d(y, z)]$.

Hence $s = 2^{1/p} > 1$.

Example 4.1.2. (V. Berinde B[13]) The space $L_p (0 < p < 1)$ of all real functions $x(t)$, $t \in [0, 1]$ such that:

$$\int_0^1 |x(t)|^p dt < \infty,$$

is a b-metric space if we take:

$$d(x, y) = (\int_0^1 |x(t) - y(t)|^p dt)^{1/p}, \text{ for each } x, y \in L_p,$$

The constant s is as in the previous example $2^{1/p}$.

The following results are important in what follows.

Lemma 4.1.1. (Bakhtin R[1], see also Czerwik R[1]) Let (X, d) be a b-metric space and let $\{x_k\}_{k=0}^n \subset X$. Then, for $n \in \mathbb{N}^*$ we have:

$$d(x_0, x_n) \leq sd(x_0, x_1) + ... + s^{n-1}d(x_{n-2}, x_{n-1}) + s^{n-1}d(x_{n-1}, x_n).$$

Using the previous lemma, by a similar approach to the contraction principle, we have:

Bakhtin's Theorem. *Let (X, d) be a complete b-metric space with constant s and let $f : X \to X$ be an α-contraction. If $\alpha s < 1$, then f is a Picard operator.*

By a similar approach, we obtain:

Berinde's Theorem. (V. Berinde B[15]) *Let (X, d) be a complete b-metric space with constant s and let $f : X \to X$ be an φ-contraction. Then, f has a unique fixed point if and only if there exists $x_0 \in X$ such that the sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations starting from x_0 (i.e. $x_{n+1} := f(x_n), n \in \mathbb{N}$) is bounded.*

4.2 Fixed point theorems in partial metric spaces

4.2.1 Partial metric spaces

Let X be a nonempty set. By definition (see S.G. Matthews R[1]), a functional $p : X \times X \to \mathbb{R}_+$ is a partial metric on X if p satisfies the following conditions:

(p_1) $d(x, x) = d(y, y) = d(x, y)$ if $x = y$;

(p_2) $p(x, x) \leq p(x, y)$, for all $x, y \in X$;

(p_3) $p(x, y) = p(y, x)$, for all $x, y \in X$;

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$, for all $x, y, z \in X$.

The following functionals are partial metrics:

1) a metric d on a set X;

2) $X = \mathbb{R}$, $p(x, y) = \max\{0, x, y\}$;

3) Let Y be a set and $X := Y^\infty$ - the set of all finite and infinite sequences in Y. Let $l : Y^\infty \times Y^\infty \to \mathbb{R}_+ \cup \{+\infty\}$ be defined by

$$l(x, y) := \begin{cases} \sup\{n \in \mathbb{N} \mid x(k) = y(k),\ k \leq n\}, & \text{if } \ x(0) = y(0) \\ 0, & \text{if } \ x(0) \neq y(0). \end{cases}$$

Then $p : X \times X \to \mathbb{R}_+$ defined by $p(x, y) := 2^{-l(x,y)}$ is a partial metric on X.

Let (X,p) be a partial metric space. By definition an element $x \in X$ is a total element if $p(x,x) = 0$, and partial if $p(x,x) > 0$.

From the definition of a partial metric we have:

Lemma 4.2.1. *Let (X,p) be a partial metric space. Then:*

(i) the functional $q_p : X \times X \to \mathbb{R}_+$, $q_p(x,y) = p(x,y) - p(x,x)$ is a quasimetric on X;

(ii) the functional $d_p : X \times X \to \mathbb{R}_+$ defined by $d_p(x,y) := q_p(x,y) + q_p(y,x)$ is a metric on X.

By definition, in a partial metric space (X,p) we have:

(a) $x_n \in X$, $x_n \to x^*$ as $n \to \infty$ if $x_n \stackrel{d_p}{\to} x^*$ as $n \to \infty$;

(b) $(x_n)_{n\in\mathbb{N}}$ is fundamental in (X,p) if $(x_n)_{n\in\mathbb{N}}$ is fundamental in (X,d_p).

(c) (X,p) is a complete partial metric space if (X,d_p) is a complete metric space.

From the above definition we have that:

$$x_n \stackrel{p}{\to} x^* \text{ as } n \to \infty \text{ iff } \lim_{n\to\infty} p(x^*,x_n) = \lim_{m,n\to\infty} p(x_n,x_m) = p(x^*,x^*).$$

In the case that x^* is a total element, then:

$$x_n \stackrel{p}{\to} x^* \text{ as } n \to \infty \text{ iff } p(x_n,x^*) \to 0 \text{ as } n \to \infty.$$

For the above considerations see S.G. Matthews R[1], R[2]. See also the references in I.A. Rus B[104].

4.2.2 Fixed point theory in partial metric spaces

Let (X,p) be a partial metric space and $f : X \to X$ be an operator. We have the following result:

Contraction Principle. (S.G. Matthews R[2]; I.A. Rus B[104]) *Let (X,p) be a complete partial metric space and $f : X \to X$ be an α-contraction. Then we have:*

(1) $F_f = F_{f^n} = \{x_f^*\}$, for all $n \in \mathbb{N}^*$ and $p(x_f^*,x_f^*) = 0$;

(2) $f^n(x) \stackrel{d_p}{\to} x_f^*$ as $n \to \infty$, i.e., f is a PO in (X,d_p);

(3) $p(f^n(x),x_f^*) \to 0$ as $n \to \infty$, for all $x \in X$;

(4) $p(x, x_f^*) \leq \dfrac{1}{1-\alpha} p(x, f(x))$, *for all* $x \in X$*;*

(5) $x_n \in X$*,* $p(x_n, f(x_n)) \to 0$ *as* $n \to \infty$ *imply that* $p(x_n, x_f^*) \to 0$ *as* $n \to \infty$*, i.e., the fixed point problem for the operator* f *is well-posed with respect to p;*

(6) $x_n \in X$*,* $p(x_{n+1}, f(x_n)) \to 0$ *as* $n \to \infty$ *imply that* $p(x_n, f^n(x)) \to 0$ *as* $n \to \infty$ *for all* $x \in X$*, i.e., the operator* f *has the limit shadowing property with respect to p;*

(7) if $g : X \to X$ *has the property that there exists* $\eta > 0$ *for which* $p(f(x), g(x)) \leq \eta$*, for all* $x \in X$*, then*

$$x_g^* \in F_g \text{ implies that } p(x_f^*, x_g^*) \leq \frac{\eta}{1-\alpha}.$$

Proof. (1)+(2)+(3). We begin our proof with some simple and useful remarks:

(a) If $x^* \in F_f$, then $p(x^*, x^*) = 0$.

Indeed, $p(x^*, x^*) = p(f^n(x^*), f^n(x^*)) \leq \alpha^n p(x^*, x^*) \to 0$ as $n \to \infty$.

(b) If $x^*, y^* \in F_f$, then $p(x^*, y^*) = 0$.

Indeed, $p(x^*, y^*) = p(f^n(x^*), f^n(y^*)) \leq \alpha^n p(x^*, y^*) \to 0$ as $n \to \infty$.

(c) From (p_1) and the above remarks we have that $card F_f \leq 1$.

(d) $p(f^n(x), f^m(x)) \leq \dfrac{\alpha^{\min(n,m)}}{1-\alpha} d(x, f(x)) \to 0$ as $n, m \to \infty$, for all $x \in X$.

From the above two estimations we have, for each $x \in X$, that:

$$d_p(f^n(x), f^m(x)) = 2p(f^n(x), f^m(x)) - p(f^n(x), f^n(x)) - p(f^m(x), f^m(x)) \to 0$$

as $n, m \to \infty$.

Since (X, d_p) is a complete metric space, it follows that

$$f^n(x) \xrightarrow{d_p} x^* \text{ as } n \to \infty, \text{ for each } x \in X.$$

But

$$d_p(f^n(x), x^*) = p(f^n(x), x^*) - p(f^n(x), f^n(x)) + p(f^n(x), x^*) - p(x^*, x^*).$$

Thus, from (p_2), we have

$$p(f^n(x), x^*) - p(f^n(x), f^n(x)) \to 0 \text{ as } n \to \infty$$

and

$$p(f^n(x), x^*) - p(x^*, x^*) \to 0 \text{ as } n \to \infty.$$

Hence

$$p(f^n(x), x^*) \to 0 \text{ as } n \to \infty$$

and

$$p(x^*, x^*) = 0.$$

From (p_4) it follows that $p(x^*, f(x^*)) = 0$.

Now we have

$$p(x^*, x^*) = p(f(x^*), f(x^*)) = p(x^*, f(x^*)).$$

This implies that $x^* = f(x^*)$.

Hence, $F_f = \{x^*_f\}$.

(4). From $p(x, x^*_f) \leq p(x, f(x)) + p(f(x), x^*_f) \leq p(x, f(x)) + \alpha p(x, x^*_f)$ we have

$$p(x, x^*_f) \leq \frac{1}{1-\alpha} p(x, f(x)), \text{ for all } x \in X.$$

(5). Let $x_n \in X$, $n \in \mathbb{N}$ such that $p(x_n, f(x_n)) \to 0$ as $n \to \infty$. From (4) it follows that

$$p(x_n, x^*_f) \leq \frac{1}{1-\alpha} p(x_n, f(x_n)) \to 0 \text{ as } n \to \infty.$$

(6). Let $x_n \in X$, $n \in \mathbb{N}$, such that

$$p(x_{n+1}, f(x_n)) \to 0 \text{ as } n \to \infty.$$

Let $x \in X$. From (3) we have that

$$p(f^n(x), x^*_f) \to 0 \text{ as } n \to \infty.$$

Now, we need to prove that

$$p(x_n, x^*_f) \to 0 \text{ as } n \to \infty.$$

We have $p(x_{n+1}, x^*_f) \leq p(x_{n+1}, f(x_n)) + p(f(x_n), x^*_f)$
$\leq p(x_{n+1}, f(x_n)) + \alpha p(x_n, x^*_f)$

$\leq p(x_{n+1}, f(x_n)) + \alpha p(x_n, f(x_{n-1})) + \alpha^2 p(x_{n-1}, x_f^*)$
$\leq p(x_{n+1}, f(x_n)) + \alpha p(x_n, f(x_{n-1})) + \cdots + \alpha^{n+1} p(x_0, x_f^*)$.
From a Cauchy type lemma we have that

$$p(x_{n+1}, x_f^*) \to 0 \text{ as } n \to \infty.$$

(7). From relation (4) we have that

$$p(x_f^*, x_g^*) \leq \frac{1}{1-\alpha} p(x_g^*, f(x_g^*)) = \frac{1}{1-\alpha} p(g(x_g^*), f(x_g^*)) \leq \frac{\eta}{1-\alpha}.$$

□

Graphic Contraction Principle. (I.A. Rus B[104]) *Let (X, p) be a complete partial metric space and $f : X \to X$ be an operator. We suppose that:*

(i) there exists $0 < \alpha < 1$ such that $p(f^2(x), f(x)) \leq \alpha p(x, f(x))$, for all $x \in X$;

(ii) $f : (X, d_p) \to (X, d_p)$ has closed graph.

Then we have:

(1) $F_f = F_{f^n} \neq \emptyset$, for all $n \in \mathbb{N}^*$;

(1') $p(x^*, x^*) = 0$, for all $x^* \in F_f$;

(2) $f : (X, d_p) \to (X, d_p)$ is a WPO;

(3) $p(f^n(x), f^\infty(x)) \to 0$ as $n \to \infty$, for each $x \in X$;

(4) $p(x, f^\infty(x)) \leq \frac{1}{1-\alpha} p(x, f(x))$, for all $x \in X$, i.e., $f : (X, p) \to (X, p)$ is a $\frac{1}{1-\alpha}$-WPO;

(5) If $g : X \to X$ is c-WPO and

$$p(f(x), g(x)) \leq \eta, \quad \text{for all } x \in X, \text{ for some } \eta > 0,$$

then

$$H_p(F_f, F_g) \leq \max\left(\frac{1}{1-\alpha}, c\right)\eta,$$

where H_p stands for the Pompeiu-Hausdorff functional.

Proof. The proof of (1)+(1')+(2)+(3) to that of (1)+(2)+(3) in the Contraction principle.

(4). Let $x \in X$. We have

$$p(x, f^\infty(x)) \leq p(x, f^n(x)) + p(f^n(x), f^\infty(x))$$

$$\leq p(x,f(x)) + p(f(x),f^2(x)) + \cdots + p(f^{n-1}(x),f^n(x)) + p(f^n(x),f^\infty(x))$$
$$\leq (1+\alpha+\cdots+\alpha^{n-1})p(x,f(x)) + p(f^n(x),f^\infty(x))$$
$$\leq \frac{1}{1-\alpha}p(x,f(x)) + p(f^n(x),f^\infty(x)), \quad \text{for all} \quad n \in \mathbb{N}^*.$$

From (3) it follows that

$$p(x,f^\infty(x)) \leq \frac{1}{1-\alpha}p(x,f(x)), \text{ for each } x \in X.$$

(5). The proof follows from (4) and the definition of H_p. □

The above results give rise to the following problems:

Problem 4.2.1. For which generalized contractions on a complete partial metric space we have a fixed point principle ?

Problem 4.2.2. If $f : (X,p) \to (X,p)$ is a generalized contraction, which condition satisfies f with respect to d_p ?

Problem 4.2.3. From Problem 4.2.2. we shall obtain some new classes of operators on a metric spaces. The problem is to give fixed point theorems for these new classes of operators.

Problem 4.2.4. How on can use the results given for the Problem 4.2.2. and Problem 4.2.3., to study the Problem 4.2.1. ?

For example if $f : (X,p) \to (X,p)$ is an α-contraction, then

$$d_p(f(x),f(y)) \leq \alpha d_p(x,y) + \alpha p(x,x) - p(f(x),f(x)) + \alpha p(y,y) - p(f(y),f(y)).$$

On the other hand we observe that (f,ψ) is a Schröder pair, where $\psi(x) := p(x,x)$.

So, a new metric conditions in a metric space is the following:

$$d_p(f(x),f(y)) \leq \alpha d_p(x,y) + \alpha\psi(x) - \psi(f(x)) + \alpha\psi(y) - \psi(f(y)), \quad \text{for all} \quad x,y \in X,$$

where (f,ψ) is a Schröder pair.

For more considerations of the Problems 4.2.1.-4.2.4. see I.A. Rus B[104]. For Schröder pairs in a metric space see I.A. Rus, A. Petruşel and M.A. Şerban B[1]. See also 1.4.

4.3 Fixed point theory in gauge spaces

4.3.1 Uniform spaces. Gauge spaces

Let X be a nonempty set. Then, the functional $d : X \times X \to \mathbb{R}_+$ is a pseudometric on X if:

(i_1) $d(x, x) = 0,$ for all $x \in X$;

(ii) $d(x, y) = d(y, x),$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y),$ for all $x, y, z \in X.$

Let us notice that, sometimes, the term "gauge" is used instead of that of "pseudometric".

The following functionals $d : X \times X \to \mathbb{R}_+$ are pseudometrics on X:

(1) $d(x, y) := |f(x) - f(y)|$, where $f : X \to \mathbb{R}_+$ is an arbitrary functional and X is a nonempty set;

(2) $X := C^1[a, b],\ d(x, y) := \max\limits_{a \leq t \leq b} |x'(t) - y'(t)|$;

(3) $X = C[a, b],\ d(x, y) := |x(a) - y(a)|$.

Let X be any set and $d_\alpha = X \times X \to \mathbb{R}$, $\alpha \in \mathcal{A}$ be a family of pseudometrics on X.

Definition 4.3.1. A family $\mathcal{D} = \{d_\alpha \mid \alpha \in \mathcal{A}\}$ of pseudometrics on X is called separating if for each pair of points $x \neq y$ there exists a $d_\alpha \in \mathcal{D}$ such that $d_\alpha(x, y) \neq 0$.

Definition 4.3.2. Let X be a set and $\mathcal{D} = \{d_\alpha \mid \alpha \in \mathcal{A}\}$ be a separating family of pseudometrics on X.

The topology $\tau_\mathcal{D}$ having for a subbasis the family

$$\mathcal{B}(\mathcal{D}) = \{B(y; d_\alpha, \varepsilon) \mid y \in X,\ d_\alpha \in \mathcal{D},\ \varepsilon > 0\}$$

of balls is called the topology in X induced by the family $\mathcal{D}$ where

$$B(y; d_\alpha, \varepsilon) = \{x \mid d_\alpha(x, y) < \varepsilon,\ \alpha \in \mathcal{A}\}$$

is the d_α-ball of radius ε centered at y.

Because we require $\mathcal{D}$ to be separating, it follows that:

a) the topology $\tau_\mathcal{D}$ is always Hausdorff

and

b) if $\mathcal{D}$ consists of one pseudometric alone, then that pseudometric must be a metric and $\tau_\mathcal{D}$ is the topology induced by that metric.

Definition 4.3.3. A gauge structure for a topological space (X,τ) is a separating family $\mathcal{D}$ of pseudometrics such that $\tau = \tau_\mathcal{D}$. A gauge space is a set X endowed with a separating family $\mathcal{D} = \{d_\alpha \mid \alpha \in \mathcal{A}\}$ of pseudometrics on X.

Also recall that a uniform structure on a set X is a family $\mathcal{U}$ of subsets of $X \times X$ such that:

(i) if $U \in \mathcal{U}$, then $\Delta \subset U$;

(ii) if $U_1, U_2 \in \mathcal{U}$, then there exists $W \in \mathcal{U}$ such that $W \subset U_1 \cap U_2$;

(iii) if $U \in \mathcal{U}$, then there exists $W \in \mathcal{U}$ such that $W \circ W^{-1} \subset U$. (where $U \circ V := \{(x,z) \mid$ there is y such that $(x,y) \in V$ and $(y,z) \in U\}$);

(iv) if $U \in \mathcal{U}$ and $U \subset V$, then $V \in \mathcal{U}$.

A family satisfying (i)-(iii) is called a base for a uniform structure on X, or simply a uniformity on X. If $(X,\mathcal{D})$ is a gauge space, then the family of sets $\{(x,y) \in X \times X \mid d(x,y) < \epsilon\}$, for all $d \in \mathcal{D}$ and each $\epsilon > 0$ is a uniformity on X, called the uniformity generated by $\mathcal{D}$.

Let $\mathcal{D} = \{d_\alpha \mid \alpha \in \mathcal{A}\}$ be a separating family of pseudometrics in X.
Let $\mathcal{D}^+$ be the family of pseudometrics

$$\{\max(d_{\alpha_1}, \ldots, d_{\alpha_n}) \mid \text{ all finite subsets } \{\alpha_1, \ldots, \alpha_n\} \subset \mathcal{A}\}.$$

Then the family $\mathcal{B}(\mathcal{D}^+)$ of all balls is a basis for $\tau_\mathcal{D}$.

Theorem 4.3.1. *(1) Let the space X have the gauge structure $\mathcal{D}$ and let A be a subspace of X. Let $\mathcal{D}_A$ be the family of pseudometrics in $\mathcal{D}$, each restricted to $A \times A$. Then $\mathcal{D}_A$ is a gauge structure for the subspace A.*

(2) Let $\{(X_\beta, \tau_{\mathcal{D}_\beta}) \mid \beta \in \mathcal{B}\}$ be any family of gauge spaces. For each $\beta \in \mathcal{B}$, let $\mathcal{D}_\beta$ be the family of pseudometrics induced on $\prod_\beta X_\beta$ by the members of $\mathcal{D}_\beta$.

Then the family $\{\mathcal{D}_\beta \mid \beta \in \mathcal{B}\}$ of pseudometrics induces a gauge structure for the cartesian product topology of the gauge spaces.

Corollary 4.3.1. *A completely regular space is metrizable if and only if it admits a countable gauge structure.*

4.3.2 Complete gauge structures

Definition 4.3.4. Let X be a nonempty set. A filterbase $\mathcal{U}$ in X is a family

$$\mathcal{U} = \{A_\alpha \mid \alpha \in \mathcal{A}\}$$

of subsets of X having the two properties:

(1) for all $\alpha \in \mathcal{A}:\ A_\alpha \neq \emptyset$

(2) for all $\alpha \in \mathcal{A},\ \beta \in \mathcal{A},\ $ there exists $\gamma \in \mathcal{A}:\ A_\gamma \subset A_\alpha \subset A_\beta$.

Let d be a pseudometric on a nonempty set X. The d-diameter of a set $A \subset X$ is defined, as for metrics, to be $\delta(A) := \sup\{d(x,y) \mid x, y \in A\}$.

Also recall that, a d-Cauchy filterbase on a set X endowed with a pseudometric d is defined exactly as in the case of a metric space, i.e. as follows:

Definition 4.3.5. A filterbase $\mathcal{U} = \{A_\alpha \mid \alpha \in \mathcal{A}\}$ in a metric space (X, d) is called a d-Cauchy filterbase if for each $\varepsilon > 0$ there is some A_α with $\delta(A_\alpha) < \varepsilon$.

Definition 4.3.6. A filterbase $\mathcal{U} = \{A_\alpha \mid \alpha \in \mathcal{A}\}$ in a gauge space $(X, \tau_\mathcal{D})$ is called a $\mathcal{D}$-Cauchy filterbase if it is d-Cauchy filterbase for each $d \in \mathcal{D}$.

Definition 4.3.7. (1) Let $\mathcal{U} = \{A_\alpha \mid \alpha \in \mathcal{A}\}$ and $\mathcal{B} = \{B_\beta \mid \beta \in \mathcal{B}\}$ be two filterbases on X. Then, $\mathcal{B}$ is subordinate to $\mathcal{U}$ (we write "$\mathcal{B} \vdash \mathcal{U}$") if:

$$\text{for all } A_\alpha, \text{ there exists } B_\beta:\ B_\beta \subset A_\alpha, \text{ for all } \alpha \in \mathcal{A} \text{ and } \beta \in \mathcal{B}.$$

(2) Let $\mathcal{U} = \{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a filterbase in X. Then:

(a) $\mathcal{U}$ converges to y_0 (we write $\mathcal{U} \to y_0$) if:

$$\text{for all } U(y_0), \text{ there exists } A_\alpha: \quad A_\alpha \tau U$$

(b) $\mathcal{U}$ accumulates at y_0 (we write $\mathcal{U} \perp y_0$) if:

$$\text{for all } U(y_0), \text{ for all } A_\alpha: \quad A_\alpha \cap U \neq \emptyset.$$

Remark 4.3.1. a) $\mathcal{U} \perp y_0$ if and only if $y_0 \in \bigcap_\alpha \overline{A_\alpha}$.

b) If $\mathcal{U} \subset \mathcal{B}$, then $\mathcal{B} \vdash \mathcal{U}$.

c) If $\mathcal{B} \vdash \mathcal{U}$, then each member of $\mathcal{B}$ meets every member of $\mathcal{U}$.

d) $\mathcal{U} \to y_0$ if and only if $\mathcal{U} \vdash U(y_0)$.

Theorem 4.3.3. *Let* $(X, \tau_\mathcal{D})$ *be a gauge space. Then:*

(1) every convergent filterbase is $\mathcal{D}$*-Cauchy*

(2) If $\mathcal{U}$ *is* $\mathcal{D}$*-Cauchy and if* $\mathcal{B} \vdash \mathcal{U}$*, then* $\mathcal{B}$ *is* $\mathcal{D}$*-Cauchy.*

(3) If $\mathcal{U}$ *is* $\mathcal{D}$*-Cauchy, and if* $\mathcal{U} \perp y_0$*, then* $\mathcal{U} \to y_0$*.*

Definition 4.3.8. A gauge structure $\mathcal{D}$ for a space X is called complete if every $\mathcal{D}$-Cauchy filterbase in X converges. A completely regular space having a complete gauge structure $\mathcal{D}$ is called $\mathcal{D}$-complete.

Theorem 4.3.4. *(1) If* X *is* $\mathcal{D}$*-complete and* $A \subset X$ *is closed, then* A *is* $\mathcal{D}_A$*-complete.*

(2) If $(X, \tau_\mathcal{D})$ *is any gauge space, and if* $A \subset X$ *is* $\mathcal{D}_A$*-complete, then* A *is closed in* X*.*

4.3.3 Fixed point theory in gauge spaces

In this section we will present the fixed point theory in a gauge space, i.e., a space X endowed with a gauge structure induced by a separating family of pseudometrics $\{d_\alpha\}_{\alpha \in I}$, where I is a directed set.

To our best knowledge, the first results in this direction were proved by Colojoară in 1961 and by Knill in 1965. In 1971, Cain and Nashed proved the following:

Theorem 4.3.5. (Cain-Nashed R[1]) *Let* $(E, (\|\cdot\|_\alpha)_{\alpha \in I})$ *be a Hausdorff locally convex topological vectorial space,* X *be a sequentially complete subset of* E *and* $f : X \to X$ *be a contraction, i.e. for all* $\alpha \in I$, $\exists\ k_\alpha \in [0, 1[$ *such that* $\|f(x) - f(y)\|_\alpha \leq k_\alpha \|x - y\|_\alpha$*, for all* $x, y \in X$*.*

Then $F_f = \{x^*\}$*.*

An interesting result for a contraction on a set with two gauge structures was established by N. Gheorghiu in 1982.

Theorem 4.3.6. (Gheorghiu B[1]) *Let* E *be a set endowed with two gauge structures* $\mathcal{D} = \{\rho_\alpha\}_{\alpha \in I}$ *and* $\mathcal{C} = \{d_\alpha\}_{\alpha \in J}$*. Suppose that* $(E, \mathcal{D})$ *is complete. Let* $f : E \to E$ *be such that:*

a) $f : (C, \mathcal{D}) \to (E, \mathcal{C})$ *is sequentially continuous;*

b) there exists $\phi : J \to J$ *such that for all* $\alpha \in J$*, there exists* $k_\alpha \in [0, 1[$

such that for every $x, y \in E$ we have:

$$d_\alpha(f(x), f(y)) \leq k_\alpha d_{\phi(\alpha)}(x, y);$$

c) $\sum_{n=1}^{\infty} k_\alpha k_{\phi(\alpha)} \ldots k_{\phi^{n-1}(\alpha)} \cdot d_{\phi^n(\alpha)}(x, y) < \infty$, *for all* $x, y \in E$

d) there exists $\psi : I \to J$ *such that for all* $\alpha \in I$ *there is* $C_\alpha > 0$ *such that*

$$\rho_\alpha(x, y) \leq C_\alpha d_{\psi(\alpha)}(x, y), \quad \text{for all } x, y \in E.$$

Then $F_f = \{x^*\}$.

Let $(E, \{d_\alpha\}_{\alpha \in I})$ be a gauge space endowed with a complete gauge structure $\{d_\alpha\}_{\alpha \in I}$ (where I is a directed set) (briefly we will call it a complete gauge space). Let $r = \{r_\alpha\}_{\alpha \in I} \in (0, \infty)^I$ and $x_0 \in E$. We denote the pseudo-ball centered at x_0 of radius r by

$$\widetilde{B}(x_0, r) := \{x \in E \mid d_\alpha(x_0, x) \leq r_\alpha, \text{ for all } \alpha \in I\}$$

For Meir-Keller type operators, the following result was established by Agarwal, O'Regan and Shahzad in 2004.

Theorem 4.3.7. (Agarwal-O'Regan-Shahzad R[1]) *Let E be a complete gauge space $x_0 \in E$, $r = \{r_\alpha\}_{\alpha \in I} \in]0, \infty[^I$ and $f : \widetilde{B}(x_0, r) \to E$ be a continuous operator with respect to d_α, for each $\alpha \in I$.*

Suppose for each $\varepsilon = \{\varepsilon_\alpha\}_{\alpha \in I} \in]0, \infty[^I$ there is $\delta = \{\delta_\alpha\}_{\alpha \in I} \in]0, \infty[^I$ such that: $x, y \in \widetilde{B}(x_0, r)$ and $\alpha \in I$ with $M_\alpha(x, y) < \varepsilon_\alpha + \delta_\alpha$ implies $d_\alpha(f(x), f(y)) < \varepsilon_\alpha$ and $d_\alpha(f(x), f(y)) \leq M_\alpha(x, y)$ if $M_\alpha(x, y) = 0$, where

$$M_\alpha(x, y) = \max\{d_\alpha(x, y), d_\alpha(x, f(x)), d_\alpha(y, f(y)), \frac{1}{2}[d_\alpha(x, f(y)) + d_\alpha(y, f(x))]\}.$$

Also suppose that, for each $\alpha \in I$, $d_\alpha(x_0, f^n(x_0)) < r_\alpha$ for each $n \in \mathbb{N}^*$ $(*)$.

Then $F_f = \{x^*\}$.

Proof. We organize the proof in several steps.

Step 1. The following two assertions are equivalent.

(1) for all $\varepsilon = \{\varepsilon_\alpha\}_{\alpha \in I} \in (0, \infty)^I$ there is $\delta = \{\delta_\alpha\}_{\alpha \in I} \in (0, \infty)^I$ such that if $x, y \in \widetilde{B}(x_0, r)$ and $\alpha \in I$ then $M_\alpha(x, y) < \varepsilon_\alpha + \delta_\alpha \Rightarrow d_\alpha(f(x), f(y)) < \varepsilon_\alpha$ and $d_\alpha(f(x), f(y)) \leq M_\alpha(x, y)$ if $M_\alpha(x, y) = 0$.

(2) for all $\varepsilon = \{\varepsilon_\alpha\}_{\alpha \in I} \in (0,\infty)^I$ there is $\delta = \{\delta_\alpha\}_{\alpha \in I} \in (0,\infty)^I$ such that if $x, y \in \widetilde{B}(x_0, r)$ and $\alpha \in I$ then $\varepsilon_\alpha \le M_\alpha(x,y) < \varepsilon_\alpha + \delta_\alpha \Rightarrow d_\alpha(f(x), f(y)) < \varepsilon_\alpha$.

Step 2. If $x, y \in \widetilde{B}(x_0, r)$, $\alpha \in I$ and $M_\alpha(x,y) \neq 0$ then $d_\alpha(f(x), f(y)) < M_\alpha(x,y)$.

Step 3. Let $x_n = f(x_{n-1})$, $n \in \mathbb{N}^*$. Let $\alpha \in I$. If $d_\alpha(x_n, x_{n+1}) = 0$ for some $n \in \mathbb{N}^*$ then $d_\alpha(x_n, x_{n+1}) \le d_\alpha(x_{n-1}, x_n)$.

If $d_\alpha(x_n, x_{n+1}) > 0$ for each $n \in \mathbb{N}^*$ then

$$d_\alpha(x_n, x_{n+1}) = d_\alpha(f(x_{n-1}), f(x_n))) < M_\alpha(x_{n-1}, x_n)$$

$$\le \max\{d_\alpha(x_{n-1}, x_n), d_\alpha(x_n, x_{n+1}), \frac{1}{2}[d_\alpha(x_{n-1}, x_n) + d_\alpha(x_n, x_{n+1})]\}$$

$$= \max\{d_\alpha(x_{n-1}, x_n), d_\alpha(x_n, x_{n+1})\} = d_\alpha(x_{n-1}, x_n).$$

Hence, in both cases $(d_\alpha(x_n, x_{n+1}))_{n \in \mathbb{N}}$ is decreasing. Consequently

$$d_\alpha(x_n, x_{n+1}) \searrow \varepsilon_\alpha$$

as $n \to \infty$, where $\varepsilon_\alpha \ge 0$. Notice that $d_\alpha(x_n, x_{n+1}) \ge \varepsilon_\alpha$.

If we suppose, by reductio ad absurdum that $\varepsilon_\alpha > 0$ then there exists $\delta_\alpha > 0$ such that $M_\alpha(x,y) < \varepsilon_\alpha + \delta_\alpha \Rightarrow d_\alpha(f(x), f(y)) < \varepsilon_\alpha$. On the other hand, there exists $N \in \mathbb{N}^*$ such that

$$d_\alpha(x_n, x_{n+1}) < \varepsilon_\alpha + \delta_\alpha, \quad \text{for all} \quad n \ge N.$$

Since $d_\alpha(x_{n-1}, x_n) = M_\alpha(x_{n-1}, x_n)$ we get for $n \ge N+1$ that

$$d_\alpha(x_n, x_{n+1}) = d_\alpha(f(x_{n-1}), f(x_n)) < \varepsilon_\alpha.$$

This is a contradiction. Hence $\varepsilon_\alpha = 0$ and $d_\alpha(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Step 4. $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to d_α and $x_n \in \widetilde{B}(x_0, r)$, $n \in \mathbb{N}^*$.

Step 5. Denote $x^* \in \widetilde{B}(x_0, r)$ the limit of the sequence $\{x_n\}_{n \in \mathbb{N}}$. Then $F_f = \{x^*\}$.

Indeed, for the continuity of f we get that $x^* \in F_f$. The uniqueness follows by contradiction from the Meir-Keeler type condition.

Remark 4.3.2. For a global result, we can drop the condition $(*)$.

A local Caristi type result is:

Theorem 4.3.8. (Agarwal-O'Regan-Shahzad R[1]) *Let E be a complete gauge space, $x_0 \in E$. $r = \{r_\alpha\}_{\alpha\in I} \in]0,\infty[^I$ and $f : \widetilde{B}(x_0, r) \to E$. Suppose for each $\alpha \in I$ there exists $\phi_\alpha : E \to [0,\infty[$ such that for each $x \in \widetilde{B}(x_0, r)$ we have $d_\alpha(x, f(x)) \leq \phi_\alpha(x) - \phi_\alpha(f(x))$ and $\phi_\alpha(x_0) < r_\alpha$.*

Also suppose that for each $\alpha \in I$, $d_\alpha(x_n, x) \to 0$ implies $d_\alpha(x, f(x)) = 0$.

Then $F_f \neq \emptyset$.

Since the notions of contraction or of Meir-Keeler type operator in a gauge space seem to be too restrictive, M. Frigon introduced the concept of generalized contraction on a gauge space, as follows.

Let $(E, \{d_\alpha\}_{\alpha\in I})$ be a complete gauge space satisfying the condition: I is a directed set such that $\alpha \leq \beta$ implies $d_\alpha(x, y) \leq d_\beta(x, y)$.

We associate to every $\alpha \in I$ a metric space $(\mathbb{E}_\alpha, d_\alpha)$ as follows:

for each $\alpha \in I$, $x \sim_\alpha y \Leftrightarrow d_\alpha(x, y) = 0$ (**).

This is an equivalence relation on E.

Denote $E_\alpha = (E/\sim_\alpha, d_\alpha)$ the quotient space and by $(\mathbb{E}_\alpha, d_\alpha)$ the completation of E_α with respect to d_α.

This construction induces a continuous map $\mu_\alpha : E \to \mathbb{E}_\alpha$.

The pseudometric d_α induces a pseudometric on $\mathbb{E}_\beta$, for every $\beta \geq \alpha$. This pseudometric is again denoted by d_α.

In a similar way the equivalence relation (**) on $\mathbb{E}_\beta$ induces a continuous mapping $\mu_{\alpha\beta} : \mathbb{E}_\beta \to \mathbb{E}_\alpha$, since $\mathbb{E}_\beta/\sim_\alpha$ can be regarded as a subset of $\mathbb{E}_\alpha$.

Let $X \subset E$ and $f : X \to E$ be an operator.

For every $\alpha \in I$ we consider the multivalued operator $f_\alpha : X_\alpha \multimap \mathbb{E}_\alpha$

$$f_\alpha(\mu_\alpha(x)) = \overline{\mu_\alpha \circ f(\{x\}_\alpha)}$$

where $\{x\}_\alpha \overset{not}{:=} \{y \in X \mid d_\alpha(x, y) = 0\}$.

When exists, the multivalued continuous extension of f_α is denoted by $\widetilde{f}_\alpha : \overline{X}_\alpha \multimap \mathbb{E}_\alpha$.

Denote by $\{H_\alpha\}_{\alpha\in I}$ a family of generalized pseudometrics on $P(E)$ and by

$$\delta_\alpha(A) := \sup\{d_\alpha(a, b) \mid a, b \in A\}$$

where $A \subset E$.

Definition 4.3.9. An operator $f : X \to E$ is a generalized contraction if:

(i) for all $\alpha \in I$, there exists $k_\alpha \in]0,1[$ such that

$$H_\alpha(f(\{x\}_\alpha), f(\{f\}_\alpha)) \le k_\alpha d_\alpha(x,y), \quad \text{for all } x, y \in X$$

(ii) for all $\varepsilon > 0$, and all $\alpha \in I$, there exists $\beta \ge \alpha$ such that

$$\delta_\beta(f(\{x\}_\alpha)) < (1 - k_\alpha)\varepsilon, \quad \text{for all } x \in X$$

Theorem 4.3.9. (Frigon R[2]) *Let $(E, \{d_\alpha\}_{\alpha \in I})$ be a complete gauge space such that $\alpha \le \beta \Rightarrow d_\alpha \sigma(x,y) \le d_\beta(x,y)$, for all $x, y \in E$. Let $f : E \to E$ be a generalized contraction. Then $F_f = \{x^*\}$.*

Since, in the above result the fixed point is not obtained as a limit of the successive approximation sequence, it was an open question to give such a result.

Positive answers were given by:

i) R. Espínola and W.A. Kirk R[2], for contractions

and

ii) R. Espínola and A. Petruşel B[1], for φ-contractions.

4.3.4 Other results

For other results see M. Frigon R[1] and the references therein.

For other extensions see:

• L. Collatz R[1]-R[2]: for fixed point results with respect to pseudo-metrics with values in a partially ordered vector space;

• S. Heikkila R[1]: for fixed point results with respect to pseudo-metrics with values in the positive cone of an abelian group;

• P. V. Subrahmanyam R[1]; for a fixed point result for contractions on quasi-gauge spaces in the sense of I.L. Reilly R[3].

For other results, see Gh. Marinescu B[2], I. Colojoară B[1], A. Deleanu and Gh. Marinescu B[1], C. Tudor B[1], O. Hadžić R[1], F. Gândac B[1]-B[10], N. Gheorghiu and E. Rotaru B[1], I.A. Rus B[87], C.M. Lee R[1], V. Angelov

R[1], R[2], R[5], R[6], V. Angelov and I.A. Rus B[1], R. Precup B[26], B[27], A. Chiş and R. Precup B[1]. See also 6.2.

4.4 Fixed point theorems in semimetric spaces

For the fixed point theory in a semimetric space, see J. Jachymski, J. Matkowski and T. Swiatkowski R[1] and the references therein (M. Cicchese (1976), T.L. Hicks (1992), T.L. Hicks and B.E. Rhoades (1992), etc.).

Chapter 5

Generalized contractions on g.m.s. $(d(x,y) \in \mathbb{R}_+ \cup \{+\infty\})$

Guidelines: W.A.J. Luxemburg (1958), A.F. Monna (1961), M. Edelstein (1964), J.B. Diaz and B. Margolis (1968), C.F.K. Jung (1969), S. Kasahara (1975).
General references: T. van der Walt R[1], C.F.K. Jung R[1], J.B. Diaz and B. Margolis R[1], I.A. Rus B[81], B[73], A. Petruşel, I.A. Rus and M.A. Şerban B[1].

5.0 Generalized metric space $(d(x,y) \in \mathbb{R}_+ \cup +\infty)$

In this chapter, by a generalized metric (or a Luxemburg metric) on a set X, we understand a functional $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ which satisfies the following axioms:

(i) $d(x,y) = 0$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$, for all $x, y \in X$;
(iii) $x, y, z \in X$, $d(x,z) < +\infty$, $d(z,y) < +\infty$ imply

$$d(x,y) \leq d(x,z) + d(z,y).$$

For example, the following functionals are Luxemburg's metrics:
(1) $X = C(\mathbb{R}_+)$ and $d(x,y) := \sup\limits_{0 \leq t < +\infty} |x(t) - y(t)|$

(2) Let $\tau > 0$. $X = C(\mathbb{R}_+)$ and $d(x,y) := \sup_{0\le t<+\infty} (|x(t) - y(t)|e^{-\tau t})$

(3) $X = C[0,1]$, $d(x,y) = \sup_{0<t\le 1} (t^{-2}|x(t) - y(t)|)$

As usual, we denote $B(x_0;r) := \{x \in X \mid d(x_0,x) < r\}$ and $\widetilde{B}(x_0;r) := \{x \in X \mid d(x_0,x) \le r\}$.

If (X,d) is a generalized metric space, then the metric topology induced on X by d is given by:

$$\tau_d := \{Y \subset X \mid y \in Y \Rightarrow \exists\, r > 0 : \ B(y;r) \subset Y\}.$$

By this definition, it follows that if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X and $x^* \in X$, then

$$x_n \xrightarrow{\tau_d} x^* \text{ as } n \to \infty \text{ if and only if } d(x_n,x^*) \to 0 \text{ as } n \to \infty.$$

Let (X,d)be an operator. Then by definition f is said to be:

(1) continuous if $x_n \to x^*$ implies $f(x_n) \to f(x^*)$;

(2) with closed graph if $x_n \to x^*$, $f(x_n) \to y^*$ imply $f(x^*) = y^*$;

(3) Lipschitz if there exists $L_f > 0$ such that:

$$x,y \in X,\ d(x,t) < +\infty \text{ imply } \rho(f(x),f(y)) \le L_f d(x,y).$$

(4) graphic α-contraction if $0 \le \alpha < 1$ and $x \in X$, $d(x,f(x)) < +\infty$ imply $d(f^2(x),f(x)) \le \alpha d(x,f(x))$.

The following result is a characterization of a generalized metric space.

Jung's Theorem. (Jung R[1]). *Let (X,d) be a generalized metric space. Then there exists a partition $X = \bigcup_{i\in I} X_i$ of X such that $d_i := d|_{X_i\times X_i}$ is a metric on X_i, for each $i \in I$. Moreover, (X,d) is complete if and only if (X_i,d_i) is complete, for each $i \in I$.*

Proof. We consider on X the following relation

$$x,y \in X,\ x \sim y \text{ if } d(x,y) < +\infty.$$

This relation is an equivalence relation and our partition is generated by this equivalence relation.

5.1 Fixed point theory in g.m.s. $(d(x,y) \in \mathbb{R}_+ \cup \{+\infty\})$

We present first some important auxiliary results.

Lemma 5.1.1. *Let (X,d) be a complete generalized metric space and $f : X \to X$ be an α-contraction. The following statements are equivalent:*

i) $F_f \neq \emptyset$;

ii) there exists $x \in X$ such that $d(x, f(x)) < +\infty$;

iii) there exist $x \in X$ and $n(x) \in \mathbb{N}$ such that $d\left(f^{n(x)}(x), f^{n(x)+1}(x)\right) < +\infty$;

iv) there exists $i \in I$ such that $X_i \in I(f)$.

Proof. $i) \Longrightarrow ii)$ Let $x^* \in F_f$. We have

$$d(x^*, f(x^*)) = d(x^*, x^*) = 0 < +\infty.$$

$ii) \Longrightarrow iii)$ We choose $n(x) = 0$;

$iii) \Longrightarrow i)$ Since f is an α-contraction we have that $(f^n(x))$ is a Cauchy sequence. This implies $f^n(x) \to x^*$, as $n \to +\infty$, for each $x \in X$. From the continuity of f it follows that $x^* \in F_f$.

$ii) \Longrightarrow iv)$ Since $d(x, f(x)) < +\infty$, there exists $i \in I$ such that $x \in X_i$. Let $y \in X_i$. Then $d(x,y) < +\infty$. We have:

$$d(x, f(y)) \leq d(x, f(x)) + d(f(x), f(y)) \leq d(x, f(x)) + \alpha \cdot d(x,y) < +\infty$$

which implies $f(y) \in X_i$.

$iv) \Longrightarrow ii)$ Let $x \in X_i$. Since $X_i \in I(f)$, we get that $f(x) \in X_i$. Therefore $d(x, f(x)) < +\infty$. □

Lemma 5.1.2. *Let (X,d) be a complete generalized metric space and $f : X \to X$ be an α-contraction. We suppose that:*

i) there exists $x \in X$ such that $d(x, f(x)) < +\infty$;

ii) if $u, v \in F_f$ then $d(u,v) < +\infty$;

Then:

a) $F_f = \{x^*\}$;

b) $f\Big|_{X_{i(x)}} : X_{i(x)} \to X_{i(x)}$ *is a Picard operator.*

Proof. From $i)$ and Lemma 5.1.1. we have that there exists $i \in I$ such that $X_i \in I(f)$, $f^n(x) \in X_i$ for every $n \in \mathbb{N}$, $F_f \neq \emptyset$, $f^n(x) \to x^* \in F_f \cap X_i$.

Let $u, v \in F_f$. Then $d(u,v) < +\infty$ and

$$d(u,v) = d(f(u), f(v)) \leq \alpha \cdot d(u,v).$$

Therefore $d(u,v) = 0$, which implies $u = v$. Hence $F_f = \{x^*\}$.

Since $X_i \in I(f)$ then $d(y, f(y)) < +\infty$ for every $y \in X_i$ and applying again Lemma 5.1.1. we get that $f\Big|_{X_{i(x)}} : X_{i(x)} \to X_{i(x)}$ is a Picard operator. □

The main fixed point result for operators on generalized complete metric space is the following.

Theorem 5.1.1. (Luxemburg R[1], A. Petruşel, I.A. Rus and M.A.Şerban B[1]). *Let (X,d) be a complete generalized metric space and $f : X \to X$. We suppose that:*

i) f is an α-contraction;

ii) for every $x \in X$ there exists $n(x) \in \mathbb{N}$ such that $d\left(f^{n(x)}(x), f^{n(x)+1}(x)\right) < +\infty$.

Then:

a) f is a weakly Picard operator. If in addition, for every $x \in X$ we have $d(x, f(x)) < +\infty$, then f is $\frac{1}{1-\alpha}$-weakly Picard;

b) If, in addition:

b_1) for every $x \in X$ we have $d(x, f(x)) < +\infty$;

b_2) $u, v \in F_f$ implies $d(u,v) < +\infty$,

then f is $\frac{1}{1-\alpha}$-Picard.

Proof. *a*) The first part follows from Lemma 5.1.1. and Lemma 5.1.2. For the second conclusion, notice that for every $x \in X$ such that $d(x, f(x)) < +\infty$ and each $n \in \mathbb{N}$ we have:

$$d(f^n(x), f^\infty(x)) \leq \frac{\alpha^n}{1-\alpha} \cdot d(x, f(x))$$

which implies

$$d(x, f^\infty(x)) \leq \frac{1}{1-\alpha} \cdot d(x, f(x)).$$

b) From b_2) we obtain $F_f = \{x^*\}$ and from *a*) we obtain that f is $\frac{1}{1-\alpha}$-Picard operator.□

The above result can be presented as the following alternative theorem:

Theorem 5.1.2. (J.B. Diaz and B. Margolis R[1]) *Let (X,d) be a generalized complete metric space and $f : X \to X$ an α-contraction. Let $x \in X$ be arbitrarily chosen in X. Then, with respect to the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations, the following alternative holds:*
either

(a) $d(f^n(x), f^{n+1}(x)) = \infty$, for all $n \in \mathbb{N}$;

or

(b) $f^n(x) \xrightarrow{d} x^* \in F_f$ as $n \to \infty$.

Other results are the following ones.

Theorem 5.1.3. *Let (X,d) be a generalized complete metric space and $f : X \to X$ be an operator. We suppose that:*

(i) f is an graphic α-contraction;

(ii) for every $x \in X$ there exists $n(x) \in \mathbb{N}$ such that $d(f^{n(x)}(x), f^{n(x)+1}(x)) < +\infty$;

(iii) f has closed graph.

Then:

(a) $F_f = F_{f^n} \neq \emptyset$, for all $n \in \mathbb{N}$;

(b) if $d(x, f(x)) < +\infty$, then

$$f^n(x) \xrightarrow{d} f^\infty(x) \in F_f \text{ as } n \to \infty$$

and

$$d(x, f^\infty(x)) \leq \frac{1}{1-\alpha} d(x, f(x)), \text{ i.e., } f \text{ is } \frac{1}{1-\alpha} - WPO.$$

Proof. The proof runs in a similar way to the proof of Theorem 5.1.1. □

Theorem 5.1.4. *Let (X,d) be a generalized complete metric space and $f : X \to X$ be an operator. We suppose that:*

(i) f is a Meir-Keeler operator, i.e., for each $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that $x, y \in X$, $\varepsilon \leq d(x,y) < \varepsilon + \eta$ imply $d(f(x), f(y)) < \varepsilon$.

(ii) there exists $x_0 \in X$ such that $d(x_0, f(x_0)) < +\infty$.

Then:

(a) $F_f \neq \emptyset$

(b) if $u, v \in F_f$ imply $d(u,v) < +\infty$, then $card F_f = 1$.

Proof. Denote $x_n := f^n(x_0)$, $n \in \mathbb{N}$.

The proof of the theorem can be organized in five steps.

Step 1. We prove that

$$d(f(x), f(y)) < d(x, y), \text{ for each } x, y \in X \text{ with } x \neq y \text{ and } d(x, y) < +\infty.$$

Let $x, y \in X$ be such that $x \neq y$ and $d(x, y) < +\infty$. Then by letting $\epsilon := d(x, y)$ in the definition of Meir-Keeler operators we get $d(f(x), f(y)) < d(x, y)$.

Step 2. We can prove, by induction, that $d(x_n, x_{n+1}) < +\infty$, for all $n \in \mathbb{N}$.

Step 3. We prove that the sequence $a_n := d(x_n, x_{n+1}) \searrow 0$ as $n \to +\infty$.

If there is $n_0 \in \mathbb{N}$ such that $a_{n_0} = 0$ then $x_{n_0} \in F_f$.

If $a_n \neq 0$, for each $n \in \mathbb{N}$, then $a_n = d(f(x_{n-1}), f(x_n)) < d(x_{n-1}, x_n) = a_{n-1}$. Hence the sequence $(a_n)_{n\in\mathbb{N}}$ converges to a certain $a \geq 0$. Suppose that $a > 0$. Then, for each $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $\epsilon \leq a_n < \epsilon + \eta$, for all $n \geq n_\epsilon$. Then, by the Meir-Keeler condition we obtain $a_{n+1} < \epsilon$, which is a contradiction with the above relation.

Step 4. We will prove that the sequence (x_n) is Cauchy.

Suppose, by contradiction, that (x_n) is not a Cauchy sequence. Then, there exists $\epsilon > 0$ such that $\limsup d(x_m, x_n) > 2\epsilon$. For this ϵ there exists $\eta := \eta(\epsilon) > 0$ such that for $x, y \in X$ with $\epsilon \leq d(x, y) < \epsilon + \eta$ we have $d(f(x), f(y)) < \epsilon$. Choose $\delta := \min\{\epsilon, \eta\}$. Since $a_n \searrow 0$ as $n \to +\infty$ it follows that there is $p \in \mathbb{N}$ such that $a_p < \frac{\delta}{3}$. Let $m, n \in \mathbb{N}^*$ with $n > m > p$ such that $d(x_n, x_m) > 2\epsilon$. For $j \in [m, n]$ we have $|d(x_m, x_j) - d(x_m, x_{j+1}| \leq a_j < \frac{\delta}{3}$. Also, $d(x_m, x_{m+1} < \epsilon$ and $d(x_m, x_n) > \epsilon + \delta$ we obtain that there exists $k \in [m, n]$ such that $\epsilon < \epsilon + \frac{2\delta}{3} < d(x_m, x_k) < \epsilon + \delta$.

On the other hand, for any $m, l \in \mathbb{N}$ we have: $d(x_m, x_l) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{l+1}) + d(x_{l+1}, x_l) = a_m + d(f(x_m), f(x_l)) + a_l < \frac{\delta}{3} + \epsilon + \frac{\delta}{3}$. The contradiction proves that (x_n) is Cauchy.

Step 5. We prove that $x^* := \lim\limits_{n\to+\infty} x_n$ is a fixed point of f.

Since f is continuous and $x_{n+1} = f(x_n)$, we get by passing to the limit that $x^* = f(x^*)$.

If $x^*, y \in F_f$ are two distinct fixed points of f then, by the contractive condition, we get the following contradiction: $d(x^*, y) = d(f(x^*), f(y)) < d(x^*, y)$. This completes the proof. $\square$

Theorem 5.1.5. *Let (X,d) be a generalized complete metric space and $f:X\to X$ an operator. We suppose that:*

(i) f is Caristi operator, i.e., there exists $\varphi : X\to\mathbb{R}_+$ such that

$$d(x,f(x))\le\varphi(x)-\varphi(f(x)),\quad \textit{for all}\ \ x\in X;$$

(ii) f has closed graph.

Then:

(a) $F_f=F_{f^n}\neq\emptyset$, for all $n\in\mathbb{N}^*$;

(b) $f^n(x)\overset{d}{\to} f^\infty(x)\in F_f$ as $n\to\infty$, i.e., f is a WPO.

Proof. Notice that, since f is a Caristi operator, then $d\left(x,f\left(x\right)\right)<+\infty$ for every $x\in X$. Denote by $x_n:=f^n(x)$, for $n\in\mathbb{N}$. Then:

$$\sum_{n=0}^{+\infty}d(x_n,x_{n+1})=\sum_{n=0}^{+\infty}d(f^n(x),f^{n+1}(x)).$$

We will prove that the series $\sum\limits_{n=0}^{+\infty}d(f^n(x),f^{n+1}(x))$ is convergent. For this purpose we need to show that the sequence of its partial sums is convergent in $\mathbb{R}_+$. Denote by $s_n:=\sum\limits_{k=0}^{n}d(f^k(x),f^{k+1}(x))$. Then $s_{n+1}-s_n=d(f^{n+1}(x),f^{n+2}(x))\ge 0$, for each $n\in\mathbb{N}$. Moreover $s_n=\sum\limits_{k=0}^{n}d(f^k(x),f^{k+1}(x))\le\varphi(x)$. Hence $(s_n)_{n\in\mathbb{N}}$ is upper bounded and increasing in $\mathbb{R}_+$. Then the sequence $(s_n)_{n\in\mathbb{N}}$ is convergent.

It follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy and, from the completeness of the space, convergent to a certain element $x^*\in X$. The conclusion follows from the fact that f has closed graph. □

We also have:

Theorem 5.1.6. *Let X be a nonempty set and $d,\rho : X\times X\to\mathbb{R}_+$ two generalized metrics on X and $f:X\to X$ be an operator. We suppose that:*

(i) (X,d) is a generalized complete metric space;

(ii) there exists $c>0$ such that

$$d(f(x),f(y))\le c\rho(x,y),\quad \textit{for all}\ x,y\in X\ \textit{with}\ \rho(x,y)<+\infty;$$

(iii) for every $x \in X$ *there exists* $n(x) \in \mathbb{N}$ *such that* $\rho(f^{n(x)}(x), f^{n(x)+1}(x)) < +\infty$*;*

(iv) $f : (X, \rho) \to (X, \rho)$ *is an* α*-contraction.*

Then:

(a) $F_f = F_{f^n} \neq \emptyset$*, for all* $n \in \mathbb{N}^*$*;*

(b) $f^n(x) \overset{d}{\to} f^\infty(x) \in F_f$ *as* $n \to \infty$*;*

(c) if $\rho(x, f(x)) < +\infty$*, then*

$$f^n(x) \overset{\rho}{\to} f^\infty(x) \text{ as } n \to \infty$$

(d) if $\rho(x, f(x)) < +\infty$*, for all* $x \in X$*, then* $f : (X, \rho) \to (X, \rho)$ *is* $\dfrac{1}{1-\alpha}$*-WPO.*

Theorem 5.1.7. *Let* X *be a nonempty set,* $\alpha \in]0, 1[$ *and* $f : X \to X$ *be an operator. Then the following statements are equivalent:*

(i) $F_f = F_{f^n} \neq \emptyset$*, for all* $n \in \mathbb{N}^*$*;*

(ii) there exists a generalized complete metric on X *such that:*

(a) $f : (X, d) \to (X, d)$ *is an* α*-contraction;*

(b) $d(x, f(x)) < +\infty$*, for all* $x \in X$*.*

Theorem 5.1.8. *Let* (X, d) *be a generalized complete metric space,* $f, g : X \to X$ *be two operators. We suppose that:*

(i) f *and* g *are* α*-contractions;*

(ii) $d(x, f(x)) < +\infty$*,* $d(x, g(x)) < +\infty$*, for all* $x \in X$*;*

(iii) there exists $\eta > 0$ *such that:*

$$d(f(x), g(x)) \leq \eta, \quad \text{for all } x \in X.$$

Then:

$$H_d(F_f, F_g) \leq \frac{\eta}{1-\alpha}.$$

Remark 5.1.1. For details concerning the above results and for other considerations on the fixed point theory in a generalized metric space ($d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$) see A. Petruşel, I.A. Rus and M.A. Şerban B[1].

Chapter 6

Generalized contractions on G-metric spaces

Guidelines: L. Kantorovich (1939), T. Wazewski (1960), A.I. Perov (1964), E. Popa (1968), A. Pelczar (1969), L.B. Ćirić (1972), I.A. Rus (1973), S. Heikkila and S. Seikkalä (1977), M. Gürtler and H. Weber (1978), M. Kwapisz (1979), P.P. Zabrejko and T.A. Makarevich (1987), S. Priess-Crampe and R. Ribenboim (1993), E. De Pascale, G. Marino and P. Pietramala (1993), P.P. Zabrejko (1997).
General references: W.A. Kirk and B. Sims (Eds.) R[1], I.A. Rus B[81], B[73], P.P. Zabrejko R[1], M. Kwapisz R[1], E. De Pascale, G. Marino and P. Pietramala R[1], M. Gürtler and H. Weber R[1], V. Berinde B[7], V. Heckmanns R[1], E. Schörner R[1], W.A. Kirk and B.G. Kang R[1], L.-G. Huang and X. Zhang R[1], J. Appell, A. Carbone and P.P. Zabrejko R[1].

6.0 Basic concepts

6.0.1 L-spaces

Let X be a nonempty set. Let

$$s(X) := \{(x_n)_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\}.$$

Let $c(X) \subset s(X)$ be a subset of $s(X)$ and $\mathrm{Lim} : c(X) \to X$ an operator. By definition (M. Fréchet (1905)) the triple $(X, c(X), \mathrm{Lim})$ is called an L-space if the following conditions are satisfied:

(i) If $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $\mathrm{Lim}(x_n)_{n\in\mathbb{N}} = x$.

(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $\mathrm{Lim}(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences, $(x_{n_i})_{i\in\mathbb{N}}$, of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $\mathrm{Lim}(x_{n_i})_{i\in\mathbb{N}} = x$.

By definition an element $(x_n)_{n\in\mathbb{N}}$ of $c(X)$ is a convergent sequence and $x = \mathrm{Lim}(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence and we shall write

$$x_n \to x \text{ as } n \to \infty.$$

In what follow we denote an L-space by $(X, \to)$.

Example 6.0.1. Let $(X, \leq)$ be a partial ordered set, $c(X) := \{(x_n)_{n\in\mathbb{N}} \mid (x_n)_{n\in\mathbb{N}}$ is an increasing sequence in X and there exists $\sup\{x_n \mid n \in \mathbb{N}\}\}$ and $\mathrm{Lim}(x_n)_{n\in\mathbb{N}} := \sup\{x_n \mid n \in \mathbb{N}\}$. The triple $(X, c(X), \mathrm{Lim})$ is an L-space. We denote this L-space by $(X, \uparrow)$.

Example 6.0.2. $(X, \leq)$ is a partial ordered set, $c(X) : \{(x_n)_{n\in\mathbb{N}} \mid (x_n)_{n\in\mathbb{N}}$ is a decreasing sequence in X and there exists $\inf\{x_n \mid n \in \mathbb{N}\}\}$ and $\mathrm{Lim}(x_n)_{n\in\mathbb{N}} := \inf\{x_n \mid n \in \mathbb{N}\}$. The triple $(X, c(X), \mathrm{Lim})$ is an L-space. We denote this L-space by $(X, \downarrow)$.

Example 6.0.3. Let $(X, \leq)$ be a partial ordered set. By definition a sequence $(x_n)_{n\in\mathbb{N}}$, in X, 0-converges to x if there exist two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that

(a) $a_n \uparrow x$ and $n \to \infty$ and $b_n \downarrow x$ as $n \to \infty$;

(b) $a_n \leq x_n \leq b_n$ for all $n \in \mathbb{N}$.

We denote this convergence by $x_n \xrightarrow{0} x$ as $n \to \infty$. The pair $(X, \xrightarrow{0})$ is an L-space.

Example 6.0.4. Let (X, d) be a metric space. Then $(X, \xrightarrow{d})$ is an L-space.

Example 6.0.5. Let (X, p) be a partial metric space. Then $(X, \xrightarrow{p})$ is an L-space.

Example 6.0.6. Let (X, d) be a semimetric space. If $d : X \times X \to \mathbb{R}_+$ is continuous, then $(X, \xrightarrow{d})$ is an L-space.

Example 6.0.7. Let $(X, \|\cdot\|)$ be a Banach space.

We denote by $\xrightarrow{\|\cdot\|}$ the strong convergence in X and by $\rightharpoonup$ the weak convergence in X. Then $(X, \xrightarrow{\|\cdot\|})$ and $(X, \rightharpoonup)$ are L-spaces.

Example 6.0.8. Let (X, d) and (Y, ρ) be two metric spaces. Let $\mathbb{M}(X, Y)$ be the set of all operators from X to Y. We denote by $\overset{p}{\to}$ the pointwise convergence on $\mathbb{M}(X, Y)$, by $\overset{unif}{\to}$ the uniform convergence on $\mathbb{M}(X, Y)$.

By definition (see M. Angrisani and M. Clavelli R[1]), a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{M}(X, Y)$ converges with continuity to f if $x_n \overset{d}{\to} x$ as $n \to \infty \Rightarrow f_n(x_n) \to f(x)$ as $n \to \infty$.

We denote by $\overset{cont}{\to}$ this convergence. Then $(\mathbb{M}(X, Y), \overset{p}{\to})$, $(\mathbb{M}(X, Y), \overset{unif}{\to})$ and $(\mathbb{M}(X, Y), \overset{cont}{\to})$ are L-spaces.

Example 6.0.9. Let (X, τ) be a Hausdorff topological space. Then, $(X, \overset{\tau}{\to})$ is an L-space.

Let $(G, +)$ be a group, $\leq$ a partial order relation on G and $\to$ an L-space structure on G. By definition, $(G, +, \leq, \to)$ is an ordered L-group if:

(a) $x_n \to x,\ y_n \to y \Rightarrow x_n + y_n \to x + y$;

(b) $x_n \to x,\ y_n \to y,\ x_n \leq y_n,\ n \in \mathbb{N} \Rightarrow x \leq y$;

(c) $x \leq y,\ u \leq v \Rightarrow x + u \leq y + v$.

If $(X, \leq, \to)$ satisfies (b), then by definition $(X, \leq, \to)$ is an ordered L-space.

Let X be a nonempty set and $(G, +, \leq, \to)$ be an ordered L-group.

By definition a functional $d : X \times X \to G$ is a G-metric if:

(i) $d(x, y) \geq 0$, for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;

(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Example 6.0.10. $(\mathbb{R}^m, +, \leq, \to)$ is an ordered L-group.

The functional $d : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m_+$ defined by

$$d(x, y) := \begin{pmatrix} |x_1 - y_1| \\ \vdots \\ |x_m - y_m| \end{pmatrix}$$

is a $\mathbb{R}^m$-metric on $\mathbb{R}^m$.

Example 6.0.11. $X = C([a, b], \mathbb{R}^m)$. The functional $d : X \times X \to \mathbb{R}^m_+$, defined by

$$d(f,g) := \begin{pmatrix} \|f_1 - g_1\|_C \\ \vdots \\ \|f_m - g_m\|_C \end{pmatrix}$$

is a $\mathbb{R}^m$-metric on $C([a,b],\mathbb{R}^m)$.

6.0.2 Ordered Banach spaces

Let $(X,+,\mathbb{R},\|\cdot\|)$ be a Banach space.

By definition, a subset $K \subset X$ is a cone if:

(a) K is a closed convex subset of X;

(b) $\lambda \in \mathbb{R}_+$ implies that $\lambda K \subset K$;

(c) $K \cap (-K) = \{0\}$;

(d) $K \neq \{0\}$.

Each cone K of a Banach space X induces a partial ordering on X by

$$x, y \in X,\ x \le y \Leftrightarrow y - x \in K.$$

A Banach space together with a cone K forms an ordered Banach space.

By definition a cone $K \subset X$ is called:

(1) reproducing (or generating) if $X = K - K$;

(2) monotonic if $0 \le x \le y \Rightarrow \|x\| \le \|y\|$;

(3) normal if there exists $\gamma > 0$ such that $0 \le x \le y \Rightarrow \|x\| \le \gamma\|y\|$;

(4) regular if every increasing sequence which is bounded from above with respect to $\le$ is convergent;

(5) fully regular if every bounded increasing sequence is convergent.

Let $\mathbb{B}$ be an ordered Banach space with the cone K. Let X be a nonempty set. A functional $d : X \times X \to K$ is a K-metric if it satisfies the Fréchet axioms (i)+(ii)+(iii).

For more considerations on ordered Banach spaces see M.A. Krasnoselskii and P.P. Zabrejko R[1], K. Deimling R[1] and D. Guo, Y.J. Cho and J. Zhu R[1]. For a recent complete survey on K-metric spaces see P.P. Zabrejko R[1].

6.0.3 Convergent to zero matrices

Let $(X, \to)$ be an L-space and

$$M_m(X) := \{(x_{ij})_m^m \mid x_{ij} \in X,\ i,j \in \overline{1,m}\}.$$

Then $(M_m(\mathbb{R}), +, \mathbb{R}, \leq, \to)$ is an ordered linear L-space, where $\to$ is the termwise convergence. By definition a matrix $S \in M_n(\mathbb{R})$ is called convergent to 0 if $S^n \to 0$ as $n \to \infty$. We have:

Theorem 6.0.1. *Let $S \in M_m(\mathbb{R}_+)$ be a square matrix. The following statements are equivalent:*

(i) S is a convergent to zero matrix;

(ii) $\det(E - S) \neq 0$ *and* $(E - S)^{-1} = E + S + \cdots + S^n + \ldots$*;*

(iii) $\lambda \in \mathbb{C}$, $\det(S - \lambda E) = 0 \Rightarrow |\lambda| < 1$*;*

(iv) $\det(E - S) \neq 0$ *and* $(E - S)^{-1}$ *has nonnegative elements.*

For the proof of this theorem see I.A. Rus B[73], p. 37-38, G.R. Belitskii and Yu.I. Lyubich R[1], pp. 38-39 and D. O'Regan and R. Precup B[5].

6.0.4 Infinite matrices

Let X be a nonempty set,

$$s(X) := \{(x_n)_{n \in \mathbb{N}^*} \mid x_n \in X,\ n \in \mathbb{N}^*\}$$

and

$$M(X) := \{(x_{ij})_1^\infty \mid x_{ij} \in X,\ i,j \in \mathbb{N}^*\}$$

where

$$(x_{ij})_1^\infty := \begin{pmatrix} x_{11} & x_{12} & x_{13} & \ldots \\ x_{21} & x_{22} & x_{23} & \ldots \\ x_{31} & x_{32} & x_{33} & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

is an infinite matrix.

If $X = \mathbb{R}$, then $(s(\mathbb{R}), +, \mathbb{R}, \leq, \to)$ and $(M(\mathbb{R}), +, \mathbb{R}, \leq, \to)$ are ordered linear L-spaces, where $\to$ is the termwise convergence.

For $A \in M(\mathbb{R})$ we denote $|A| := \sup\limits_{1 \leq i \leq \infty} \sum\limits_{j \in \mathbb{N}} |a_{ij}|$.

The functional

$$|\cdot| : M(\mathbb{R}) \to \mathbb{R}_+ \cup \{+\infty\}, \quad A \mapsto |A|$$

is a generalized norm on $M(\mathbb{R})$.

A matrix $A \in M(\mathbb{R})$ is called:

(1) row-column-finite if there are only a finite number of nonzero elements in each row and each column;

(2) Neumann matrix if A^n is defined for all $n \in \mathbb{N}$ and $\sum_{n\in\mathbb{N}} A^n$ termwise converges.

Remark 6.0.1. In general, product of matrices is not associative. Nevertheless, for row-column-finite matrices the product is associative.

We have

Theorem 6.0.2. *Let $A \in M(\mathbb{R})$ be a matrix. We suppose that:*

(i) A is row-column-finite matrix;

(ii) $|A| < 1$.

Then:

(a) S is a Neumann matrix;

(b) $(E - S)^{-1} = \sum_{n\in\mathbb{N}} S^n$.

6.1 Fixed point theorems in $\mathbb{R}^m$-metric spaces

Let X be a nonempty set and $d : X \times X \to \mathbb{R}^m$ be a $\mathbb{R}^m$-metric on X.

Remark 6.1.1. A functional $d : X \times X \to \mathbb{R}^m$,

$$(x, y) \mapsto (d_1(x, y), \dots, d_m(x, y))$$

is a $\mathbb{R}^m$-metric on X if:

(a) d_k is a pseudometric, for all $k = \overline{1, m}$;

(b) for all $x, y \in X$, $x \neq y$, there exists $k \in \{1, \dots, m\} : d_k(x, y) \neq 0$.

Let (X, d) be a $\mathbb{R}^m$-metric space. By definition an operator $f : X \to X$ is an S-contraction if there exists $S \in M_m(\mathbb{R}_+)$ such that:

(i) S is a convergent to zero matrix;

(ii) $d(f(x), f(y)) \leq S d(x, y), \quad \text{for all} \ \ x, y \in X$.

We have:

Theorem 6.1.1. (Perov R[1]). *Let (X, d) be a complete $\mathbb{R}^m$-metric space and $f : X \to X$ be an S-contraction. Then:*

(i) $F_f = F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$; i.e., f is a Bessaga operator;

(ii) $f^n(x) \xrightarrow{d} x^*$ as $n \to \infty$, for all $x \in X$, i.e. f is a PO in (X, d);

(iii) $d(f^n(x), x^*) \leq (E - S)^{-1} S^n d(x, f(x))$, for all $x \in X$ and for all $n \in \mathbb{N}^*$;

(iv) $d(x, x^*) \leq (E - S)^{-1} d(x, f(x))$, for all $x \in X$.

Proof. (i)+(ii)+(iii). First of all we remark that if $x^*, y^* \in F_f$, then

$$d(x^*, y^*) = d(f^n(x^*), f^n(y^*)) \leq S^n d(x^*, y^*) \to 0 \text{ as } n \to \infty.$$

Thus, $cardF_f \leq 1$.

On the other hand, for each $x \in X$ we have:

$$\begin{aligned} d(f^n(x), f^{n+p}(x)) &\leq (S^n + \cdots + S^{n+p} + \dots) d(x, f(x)) \\ &= (E - S)^{-1} S^n d(x, f(x)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

This implies that $(f^n(x))_{n \in \mathbb{N}}$ converges, for each $x \in X$. Let x^* be its limit. From the continuity of f it follows that $x^* \in F_f$. So, $F_f = \{x^*\}$. Hence we have (ii) and (iii). The conclusion (i) follows from (ii).

(iv). From

$$\begin{aligned} d(x, x^*) &\leq d(x, f(x)) + d(f(x), x^*) \\ &\leq d(x, f(x)) + S d(x, x^*) \end{aligned}$$

it follows that

$$d(x, x^*) \leq (E - S)^{-1} d(x, f(x)).$$

Theorem 6.1.2. *Let f be as in Theorem 6.1.1. Then:*

(v) if $x_n \in X$, $n \in \mathbb{N}$ are such that

$$d(x_n, f(x_n)) \to 0 \text{ as } n \to \infty,$$

then, $x_n \to x^*$ as $n \to \infty$, i.e. the fixed point problem for f is well posed;

(vi) if $x_n \in X$, $n \in \mathbb{N}$ are such that

$$d(x_{n+1}, f(x_n)) \to 0 \text{ as } n \to \infty,$$

then for all $x \in X$ *we have*

$$d(x_n, f^n(x)) \to 0 \text{ as } n \to \infty,$$

i.e. the operator f *has the limit shadowing property;*

(vii) if $(x_n)_{n\in\mathbb{N}}$ *is a bounded sequence in* (X, d)*, then*

$$f^n(x_n) \to x^* \text{ as } n \to \infty;$$

(viii) if $g : X \to X$ *is such that there exists* $\eta \in \mathbb{R}^m_+$ *with*

$$d(f(x), g(x)) \leq \eta, \quad \text{for all } \; x \in X,$$

then

$$x^*_g \in F_g \Rightarrow d(x^*, x^*_g) \leq (E - S)^{-1}\eta.$$

Proof. (v). From (iv) it follows that

$$d(x_n, x^*) \leq (E - S)^{-1} d(x_n, f(x_n)) \to 0 \text{ as } n \to \infty.$$

(vi). We have

$$\begin{aligned} d(x_n, x^*) &\leq d(x_n, f(x_{n-1})) + d(f(x_{n-1}), x^*) \\ &\leq d(x_n, f(x_{n-1})) + Sd(x_{n-1}, x^*) \leq \cdots \leq \\ &\leq d(x_n, f(x_{n-1})) + Sd(x_{n-1}, f(x_{n-2})) + \cdots + \\ &+ S^{n-1} d(x_1, f(x_0)) + S^n d(x_0, x^*) \end{aligned}$$

From the above estimations it follows

$$\begin{aligned} |d(x_n, x^*)| &:= \max_{1\leq i\leq m} d_i(x_n, x^*) \\ &\leq |d(x_n, f(x_{n-1}))| + |S||d(x_{n-1}, f(x_{n-2}))| + \cdots + \\ &+ |S^{n-1}||d(x_1, f(x_0))| + |S^n||d(x_0, x^*)| \\ &\leq |d(x_n, f(x_{n-1}))| + |S||d(x_{n-1}, f(x_{n-2}))| + \cdots + \\ &+ |S|^{n-1}|d(x_1, f(x_0))| + |S|^n|d(x_0, x^*)| \to 0 \end{aligned}$$

as $n \to \infty$ by a Cauchy Lemma (see I.A. Rus B[6]).

So,

$$d(x_n, f^n(x)) \le d(x_n, x^*) + d(x^*, f^n(x)) \to 0$$

as $n \to \infty$.

(vii). Let $\eta \in \mathbb{R}^m_+$ be such that

$$d(x_n, x_m) \le \eta, \quad \text{for all} \quad n, m \in \mathbb{N}^*.$$

We have

$$\begin{aligned} d(x_n, f(x_n)) &\le d(x_n, x^*) + Sd(x_n, x^*) \\ &\le (E+S)[\eta + d(x_1, x^*)]. \end{aligned}$$

From (iii) it follows that

$$d(f^n(x_n), x^*) \le (E-S)^{-1} S^n (E+S)[\eta + d(x_1, x^*)] \to \text{ as } n \to \infty.$$

(viii) From (iv) we have

$$\begin{aligned} d(x_g^*, x^*) &\le (E-S)^{-1} d(x_g^*, f(x_g^*)) \\ &= (E-S)^{-1} d(g(x_g^*), f(x_g^*)) \le (E-S)^{-1}\eta. \end{aligned}$$

Theorem 6.1.3. *Let (X, d) be a complete $\mathbb{R}^m$-metric space and $f : X \to X$ be an orbitally continuous graphic S-contraction. Then:*

(i) $F_f = F_{f^n} \neq \emptyset$, for all $n \in \mathbb{N}^*$;

(ii) f is a WPO;

(iii) $d(f^n(x), f^\infty(x)) \le (E-S)^{-1} S^n d(x, f(x))$, for all $x \in X$ and for all $n \in \mathbb{N}^*$;

(iv) f is a $(E-S)^{-1}$-WPO;

(v) let $g : X \to X$ be such that:

(a) g is $(E-S)^{-1}$-WPO;

(b) there exists $\eta \in \mathbb{R}^m_+$ such that

$$d(f(x), g(x)) \le \eta.$$

Then:

$$H_d(F_A, F_B) \le (E-S)^{-1}\eta.$$

Here

$$H_d := \begin{pmatrix} H_{d_1} \\ \vdots \\ H_{d_m} \end{pmatrix}$$

stands for Pompeiu-Hausdorff functional.

Proof. The proof are similar with that of Theorem 6.1.2. □

For other results on fixed point theory in a $\mathbb{R}^m$-metric space see M. Albu B[1], M. Turinici B[30], V. Berinde B[18], S. András B[1], B[4], G. Dezsö B[1], M.A. Şerban B[2], I.A. Rus B[70], B[73], B[84], D. O'Regan and R. Precup B[5], R. Precup B[1].

6.2 Fixed point theorems in a $s(\mathbb{R})$-metric spaces

Let X be a nonempty set. A functional $d : X \times X \to s(\mathbb{R})$ is a $s(\mathbb{R})$-metric if it satisfies the Fréchet axioms (i)+(ii)+(iii).

Remark 6.2.1. A functional $d : X \times X \to s(\mathbb{R}_+)$, $(x, y) \mapsto (d_k(x, y))_{k\in\mathbb{N}^*}$ is a metric on X if:

(a) d_k is a pseudometric, for all $k \in \mathbb{N}^*$;

(b) for all $x, y \in X$, $x \neq y$, there exists $k \in \mathbb{N}^*$: $d_k(x, y) \neq 0$.

Definition 6.2.1. A $s(\mathbb{R})$-metric space is complete (in the Weierstrass sense) if $x_n \in X$, $\sum\limits_{n\in\mathbb{N}^*} d(x_n, x_{n+1})$ converges $\Rightarrow (x_n)_{n\in\mathbb{N}^*}$ converges.

Definition 6.2.2. Let (X, d) be a $s(\mathbb{R})$-metric space, $f : X \to X$ and $S \in M(\mathbb{R}_+)$. The operator A is a S-contraction if:

(i) S is row and column finite;

(ii) S is a Neumann matrix;

(iii) $\sum\limits_{n\in\mathbb{N}} S^n d(x, y)$ converges, for all $x, y \in X$;

(iv) $d(A(x), A(y)) \leq Sd(x, y)$, for all $x, y \in X$.

Remark 6.2.2. In the case of E. Tarafdar's contractions (Tarafdar R[1]), $S = (s_{ij})_1^\infty$, $s_{ij} = 0$ if $i \neq j$ and $s_{ii} = \lambda_i < 1$, and in the case of I. Colojoară's contractions (see Colojoară B[1]), $s_{ij} = 0$, $j \neq \varphi(i)$, $s_{i\varphi(i)} = \lambda_i < 1$, where $\varphi : \mathbb{N}^* \to \mathbb{N}^*$.

For other examples of S-contractions see V.G. Angelov R[2], M. Frigon

R[1], N. Gheorghiu B[1], P.P. Zabrejko and T.A. Makarevich R[1], I.A. Rus B[108], etc.

In what follows we shall present some fixed point theorems in a $s(\mathbb{R})$-metric spaces (see I.A. Rus B[108]).

Theorem 6.2.1. *Let (X, d) be a complete $s(\mathbb{R})$-metric space and $f : X \to X$ an S-contraction. Then:*

(i) $F_f = F_{f^n} = \{x^*\}$ for all $n \in \mathbb{N}^*$;

(ii) $f^n(x) \xrightarrow{d} x^*$ as $n \to \infty$, for all $x \in X$;

(ii) $d(f^n(x), x^*) \leq (E-S)^{-1}S^n d(x, f(x))$, for all $x \in X$ and for all $n \in \mathbb{N}^*$;

(iv) $d(x, x^*) \leq (E - S)^{-1}d(x, f(x))$, for all $x \in X$;

(v) the fixed point problem for the operator f is well posed;

(vi) if $|S| < 1$, then the operator f has the limit shadowing property;

(vii) if $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in (X, d), then

$$f^n(x_n) \to x^* \text{ as } n \to \infty;$$

(viii) if $g : X \to X$ is such that there exists $\eta \in s(\mathbb{R}_+)$ with

$$d(f(x), g(x)) \leq \eta, \quad \text{for all } x \in X,$$

then

$$x_g^* \in F_g \Rightarrow d(x^*, x_g^*) \leq (E - S)^{-1}\eta.$$

Remark 6.2.3. The above S-contraction principle ((i)+(ii)+(iii)+(iv)) is a generalization of Perov's fixed point theorem. On the other hand, it is a particular case of some fixed point theorems in gauge spaces (see Chapter 5).

Definition 6.2.3. An operator $f : X \to X$ is a graphic S-contraction if:

(i) $S \in M(\mathbb{R}_+)$ is a row-column-finite matrix;

(ii) S is a Neumann matrix;

(iii) $\sum_{n\in\mathbb{N}} S^n d(x, y)$ converges, for all $x, y \in X$;

(iv) $d(f^2(x), f(x)) \leq Sd(x, f(x))$, for all $x \in X$.

Theorem 6.2.2. *Let (X, d) be a complete $s(\mathbb{R})$-metric space and $f : X \to X$ be an orbitally continuous graphic S-contraction. Then:*

(i) $F_f = F_{f^n} \neq \emptyset$ *for all* $n \in \mathbb{N}^*$*;*

(ii) $f^n(x) \to f^\infty(x)$ *as* $n \to \infty$, *for all* $x \in X$*;*

(iii) $d(f^n(x), f^\infty(x)) \leq (E-S)^{-1}S^n d(x, f(x))$, *for all* $x \in X$ *for all* $n \in \mathbb{N}^*$*;*

(iv) $d(x, f^\infty(x)) \leq (E-S)^{-1} d(x, f(x))$, *for all* $x \in X$*;*

(v) Let $g : X \to X$ *be such that:*

(a) g *is* $(E-S)^{-1}$*-WPO;*

(b) there exists $\eta \in s(\mathbb{R}_+)$ *such that*

$$d(f(x), g(x)) \leq \eta, \quad \textit{for all } \; x \in X.$$

Then

$$H_d(F_f, F_g) \leq (E-S)^{-1}\eta.$$

Here, $H_d = (H_{d_1}, \dots, H_{d_n}, \dots)$ stands for Pompeiu-Hausdorff functional.

For other results see also I. Gohberg, S. Goldberg and M.A. Kaashoek R[1].

6.3 Other results

The above results can be generalized to K-metric spaces.

For example, we present here some results of this type (for details and other results, see I.A. Rus, A. Petruşel and M.A. Şerban B[1]).

Let (X, d) be a complete K-metric space, where K is a generating and normal cone of an ordered Banach space Y. Let $Q : Y \to Y$ be a linear positive operator. Then $A : X \to X$ is said to be a Q-contraction if $\|Q\| < 1$ and $d(A(x_1), A(x_2)) \leq Q(d(x_1, x_2))$, for each $x_1, x_2 \in X$.

We have:

Theorem 6.3.1. (I.A. Rus, A. Petruşel and M.A. Şerban B[1]) *Let* (X, d) *be a complete K-metric space, where* K *is a generating and regular cone. Let* $A : X \to X$ *be a* Q*-contraction. Then we have:*

(i) $F_A = F_{A^n} = \{x^*\}$, *for* $n \in \mathbb{N}^*$*;*

(ii) If (X, d) *is bounded then* $\bigcap_{n \in \mathbb{N}} A^n(X) = \{x^*\}$*;*

(iii) If we consider $\psi : X \to K$, $\psi(x) := d(x, x^*)$ *then the pair* (A, ψ) *is a Schröder pair;*

(iv) $A^n(x) \to x^*$, *as* $n \to +\infty$, *for each* $x \in X$;

(v) $d(A^n(x), x^*) \leq (I - Q)^{-1} \cdot Q^n \cdot d(x, A(x))$, *for each* $n \in \mathbb{N}^*$ *and each* $x \in X$;

(vi) $d(A^n(x), x^*) \leq Q^n \cdot d(x, x^*)$, *for each* $n \in \mathbb{N}^*$ *and each* $x \in X$;

(vii) $d(A^n(x), x^*) \leq (I - Q)^{-1} \cdot d(A^n(x), A^{n+1}(x))$, *for each* $n \in \mathbb{N}^*$ *and each* $x \in X$;

(viii) $d(x, x^*) \leq (I - Q)^{-1} \cdot d(x, A(x))$, *for each* $x \in X$;

(ix) $\sum_{n=0}^{+\infty} d(A^n(x), A^{n+1}(x)) \leq (I - Q)^{-1} \cdot d(x, A(x))$, *for each* $x \in X$;

(x) there exists a neighborhood U *of* x^* *such that* $A^n(U) \to \{x^*\}$, *as* $n \to +\infty$.

The main abstract result for Picard operators is:

Theorem 6.3.2. (I.A. Rus, A. Petruşel and M.A. Şerban B[1]) *Let* X *be a nonempty set and* $A : X \to X$ *be an operator. Then the following statements are equivalent:*

(P_1) *there exists an L-space structure on the set* X, *denoted by* $\to$, *such that* $A : (X, \to) \to (X, \to)$ *is a PO;*

(P_2) *the operator* A *is a Bessaga;*

(P_3) *there exist* $\alpha \in]0, 1[$ *and* $\chi : X \to \mathbb{R}_+$ *such that:*

(i) $card(Z_\chi) = 1$

(ii) $\chi(A(x)) \leq \alpha \cdot \chi(x)$, *for each* $x \in X$;

(P_4) *there exist* $\alpha \in]0, 1[$ *and a complete metric* d *on* X *such that* $A : (X, d) \to (X, d)$ *is an* α*-contraction;*

(P_5) *there exist* $x^* \in F_A$, $\alpha \in]0, 1[$ *and a metric* d *on* X *such that* $d(A(x), x^*) \leq \alpha \cdot d(x, x^*)$, *for each* $x \in X$;

(P_6) *there exist* $x^* \in F_A$ *and a Hausdorff topology on* X *such that if* $Y \in I_{cl}(A)$ *then* $x^* \in Y$;

(P_7) *there exists* $n_0 \in \mathbb{N}^*$ *such that* $A^{n_0} : (X, d) \to (X, d)$ *is a Bessaga operator;*

(P_8) *there exist* $n_0 \in \mathbb{N}^*$, $\alpha \in]0, 1[$ *and a complete metric* d *on* X *such that* $A^{n_0} : (X, d) \to (X, d)$ *is an* α*-contraction;*

(P_9) *there exist* $n_0 \in \mathbb{N}^*$ *and an L-space structure on* X, *denoted by* $\to$, *such that* $A^{n_0} : (X, \to) \to (X, \to)$ *is a PO.*

Proof.

$(P_1) \Rightarrow (P_2)$ Let $F_A = \{x^*\}$ and $y^* \in F_{A^m}$. Then $A^n(y^*) \to x^*$, as $n \to +\infty$. Since $A^{km}(y^*) = y^*$, for $k \in \mathbb{N}$ we have $x^* = y^*$.

$(P_2) \Rightarrow (P_3)$ This implication is a theorem by J. Jachymski R[1].

$(P_3) \Rightarrow (P_4)$ Let $Z_\chi = \{x^*\}$. We define $d(x, y) := \chi(x) + \chi(y)$.

$(P_4) \Rightarrow (P_5)$ Let (X, d) be a complete metric space and $A : X \to X$ be an α-contraction. Then $F_A = \{x^*\}$ and $d(A(x), x^*) \leq \alpha \cdot d(x, x^*)$, for each $x \in X$.

$(P_5) \Rightarrow (P_6)$ We consider on X the topology defined by the metric d. Let $Y \in I_{cl}(A)$ $and x \in Y$. We have $A^n(x) \in Y$ and $d(A^n(x), x^*) \leq \alpha^n \cdot d(x, x^*)$, for each $\mathbb{N}$. Hence $A^n(x) \to x^*$, as $n \to +\infty$ and $x^* \in Y$.

$(P_6) \Rightarrow (P_7)$ Let us remark first that $F_A = \{x^*\}$. Indeed, if there exists $y^* \in F_A$ with $x^* \neq y^*$ then taking $Y := \{y^*\}$ and using (P_6) we get $x^* = y^*$. Further let $y^* \in F_{A^n}$ with $n > 1$ and $x^* \neq y^*$. Then if we choose $Y := \{y^*, A(y^*), A^2(y^*), \cdots, A^{n-1}(y^*)\}$ we obtain again $x^* = y^*$. Hence A^n is a Bessaga operator.

$(P_7) \Rightarrow (P_8)$ This implication follows from Bessaga's theorem.

$(P_8) \Rightarrow (P_9)$ Define $\to := \xrightarrow{d}$. From the contraction principle the operator $A^{n_0} : (X, d) \to (X, d)$ is Picard.

$(P_9) \Rightarrow (P_2)$ $F_A^{n_0} = \{x^*\}$. We have $A^{n_0}(x) \to x^*$, as $n \to +\infty$, for each $x \in X$. Obviously $x^* \in F_A$. Since $F_A \subset F_{A^{n_0}}$ and $F_{A^n} \subset F_{A^{nn_0}}$ we get that A is Bessaga.

$(P_4) \Rightarrow (P_1)$ Let us define $\to := \xrightarrow{d}$. Then the proof follows from the contraction principle. □

The main abstract result for weakly Picard operators is the following.

Theorem 6.3.3. (I.A. Rus, A. Petruşel and M.A. Şerban B[1]) *Let* X *be a nonempty set and* $A : X \to X$ *an operator. Then the following statements are equivalent:*

(WP_1) *there exists an L-space structure on the set* X, *denoted by* $\to$, *such that* $A : (X, \to) \to (X, \to)$ *is WPO;*

(WP_2) $F_A = F_{A^n} \neq \emptyset$, *for each* $n \in \mathbb{N}^*$;

(WP_3) *there exists* $\leq$ *a partial ordering, such that the set of all maximal elements of* X*, denoted by* $Max(X)$*, is nonempty and* $A : (X, \leq) \to (X, \leq)$ *is progressive;*

(WP_4) *there exists a complete metric* d *on* X *and a number* $\alpha \in]0, 1[$ *such that:*

(i) $A : (X, d) \to (X, d)$ *has closed graph;*

(ii) $d(A^2(x), A(x)) \leq \alpha \cdot d(A(x), x)$*, for each* $x \in X$*.*

(WP_5) *there exist a complete metric* d *on* X *and a lower semicontinuous functional* $\varphi : X \to \mathbb{R}_+$ *such that* $d(x, A(x)) \leq \varphi(x) - \varphi(A(x))$*, for each* $x \in X$*;*

(WP_6) *there exist a complete metric* d *on* X *and a functional* $\varphi : X \to \mathbb{R}_+$ *such that:*

(i) A *has closed graph;*

(ii) $d(x, A(x)) \leq \varphi(x) - \varphi(A(x))$*, for each* $x \in X$*.*

(WP_7) *there exists a partition* $X = \bigcup_{i \in I} X_i$ *of* X *such that* $A(X_i) \subset X_i$ *and* $A|_{X_i} : X_i \to X_i$ *is a Bessaga operator for all* $i \in I$*;*

(WP_8) *there exists a partition* $X = \bigcup_{i \in I} X_i$ *of* X *such that* $A(X_i) \subset X_i$ *and* $A|_{X_i} : X_i \to X_i$ *satisfies* (P_3) *in Theorem 6.3.2.;*

$(WP9)$ *there exists a complete metric* d *on* X *and a number* $\alpha \in]0, 1[$ *such that:*

(i) $A : (X, d) \to (X, d)$ *is continuous;*

(ii) $d(A^2(x), A(x)) \leq \alpha \cdot d(A(x), x)$*, for each* $x \in X$*.*

(WP_{10}) *there exists a complete metric* d *on* X *such that:*

(i) $A : (X, d) \to (X, d)$ *is continuous;*

(ii) $\sum_{n \in \mathbb{N}} d(A^n(x), A^{n+1}(x) < +\infty$*, for each* $x \in X$*.*

(WP_{11}) *there exists a complete metric* d *on* X *such that:*

(i) $A : (X, d) \to (X, d)$ *has closed graph;*

(ii) $\sum_{n \in \mathbb{N}} d(A^n(x), A^{n+1}(x) < +\infty$*, for each* $x \in X$*.*

(WP_{12}) *there exist a complete metric* d *on* X *and a functional* $\varphi : X \to \mathbb{R}_+$ *such that:*

(i) A *is continuous;*

(ii) $d(x, A(x)) \leq \varphi(x) - \varphi(A(x))$*, for each* $x \in X$*.*

Proof. $(WP1) \Rightarrow (WP2)$. The definition of the weakly Picard operator implies that $F_A \neq \emptyset$. The convergence of all sequences of successive approximation with the limits in F_A, implies that $F_A = F_{A^n}$, for all $n \in \mathbb{N}^*$.

$(WP2) \Rightarrow (WP4)$ Since $F_A = F_{A^n}$, for all $n \in \mathbb{N}^*$, then there exist a partition of X, $X = \bigcup\limits_{i \in I} X_i$ such that $X_i \in I(A)$, $card\,(F_A \cap X_i) = 1$ and $A\,|_{X_i}$ is a Bessaga mapping (see Rus [49]). From Bessaga's theorem, there exists a complete metric d_i on X_i such that $A\,|_{X_i}$ is an $\alpha-$contraction for all $i \in I$. We define a complete metric on X. Let $x_i^* \in X_i \cap F_A$, $i \in I$. Then, we define

$$d : X \times X \to X$$
$$d\,(x, y) = \begin{cases} d_i\,(x, y)\,, \text{ if } x, y \in X_i \\ d_i(x, x_i^*) + d_j(y, x_j^*) + 1, \text{ if } x \in X_i,\ y \in X_j,\ i \neq j \end{cases}$$

The completeness of (X, d) follows from the following remark:

$$d\,(x, y) < 1 \Rightarrow \exists i \in I,\ x, y \in X_i.$$

If $x \in X_i$ then $A\,(x)\,, A^2\,(x)\,, ..., A^n\,(x) \in X_i$ since $X_i \in I\,(A)$ and

$$d\left(A^2\,(x)\,, A\,(x)\right) = d_i\left(A^2\,(x)\,, A\,(x)\right) \leq \alpha \cdot d_i\,(A\,(x)\,, x) = \alpha \cdot d\,(A\,(x)\,, x)\,.$$

The conclusion (i) follows from the remark that $A\,|_{X_i}$ is continuous.

$(WP4) \Rightarrow (WP6)$ We define $\varphi : X \to \mathbb{R}_+, \varphi(x) = \frac{1}{1-\alpha} \cdot d(x, A(x))$.

$(WP6) \Rightarrow (WP3)$ See J. Jachymski R[6].

$(WP3) \Rightarrow (WP2)$ See J. Jachymski R[6].

$(WP4) \Rightarrow (WP1)$ We take on X, $\to := \overset{d}{\to}$.The proof follows from the conditions (ii) and (i).

$(WP4) \Rightarrow (WP5)$ We take $\varphi : X \to \mathbb{R}_+, \varphi(x) = \frac{1}{1-\alpha} \cdot d(x, A(x))$.

$(WP5) \Rightarrow (WP1)$ Follows from Caristi's theorem and a remark of A. Brøndsted R[2].

$(WP1) \Rightarrow (WP7)$ see I.A. Rus B[61].

$(WP7) \Rightarrow (WP8)$. The condition $(WP7)$ implies $(WP2)$ and thus we obtain $(WP4)$. Now we define

$$\chi : X \to \mathbb{R}_+$$
$$\chi\,(x) = d\,(x, A\,(x))$$

It is obvious to see that $A\left|_{X_i}\right.$ satisfies the condition $(P3)$ from Theorem 6.3.2.

$(WP8) \Rightarrow (WP1)$. This is obvious.

$(WP7) \Rightarrow (WP9)$. We know that $(WP7)$ implies $(WP2)$. The proof is similar to $(WP2) \Rightarrow (WP4)$, but we will additionally prove that the operator A is nonexpansive with respect to d. Since $A\left|_{X_i}\right.$ is an $\alpha-$contraction for all $i \in I$, hence nonexpansive, it suffices to consider the case for $x \in X_i$ and $y \in X_j$, $i \neq j$. Since $x \in X_i$ and $y \in X_j$ then $A(x) \in X_i$ and $A(y) \in X_j$, hence

$$\begin{aligned} d\left(A\left(x\right), A\left(y\right)\right) & = d_i\left(A\left(x\right), x_i^*\right) + d_j(A(y), x_j^*) + 1 \leq \\ & \leq \alpha \cdot d_i\left(x, x_i^*\right) + \alpha \cdot d_j(y, x_j^*) + 1 \leq \\ & \leq d_i\left(x, x_i^*\right) + d_j(y, x_j^*) + 1 = \\ & = d\left(x, y\right) \end{aligned}$$

thus the operator A is continuous.

$(WP9) \Rightarrow (WP8)$ It is obvious, since the continuity of the operator A implies that A has closed graph.

$(WP9) \Rightarrow (WP10)$ Condition (ii) from $(WP9)$ implies

$$\sum_{n\in\mathbb{N}} d\left(A^n\left(x\right), A^{n+1}\left(x\right)\right) \leq \frac{1}{1-\alpha} \cdot d\left(x, A\left(x\right)\right), \quad \text{for all } x \in X$$

which proves $(WP10)$.

$(WP10) \Rightarrow (WP11)$ It is obvious.

$(WP11) \Rightarrow (WP1)$ Condition (ii) from $(WP11)$ implies that every sequence of successive approximations is Cauchy and therefore convergent to $x^* \in X$. From the condition that the operator A has closed graph, we get that $x^* \in F_A$. Thus, A is a WPO. The $L-$space structure is generated by the metric d.

$(WP9) \Rightarrow (WP12)$ The proof is the same as in $(WP4) \Rightarrow (WP6)$.

$(WP12) \Rightarrow (WP11)$ From condition (ii) of $(WP12)$ we obtain

$$\sum_{n\in\mathbb{N}} d\left(A^n\left(x\right), A^{n+1}\left(x\right)\right) \leq \varphi\left(x\right) < \infty, \quad \text{for all } x \in X$$

and thus the proof is complete. $\square$

For other results see S.P. Singh and C.W. Norris R[1], B. Rzepecki R[3], etc.

Chapter 7

Generalized contractions on probabilistic metric spaces

Guidelines: K. Menger (1942), V.M. Sehgal and A.T. Bharucha-Reid (1972), H. Sherwood (1971), V.I. Istrăţescu and I. Săcuiu (1971), O. Hadžić (1978), V. Radu (1987), R.M. Tardiff (1992).
General references: V.I. Istrăţescu and I. Săcuiu B[1], O. Hadžić R[1], R[2], V. Radu B[2], Gh. Constantin B[2], Gh. Constantin and V. Radu B[1], Gh. Constantin, Gh. Bocşan and V. Radu B[4], S. Heikkilä and S. Seikkalä R[1], R.M. Tardiff R[1], D.H. Tan R[1], Y.J. Cho, M. Grabiec and V. Radu B[1].

7.0 Probabilistic metric spaces

The very first idea to extend the notion of metric space to a probabilistic setting belongs to K. Menger. He replaced the distance $d(x, y)$ between two elements $x, y \in X$, by a distribution function $F_{x,y}$, where $F_{x,y}(p)$ can be interpreted as the probability that the distance between x and y is less than p.

A distribution function on $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ is a function $F : \overline{\mathbb{R}} \to [0, 1]$ which is left-continuous on $\mathbb{R}$, non-decreasing and $F(-\infty) = 0$, $F(+\infty) = 1$. A distance distribution function $F : \overline{\mathbb{R}} \to [0, 1]$ is a distribution function with support contained in $[0, +\infty]$. The family of all distance distributions will be denoted by Δ^+. A triangle function τ is a binary operation on Δ^+, that is commutative, associative, non-decreasing in each variable and has the Heaviside function H_0 as identity. Denote $\mathcal{D}^+ := \{F \in \Delta^+ | lim_{x\to\infty} F(x) = 1\}$.

Definition 7.0.1. A probabilistic metric space (in the the sense of Schweizer and Sklar) is an ordered pair $(S, \mathcal{F})$, where S is a nonempty set and $\mathcal{F} : S \times S \to \Delta^+$ satisfies the following assertions:

i) $F_{x,y}(0) = 0$ (where $F_{x,y}$ denotes $\mathcal{F}(x, y)$), for each $x, y \in S$;

ii) $F_{x,y} = F_{y,x}$, for each $x, y \in S$;

iii) $F_{x,y}(u) = 1$ for every $u > 0$ if and only if $x = y$;

iv) for every $x, y, z \in S$ and every $u, v > 0$ the following implication holds:

$$F_{x,y}(u) = 1 \text{ and } F_{y,z} = 1 \text{ implies } F_{x,z}(u + v) = 1.$$

Definition 7.0.2. A probabilistic metric space (in the sense of Serstnev) is a triple $(S, \mathcal{F}, \tau)$, where S is a nonempty set, $\mathcal{F} : S \times S \to \Delta^+$ and τ is a triangle function, such that the following conditions are satisfied for each $x, y, z \in S$:

i) $F_{x,x} = H_0$;

ii) $F_{x,y} \neq H_0$, for $x \neq y$;

iii) $F_{x,y} = F_{y,x}$;

iv) $F_{x,z} \geq \tau(F_{x,y}, F_{y,z})$.

Important classes of probabilistic metric spaces in Serstnev sense are Menger, Drossos and Wald spaces. For example, $(S, \mathcal{F}, T)$ is called a Menger space if $(S, \mathcal{F}, \tau)$ is a probabilistic metric space with $\tau = \tau_T$ defined by:

$$\tau_T(F, G)(x) = sup\{T(F(u), G(u)) | u + v = x\},$$

for a t-norm T.

Definition 7.0.3. A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a Menger norm (briefly M-norm) if it satisfies the following conditions:

i) $T(a, b) = T(b, a)$, for each $a, b \in [0, 1]$;

ii) $a \leq c$ and $b \leq d$ implies $T(a, b) = T(c, d)$;

iii) $T(a, 1) = a$, for each $a \in [0, 1]$.

Definition 7.0.4. A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (briefly t-norm) if it is an associative M-norm, i.e.,

iv) $T(a, T(b, c)) = T(T(a, b), c)$, for each $a, b, c \in [0, 1]$.

Also, a t-norm is Archimedean if for each $a \in [0, 1]$, we have $\lim_{n \to +\infty} a^n = 0$. For example the t-norms $T_P(a, b) := a \cdot b$ and $T_L(a, b) := max(a + b - 1, 0)$ are Archimedean, while the minimum t-norm $T_M(a, b) := min(a, b)$ is not.

For every probabilistic metric space $(S, \mathcal{F})$ we can consider the sets

$$U_{\varepsilon,\lambda} := \{(x, y) \in S \times S | F_{x,y}(\varepsilon) > 1 - \lambda\},$$

(where $\varepsilon > 0$ and $\lambda \in (0, 1)$), which generate a topology, named the (ε, λ)-topology.

For generalized metrics we have the following result of V. Radu:

Theorem 7.0.1. (V. Radu, B[20]) *Let $\mathcal{F} : S \times S \to \mathcal{D}^+$, F be a fixed element of $\mathcal{D}^+$ and $d_F : S \times S \to [0, +\infty]$ given by:*

$$d_F(x, y) = inf\{a > 0 | F_{x,y}(at) \geq F(t), \text{ for all } t > 1\}.$$

If $(S, \mathcal{F}, T_M)$ is a Menger space then:

i) d_F is a Luxemburg metric on S (with the convention $inf\emptyset = +\infty$);

ii) (S, d_F) is complete if and only if $(S, \mathcal{F})$ is complete;

iii) The d_F-topology is stronger than the (ϵ, λ)-topology.

Let us remark also, that V. Radu introduced a family of deterministic metrics on a Menger space, (see V. Radu B[4], B[14], B[24]).

Definition 7.0.5. A t-norm T is called a Hadžić type t-norm (briefly H-t-norm) if the family $\mathcal{H}_+ = (T_m)_{m \in \mathbb{N}}$ defined on $I := [0, 1]$ by $T_m(x) = T^m(x, x, ..., x)$ is equicontinuous at $x = 1$, where $T^m : I^m \to I$ is defined by: $T^1(x) = x$, $T^{m+1}(x_1, ..., x_{m+1}) = T(T^m(x_1, ..., x_m), x_{m+1})$.

For arbitrary H-t-norms we have the following characterization theorem of V. Radu:

Theorem 7.0.2. (V. Radu, B[25]) *Let T be a t-norm. Then:*

i) Suppose that there exists a strictly increasing sequence $(b_n)_{n \in \mathbb{N}}$ from $[0, 1)$ such that $\lim_{n \to +\infty} b_n$ and $T(b_n, b_n) = b_n$. Then T is an H-t-norm.

ii) If T is continuous and of Hadžić type, then there exists a sequence $(b_n)_{n \in \mathbb{N}}$ as in (i).

7.1 Contractions on probabilistic metric spaces

Sehgal and Bharucha-Reid introduced in 1972 the notion of probabilistic q-contraction as follows.

Definition 7.1.1. Let $(S, \mathcal{F})$ be a probabilistic metric space. A mapping $f : X \to X$ is said to be a probabilistic q-contraction if $q \in [0, 1)$ and $F_{f(p_1),f(p_2)}(x) \geq F_{p_1,p_2}(\frac{x}{q})$, for every $p_1, p_2 \in S$ and every $x \in \mathbb{R}$.

The first fixed point theorem in probabilistic metric space was also proved by Sehgal and Bharucha-Reid in 1972.

Theorem 7.1.1. *Let $(S, \mathcal{F}, T_M)$ be a complete Menger space and $f : S \to S$ be a probabilistic q-contraction. Then f has a unique fixed point $u^* \in S$ and $u^* = \lim_{n\to+\infty} f^n(p)$, for every $p \in S$.*

The following result (see Hadžić-Pap R[1]) is an extension of Sehgal and Bharucha-Reid theorem.

We need first two definitions.

Let $(S, \mathcal{F})$ be a probabilistic metric space and $f : S \to S$. For every $x_0 \in S$ we denote the orbit of the mapping f at x_0 by:

$$\mathcal{O}(x_0, f) := \{f^n(x_0) \mid n \in \mathbb{N}\}.$$

Let

$$D_{\mathcal{O}(x_0,f)}(x) := \sup_{s<x} \inf_{u,v\in\mathcal{O}(x_0,f)} F_{u,v}(s),$$

the diameter of $\mathcal{O}(x_0, f)$.

If $\sup_{x\in\mathbb{R}} D_{\mathcal{O}(x_0,f)}(x) = 1$, then the orbit $\mathcal{O}(x_0, f)$ is a probabilistic bounded subset of S.

Theorem 7.1.2. (Sherwood (1971), Hadžić (1995)) *Let $(S, \mathcal{F}, T)$ be a complete Menger space, let T be a $H - t$-norm and $f : S \to S$ be a probabilistic q-contraction. Then f has a unique fixed point $u^* \in S$ and $u^* = lim_{n\to+\infty} f^n(p)$, for every $p \in S$.*

Proof. It is easy to prove that if T is a $H-t$-norm, then the orbit $\mathcal{O}(x_0, f)$ is probabilistic bounded for each $x_0 \in S$.
Let $x_n := f^n(x_0), n \in \mathbb{N}^*$. We will prove first that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Let $n \in \mathbb{N}$, $p \in \mathbb{N}^*$, $\epsilon > 0$ and $\lambda \in]0, 1[$. Then we

have $F_{x_{n+p},x_n}(\epsilon) = F_{f^{n+p}(x_0),f^n(x_0)}(\epsilon) \geq F_{f^{n+p-1}(x_0),f^{n-1}(x_0)}(\frac{\epsilon}{q}) \geq \cdots \geq F_{f^p(x_0),x_0}(\frac{\epsilon}{q^n}) \geq D_{\mathcal{O}(x_0,f)}(\frac{\epsilon}{q^n})$. Since the orbit is probabilistic bounded we get that $D_{\mathcal{O}(x_0,f)}(\frac{\epsilon}{q^n}) \to 1$ as $n \to +\infty$, it follows that there exists $n_0(\epsilon,\lambda) \in \mathbb{N}$ such that for every $n \geq n_0(\epsilon,\lambda)$ and every $p \in \mathbb{N}^*$ we have $F_{x_{n+p},x_n}(\epsilon) > 1-\lambda$. Hence, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and since the space S is complete, the sequence converges to a certain element $u^* \in S$. By the continuity of f we obtain that u^* is a fixed point for f. The uniqueness follows from the q-contraction condition. □

Notice that, since the t-norm T_M is of H-type, the fixed point theorem of Sehgal and Bharucha-Reid is an immediate consequence of the previous theorem.

Some extensions and generalizations of the previous theorem were given by Radu, Părău-Radu and Miheţ.

Definition 7.1.2. (V. Radu, 1984) A t-norm T has the fixed point property (shortly f.p.p.) if each probabilistic q-contraction on every complete Menger space $(S,\mathcal{F},T)$ has a unique fixed point.

Theorem 7.1.3. (V. Radu, B[25]) *Every H-t-norm has the f.p.p.*

Theorem 7.1.4. (V. Radu, B[25]) *Let T be a continuous t-norm. Then the following assertions are equivalent:*

i) T is of H-type

ii) T has f.p.p.

iii) for each $a \in (0,1)$ there exists $b \geq a$ such that $T(b,b) = b < 1$.

Theorem 7.1.5. (Părău-Radu, B[1]) *Let $(S,\mathcal{F},T)$ be a complete Menger space such that $T \geq T_L$ and $f : S \to S$ be a probabilistic q-contraction. Then:*

i) If for some $p \in S$ we have $p = f(p)$, then for every $k > 0$

$$E_k(p) = \sup_{u>0} u^k(1 - F_{p,f(p)}(u)) < +\infty.$$

ii) If there exist $p \in S$ and $k > 0$ such that:

$$E_k(p) = \sup_{u>0} u^k(1 - F_{p,f(p)}(u)) < +\infty,$$

then f has a fixed point $u^* \in S$ and the following error estimation holds:

$$\vartheta_k(u^*, f^n(p)) \leq (\sum_{i=n}^{\infty}(q^{\frac{k}{k+1}})^i)(E_k(p))^{\frac{1}{k+1}},$$

for every $n \in \mathbb{N}$. *(where* $\vartheta_k(x,y) := (\sup_{u>0} u^k(1 - F_{p,f(p)}(u)))^{\frac{1}{k+1}}$ *)*

Definition 7.1.3. (Miheţ, B[2]) Let $(S, \mathcal{F})$ be a probabilistic metric space. A mapping $f : S \to S$ is said to be a q-contraction of (ε, λ)-type if $q \in (0,1)$ and for each $\varepsilon > 0$ and each $\lambda \in (0,1)$ the following implication holds:

$$F_{x,y}(\varepsilon) > 1 - \lambda \text{ implies } F_{f(x),f(y)}(q\varepsilon) > 1 - q\lambda.$$

Theorem 7.1.6. (Miheţ, B[2]) *Let* $(S, \mathcal{F}, T_L)$ *be a complete Menger space and* $f : S \to S$ *be a q-contraction of* (ε, λ)*-type. Then* f *has a unique fixed point* $u^* \in S$ *and* $u^* = lim_{n\to+\infty} f^n(p)$, *for every* $p \in S$.

For other results on this field, see also V. Radu B[2], B[24], Părău-Radu B[2], Miheţ B[2], Bocşan B[7], Bocşan-Roventa B[1], Gh. Constantin B[2], Gh. Constantin-Bocşan-Radu B[4], etc.

7.2 Fixed point principles for multivalued operators

If $(S, \mathcal{F}, T)$ is a Menger space, we shall denote by CB(S) the family of all nonempty closed (in the (ε, λ)-topology) and probabilistic bounded subsets of S. The probabilistic distance between two sets A, B from CB(S) will be denoted by $\widetilde{F}_{A,B}$. Recall also, that a t-norm $T \in \mathcal{H}$ if there exists a non-decreasing sequence $(b_n)_{n\in\mathbb{N}}$ from $(0,1)$ such that $b_n \to 1$, as $n \to +\infty$ and the following implication holds:

$$\text{for every } n \in \mathbb{N}\ ,\ 1 \geq x \geq b_n, 1 \geq y \geq b_n \text{ implies } T(x,y) > b_n.$$

For the proof of the main theorem of this section we need an auxiliary result (see for example Hadžić-Pap R[1]).

Lemma 7.2.1 *Let* X *be a nonempty compact uniform space such that the family* $(d_i)_{i\in I}$ *generates the uniformity of* X. *Let* $G : X \to X$ *be a multivalued operator such that for every* $i \in I$ *there exists a constant* $k_i \in]0,1[$ *with the following property:*

$$H_i(G(x), G(y)) \leq k_i d_i(x,y),\ \text{for every } x, y \in X,$$

where H_i *is the Pompeiu-Hausdorff functional generated by* d_i.
Then, $F_G \neq \emptyset$.

The following theorem was proved by D. Miheţ:

Theorem 7.2.1. (Miheţ, B[14]) *Let $(S,\mathcal{F},T)$ be a compact Menger space, $T\in\mathcal{H}$ and $G:S\to CB(S)$ be a multivalued mapping. If for every $n\in\mathbb{N}$ there exists a constant $k_n\in(0,1)$ such that for every $p,q\in S$ and every $s>0$,*

$$F_{p,q}(s)>b_n\Rightarrow \widetilde{F}_{Gp,Gq}(k_n s)>b_n, \tag{7.1}$$

then there exists $x\in S$ such that $x\in Gx$.

Proof. If $T\in\mathcal{H}$ then the family of pseudo-metrics $(d_n)_{n\in\mathbb{N}}$, defined by

$$d_n(p,q)=\sup\{t\mid F_{p,q}(t)\le b_n\},\quad p,q\in S,$$

generates the (ε,λ)-uniformity.

We prove that (7.1) implies (7.2), where

$$H_n(Gp,Gq)\le k_n d_n(p,q), \tag{7.2}$$

for every $n\in\mathbb{N}$ and every $p,q\in S$.

If (7.2) does not hold, there exist $n\in\mathbb{N}$ and $p,q\in S$ such that

$$H_n(Gp,Gq)>k_n d_n(p,q). \tag{7.3}$$

Let $s=\dfrac{1}{k_n}H_n(Gp,Gq)$. Then (7.3) implies that $s>d_n(p,q)$ and therefore $F_{p,q}(s)>b_n$. Using (7.1) we conclude that $\widetilde{F}_{Gp,Gq}(k_n s)>b_n$, i.e., $\widetilde{F}_{Gp,Gq}(H_n(Gp,Gq))>b_n$. We shall prove that for all $A,B\in CB(S)$, $\widetilde{F}_{A,B}(H_n(A,B))\le b_n$ by showing that

$$H_n(A,B)\le\sup\{s\mid \widetilde{F}_{A,B}(s)\le b_n\}. \tag{7.4}$$

In order to prove (7.4), we shall prove that

$$\sup\{s\mid \widetilde{F}_{A,B}(s)\le b_n\}=\sup\left\{s\mid T\left(\inf_{p\in A}\sup_{q\in B}F_{p,q}(s),\inf_{q\in B}\sup_{p\in A}F_{p,q}(s)\right)<b_n\right\} \tag{7.5}$$

and

$$\sup\left\{s\mid T\left(\inf_{p\in A}\sup_{q\in B}F_{p,q}(s),\inf_{q\in B}\sup_{p\in A}F_{p,q}(s)\right)\le b_n\right\}$$

$$= \max\left\{\sup\left\{s \mid inf_{p\in A}\sup_{q\in B} F_{p,q}(s) \le b_n\right\}, \sup\left\{s \mid \inf_{q\in B}\sup_{p\in A} F_{p,q}(s) \le b_n\right\}\right\}. \tag{7.6}$$

Let

$$G(s) = T\left(\inf_{p\in A}\sup_{q\in B} F_{p,q}(s), \inf_{q\in B}\sup_{p\in A} F_{p,q}(s)\right).$$

It is easy to see that

$$\sup\{s \mid G(s) \le b_n\} = \sup\left\{s \mid \sup_{u<s} G(u) \le b_n\right\}.$$

Let

$$P(s) = \inf_{p\in A}\sup_{q\in B} F_{p,q}(s), \quad R(s) = \inf_{q\in B}\sup_{p\in A} F_{p,q}(s).$$

Since $T \le T_M$ we have that

$$\{s \mid P(s) \le b_n\} \subset \{s \mid T(P(s), R(s)) \le b_n\}$$

and

$$\{s \mid R(s) \le b_n\} \subset \{s \mid T(P(s), R(s)) \le b_n\},$$

which implies that

$$\max\{\sup\{s \mid P(s) \le b_n\}, \sup\{s \mid R(s) \le b_n\}\} \le \sup\{s \mid T(P(s), R(s)) \le b_n\}.$$

In order to prove (7.6) we shall suppose that there exists $\delta > 0$ such that

$$\max\{\sup\{s \mid P(s) \le b_n\}, \sup\{s \mid R(s) \le b_n\}\} < \delta < \sup\{s \mid T(P(s), R(s)) \le b_n\}.$$

Then

$$P(\delta) > b_n, \quad R(\delta) > b_n, \quad T(P(\delta), R(\delta)) \le b_n,$$

which is a contradiction since $T \in \mathcal{H}$. Hence

$$\sup\{s \mid \widetilde{F}_{A,B}(s) \le b_n\} = \max\{\sup\{s \mid P(s) \le b_n\}, \sup\{s \mid R(s) \le b_n\}\}.$$

From

$$\sup_{q\in B}\inf_{p\in A} d_n(p,q) \le \sup\{s \mid P(s) \le b_n\}$$

$$\sup_{p\in A}\inf_{q\in B} d_n(p,q) \le \sup\{s \mid R(s) \le b_n\},$$

we obtain that

$$H_n(A, B) \leq \sup\{s \mid \widetilde{F}_{A,B}(s) \leq b_n\}.$$

Since (7.2) holds, the theorem follows from Lemma 7.2.1. □

Remark 7.2.1. For other aspects of the probabilistic structures in connection to fixed point theory for single-valued and multivalued operators, see the books of O. Hadžić and E. Pap R[1] and that of Y.J. Cho, M. Grabiec and V. Radu B[1].

Chapter 8

Nonexpansive operators

Guidelines: M.S. Brodskii and D.P. Milman (1948), M.A. Krasnoselskii (1955), F.E. Browder (1965), D. Göhde (1965), W.A. Kirk (1965), Z. Opial (1967), W. Takahashi (1970), M. Edelstein (1972), W.V. Petryshyn and T.E. Williamson (1972), W.G. Dotson (1973), R.E. Bruck (1974), K. Goebel (1975), L.A. Karlovitz (1976), S. Ishikawa (1976), S. Reich (1980; 1983), D.S. Jaggi (1982), G. Kassay (1986), J.B. Baillon (1988), M.A. Khamsi (1996).
General references: K. Goebel and W.A. Kirk R[1], W.A. Kirk and B. Sims (Eds.) R[1], P.L. Papini R[1], R. C. Sine (ed.) R[1], S.P. Singh, S. Thomeier and B. Watson (Eds.) R[1], M.A. Théra and J.B. Baillon (Eds.) R[1], R.E. Bruck R[2], A. Petruşel B[21], V. Berinde B[6], G. Kassay B[1]-B[3], R. Precup B[7], B[11].

8.0 Preliminaries

8.0.1 The geometry of the Banach spaces

Let $(X, +, \mathbb{R}, \|\cdot\|)$ be a Banach space over the real field, i.e.,

(a) $(X, +, \mathbb{R})$ is a real linear space;

(b) $\|\cdot\| : X \to \mathbb{R}_+$ is a norm on X;

(c) the metric space $(X, d_{\|\cdot\|})$ is complete, where $d_{\|\cdot\|} : X \times X \to \mathbb{R}_+$ is defined by $d_{\|\cdot\|}(x, y) = \|x - y\|$.

By definition, a Banach space $(X, \|\cdot\|)$ is:

(1) strictly convex if

$$x, y \in X, \quad \|x+y\| = \|x\| + \|y\| \Rightarrow x = 0 \quad \text{or} \quad y = 0 \quad \text{or} \quad y = \lambda x$$

for some $\lambda > 0$;

(2) uniformly convex if for each $\varepsilon \in]0,2]$ there exists $\delta(\varepsilon) > 0$ such that

$$x, y \in X, \quad \|x\| = \|y\| = 1, \ \|x-y\| \geq \varepsilon \Rightarrow \left\| \frac{x+y}{2} \right\| \leq 1 - \delta(\varepsilon).$$

For a better understanding the above notions we present

Theorem 8.0.1. *If $(X, \|\cdot\|)$ is a Banach space, then the following statements are equivalent:*

(i) X is a strict convex Banach space.

(ii) $x, y \in X$, $\|x\| = \|y\| = 1$, $x \neq y \Rightarrow \left\| \frac{x+y}{2} \right\| < 1$.

(iii) $x, y \in X$, $\|x\| = \|y\| = 1$, $x \neq y \Rightarrow \|\lambda x + (1-\lambda)y\| < 1$, for all $\lambda \in]0,1[$.

(iv) $\{z \in X \mid \|x-z\| + \|z-y\| = \|x-y\|\} = \{\lambda x + (1-\lambda)y \mid \lambda \in [0,1]\}$, for all $x, y \in X$.

Example 8.0.1. The Hilbert spaces are uniformly convex.

Example 8.0.2. The Banach space $(\mathbb{R}^m, \|\cdot\|)$, where $\|x\| := \sum_{i=1}^{m} |x_i|$ isn't uniformly convex.

Let Y be a bounded subset of a Banach space $(X, \|\cdot\|)$. An element $y_0 \subset Y$ is a nondiametral point if

$$\sup\{\|y_0 - y\| \mid y \in X\} < \delta(Y).$$

By definition, a closed convex subset Y of a Banach space X has normal structure if any bounded convex subset $Z \subset Y$ which contains more than one point contains a nondiametral point.

For the theory of Banach spaces, see V. Barbu and T. Precupanu R[1], B. Beauzamy R[1], C. Foiaş R[1], V.I. Istrăţescu R[1], M.A. Krasnoselskii and P. Zabrejko R[1], Gh. Marinescu R[1], L. Nirenberg R[2], A. Pietsch R[1], J.T. Schwartz R[1], I. Singer R[1], W. Takahashi R[3].

8.0.2 Averaged operators

Let X be a Banach space and Y a convex subset of X. Let $f : Y \to Y$ be an operator and $\lambda \in]0, 1[$. We consider the operator $f_\lambda : Y \to Y$, given by

$$f_\lambda := \lambda \cdot 1_Y + (1 - \lambda) \cdot f.$$

We have:

(1) $F_f = F_{f_\lambda}$, for each $\lambda \in]0, 1[$;

(2) if f is nonexpansive then f_λ is nonexpansive too, for each $\lambda \in]0, 1[$;

A deep result for the averaged operator f_λ is the following:

Ishikawa's Theorem. *Let X be a Banach space, Y be a bounded convex subset of X and $f : Y \to Y$ be a nonexpansive operator. Then f_λ is asymptotically regular, for each $\lambda \in]0, 1[$.*

8.1 Fixed point theory of nonexpansive operators

We begin our consideration with some examples.

Example 8.1.1. Let X be a Banach space and $x_0 \in X$, $x_0 \neq 0$. We consider the following nonexpansive operators:

$$f : X \to X,\ f(x) := x + x_0, \text{ for all } x \in X,$$
$$1_X : X \to X,\ f(x) := x, \text{ for all } x \in X,$$
$$g : X \to X \text{ be an } \alpha - \text{contraction.}$$

In these cases we have:

$$F_f = \emptyset, \quad F_{1_X} = X, \quad cardF_g = 1.$$

Example 8.1.2. Let Y be a nonempty compact subset of a Banach space X and $f : Y \to Y$ be a contractive operator. Then f is nonexpansive and $cardF_f = 1$.

Example 8.1.3. Let $(c_0(\mathbb{R}), \|\cdot\|)$ be the Banach space of all real sequences, $x = (x_n)_{n\in\mathbb{N}}$, tending to zero, with norm, $\|x\| := \sup\limits_{n\in\mathbb{N}} |x_n|$.

Let $Y \subset X$, $Y := \{(x_n)_{n\in\mathbb{N}} \in c_0(\mathbb{R}) \mid 0 \leq x_n \leq 1, \text{ for all } n \in \mathbb{N}\}$ and the nonexpansive operator $f : Y \to Y$ defined by $f((x_n)_{n\in\mathbb{N}}) := (1, x_1, x_2, \ldots)$. The set Y is a bounded closed convex subset of $c_0(\mathbb{R})$ and $F_f = \emptyset$.

Example 8.1.4. (F.E. Browder R[5]). *Let X be a Hilbert space, $Y \subset X$ a bounded closed convex subset of X and $f : X \to X$ be a nonexpansive operator. Then $F_f \neq \emptyset$.*

Example 8.1.5. (W.G. Dotson R[1]). *Let X be a Banach space and $Y \subset X$ a compact starshaped subset of X and $f : Y \to Y$ be a nonexpansive operator. Then $F_f \neq \emptyset$.*

Indeed, let $y_0 \in Y$ be such that $(1-\lambda)y_0 + \lambda y \in Y$ for all $y \in Y$ and $\lambda \in [0,1]$. Let us consider the following operators

$$f_n : Y \to Y, \quad f_n(y) := \frac{1}{n}y_0 + \left(1 - \frac{1}{n}\right) f(y), \quad n \in \mathbb{N}^*.$$

We remark that f_n is a contraction for all $n \in \mathbb{N}^*$ and $(Y, d_{\|\cdot\|})$ is a complete metric space. Let we denote by y_n the unique fixed point of f_n. Since Y is compact, there is a subsequence $(y_{n_i})_{i\in\mathbb{N}^*}$ of $(y_n)_{n\in\mathbb{N}^*}$ which converges to some $y^* \in Y$. From the continuity of f we have that $f(y_{n_i}) \to f(y^*)$. From the relation

$$y_{n_i} = \frac{1}{n_i}y_0 + \left(1 - \frac{1}{n_i}\right) f(y_{n_i})$$

it follows that y^* is a fixed point of f.

Example 8.1.6. *Let $(X, \|\cdot\|)$ be a strictly convex Banach space, $Y \subset X$ a nonempty convex subset of X and $f : X \to X$ be a nonexpansive operator. Then F_f is a convex subset of Y, possible empty. See also Example 8.1.1.*

Indeed, let $y_1, y_2 \in F_f$, $y_1 \neq y_2$ and $y \in [y_1, y_2]$ (see Theorem 8.0.1(iv)). We have

$$\|y_1 - y_2\| \leq \|y_1 - f(y)\| + \|f(y) - y_2\| \leq \|y_1 - y\| + \|y - y_2\| = \|y_1 - y_2\|.$$

Thus, $f(y) = y$.

Example 8.1.7. *Let Y be a closed convex subset of a Banach space X and $f : Y \to Y$ a nonexpansive operator. Let $\lambda \in]0,1[$ and $f_\lambda : Y \to Y$ be defined by $f_\lambda(x) = \lambda x + (1-\lambda)f(x)$. Then:*

(i) f_λ is nonexpansive, for all $\lambda \in]0,1[$;

(ii) $F_f = F_{f_\lambda}$, *for all* $\lambda \in [0,1]$*;*

(iii) f_λ *is asymptotically regular, for all* $\lambda \in]0,1[$.

One of the basic results of the fixed point theory of nonexpansive operators is the following:

Browder-Göhde-Kirk's Theorem. (F.E. Browder R[5], D. Göhde R[1], W.A. Kirk R[2]) *Let* X *be a reflexive Banach space,* $Y \subset X$ *be a nonempty closed convex bounded subset of* X *with normal structure and* $f : Y \to Y$ *be a nonexpansive operator. Then* $F_f \neq \emptyset$.

More general we have:

Kirk's Theorem. (W.A. Kirk R[2]; see also W.A. Kirk and B. Sims R[1], pp. 629) *Let* Y *be a nonempty weakly compact and convex subset of a Banach space* X *and* $f : Y \to Y$ *be a nonexpansive operator. If* Y *has normal structure then* $F_f \neq \emptyset$.

Another type of result is the following

Belluce-Kirk's Theorem. (L.P. Belluce and W.A. Kirk R[1]) *Let* Y *be a nonempty, weakly compact, convex subset of a Banach space* X*. Suppose* $f : Y \to Y$ *satisfies the following conditions:*

(i) f *is a nonexpansive operator;*

(ii) $1_Y - f$ *is convex, i.e.,*

$$\left\|(1 - Y - f)\left(\frac{x+y}{2}\right)\right\| \leq \frac{1}{2}(\|(1_Y - f)(x)\| + \|(1_Y - f)(y)\|)$$

for all $x, y \in Y$.

Then f *has at least a fixed point.*

The above considerations give rise to the following problems.

Problem 8.1.1. Which are the Banach spaces X with the following property:

$$Y \in P_{cl,b,cv}(X), \quad f : Y \to Y \text{ nonexpansive} \Rightarrow F_f \neq \emptyset?$$

Problem 8.1.2. Let X a Banach space, $Y \in P_{cl,b,cv}(X)$ and $f : X \to X$ be a nonexpansive operator. In which additional conditions on f we have that $F_f \neq \emptyset$?

For the above definitions, examples, theorems and problems see: F.E. Browder R[13], K. Goebel and W.A. Kirk R[1] and R[2], J.B. Baillon R[1], W.A.

Kirk and B. Sims R[1] (the papers by B. Sims, K. Goebel and W.A. Kirk, S. Prus, E. Llorens-Fuster, W. Kaczor and M. Koter-Mórgowska, J. Jachymski,...), G. Marino and P. Pietramala R[1], J. García-Falset, E. Llorens-Fuster and E.M. Mazcunan-Navarro R[1], etc.

8.2 Jaggi-nonexpansive operators

Let X be a normed space and $Y \in P_{b,cl,cv}(X)$. An operator $f : Y \to Y$ is Jaggi-nonexpansive if

$$\sup\{\|f(x) - f(y)\| \mid y \in Z\} \leq \sup\{\|x - y\| \mid y \in Z\},$$

for all $Z \in I_{b,cl,cv}(f)$ and all $x \in Z$.

A normed space X has Jaggi fixed point property if every $Y \in P_{b,cl,cv}(X)$ has the fixed point property with respect to all Jaggi-nonexpansive operators $f : Y \to Y$.

D.S. Jaggi obtained the following result:

Jaggi's Theorem. *Every reflexive Banach space with normal structure has Jaggi's fixed point property.*

On the other hand we have:

Theorem 8.2.1. (G. Kassay, B[3]). *Every normed space with Jaggi's fixed point property has normal structure.*

Thus, we have:

Theorem 8.2.2. (Jaggi-Kassay) *A reflexive Banach space has normal structure if and only if it possesses Jaggi's fixed point property.*

8.3 Nonexpansive operators on nonconvex sets

Let X be a nonempty set and $F : [0,1] \times X \times X \to X$ an operator. By definition (X, F) is a semi-convex structure in Gudder' sense if:

(i) $F(\lambda, x, F(\mu, y, z)) = F(\lambda + (1-\lambda)\mu, F(\lambda(\lambda + (1-\lambda)\mu)^{-1}, x, y), z)$, for all $x, y, z \in X$ and all $\lambda, \mu \in [0,1]$;

(ii) $F(\lambda, x, x) = x$, for any $x \in X$ and $\lambda \in [0,1]$.

A subset $Y \subset X$ is F-starshaped if there exists $p \in Y$ such that, that for any $x \in Y$ and $\lambda \in [0,1]$ we have $F(\lambda, x, p) \in Y$.

We have:

Theorem 8.3.1. (A. Petruşel, B[21]). *Let X be a Banach space with a semi-convex structure F such that:*

(a) there exists $\varphi : [0,1] \to [0,1]$ such that

$$\|F(\lambda x, p) - F(\lambda, y, p)\| \leq \varphi(\lambda)\|x - y\|, \quad \textit{for all } x, y, p \in X, \quad \textit{for all } \lambda \in [0,1[;$$

(b) F is continuous with respect to its first argument.

Let $Y \subset X$ be a compact and F-semistarshaped subset of X and $f : Y \to Y$ be a nonexpansive operator. Then $F_f \neq \emptyset$.

Let X be a Banach space. Let $Y \subset X$ and $(f_\alpha)_{\alpha \in Y}$ be a family of functions $f_\alpha : [0,1] \to Y$, having the property that for each $\alpha \in Y$ we have $f_\alpha(1) = \alpha$. Such a family is said to be φ-contractive provided, for all α and β in Y and $t \in]0,1[$ there exists a comparison function φ_t such that

$$\|f_\alpha(t) - f_\beta(t)\| \leq \varphi_t(\|\alpha - \beta\|).$$

We have:

Theorem 8.3.2. (V. Berinde, B[6]). *Let Y be a compact subset of a Banach space X and suppose there exists a φ-contractive family such that:*

$$f_\alpha(t) \to f_{\alpha_0}(t_0), \ \textit{as } \alpha \to \alpha_0, \ t \to t_0.$$

Then any nonexpansive operator $f : Y \to Y$ has a fixed point in Y.

8.4 Nonexpansive operators on convex metric spaces

Let (X, d) be a metric space. We suppose that X has a convex structure in Takahashi' sense, defined by:

$$W : [0,1] \times X \times X \to X$$

such that

$$d(u, w(\lambda, x, y)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y),$$

for all $x, y, u \in X,\ \lambda \in [0, 1]$.

By definition, a convex metric space has the property (c) if every decreasing net of nonempty closed convex subsets of X has nonempty intersection.

We have:

Theorem 8.4.1. (G. Kassay, B[2]). *Let (X, d, W) be a uniformly convex metric space with property (c). If $f : X \to X$ is a nonexpansive operator, then $F_f \neq \emptyset$.*

8.5 Other results

There are many generalizations of Browder-Göhde-Kirk theorem. For example we have:

Goebel-Kirk-Shimi's Theorem. (K. Goebel, W.A. Kirk and T.N. Shimi R[1]) *Let Y be a nonempty closed convex bounded subset of a uniformly convex Banach space and $f : Y \to Y$ be an operator. We suppose that:*

(i) there exist $a_i \in \mathbb{R}_+$, $i = \overline{1, 5}$, $\sum_{i=1}^{5} a_i = 1$, such that

$$\|f(x) - f(y)\| \leq a_1\|x - y\| + a_2\|x - f(x)\| + a_3\|y - f(y)\| + a_4\|x - f(y)\| + a_5\|y - f(x)\|, \quad \textit{for all} \ \ x, y \in Y;$$

Then, f has at least a fixed point.

For other generalizations see D. Roux and P. Soardi R[1], J.S. Bae R[2] and the references therein (J. Bogin (1976), R. Kannan (1973), M. Gregus (1980), etc.).

For other results in connection with the theory of nonexpansive operators, see G. Kassay B[2], B[3], A. Petruşel B[21], V. Berinde B[6], R. Precup B[7], B[11], J. Gornicki R[1], R.P. Agarwal, D. O'Regan and D.R. Sahu R[1], J.-C. Yao and L.-C. Zeng R[1], N.M. Gulevich R[1], etc.

Chapter 9

Expansive, noncontractive and dilating operators

Precursors: K. Borsuk (1933), H. Freundental and W. Hurewicz (1936).
Guidelines: M. Altman (1970), C. Avramescu (1972), I. Rosenholtz (1975), B. Fisher (1976), T. Hu (1980), S. Leader (1982).
General references: M. Altman R[1], C. Avramescu B[3], I. Rosenholtz R[1], B. Fisher R[1], I.A. Rus B[66], T. Hu R[1], A.A. Gillespie and B.B. Williams R[1], V. Popa B[14], B[20], S. Wang, B. Li, Z. Gao and K. Iseki R[1], M.A. Khan, M.S. Khan and S. Sessa R[1], S. Wang, B. Li, Z. Gas and K. Iseki R[1], I.A. Rus B[4].

9.0 Basic notions and results

Let (X, d) be a metric space. By definition, an operator $f : X \to X$ is called:

(i) noncontractive if $d(f(x), f(y)) \geq d(x, y)$, for all $x, y \in X$;

(ii) dilatation if there exists $l > 1$ such that $d(f(x), f(y)) \geq l d(x, y)$, for all $x, y \in X$;

(iii) nonlipschitzian if there exists $l > 0$ such that $d(f(x), f(y)) \geq l d(x, y)$, for all $x, y \in X$;

(iv) similarity if there exists $l > 0$ such that $d(f(x), f(y)) = l d(x, y)$, for all $x, y \in X$;

(v) isometry if $d(f(x), f(y)) = d(x, y)$, for all $x, y \in X$;

(vi) expansive if $d(f(x), f(y)) > d(x, y)$, for all $x, y \in X$ with $x \neq y$;

(vii) locally noncontractive (respectively dilatation, etc.) if each $x \in X$ admits a neighborhood $U(x)$ such that $f : U(x) \to X$ is noncontractive respectively dilating, etc.)

(viii) open if $(U \in P_{op}(X))$ implies $(f(U) \in P_{op}(X))$;

(ix) closed if $(U \in P_{cl}(X))$ implies $(f(U) \in P_{cl}(X))$;

We have the following fundamental results.

Rosenholtz's Theorem. (Rosenholtz R[2]) *Let (X, d) be a compact connected metric space and $f : X \to X$ be an operator. Suppose:*

(i) f is continuous;

(ii) f is open;

(iii) f is a locally dilatation.

Then f has at least one fixed point.

Browder's Theorem. (F.E. Browder R[2]) *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces and $f : X \to Y$ be an operator. Suppose:*

(i) f is continuous;

(ii) f is open;

(iii) f is locally nonlipschitzian.

Then f is a topological isomorphism.

Gillespie-Williams's Theorem. (A.A. Gillespie and B.B. Williams R[1])

Let (X, d) be a complete metric space and $f : X \to X$ be an operator. Suppose:

(i) f is an l-dilatation;

(ii) $f(X)$ is a closed subset of X.

Then the following statements are equivalent:

(a) $cardF_f = 1$;

(b) $\bigcap\limits_{n \in \mathbb{N}^*} f^n(X) \neq \emptyset$;

(c) there exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that $d(x_n, f(x_n)) \to 0$ as $n \to +\infty$.

Proof. $(a) \Rightarrow (b)$. Notice that if $F_f = \{x^*\}$, then $x^* \in \bigcap_{n\in\mathbb{N}^*} f^n(X) \neq \emptyset$.

$(b) \Rightarrow (c)$. Let $x_0 \in \bigcap_{n\in\mathbb{N}^*} f^n(X) \neq \emptyset$. Since each dilatation is injective, we consider the operator $f^{-1} : f(X) \to X$. Let $x_n := (f^{-1})^n(x_0)$. We have: $d(x_n, f(x_n)) = d((f^{-1})^n(x_0), (f^{-1})^{n-1}(x_0)) \leq \frac{1}{l^n} d(x_0, f^{-1}(x_0)) \to 0$ as $n \to +\infty$.

$(c) \Rightarrow (a)$. Let $(x_n)_{n\in\mathbb{N}}$ be such that $d(x_n, f(x_n)) \to 0$ as $n \to +\infty$. Then $f(x_n)$ is a Cauchy sequence in $f(X)$. Let $u \in X$ such that $f(x_n) \to f(u)$ as $n \to +\infty$. From the continuity of f^{-1} it follows that $x_n \to u$ as $n \to +\infty$. Thus, we have $d(f(u), u) \leq d(f(u), f(x_n)) + d(f(x_n), x_n) \to 0 + d(x_n, u)$ as $n \to +\infty$. Hence $f(u) = u$. Moreover, since f^{-1} is a contraction, we obtain that $F_f = \{u\}$. □

For more considerations on the above classes of operators see M. Altman R[1], F.E. Browder R[2], T. Hu and W.A. Kirk R[1], I. Rosenholtz R[1], R[2], I. Rosenholtz and W.O. Ray R[1], S. Miklos R[1], A.A. Gillespie and B. Williams R[1], I.A. Rus B[4], B[6], etc.

9.1 Dilating operators

In 1972, C. Avramescu (see B[3]) proved the following result:

Theorem 9.1.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a dilating surjection. Then f has a unique fixed point.*

More general we have:

Theorem 9.1.2. (I.A. Rus B[70]). *Let (X, d) be a complete metric space and $f : X \to X$ a surjective operator. We suppose that there exists $l > 1$, such that*

$$\max\{d(f(x), f(y)), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\} \geq l d(x, y),$$

for all $x, y \in X$.
Then f has a unique fixed point.

For other results see C. Avramescu B[3], I.A. Rus B[45], A. Buică B[2], A. Buică and A. Domokos B[1] and F. Aldea B[3].

9.2 Noncontractive operators

The main result for noncontractive operator is the following theorem given by H. Freundental and W. Hurewicz (1936) (see T. Hu R[1]):

Freundental-Hurewicz's Theorem. *Let* (X, d) *be a compact metric space and* $f : X \to X$ *be an operator such that*

$$d(f(x), f(y)) \geq d(x, y), \quad \textit{for all} \ \ x, y \in X.$$

Then the operator f *is an isometry.*

On the other hand, we have: (see Y. Benyamini and J. Lindenstrauss R[1])

Mazur-Ulam's Theorem. *A surjective isometry between two Banach spaces which takes 0 to 0 is necessarily linear.*

If we consider generalized noncontractive operators then, in general, we have degeneracy cases.

Theorem 9.2.1. (I.A. Rus, B[70]). *Let* (X, d) *be a metric space and* $f : X \to X$ *be an operator. If there exists* $a > 0$ *such that*

$$d(f(x), f(y)) \geq a \min\{d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\},$$

for all $x, y \in X$, *then* $f = 1_X$.

For other results see V. Popa B[14], B[20], D. Trif B[2]. see also 3.1.2.

9.3 Fixed points, zeros and surjectivity

Let $(X, +)$ be an abelian group and $\mathcal{F} \subset \mathbb{M}(X)$. By definition:

(i) X has the fixed point property with respect to $\mathcal{F}$, if $f \in \mathcal{F}$ implies $F_f \neq \emptyset$.

(ii) X has the zero point property with respect to $\mathcal{F}$, if $f \in \mathcal{F}$ implies $Z_f \neq \emptyset$.

(iii) X has the surjectivity property with respect to $\mathcal{F}$, if $f \in \mathcal{F}$ implies f is surjective.

We have:

Theorem 9.3.1. (I.A. Rus, F. Aldea, B[1]). *We suppose that:*

(i) $f \in \mathcal{F}$ *implies* f *is injective operator;*

(ii) $f \in \mathcal{F}$ *and* $f(X) = X$ *implies* $F_{f^{-1}} \neq \emptyset$*;*

(iii) For all $f \in \mathcal{F}$ *there is* $n_0(f) \in \mathbb{N}$ *such that* $f^{n_0} + 1_X \in \mathcal{F}$*;*

(iv) $f \in \mathcal{F}$ *and* $y_0 \in X$ *imply* $f + y_0 \in \mathcal{F}$*.*

Then the following statement are equivalent:

(a) X *has the f.p.p. with respect to* $\mathcal{F}$*;*

(b) X *has the z.p.p. with respect to* $\mathcal{F}$*;*

(c) X *has the s.p.p. with respect to* $\mathcal{F}$*.*

From this theorem, we have:

Theorem 9.3.2. (I.A. Rus, F. Aldea, B[1]). *Let* X *be a Banach space,* $\varphi : R_+ \to R_+$ *be a function. We suppose that:*

(i) $\varphi(0) = 0$*;*

(ii) φ *is bijective;*

(iii) φ^{-1} *is a comparison function;*

(iv) there exists $n \in \mathbb{N}$ *such that* $\varphi^n(t) - t \geq \varphi(t)$*, for all* $t > 0$*.*

Let $\mathcal{F} := \{f : X \to X | \ f$ *is* φ*-dilating operator*$\}$*.*

Then the conclusion of the Theorem 9.3.1. holds.

Remark 9.3.1. From Theorem 9.3.2., we have the following result given by A.A. Gillespie and B B. Williams R[1]:

Gillespie-Williams's Theorem. *Let* X *be a Banach space and* $\mathcal{F} := \{f : X \to X | \ f$ *is a continuous dilating operator* $\}$*.*

Then the conclusion of Theorem 9.3.2. holds.

For other results see 24.29.

Chapter 10

Picard and weakly Picard operators

Precursors: E. Picard (1890), T. Lalescu (1908), S. Banach (1922), R. Caccioppoli (1930), L. Kantorovich (1939), M.A. Kraskoselskii (1955), C. Bessaga (1959), L. Janos (1967), P.R. Meyers (1967), Z. Opial (1967), M.W. Hirsch and C.C. Pugh (1970), V. Barbuti and S. Guerra R[1], L. Losonczi (1973), K. Valeev (1973), K. Iseki (1975), B. Fuchssteiner (1977), A. Pazy (1977), V.Yu. Stetsenko and M. Shaaban (1986).
Guidelines: I.A. Rus (1983), I.A. Rus (1993), I.A. Rus (2001), I.A. Rus (2003), I.A. Rus, A. Petruşel and M.A. Şerban (2006).
General references: I.A. Rus B[102], B[5], B[14], B[16], B[30], B[34], B[41], B[100], I.A. Rus, A. Petruşel and M.A. Serban B[1], V. Berinde B[2], B[7], M.A. Serban B[3], B[5], J. Dugundji and A. Granas R[1], W.A. Kirk and B. Sims R[1], D.R. Smart R[1], K. Goebel and W.A. Kirk R[1], R.P. Agarwal, M. Meehan and D. O'Regan R[1], F.E. Browder and W.V. Petryshyn R[1], G. Gabor R[1], S. Heikkilä and V. Lakshmikantham R[1], M.A. Krasnoselskii and P.P. Zabrejko R[1], Şt. Măruşter R[1], F. Robert R[1], R. Sine (ed.) R[1].

10.0 Basic notions

Let (X, d) be a metric space and $f : X \to X$ be an operator. By definition,

(i) the operator f is said to be weakly Picard (briefly WPO) if the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit is a fixed point of f;

(ii) if f is a WPO and $F_f = \{x^*\}$, then f is a Picard operator (briefly, PO).

Remark 10.0.1. If f is a PO, then f is a Bessaga operator, i.e.,

$$F_{f^n} = F_f = \{x^*\}, \quad \text{for all} \quad n \in \mathbb{N}^*.$$

Remark 10.0.2. An operator f is Picard if and only if $F_f = \{x^*\}$ and $\{x^*\}$ is a global attractor for the discrete dynamic generated by the operator f.

If f is a WPO, then we consider the operator f^∞ defined by

$$f^\infty : X \to X, \quad f^\infty(x) := \lim_{n\to\infty} f^n(x).$$

It is obvious that

$$f^\infty(X) = F_f \text{ and } \omega_f(x) = \{f^\infty(x)\}, \text{ for all } x \in X.$$

Let f be a WPO and $c > 0$. By definition, f is called c-WPO if

$$d(x, f^\infty(x)) \le c d(x, f(x)), \quad \text{for all} \quad x \in X.$$

For some examples of POs and WPOs see Chapters 3-8.

Notice that it is a very difficult problem to establish that a given operator is or isn't Picard or weakly Picard. For example we have:

Markus-Yamabe Conjecture. (see G. Meisters R[1], A. Cima, A. Gasull and F. Mañosas R[1]). Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-function such that $f(0) = 0$, and $Jf(x)$ (the Jacobian matrix of f at x) has all its eigenvalues with modulus less than one. Then, f is a Picard operator.

10.1 The structure theorem of WPOs

We begin our considerations with an example.

Let (X_i, d_i), $i \in I$, be a family of metric space, $f_i : X_i \to X_i$, $i \in I$, a family of POs. Let x_i^* be the unique fixed point of f_i. Let $X := \bigcup_{i \in I} X_i$ be the disjoint union of the family $(X_i)_{i \in I}$. Let $d : X \times X \to R_+$,

$$d(x,y) := \begin{cases} d_i(x,y), & \text{if } \ x, y \in X_i, \ i \in I \\ d_i(x, x_i^*) + d_j(y, x_j^*), & \text{if } \ i \neq j, \ x \in X_i, \ y \in X_j \end{cases}$$

Then, the functional d is a metric on X and the operator

$$f : X \to X, \quad f(x) = f_i(x), \text{ if } x \in X_i, \ i \in I,$$

is a WPO.

Moreover, we have the following characterization of the WPOs.

Theorem 10.1.1. (I.A. Rus, B[16]). *Let (X,d) be a metric space and $f : X \to X$ be an operator. Then, f is WPO (respectively c-WPO) if and only if there exists a partition of X, $X = \bigcup_{\lambda \in \Lambda} X_\lambda$, such that:*

(a) $X_\lambda \in I(f)$, $\lambda \in \Lambda$;

(b) $f|_{X_\lambda} : X_\lambda \to X_\lambda$ is a PO (respectively c-PO), for all $\lambda \in \Lambda$.

From Theorem 10.1.1. we have (see I.A. Rus B[1])

Theorem 10.1.2. *We consider the Bernstein operators*

$$B_n : C[0,1] \to C[0,1], \ B_n(f)(x) := \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k}.$$

If $n \in \mathbb{N}^*$, then, B_n is a WPO and

$$B_n^\infty(f)(x) = f(0) + (f(1) - f(0))x.$$

Proof. Let $X_{\lambda,\mu} := \{f \in C[0,1] \mid f(0) = \lambda, \ f(1) = \mu\}$, $\lambda, \mu \in \mathbb{R}$. We remark that:

(i) $X_{\lambda,\mu}$ is a closed subset of $(C[0,1], \|\cdot\|_C)$, for all $\lambda, \mu \in \mathbb{R}$;

(ii) $X_{\lambda,\mu}$ is an invariant subset of B_n, for all $\lambda, \mu \in \mathbb{R}$ and all $n \in \mathbb{N}^*$;

(iii) $C[0,1] = \bigcup_{\lambda,\mu \in \mathbb{R}} X_{\lambda,\mu}$ is a partition of $C[0,1]$.

On the other hand

$$\|B_n(f) - B_n(g)\|_C \leq \left(1 - \frac{1}{2^{n-1}}\right) \|f - g\|_C, \text{ for all } f, g \in X_{\lambda,\mu}$$

But $B_n(\lambda + (\mu - \lambda)(\cdot)) = \lambda + (\mu - \lambda)(\cdot)$.

Hence B_n is a WPO and $B_n^\infty(f) = f(0) + (f(1) - f(0))(\cdot)$. □

Remark. 10.1.1 Since B_n is a contraction on $X_{\lambda,\mu}$, we have that B_n is a graphic contraction on $C[0,1]$.

For other applications to linear positive operators, see I.A. Rus B[2], O. Agratini and I.A. Rus B[1], B[2].

10.2 Data dependence of the fixed point set

For the class of c-WPOs we have the following data dependence result:

Theorem 10.2.1. (I.A. Rus and S. Mureşan, B[1]). *Let (X, d) be a metric space and $f_1, f_2 : X \to X$. We suppose that:*

(i) the operator f_i is $c_i - WPO$ for $i \in \{1, 2\}$;

(ii) there exists $\eta > 0$ such that

$$d(f_1(x), f_2(x)) \leq \eta, \quad \textit{for all} \quad x \in X.$$

Then:

$$H(F_{f_1}, F_{f_2}) \leq \eta \max(c_1, c_2).$$

Proof. Since f_1 is a c_1-WPO we have that

$$d(x, f_1^\infty(x)) \leq c_1 d(x, f_1(x)), \quad \text{for all} \quad x \in X.$$

If we take $x = x_2^* \in F_{f_2}$, then

$$d(x_2^*, f_1^\infty(x_2^*)) \leq c_1 d(x_2^*, f_1(x_2^*)) = c_1 d(f_2(x_2^*), f_1(x_2^*)) \leq c_1 \eta.$$

So, for each $x_2^* \in F_{f_2}$ and $f_1^\infty(x_2^*) \in F_{f_1}$,

$$d(x_2^*, f_1^\infty(x_2^*)) \leq c_1 \eta.$$

In a similar way we have that for each $x_1^* \in F_{f_1}$ and $f_2^\infty(x_1^*) \in F_{f_2}$,

$$d(x_1^*, f_2^\infty(x_1^*)) \leq c_2\eta.$$

From the definition of Pompeiu-Hausdorff functional it follows that

$$H_d(F_{f_1}, F_{f_2}) \leq \eta \max(c_1, c_2).$$

□

For the case of multivalued operators see 11.6.

10.3 Picard operators on ordered metric spaces

Throughout this section, $(X, d, \leq)$ be an ordered metric space (i.e. a set X endowed with a metric d and a partially order relation $\leq$ which is closed w,r.t d) and $f : X \to X$ is an operator.

Then, we have:

Abstract Gronwall Lemma. *We suppose that:*

(i) f is a PO ($F_f = \{x_f^*\}$);

(ii) f is increasing.

Then:

(a) $x \leq f(x) \Rightarrow x \leq x_f^*$;

(b) $x \geq f(x) \Rightarrow x \geq x_f^*$.

Proof. (a) Let $x \in X$ be such that $x \leq f(x)$. From (ii) we have that

$$x \leq f(x) \leq f^2(x) \leq \cdots \leq f^n(x), \quad \text{for all} \quad n \in \mathbb{N}^*.$$

From (i) $f^n(x) \to x_f^*$ as $n \to \infty$. Since the ordered relation is closed, hence $x \leq x_f^*$.

It is important to remark that:

(1) in the above lemma, instead of $(X, d, \leq)$ we can put one of the following:

- $(X, d, \leq)$ an ordered complete metric space
- $(X, d, \leq)$ an ordered complete generalized metric space.

Instead of condition (i) we can put a condition which implies that f if a PO.

(2) Let $(X, \leq)$ be an ordered set and $f : X \to X$ be an operator. For to have a concrete Gronwall lemma we follow the following algorithm:

- we examine if f is increasing;
- we choose a metric d on X with respect to which $\leq$ is closed;
- we examine if f is PO in (X, d);
- we "determine" the unique fixed point of f, x_f^*.

The last step in the above algorithm is a difficult problem in the way to a concrete Gronwall lemma.

For abstract and concrete Gronwall lemmas see I.A. Rus B[97], D.S. Mitrinović, J.E. Pečarić and A.M. Fink R[1], V. Lakshmikantham, S. Leela and A.A. Martynyuk R[1], D. Bainov and P. Simeonov R[1], L. Losonczi R[1], V.Ya. Stetsenko and M. Shaaban R[1], T.M. Flett R[1], P. Ver Eecke R[1], K. Valeev R[1], S. András B[2], A. Buică R[2], R[5], V. Dincuţă B[1], N. Lungu B[1], B[2], V. Mureşan R[5], R[13], etc.

10.4 WPOs on ordered metric spaces

Throughout this section $(X, d, \leq)$ is an ordered metric space and $f, g, h : X \to X$ are operators.

We have the following results:

Theorem 10.4.1. (I.A. Rus, B[5]). *We suppose that:*

(i) f is increasing operator

(ii) f is WPO.

Then, the operator f^∞ is monotone increasing.

Theorem 10.4.2. (I.A. Rus, B[5]). *We suppose that:*

(i) $f \leq g \leq h$

(ii) the operators f, g and h are WPOs

(iii) the operator g is increasing.

Then:

$$x \leq y \leq z \;\Rightarrow\; f^\infty(x) \leq g^\infty(y) \leq h^\infty(z).$$

Theorem 10.4.3. (I.A. Rus, B[5]). *We suppose that:*

(i) there exists $x, y \in X$ *such that* $x \leq f(x)$, $y \geq f(y)$*;*

(ii) f *is WPO;*

(iii) f *is increasing.*

Then:

(a) $x \leq f^\infty(x) \leq f^\infty(y) \leq y$

(b) $f^\infty(x)$ *is the minimal fixed point of* f *in* $[x, y]$ *and* $f^\infty(y)$ *is the maximal fixed point of* f *in* $[x, y]$.

Remark 10.4.1. The above theorems are in connection with some results of the same type given by S. Carl and S. Heikkilä R[1], L. Losonczi R[1], V.Ya. Stetsenko and M. Shaaban R[1], K. Valeev R[1].

10.5 Fiber WPOs

Let (X, d) and (Y, ρ) be two metric spaces and $g : X \to X$, $h : X \times Y \to Y$. Let $f : X \times Y \to X \times Y$ defined by:

$$f(x, y) = (g(x), h(x, y)), \quad \text{for all } \ x \in X, \ y \in Y.$$

We have the following results by I.A. Rus B[5]:

Theorem 10.5.1. *We suppose that:*

(i) (Y, ρ) *is a complete metric space;*

(ii) g *is a WPO;*

(iii) $h(x, \cdot)$ *is an* a*-contraction, for all* $x \in X$*;*

(iv) if $(x^*, y^*) \in F_f$*, then* $h(\cdot, y^*)$ *is continuous in* x^*.

Then, the operator f *is WPO. Moreover, if* g *is a PO, then* f *is a PO too.*

Remark 10.5.1. For other results on fiber WPO's see S. András B[1], C. Bacoţiu B[1], I.A. Rus B[6], B[8] and B[9], M.A. Şerban B[5].

Remark 10.5.2. Theorem 10.5.1. generalizes a result given by M. W. Hirsch and C. C. Pugh R[1].

Remark 10.5.3. For asymptotic regularity and Picard operators see I.A. Rus B[30], B[34] and B[91], L. Coroian B[1] and B[2].

Remark 10.5.4. It is a very difficult problem to establish if a given operator is or isn't a PO or a WPO. See I.A. Rus B[5]. See also A. Cima, A. Gasal and F. Mañosas R[1], M. -H. Sich and J.W. Wu R[1].

Remark 10.5.5. Theorem 10.5.1. is very useful in order to prove that solutions of some operator equations are differentiable with respect to parameters. See for example: T. Lalescu R[1], J.K. Hale R[1], J.K. Hale and L.A.C. Ladeira R[1], G. Dezsö B[1] and R[2], V. Mureşan B[1]-B[4], I.A. Rus B[1], B[2], B[4] and R[12], A. Tămăşan R[1], M.A. Şerban B[2], R[6], R. Gabor R[1], C. Bacoţiu B[4], V. Olaru R[1], G. Petruşel B[4], E. Egri R[1]., D. Otrocol R[1].

Chapter 11

Multivalued generalized contractions on metric spaces

Guidelines: S.B. Nadler jr. (1967), J.T. Markin (1968), S.B. Nadler jr. (1969), R.B. Fraser and S.B. Nadler jr. (1969), C. Avramescu (1970), H. Schirmer (1970), R.E. Smithson (1971), S. Reich (1972), I.A. Rus (1972), A. S. Finbow (1972), I.A. Rus (1975), T.C. Lim (1980), I.A. Rus, A. Petruşel and A. Sîntămărian (2003).
General references: W.A. Kirk and B. Sims (Eds.) R[1], K. Goebel ans W.A. Kirk R[1], M.A. Khamsi ans W.A. Kirk R[1], R.P. Agarwal, M. Meehan and D. O'Regan R[1], A. Petruşel B[26], I.A. Rus B[18], R.P. Agarwal, D. O'Regan and M. Sambandham R[1], R.P. Agarwal, D. O'Regan and N. Shahzad R[1], J. Andres and L. Górniewicz R[1], V.G. Angelov R[6], L.B. ćirić R[2], R[4], R. Espínola and W.A. Kirk R[2], K. Iseki R[2], R. Mańka R[3], R. Wegrzyk R[1], H.K. Xu R[6], T. Donchev and V. Angelov R[1].

11.0 Preliminaries

11.0.1 Functionals on $P(X)$

Let us consider now the following sets of subsets of a metric space (X, d):

$P(X) = \{Y \in \mathcal{P}(X) |\ Y \neq \emptyset\}$; $P_b(X) = \{Y \in P(X) |\ \text{Y is bounded }\}$;

$$P_{op}(X) = \{Y \in P(X) |\ Y \text{ is open}\};\ P_{cl}(X) = \{Y \in P(X) |\ Y \text{ is closed}\};$$

$$P_{b,cl}(X) = P_b(X) \cap P_{cl}(X);\ P_{cp}(X) = \{Y \in P(X) |\ Y \text{ is compact}\};$$

If X is a normed space, then we denote:

$$P_{cv}(X) = \{Y \in P(X) |\ Y \text{ convex}\};\ P_{cp,cv}(X) = P_{cp}(X) \cap P_{cv}(X).$$

Let us define the following generalized functionals:

(1) $D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$

$$D(A, B) = \begin{cases} \inf\{d(a, b) |\ a \in A,\ b \in B\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset = B \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

D is called the gap functional between A and B.

In particular, $D(x_0, B) = D(\{x_0\}, B)$ (where $x_0 \in X$) is called the distance from the point x_0 to the set B.

(2) $\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\delta(A, B) = \begin{cases} \sup\{d(a, b) |\ a \in A,\ b \in B\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{otherwise} \end{cases}$$

In particular, $\delta(A) := \delta(A, A)$ is the diameter of the set A.

(3) $\rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\rho(A, B) = \begin{cases} \sup\{D(a, B) |\ a \in A\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset \\ +\infty, & \text{if } B = \emptyset \neq A \end{cases}$$

ρ is called the excess functional of A over B.

(4) $H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$H(A, B) = \begin{cases} \max\{\rho(A, B), \rho(B, A)\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset = B \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

H is called the generalized Pompeiu-Hausdorff functional of A and B.

Then we have:

Lemma 11.0.1. *$D(b, A) = 0$ if and only if $b \in \overline{A}$.*

Theorem 11.0.1. *Let (X, d) be a metric space. Then the pair $(P_{b,cl}(X), H)$ is a metric space.*

Remark 11.0.1. H is called the Pompeiu- Hausdorff metric induced by the metric d. Occasionally, we will denote by H_d the Pompeiu-Hausdorff functional generated by the metric d of the space X.

Lemma 11.0.2. *Let (X, d) a metric space. Then we have:*

i) If $Y, Z \in P(X)$ then $\delta(Y, Z) = 0$ if and only if $Y = Z = \{x_0\}$

ii) $\delta(Y, Z) \leq \delta(Y, W) + \delta(W, Z)$, for all $Y, Z, W \in P_b(X)$.

iii) Let $Y \in P_b(X)$ and $q \in]0, 1[$. Then, for each $x \in X$ there exists $y \in Y$ such that $q\delta(x, Y) \leq d(x, y)$.

Let us define now the notion of neighborhood for a nonempty set.

Definition 11.0.1. Let (X, d) be a metric space, $Y \in P(X)$ and $\varepsilon > 0$. An open neighborhood of radius ε for the set Y is the set denoted $V^0(Y, \varepsilon)$ and defined by $V^0(Y, \varepsilon) = \{x \in X |\ D(x, Y) < \varepsilon\}$. We also consider the closed neighborhood for the set Y, defined by $V(Y, \varepsilon) = \{x \in X |\ D(x, Y) \leq \varepsilon\}$.

Lemma 11.0.3. *Let (X, d) be a metric space and $Y, Z, V, W \in P(X)$. Then we have:*

i) $H(Y, Z) = 0$ if and only if $\overline{Y} = \overline{Z}$

ii) $H(Y, Z) = H(Y, \overline{Z})$.

iii) $H(Y \bigcup V, Z \bigcup W) \leq max\{H(Y, Z), H(V, W)\}$.

Lemma 11.0.4. *Let (X, d) be a metric space. Then we have:*

i) Let $Y, Z \in P(X)$. Then $H(Y, Z) = \sup_{x \in X} D(x, Y) - D(x, Z)$

ii) Let $Y, Z \in P(X)$ and $\varepsilon > 0$.Then for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq H(Y, Z) + \varepsilon$.

iv) Let $Y, Z \in P(X)$ and $q > 1$. Then for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq qH(Y, Z)$.

v) If $Y, Z \in P_{cp}(X)$ then for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq H(Y, Z)$.

vi) If $Y, Z \in P(X)$ are such that for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq \varepsilon$ and for each $z \in Z$ there exists $y \in Y$ with $d(y, z) \leq \varepsilon$, then

$H(Y, Z) \leq \varepsilon$.

vii) Let $\varepsilon > 0$. *If* $Y, Z \in P(X)$ *are such that* $H(Y, Z) < \varepsilon$ *then for each* $y \in Y$ *there exists* $z \in Z$ *such that* $d(y, z) < \varepsilon$.

Some very important properties of the metric space $(P_{cl}(X), H_d)$ are contained in the following result:

Theorem 11.0.2. *i) If* (X, d) *is a complete metric space, then* $(P_{cl}(X), H_d)$ *is a complete metric space.*

ii) If (X, d) *is a totally bounded metric space, then* $(P_{cl}(X), H_d)$ *is a totally bounded metric space.*

iii) If (X, d) *is a compact metric space, then* $(P_{cl}(X), H_d)$ *is a compact metric space.*

iv) If (X, d) *is a separable metric space, then* $(P_{cp}(X), H_d)$ *is a separable metric space.*

v) Let $\varepsilon > 0$. *If* (X, d) *is a* ε*-chainable metric space (i.e., for each* $x, y \in X$ *there exists* $x = x_0, x_1, \cdots, x_n = y$ *in* X *such that* $d(x_{k-1}, x_k) < \varepsilon$*, for all* $k \in \{1, 2, \cdots, n\}$*), then* $(P_{cp}(X), H_d)$ *is also an* ε*-chainable metric space.*

11.0.2 Multivalued operators on topological spaces

Let (X, τ_X) and (Y, τ_Y) be two topological spaces. A multivalued operator $T \multimap Y$ is called:

(i) upper semicontinuous (u.s.c.) in $x_0 \in X$ if for each open subset $U \subset Y$ for which $T(x_0) \subset U$ there exists an open neighborhood V of x_0 such that for each $x \in V \Rightarrow T(x) \subset U$. T is called u.s.c. on X if it is u.s.c. in each point $x_0 \in X$.

(ii) lower semicontinuous (l.s.c.) in $x_0 \in X$ if for each open subset $U \subset Y$ for which $T(x_0) \cap U \neq \emptyset$ there exists a neighborhood V of x_0 such that $T(x) \cap U \neq \emptyset$, for every $x \in V$. T is said to be l.s.c on X if it is l.s.c. in each point $x_0 \in X$.

(iii) continuous if it is u.s.c and l.s.c.;

(iv) with closed graph if the set $Graph(T) := \{(x, y) \in X \times Y \mid y \in T(x)\}$ is closed in $X \times Y$, i.e., if $(x_n)_{n\in\mathbb{N}} \to x, y_n \in T(x_n), n \in \mathbb{N}$ and $(y_n)_{n\in\mathbb{N}} \to y$ imply $y \in T(x)$;

(v) closed if $A \in P_{cl}(X)$ implies $T(A) \in P_{cl}(Y)$;

(vi) open if $A \in P_{op}(X)$ implies $T(A) \in P_{op}(Y)$.

In the particular case of the metric spaces (X, d) and (Y, d') we say that the multivalued operator $T \multimap Y$ is called:

(vii) bounded $A \in P_b(X)$ implies $T(A) \in P_b(Y)$;

(viii) compact $A \in P_b(X)$ implies $\overline{T(A)} \in P_{cp}(Y)$;

(ix) completely continuous if it is continuous and compact.

If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two linear normed spaces, then the multivalued operator $T \multimap Y$ is called quasibounded if there exist $m, M \in \mathbb{R}_+^*$ such that

$$(*) \ \ \|y\| \leq m \cdot \|x\| + M, \ \text{for all } (x, y) \in Graph(T).$$

The number

$$|T| := \inf\{m > 0| \text{ there exists } M > 0 \text{ such that the condition (*) holds } \},$$

is called the quasi-norm of T. If $|T| < 1$, then T is said to be a multivalued norm-contraction Recall that $\|T(x)\| := H(T(x), \{0\}), x \in X$.

11.0.3 Multivalued generalized contractions

Let (X, d) be a metric space. The multivalued operator $T : X \to P(X)$ (or $T : X \to P_{cl}(X)$) is said to be:

1) a-Lipschitz if $a > 0$ and $H(T(x_1), T(x_2)) \leq a \cdot d(x_1, x_2)$, for each $x_1, x_2 \in X$.

2) a-contraction if it is a-Lipschitz with $a < 1$.

3) graphic contraction if there exists $\alpha \in \mathbb{R}_+$ with $\alpha < 1$ such that $H(T(x_1), T(x_2)) \leq \alpha d(x_1, x_2)$, for each $(x_1, x_2) \in Graph(T)$.

3) contractive if $H(T(x_1), T(x_2)) < d(x_1, x_2)$, for each $x_1, x_2 \in X$, with $x_1 \neq x_2$.

4) nonexpansive if $H(T(x_1), T(x_2)) \leq d(x_1, x_2)$, for each $x_1, x_2 \in X$.

5) Kannan if there exists $a \in [0, \frac{1}{2}[$ such that $H(T(x_1), T(x_2)) \leq a \cdot [D(x_1, T(x_1)) + D(x_2, T(x_2))]$, for each $x_1, x_2 \in X$.

6) Reich if there exist $a, b, c \geq 0$, with $a + b + c < 1$ such that $H(T(x_1), T(x_2)) \leq a \cdot d(x_1, x_2) + b \cdot D(x_1, T(x_1)) + cD(x_2, T(x_2))$, for each $x_1, x_2 \in X$.

7) Ćirić if there exists $a \in [0,1[$ such that $H(T(x_1),T(x_2)) \leq a \cdot \max\{d(x_1,x_2), D(x_1,T(x_1)), D(x_2,T(x_2)), \frac{1}{2}[D(x_1,T(x_2))+D(x_2,T(x_1))]\}$, for all $x_1,x_2 \in X$.

8) Mánka if for all $x_1,x_2 \in X$ there exist $a_i := a_i(x_1,x_2) \in \mathbb{R}_+$ with $\sum_{i=1}^{5} a_i < 1$ such that $H(T(x_1),T(x_2)) \leq a_1 d(x_1,x_2) + a_2 D(x_1,T(x_1)) + a_3 D(x_2,T(x_2)) + a_4 D(x_1,T(x_2)) + a_5 D(x_2,T(x_1))$.

9) φ-contraction if there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $H(T(x_1),T(x_2)) \leq \varphi(d(x_1,x_2))$, for each $x_1,x_2 \in X$.

10) Meir-Keeler if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x,y \in X$ with $\varepsilon \leq d(x,y) < \varepsilon + \delta \Rightarrow H(T(x),T(y)) < \varepsilon$.

11) topological contraction if T is u.s.c. with closed values and

$$A \in P_{cl}(X) \text{ with } T(A) = A \text{ implies } A = \{x^*\}.$$

12) noncontractive (respectively expansive, dilatation, isometry) if $T : (X,d) \to (P(X),H)$ satisfies the corresponding conditions for singlevalued operators.

Definition 11.0.2. Let(X,d) be a metric space and $Y \in P(X)$. Then, the multivalued operator $T : Y \to P(X)$ is said to be:

1) Caristi if T admits a Caristi selection;

2) strong Caristi if T admits a Caristi selection, with a lower semicontinuous functional $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$.

Definition 11.0.3. Let (X,d) be a metric space. Then, $T : X \to P(X)$ is called a (δ,a)-contraction if $a \in [0,1[$ and $\delta(T(A)) \leq a \cdot \delta(A)$, for each $A \in I_{b,cl}(T)$.

11.1 Basic fixed point principles for multivalued operators

Let us recall first some basic notations and concepts.

Definition 11.1.1. Let X be a metric space. If $T : X \to P(X)$ is a multivalued operator and $x_0 \in X$ is an arbitrary point, then the sequence $(x_n)_{n\in\mathbb{N}}$ is, by definition, the successive approximations sequence of T starting from x_0 if and only if $x_k \in T(x_{k-1})$, for all $k \in \mathbb{N}^*$. Let us remark that in the theory of dynamical systems, the sequence of successive approximations is called the motion of the system T at x_0 or a dynamic process of T starting at x_0.

The following result is known in the literature as Nadler theorem.

Theorem 11.1.1. (Nadler R[1], Covitz-Nadler R[1]) *Let (X, d) be a complete metric space and $x_0 \in X$ be arbitrary. If $T : X \to P_{cl}(X)$ is a multivalued a-contraction, then there exists a sequence of successive approximations of T starting from x_0 which converges to a fixed point of T.*

Proof. Let $1 < q < \frac{1}{a}$, and $(x_0, x_1) \in Graph(T)$ be arbitrary. Then, for $x_1 \in T(x_0)$ there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) \le qH(T(x_0), T(x_1)) \le qad(x_0, x_1)$. By this procedure, we can construct inductively a sequence $(x_n)_{n\in\mathbb{N}}$ in X having the properties:

(i) $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$;

(ii) $d(x_n, x_{n+1}) \le (qa)^n d(x,x_1)$, for each $n \in \mathbb{N}$.

From (ii), we get that the sequence is Cauchy, hence it converges to a certain $x^* \in X$, while from (i), taking account of the fact that T has closed graph, we obtain that $x^* \in T(x^*)$. □

Theorem 11.1.2. (Reich R[1], R[2]) *Let (X, d) be a complete metric space and $T : X \to P_{b,cl}(X)$ be a Reich type multivalued operator (with constants a, b, c). Then $F_T \neq \emptyset$.*

A generalization of Reich's theorem was established by L.B. Ćirić.

If (X, d) is a metric space and $T : X \to P_{cl}(X)$ is a multivalued operator, then for $x, y \in X$, we will denote

$$M^T(x, y) := \max\{d(x, y), D(x, T(x)), D(y, T(y)), \frac{1}{2}[D(x, T(y))+D(y, T(x))]\}.$$

Theorem 11.1.3. (Ćirić R[7]) *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a Ćirić type multivalued operator, i.e., T satisfies the following condition:*

there exists $\alpha \in [0, 1[$ such that $H(T(x), T(y)) \le \alpha \cdot M^T(x, y)$, for each $x, y \in X$.

Then $F_T \neq \emptyset$ and for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(1) $x_0 = x,\ x_1 = y$;

(2) $x_{n+1} \in T(x_n),\ n \in \mathbb{N}$;

(3) $x_n \xrightarrow{d} x^* \in T(x^*)$, *as* $n \to \infty$;

(4) $d(x_n, x^*) \leq \frac{(\alpha p)^n}{1-\alpha p} \cdot d(x_0, x_1)$, *for each* $n \in \mathbb{N}$ *(where* $p \in]1, \frac{1}{\alpha}[$ *is arbitrary).*

If the multivalued operator is contractive and the space is compact, then we have the following result:

Theorem 11.1.4. (Smithson R[1]) *Let* (X, d) *be a compact metric space and* $T : X \to P_{cl}(X)$ *be a contractive multivalued operator. Then* $F_T \neq \emptyset$.

Proof. The contractive condition implies that T is upper semicontinuous. Then, the mapping $\varphi : X \to \mathbb{R}_+$ defined by $\varphi(x) := D(x, T(x))$, $x \in X$ is lower semicontinuous. Since the space X is compact, there exists $x^* \in X$ such that $h(x^*) = \min_{x \in X} h(x)$. Suppose, by contradiction, that $h(x^*) > 0$. By the compactness of the set $T(x^*)$ there exists $y^* \in T(x^*)$ such that $d(x^*, y^*) = D(x^*, T(x^*))$. Then:

$$h(y^*) \leq H(T(x^*), T(y^*)) < d(x^*, y^*) = D(x^*, T(x^*)) = h(x^*) \leq h(y^*),$$

which is a contradiction. Then $h(x^*) = 0$ and thus $x^* \in F_T$. □

Another generalization of the Covitz-Nadler principle (see also P.V. Semenov R[1]) is:

Theorem 11.1.5. (Mizoguchi-Takahashi R[1]) *Let* (X, d) *be a complete metric space and* $T : X \to P_{cl}(X)$ *a multifunction such that* $H(T(x), T(y)) \leq k(d(x, y))d(x, y)$, *for each* $x, y \in X$ *with* $x \neq y$, *where* $k :]0, \infty[\to [0, 1[$ *satisfies* $lim_{r \to t^+} k(r) < 1$, *for every* $t \in [0, \infty[$. *Then* $F_T \neq \emptyset$.

The following result is known in the literature as Węgrzyk's theorem.

Theorem 11.1.6. (Węgrzyk R[1]) *Let* (X, d) *be a complete metric space and* $T : X \to P_{cl}(X)$ *be a multivalued* φ*-contraction, where* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a strong comparison function (i.e.,* φ *is strictly increasing and* $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$, *for each* $t > 0$*). Then* F_T *is nonempty and for any* $x_0 \in X$ *there exists a sequence of successive approximations of* T *starting from* x_0 *which converges to a fixed point of* T.

A basic result for multivalued Meir-Keeler type operators was established by S. Reich R[1].

Theorem 11.1.7. *Let (X, d) be a complete metric space and $T : X \to P_{cp}(X)$ be a multivalued Meir-Keeler type operator. Then, T has at least one fixed point.*

Proof. Define the set-to-set operator

$$G : P_{cp}(X) \to P_{cp}(X), \text{ defined by } G(Y) := \bigcup_{x \in Y} T(x).$$

Then, G satisfies the Meir-Keeler condition, i.e., for each $\varepsilon > 0$ there exists $\delta > 0$ such that $A, B \in P_{cp}(X)$ with $\varepsilon \leq H(A, B) < \varepsilon + \delta \Rightarrow H(G(A), G(B)) < \varepsilon$. By the completeness of the space (X, d) we get that $(P_{cp}(X), H)$ is complete too. Thus, by Meir-Keeler Theorem, there exists a unique $A^* \in P_{cp}(X)$ such that $G(A^*) = A^*$. Since A^* is compact, there exists $a \in A^*$ and $b \in F(a)$ such that $\inf_{x \in A^*} D(x, T(x)) = d(a, b)$. Suppose, by contradiction, that $d(a, b) > 0$. Then:

$$D(b, T(b)) \leq H(T(a), T(b)) < d(a, b),$$

which is a contradiction with the minimality of $d(a, b)$. The proof is complete $\square$

For an extension of this result see S. Leader R[2].

There are several concepts of multivalued directional contractions, see Sehgal and Smithson R[1], Uderzo R[1], S. Park R[8]. We present here the direct generalization of Clarke's theorem, extension to the multivalued case given by H.K. Xu R[2].

Recall that, by $]x, y[$ we denote the open metric segment defined by x and y, i.e.,

$$]x, y[:= \{z \in X \mid z \neq x, z \neq y \text{ and } d(x, z) + d(z, y) = d(x, y)\}.$$

Definition 11.1.2. Let (X, d) be a metric space. Then $T : X \to P_{b,cl}(X)$ is said to be a multivalued directional contraction provided T is u.s.c. and there exists $k \in]0, 1[$ with the following property: whenever $x \in X$ is such that $x \notin T(x)$ and $z \in T(x)$ there exists $y \in]x, z[$ such that $H(T(x), T(y)) \leq kd(x, y)$.

Theorem 11.1.8. *Let (X, d) be a complete metric space and $T : X \to P_{cp}(X)$ be a multivalued directional contraction. Then $F_T \neq \emptyset$.*

Proof. Define $\varphi : X \to \mathbb{R}_+$ by $\varphi(x) := D(x, T(x))$. Then φ is l.s.c. By Ekeland variational principle, applied for φ with $\epsilon := \frac{1-k}{2}$ there exists an element $v \in X$ such that, for all $w \in X$ we have:

$$D(v, T(v)) \leq D(w, T(v)) + \frac{1-k}{2} d(w, v).$$

Suppose, by contradiction that $v \notin T(v)$. Then, since $T(v)$ is compact, there exists $u \in T(v)$ such that $d(u, v) = D(v, T(v))$. By the directional contraction condition, there exists $w \in]u, v[$ (i.e., $d(u, w) + d(w, v) = d(u, v)$) such that $H(T(v), T(w)) \leq kd(v, w)$. By the triangle inequality we also have that

$$D(w, T(w)) \leq D(w, T(v)) + H(T(v), T(w)).$$

Then we obtain:

$$\begin{aligned} 0 &\leq kd(v, w) - H(T(v), T(w)) \\ &\leq kd(v, w) - D(w, T(w)) + D(w, T(v)) \\ &\leq kd(v, w) - D(w, T(w)) + d(w, u) \\ &= (k-1)d(v, w) - D(w, T(w)) + d(v, u) \\ &= kd(v, w) - D(w, T(w)) + D(v, T(v)) \\ &\leq \tfrac{k-1}{2} d(v, w). \end{aligned}$$

The contradiction shows that $v \in F_T$. □

Other fixed point theorems for multivalued generalized contractions on metric spaces are given in D. Azé and J.-P. Penot R[1] (see also Chapter 12.0), M. Berinde and V. Berinde B[1], Y. Feng and S. Liu R[2], etc.

For fixed point theorems for multivalued operators on ϵ-chainable metric spaces see S.B. Nadler jr. R[1], S. Reich R[1], H.K. Xu R[2], etc.

11.2 Basic strict fixed point principles for multivalued operators

We present first a strict fixed point theorem given by Reich R[1]. The proof presented here is based on the construction of a successive approximations sequence which converges to the strict fixed point.

Theorem 11.2.1. *Let (X, d) be a complete metric space and $T : X \to P_b(X)$ be a multivalued operator, for which there exist $a, b, c \in \mathbb{R}_+$ with $a+b+c < 1$ such that*

$$\delta(T(x), T(y)) \leq ad(x, y) + b\delta(x, T(x)) + c\delta(y, T(y)), \text{ for all } x, y \in X.$$

Then, T has a unique strict fixed point in X, i.e., $(SF)_T = \{x^*\}$ and there exists a sequence of successive approximations for T starting from arbitrary $x_0 \in X$ such that $(x_n) \to x^*$ as $n \to +\infty$ and

$$d(x_n, x^*) \leq \frac{\alpha^n}{1-\alpha} d(x_0, x_1),$$

where $\alpha := \frac{a+bq}{1-c}$, with arbitrary $q < \frac{1-a-c}{1-b}$.

Proof. Let $q > 1$ and $x_0 \in X$ be arbitrarily chosen. Then, there exists $x_1 \in T(x_0)$ such that

$$\delta(x_0, T(x_0)) \leq qd(x_0, x_1).$$

We have:

$$\delta(x_1, T(x_1)) \leq \delta(T(x_0), T(x_1)) \leq ad(x_0, x_1) + b\delta(x_0, T(x_0)) + c\delta(x_1, T(x_1))$$

$$\leq (a + bq)d(x_0, x_1) + c\delta(x_1, T(x_1))$$

It follows that

$$\delta(x_1, T(x_1)) \leq \frac{a + bq}{1 - c} d(x_0, x_1).$$

For $x_1 \in T(x_0)$ there exists $x_2 \in T(x_1)$ such that

$$\delta(x_1, T(x_1)) \leq qd(x_1, x_2).$$

Then:

$$\delta(x_2, T(x_2)) \leq \delta(T(x_1), T(x_2)) \leq ad(x_1, x_2) + b\delta(x_1, T(x_1)) + c\delta(x_2, T(x_2))$$

$$\leq (a + bq)d(x_1, x_2) + c\delta(x_2, T(x_2))$$

It follows that:

$$\delta(x_2, T(x_2)) \leq \frac{a + bq}{1 - c} d(x_1, x_2) \leq \frac{a + bq}{1 - c} \delta(x_1, T(x_1))$$

$$\leq \left(\frac{a+bq}{1-c}\right)^2 d(x_0,x_1).$$

Inductively we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ having the properties:

(1) (α) $x_n \in T(x_{n-1}),\ n \in \mathbb{N}^*$;

(2) (β) $d(x_n,x_{n+1}) \leq \delta(x_n,T(x_n)) \leq \left(\frac{a+bq}{1-c}\right)^n d(x_0,x_1)$.

We will prove now that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

Let us denote $\alpha := \frac{a+bq}{1-c}$. Then

$$\begin{aligned} d(x_n,x_{n+p}) &\leq \alpha^n(1+\alpha+\cdots+\alpha^{p-1})d(x_0,x_1) \\ &= \alpha^n\frac{\alpha^p-1}{\alpha-1}d(x_0,x_1) \end{aligned}$$

If we chose $q < \frac{1-a-c}{b}$, then $\alpha < 1$.

Letting $n \to \infty$, since $\alpha^n \to 0$, it follows that:

$$d(x_n,x_{n+p}) \to 0 \quad asn \to \infty.$$

Hence $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

By the completeness of the space (X,d) we get that there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Next, we will prove that $x^* \in (SF)_T$.

We have:

$$\delta(x^*,T(x^*)) \leq d(x^*,x_n) + \delta(x_n,T(x_n)) + \delta(T(x_n),T(x^*))$$

$$\leq d(x^*,x_n) + \delta(x_n,T(x_n)) + ad(x_n,x^*) + b\delta(x_n,T(x_n)) + c\delta(x^*,T(x^*))$$

Then:

$$\delta(x^*,T(x^*)) \leq \frac{1+a}{1-c}d(x^*,x_n) + \frac{1+b}{1-c}\delta(x_n,T(x_n))$$

because $\delta(x_n,T(x_n)) \leq \alpha^n d(x_0,x_1) \ \Rightarrow\ \delta(x^*,T(x^*)) = 0 \ \Rightarrow\ T(x^*) = \{x^*\}$ (i.e. $x^* \in (SF)_T$)

For the last part of our proof, we will show the uniqueness of the strict fixed point.

Suppose there exist $x^*, y^* \in (SF)_T$. Then:

$$d(x^*, y^*) = \delta(T(x^*), T(y^*)) \leq ad(x^*, y^*) + b\delta(x^*, T(x^*)) + c\delta(y^*, T(y^*))$$

If x^* and y^* are distinct points, then we get that $a \geq 1$, which contradicts our hypothesis. Thus $x^* = y^*$. The proof is complete. □

Notice that another proof relies on the construction of a Reich type singlevalued selection of T, see I.A. Rus B[4]. The conclusion then follows from Ćirić-Reich-Rus's Theorem.

Another strict fixed point result was given by I.A. Rus, see B[4] and B[101]. We present here this strict fixed point theorem for Reich type operators.

Theorem 11.2.2. *Let (X, d) be a complete metric space and $T : X \to P_{b,cl}(X)$ be a Reich type multivalued operator with constants a, b, c. Suppose $(SF)_T \neq \emptyset$. Then, $F_T = (SF)_T = \{x^*\}$.*

Proof. Let $x^* \in (SF)_T$ and $x \in F_T$. Then we have:

$$\begin{aligned} d(x, x^*) &= D(x, T(x^*)) \leq H(T(x), T(x^*)) \\ &\leq ad(x, x^*) + bD(x, T(x)) + cD(x^*, T(x^*)) = ad(x, x^*). \end{aligned}$$

Hence $x = x^*$ and the proof is complete. □

Remark 11.2.1. For some extensions of this result see A. Sîntămărian B[6] and B[7]. For other results of this type see C. Chifu and G. Petruşel B[2].

Let us remark now that if $T : X \to P(X)$ and we define the following sequence of multivalued operators: $T^0(x) = \{x\}$, $T^1(x) = T(T^0(x)) = T(x)$, $T^2(x) = T(T^1(x)) = \bigcup\limits_{y \in T^1(x)} T(y), \ldots, T^n(x) = T(T^{n-1}(x)) = \bigcup\limits_{y \in T^{n-1}(x)} T(y)$, for $x \in X$, then a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in T^n(x)$, $x \in X$ for $n \in \mathbb{N}$ is, by definition, a generalized sequence of successive approximations of T starting from $x \in X$. Obviously, each sequence of successive approximations of T starting from arbitrary $x \in X$ is a generalized sequence of successive approximations, but the converse may not be true, since $T^n(x)$ is, in general, bigger than $T(x_{n-1})$, i.e. $T(x_{n-1}) \subset T^n(x)$ but not conversely.

Another strict fixed point theorem was given by Tarafdar-Vyborny.

Theorem 11.2.3. (Tarafdar-Vyborny, see Yuan R[1]) *Let (X, d) be a complete metric space and $Y \in P_{b,cl}(X)$. Let $T : Y \to P(Y)$ be a multivalued (δ, a)-contraction.*

Then:

i) $(SF)_T = \{x^*\}$

ii) for each $x_0 \in X$, there exists a generalized sequence of successive approximations of T starting from x_0, such that $x_n \to x^*$.

Remark 11.2.2. X be a nonempty set and $T : X \to P(X)$ be a multivalued operator. Then $(SF)_T \subset F_T \subset \bigcap_{n\in\mathbb{N}} T^n(X)$, where $T^0(X) = X$ and $T^n(X) = T(T^{n-1}(X)) = \bigcup_{y\in T^{n-1}(X)} T(y)$.

Proof. First inclusion is quite obviously. For the second one let $x \in F_T$. Then $x \in T(x) \subset T(X) \subset T^2(X) \subset \cdots \subset T^n(X) \subset \ldots$ Hence $x \in \bigcap_{n\in\mathbb{N}} T^n(X)$. □

Other strict fixed point results are in connection with the concept of multivaued Janos operator.

Definition 11.2.1. Let (X, d) be a metric space. Then $T : X \to P(X)$ is called a multivalued Janos operator if $\bigcap_{n\in\mathbb{N}} T^n(X) = \{x^*\}$.

When T is a singlevalued operator we get the notion of singlevalued Janos operator.

Remark 11.2.3. If $T : X \to P(X)$ is a multivalued Janos operator then $(SF)_T = F_T = \{x^*\}$.

Theorem 11.2.4. (Tarafdar-Vyborny, see Yuan R[1]) *Let X be a compact Hausdorff topological space and $T : X \to P_{cl}(X)$ be a topological contraction.*

Then T is a multivalued Janos operator.

Theorem 11.2.5. *Let (X, d) be a compact metric space and $T : X \to P_{cl}(X)$ be a multivalued (δ, a)-contraction. Then T is a multivalued Janos operator.*

Proof. Each multivalued (δ, a)-contraction on a bounded metric space is a topological contraction. □

We conclude this section with another strict fixed point theorem.

Theorem 11.2.6. *Let* (X, d) *be a complete metric space, and* $T : X \to P_b(X)$ *be a set-valued operator. Suppose that there exist* $a, b \in \mathbb{R}_+$ *with* $a+b < 1$ *such that for each* $x \in X$ *there exists* $y \in T(x)$ *with*

$$\delta(y, T(y)) \leq a \cdot d(x, y) + b \cdot \delta(x, T(x)).$$

If the map $f : X \to \mathbb{R}_+$*, defined by* $f(x) := \delta(x, T(x))$ *is lower semicontinuous, then* $SF_T \neq \emptyset$*.*

Proof. From the hypothesis we have that for each $x \in X$ there is $y \in T(x)$ such that $\delta(y, T(y)) \leq (a + b) \cdot \delta(x, T(x))$. Then, for each $x_0 \in X$ we can construct inductively a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from x_0, having the property $\delta(x_n, T(x_n)) \leq (a + b)^n \cdot \delta(x_0, T(x_0))$. Hence, we will obtain $d(x_n, x_{n+1}) \leq \delta(x_n, T(x_n)) \to 0$, as $n \to +\infty$. As consequence, the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Denote by $x^* \in X$ the limit of this sequence.

If we denote $f(x_n) := \delta(x_n, T(x_n))$, then using the lower semicontinuity of f we can write:

$$0 \leq f(x^*) \leq \liminf_{n\to+\infty} f(x_n) = 0.$$

So, $f(x^*) = 0$ and the conclusion $\{x^*\} = T(x^*)$ follows. □

Remark 11.2.4. If, instead of the lower semicontinuity of f, we suppose that the graph of T is closed, then, since $(x_n)_{n\in\mathbb{N}}$ is a sequence of successive approximations for T, we immediately get that $x^* \in T(x^*)$. So, the conclusion of the above result is $F_T \neq \emptyset$. It is an open question if the above fixed point is a strict fixed point for T.

For other strict fixed point theorems see J.-P. Aubin, J. Siegel R[1], G. Gabor R[3], K. Włodarczyk, D. Klim, R. Plebaniak R[1], I.A. Rus B[101], etc.

11.3 Properties of the fixed point set

We briefly present here some topological properties of the fixed point set of multivalued generalized contractions.

To our best knowledge, the first result concerning the compactness and the convexity of the fixed point set of a multivalued contraction was given by H. Schirmer in 1970. Schirmer proved that for a self multivalued contraction on $\mathbb{R}$ with compact connected values, the fixed point set is compact and connected, with the remark that the word "connected" can be replaced by "convex".

An extension of the above result is the following theorem, proved by M. C. Anisiu (Alicu)- O. Mark in 1980.

Theorem 11.3.1. (M. C. Anisiu (Alicu)- O. Mark, B[1]) *Let $T : [a, b] \to P_{cp,cv}([a, b])$ be a contractive multivalued operator (i.e, for each $x, y \in X$, with $x \neq y$ we have $H(T(x), T(y)) < d(x, y)$).*
Then the fixed point set of T is compact and convex.

Regarding the compactness of the fixed point set of a multivalued contraction, the basic result was established by J. Saint-Raymond R[1].

Theorem 11.3.2. *Let (X, d) be a complete metric space and $T : X \to P_{cp}(X)$ be a multivalued contraction. Then F_T is compact.*

Remark 11.3.1. Another proof of the above result, by using the fractal operator technique, was proved by A. Petruşel and I.A. Rus in B[5]. For other similar results, see A. Petruşel and I.A. Rus in B[5], R. Espínola and A. Petruşel B[1].

An extension of the above mentioned result is the following theorem, proved by A. Petruşel.

Theorem 11.3.3. (A. Petruşel, B[2]) *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let us suppose that $T : \tilde{B}(x_0; r) \to P_{cp}(X)$ satisfies the following two conditions:*

i) there exist $\alpha, \beta \in \mathbb{R}_+$, $\alpha + 2\beta < 1$ such that

$$H(T(x), T(y)) \leq \alpha d(x, y) + \beta[D(x, T(x)) + D(y, T(y))], \text{ for each } x, y \in \tilde{B}(x_0; r)$$

ii) $D(x_0, T(x_0)) < [1 - (\alpha + 2\beta)](1 - \beta)^{-1} r$.

Then the fixed points set F_T is compact.

Other interesting results given by J. Saint-Raymond R[1], R[2] are the following.

Theorem 11.3.4. *Let X be a Banach space, $Y \in P_{cl,cv}(X)$ and $T : Y \to$*

$P_{cl}(Y)$ *be a multivalued a-contraction. Let* $x^* \in F_T$. *Then*

$$\delta(F_T) \geq \frac{1-a}{2}\delta(T(x^*)).$$

As an immediate consequence, we get:

Corollary 11.3.1. *Let* X *be a Banach space,* $Y \in P_{cl,cv}(X)$ *and* $T : Y \to P_{cl}(Y)$ *be a multivalued a-contraction. If there exists some* $x_0 \in X$ *such that the set* $T(x_0)$ *is unbounded, then* F_T *is unbounded too.*

In what follows, the symbol $\mathcal{M}$ will indicate the family of all metric spaces. Let $X \in \mathcal{M}$. The space X is called an absolute retract for metric spaces (briefly $X \in AR(\mathcal{M})$) if, for any $Y \in \mathcal{M}$ and any $Y_0 \in P_{cl}(X)$, every continuous function $f_0 : Y_0 \to X$ has a continuous extension over Y, that is $f : Y \to X$. Obviously, any absolute retract is arcwise connected.

The basic result concerning the absolute retract property of the fixed point set of a multivalued contraction was proved 1n 1987, by B. Ricceri R[1]. Using a similar approach, we have:

Theorem 11.3.5.(A. Petruşel, B[2]) *Let* E *be a Banach space,* $X \in P_{clc,cv}(E)$ *and* $T : X \to P_{cl,cv}(X)$ *be a l.s.c. multivalued Reich type operator. Then* $F_T \in AR(\mathcal{M})$.

For some extensions of the above results see L. Górniewicz and S.A. Marano R[1].

Let (X, d) be a complete separable metric space and (Ω, Σ) be a measurable space. A multivalued operator $T : \Omega \to P(X)$ is said to be measurable if, for any open subset B of X we have that

$$T^{-1}(B) := \{\omega \in \Omega : \ f(\omega) \cap B \neq \emptyset\} \in \Sigma.$$

Recall also that a multivalued operator $T : \Omega \times X \to \mathcal{P}(X)$ is said to be a random operator if, for any $x \in X$ $T(\cdot, x) : \Omega \to P(X)$ is measurable. We will denote by $F(\omega)$ the fixed points set of $T(\omega, \cdot)$, i.e. $F(\omega) := \{x \in X|\ x \in T(\omega, x)\}$. A random fixed point of T is a measurable function $x : \Omega \to X$ such that $x(\omega) \in T(\omega, x(\omega))$, for all $\omega \in \Omega$, or equivalently, x is a measurable selection for F.

If $T : \Omega \times X \to P_{b,cl}(X)$ is a random contraction (that is, for each $x \in X$, $T(\cdot, x)$ is measurable and for each $\omega \in \Omega$ there exists a number $k(\omega) \in [0, 1[$ such that

$$H(T(\omega, x), T(\omega, y)) \leq k(\omega)d(x, y), \text{ for all } x, y \in X)$$

then Xu and Beg(see R[1]) proved that the multivalued operator T is measurable and hence T admits a random fixed point.

The following result is an extension of Xu and Beg's theorem:

Theorem 11.3.6. (A. Petruşel, B[2]) *Suppose that (X, d) is a complete separable metric space, (Ω, Σ) is a measurable space and $T : \Omega \times X \to P_{b,cl}(X)$ is a random continuous Reich-type operator, that is, for each $x \in X$, $T(\cdot, x)$ is measurable and for each $\omega \in \Omega$ there exist $\alpha(\omega), \beta(\omega), \gamma(\omega) \in \mathbb{R}_+$ with $\alpha(\omega) + \beta(\omega) + \gamma(\omega) < 1$ such that*

$$H(T(\omega, x), T(\omega, y)) \leq \alpha(\omega)d(x, y) + \beta(\omega)D(x, T(\omega, x)) + \gamma(\omega)D(y, T(\omega, y))$$

for each $x, y \in X$.

Then, the fixed point set of the multivalued operator T is measurable.

Remark 11.3.2. From the above theorem, using the well-known Kuratowski-Ryll-Nardzewski theorem, it follows that T has a random fixed point.

11.4 Fixed point theorems on a set with two metrics

Our first result is a multivalued version of Maia's fixed point theorem.

Theorem 11.4.1. *Let X be a nonempty set, d and ρ two metrics on X and $T : X \to P(X)$ be a multivalued operator. We suppose that:*

(i) (X, d) is a complete metric space;

(ii) there exists $c > 0$ such that $d(x, y) \leq c\rho(x, y)$, for each $x, y \in X$;

(iii) $T : (X, d) \to (P(X), H_d)$ has closed graph;

(iv) there exists $\alpha \in [0, 1[$ such that $H_\rho(F(x), F(y)) \leq \alpha\rho(x, y)$, for each $x, y \in X$.

Then we have:

(a) $F_T \neq \emptyset$;

(b) for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(1) $x_0 = x,\ x_1 = y$;

(2) $x_{n+1} \in T(x_n),\ n \in \mathbb{N}$;

(3) $x_n \xrightarrow{d} x^* \in T(x^*)$, as $n \to \infty$.

Proof. The hypothesis (iv) implies that there exists a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X,ρ), such that (1) and (2) hold. From (ii) it follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X,d). Denote by $x^* \in X$ the limit of this sequence. From (i) and (iii) we get that $x_n \xrightarrow{d} x^* \in T(x^*)$, as $n \to \infty$. The proof is complete. □

The second main result of this section is the following theorem.

Theorem 11.4.2. *Let X be a nonempty set, d and ρ two metrics on X and $T : X \to P(X)$ be a multivalued operator. We suppose that:*

(i) (X,d) is a complete metric space;

(ii) there exists $c > 0$ such that $d(x,y) \le c\rho(x,y)$, for each $x,y \in X$;

(iii) $T : (X,d) \to (P(X),H_d)$ is closed;

(iv) there exists $\alpha \in [0,1[$ such that $H_\rho(T(x),T(y)) \le \alpha\rho(x,y)$, for each $x,y \in X$;

(v) $(SF)_T \neq \emptyset$.

Then we have:

(a) $F_T = (SF)_T = \{x^*\}$;

(b) $H_\rho(T^n(x),x^*) \le \alpha^n \cdot \rho(x,x^*)$, for each $n \in \mathbb{N}$ and each $x \in X$;

(c) $\rho(x,x^*) \le \frac{1}{1-\alpha} \cdot H_\rho(x,T(x))$, for each $x \in X$;

(d) the fixed point problem is well-posed for T with respect to D_ρ.

Proof. (a)-(b) From (iv) we have that if $x^* \in (SF)_T$ then $(SF)_T = \{x^*\}$. Also, by taking $y := x^*$ in (iv) we have that $H_\rho(T(x),x^*) \le \alpha\rho(x,x^*)$, for each $x \in X$. By induction we get that $H_\rho(T^n(x),x^*) \le \alpha^n\rho(x,x^*)$, for each $x \in X$. Consider now $y^* \in F_T$. Then

$\rho(y^*,x^*) \le H_\rho(T^n(x),x^*) \le \alpha^n\rho(x,x^*) \to 0$, as $n \to \infty$. Hence $y^* = x^*$.

(c) We successively have: $\rho(x,x^*) \le H_\rho(x,T(x)) + H_\rho(T(x),x^*) \le H_\rho(x,T(x)) + \alpha\rho(x,x^*)$. Hence $\rho(x,x^*) \le \frac{1}{1-\alpha} \cdot H_\rho(x,T(x))$, for each $x \in X$.

(d) Let $(x_n)_{n\in\mathbb{N}}$ be such that $D_\rho(x_n,T(x_n)) \to 0$, as $n \to \infty$. We have to prove that $\rho(x_n,x^*) \to 0$, as $n \to \infty$.

Then we have:

$$\rho(x_n, x^*) \leq D_\rho(x_n, T(x_n)) + H_\rho(T(x_n), T(x^*)) \leq D_\rho(x_n, T(x_n)) + \alpha\rho(x_n, x^*).$$

Hence we get $\rho(x_n, x^*) \leq \frac{1}{1-\alpha} \cdot D_\rho(x_n, T(x_n)) \to 0$, as $n \to \infty$. □

Next, we will prove a data dependence result.

Theorem 11.4.3. *Let X be a nonempty set, d and ρ two metrics on X and $T, S : X \to P(X)$ be two multivalued operators. We suppose that:*

(i) (X, d) is a complete metric space;

(ii) there exists $c > 0$ such that $d(x, y) \leq c\rho(x, y)$, for each $x, y \in X$;

(iii) $T : (X, d) \to (P(X), H_d)$ has closed graph;

(iv) there exists $\alpha \in [0, 1[$ such that $H_\rho(T(x), T(y)) \leq \alpha\rho(x, y)$, for each $x, y \in X$

(v) $(SF)_T \neq \emptyset$;

(vi) $F_S \neq \emptyset$

(vii) there exists $\eta > 0$ such that $H_\rho(T(x), S(x)) \leq \eta$, for each $x \in X$.

Then $H(F_T, F_S) \leq \frac{\eta}{1-\alpha}$.

Proof. Let $y^* \in F_S$. From the conclusion (c) of the previous theorem we have that:

$\rho(y^*, x^*) \leq H_\rho(S(y^*), x^*) \leq H_\rho(S(y^*), T(y^*)) + H_\rho(T(y^*), x^*) \leq \eta + \alpha\rho(y^*, x^*)$. Thus, $\rho(y^*, x^*) \leq \frac{\eta}{1-\alpha}$.

Hence $H(F_T, F_S) = \sup\limits_{y^* \in F_S} \rho(y^*, x^*) \leq \dfrac{\eta}{1-\alpha}$. The proof is complete. □

Remark 11.4.1. For other similar results, see A. Petruşel and I.A. Rus B[3].

11.5 Fixed point theorems for multivalued nonexpansive operators

We consider now the case of a nonexpansive multivalued operator T (i.e. 1-Lipschitz) on a Banach space. The first results in this setting were obtained by Markin R[3] (see also R[4]) and F.E. Browder R[2].

The main result of this section was proved by T.C. Lim in 1980 (see also Lim R[2]). For this result we need some preliminaries.

Let X be a Banach space. For $x \in X$ and a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X, the asymptotic center of $(x_n)_{n\in\mathbb{N}}$ at x is the real number:

$$r(x,(x_n)) := \limsup_{n\to+\infty} \|x - x_n\|.$$

Also, if Y is a nonempty closed subset of X, then the asymptotic radius of $(x_n)_{n\in\mathbb{N}}$ relative to Y is defined by:

$$r(Y,(x_n)) := \inf_{x\in Y} r(x,(x_n)).$$

A bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be regular with respect to a subset Y of X if the asymptotic radii relative to Y of all subsequences of $(x_n)_{n\in\mathbb{N}}$ are the same.

For $p \geq 0$ the level sets are defined by:

$$A_p(Y,(x_n)) := \{x \in Y \| \ r(x,(x_n)) \leq r(Y,(x_n) + p)\}.$$

The set $A_0(Y,(x_n))$ is called the asymptotic center of $(x_n)_{n\in\mathbb{N}}$ in Y.

Theorem 11.5.1. (T.C. Lim R[3]) *Let X be an uniformly convex Banach space, $Y \in P_{b,cl,cv}(X)$ and $F : Y \to P_{cp}(Y)$ be nonexpansive. Then $F_T \neq \emptyset$.*

Proof. The proof can be organized (see K. Goebel and W.A. Kirk R[1]) in several steps:

Step 1. Any bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X admits a regular subsequence with respect to Y;

Step 2. There exists a regular sequence $(x_n)_{n\in\mathbb{N}}$ with respect to Y, such that $\lim\limits_{n\to+\infty} D(x_n, T(x_n)) = 0$;

Step 3. Let $\{x^*\} = A_0(Y,(x_n))$ and $r = r(Y,(x_n))$. Then, there exists $y_n \in T(x_n)$ and $z_n \in T(v)$ such that $\lim\limits_{n\to+\infty} \|y_n - x_n\| = 0$ and $\|y_n - z_n\| \leq \|x_n - x^*\|$.

Step 4. By the compactness of the $T(x^*)$, there exists a subsequence $(z_{n_k})_{k\in\mathbb{N}}$ of $(z_n)_{n\in\mathbb{N}}$ that converges to some $y^* \in T(x^*)$ and $\limsup\limits_{k\to+\infty} \|y^* - x_{n_k}\| \leq r$;

Step 5. We have that $y^* = x^*$ (by the regularity of the sequence $(x_n)_{n\in\mathbb{N}}$ and thus $x^* \in F_T$. □

For other problems and results see K. Goebel and W.A. Kirk R[1], R.K. Bose and R.N. Mukherjee R[1], G. Marino and H.-K. Xu R[1], S. Itoh and W. Takahashi R[1], S. Massa R[1], B. Sims, H.-K. Xu and G.X.-Z. Yuan R[1], S. Reich and A.J. Zaslavski R[2], Dominguez Benavides and P. Lorenzo Ramirez R[1], T. Dominguez Benavides and B. Gavira R[1], B. Gavira R[1], etc.

11.6 Multivalued weakly Picard operators

The following notions appear in Rus - Petruşel - Sîntămărian B[1] and B[2].

Definition 11.6.1. Let (X, d) be a metric space and $T : X \to P(X)$ a multivalued operator. By definition, T is a multivalued weakly Picard (briefly MWP) operator if and only if for all $x \in X$ and all $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

i) $x_0 = x$, $x_1 = y$

ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$

iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of the multivalued operator T.

We can illustrate this notions by several examples.

Example 11.6.1. (Covitz-Nadler, R[1]) Let (X, d) be a complete metric space and let $T : X \to P_{cl}(X)$ be a multivalued a-contraction. Then T is a MWP operator.

Example 11.6.2. (Reich, R[1], R[2]) Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued Reich-type operator. Then T is a MWP operator.

Example 11.6.3. (Petruşel, B[2]) Let (X, d) be a complete metric space. A multivalued operator $T : X \to P_{cl}(X)$ is said to be a multivalued Rus-type graphic-contraction if G(T) is closed and the following condition is satisfied: there exist $\alpha, \beta \in \mathbb{R}_+$, $\alpha + \beta < 1$ such that: $H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(y, T(y))$, for every $x \in X$ and every $y \in T(x)$.

Then T is a MWP operator.

Example 11.6.4. (Petruşel, B[23]) Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. The multivalued operator T is called a Frigon-Granas-type operator if $T : \tilde{B}(x_0; r) \to P_{cl}(X)$ and satisfies the following assertion:

i) there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$, $\alpha + \beta + \gamma < 1$ such that:

$$H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(x, T(x)) + \gamma D(y, T(y)), \text{ for all } x, y \in \tilde{B}(x_0; r)$$

If T is a Frigon-Granas-type operator such that:

ii) $\delta(x_0, T(x_0)) < [1 - (\alpha + \beta + \gamma)](1 - \gamma)^{-1} r$,

then T is a MWP operator.

Definition 11.6.2. Let (X, d) be a metric space and $T : X \to P(X)$ a MWP operator. Then we define the multivalued operator $T^\infty : G(T) \to P(F_T)$ by the formula:

$T^\infty(x, y) := \{z \in F_T |$ there exists a sequence of successive approximations of T starting from (x, y) that converges to $z\}$.

An important abstract concept in this approach is the following:

Definition 11.6.3. Let (X, d) be a metric space and $T : X \to P(X)$ a MWP operator. Then T is a c-multivalued weakly Picard operator (briefly c-MWP operator) if there is a selection t^∞ of T^∞ such that: $d(x, t^\infty(x, y)) \leq cd(x, y)$, for all $(x, y) \in G(T)$.

Further on we shall present several examples of c-MWP operators.

Example 11.6.4. A multivalued α-contraction on a complete metric space is a c-MWP operator with $c = (1 - \alpha)^{-1}$.

Example 11.6.5. A multivalued Reich-type operator on a complete metric space is a c-MWP operator with $c = [1 - (\alpha + \beta + \gamma)]^{-1}(1 - \gamma)$.

Let us recall that in 1985, T. C. Lim R[1] proved that if T_1 and T_2 are multivalued contractions on a complete metric space X with a same contraction constant $\alpha < 1$ and if $H(T_1(x), T_2(x)) \leq \eta$, for all $x \in X$, then the data dependence phenomenon for the fixed point set holds, i.e. $H(F_{T_1}, F_{T_2}) \leq \eta\{1 - a\}^{-1}$.

An important abstract result of is the following:

Theorem 11.6.1. (Rus-Petruşel-Sîntămărian, B[1]) *Let (X, d) be a metric space and $T_1, T_2 : X \to P(X)$. We suppose that:*

i) T_i is a c_i-MWP operator for $i \in \{1, 2\}$

ii) there exists $\eta > 0$ such that $H(T_1(x), T_2(x)) \leq \eta$, for all $x \in X$.

Then $H(F_{T_1}, F_{T_2}) \leq \eta \max\{c_1, c_2\}$.

Remark 11.6.1. As consequences of this abstract principle, we deduce that the data dependence phenomenon regarding the fixed points set for some generalized multivalued contractions (such as Reich-type operators, Rus-type graphic contractions, Frigon-Granas-type operators) holds.

11.7 Well-posedness of the fixed point problems

Let us present first the notion of well-posedness in the generalized sense for a fixed point problem, see A. Petruşel, I.A. Rus and J.-C. Yao B[1].

Definition 11.7.1. Let (X,d) be a metric space, $Y \in P(X)$ and $T : Y \to P_{cl}(X)$ be a multivalued operator. Then the fixed point problem for T with respect to D is well-posed in the generalized sense (respectively well-posed, see A. Petruşel, I.A. Rus B[2]) if

(a_1) $F_T \neq \emptyset$ (respectively $F_T = \{x^*\}$);

(b_1) If $x_n \in Y$, $n \in \mathbb{N}$ and $D(x_n, T(x_n)) \to 0$ as $n \to +\infty$, then there exists a subsequence (x_{n_i}) of (x_n) such that x_{n_i} converges to $x^* \in F_T$ as $i \to +\infty$ (respectively $x_n \to x^*$ as $n \to +\infty$).

Definition 11.7.2. Let (X,d) be a metric space, $Y \in P(X)$ and $T : Y \to P_{cl}(X)$ be a multivalued operator. Then the fixed point problem for T with respect to H is well-posed in the generalized sense (respectively well-posed, see A. Petruşel, I.A. Rus B[2]) if

(a_2) $(SF)_T \neq \emptyset$ (respectively $(SF)_T = \{x^*\}$);

(b_2) If $x_n \in Y$, $n \in \mathbb{N}$ and $H(x_n, T(x_n)) \to 0$ as $n \to +\infty$, then there exists a subsequence (x_{n_i}) of (x_n) such that x_{n_i} converges to $x^* \in (SF)_T$ as $i \to +\infty$ (respectively $x_n \to x^*$ as $n \to +\infty$).

If in the above definitions we consider the setting of a normed space, then a fixed point problem is weakly well-posed in the generalized sense (respectively weakly well-posed) if the convergence of the subsequence (x_{n_i}) (respectively of the sequence (x_n)) to x^* is weakly.

Remark 11.7.1. It's easy to see that if the fixed point problem is well-posed (in the generalized sense) for T with respect to D and $F_T = (SF)_T$, then the fixed point problem is well-posed (in the generalized sense) for T with respect to H.

Also, when the operator T is singlevalued, then the above definitions coincide with the concept given by F.S. De Blasi and J. Myjak R[2]. See also I.A.

Rus B[106] and B[108].

An abstract general result (A. Petruşel, I.A. Rus and J.-C. Yao B[1]) is

Theorem 11.7.1. *Let (X, d) be a compact metric space. If $T : X \to P(X)$ is a multivalued operator with closed graph, such that $F_T \neq \emptyset$, then the fixed point problem is well-posed in the generalized sense for T with respect to D. Moreover, if, additionally, T is lower semicontinuous and $(SF)_T \neq \emptyset$, then the fixed point problem is well-posed in the generalized sense for T with respect to H.*

Proof. Let $x_n \in X$, $n \in \mathbb{N}$ be such that $D(x_n, T(x_n)) \to 0$ as $n \to +\infty$. Let $(x_{n_i})_{i\in\mathbb{N}}$ be a convergent subsequence of $(x_n)_{n\in\mathbb{N}}$. Suppose $x_{n_i} \stackrel{d}{\to} \widetilde{x}$ as $i \to +\infty$. Then there exists $y_{n_i} \in T(x_{n_i})$, $i \in \mathbb{N}$, such that $y_{n_i} \stackrel{d}{\to} \widetilde{x}$ as $i \to +\infty$. Since T has closed graph, we obtain that $\widetilde{x} \in F_T$.

For the second conclusion, let $x_n \in X$, $n \in \mathbb{N}$ be such that $H(x_n, T(x_n)) \to 0$ as $n \to +\infty$. Let $(x_{n_i})_{i\in\mathbb{N}}$ be a convergent subsequence of $(x_n)_{n\in\mathbb{N}}$. Suppose $x_{n_i} \stackrel{d}{\to} \widetilde{x}$ as $i \to +\infty$. Since T is continuous, we immediately get that $H_d(\widetilde{x}, T(\widetilde{x})) = 0$ and hence $\widetilde{x} \in (SF)_T$. □

Some well-posedness results are:

Theorem 11.7.2. *If (X, d) is a compact metric space, then for any multivalued contractive operator $T : X \to P_{cl}(X)$, the fixed point problem is well-posed in the generalized sense with respect to D. Moreover, if additionally $(SF)_T \neq \emptyset$, then the fixed point problem is well-posed in the generalized sense with respect to H too.*

Proof. By a theorem of Smithson R[1], we have that $F_T \neq \emptyset$. Since T is contractive, it is upper semicontinuous and hence it has closed graph. The conclusion follows by Theorem 11.7.1. □

Theorem 11.7.3. *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued a-contraction. Suppose that $(SF)_T \neq \emptyset$. Then the fixed point problem is well-posed for T with respect to D and with respect to H too.*

Proof. Since $(SF)_T \neq \emptyset$ and T is an a-contraction we have that $F_T = (SF)_T = \{x^*\}$. Suppose $D(x_n, T(x_n)) \to 0$, as $n \to +\infty$. Then $d(x_n, x^*) \leq D(x_n, T(x_n)) + H(T(x_n), T(x^*)) \leq D(x_n, T(x_n)) + a \cdot d(x_n, x^*)$. Hence $d(x_n, x^*) \leq \frac{1}{1-a} \cdot D(x_n, T(x_n))$ and the conclusion follows. The second conclusion follows from Remark 11.7.1. □

Theorem 11.7.4. *Let (X,d) be a bounded and complete metric space and let $T : X \to P_{b,cl}(X)$ be a condensing multivalued operator with respect to α_K or α_H (i.e., $\alpha(T(A)) < \alpha(A)$, for each $A \in P_b(X) \cap I(T)$ with $\alpha(A) > 0$), such that the functional $x \mapsto D(x,T(x))$ is continuous. Suppose that $\inf\limits_{x\in X} D(x,T(x)) = 0$. Then, any bounded sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that $D(x_n,T(x_n)) \to 0$ as $n \to +\infty$, has a convergent subsequence and all the limit points of $(x_n)_{n\in\mathbb{N}}$ are fixed points of T. Moreover, in this case, the fixed point problem is well-posed in the generalized sense for T with respect to D.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \in X$ be a bounded sequence such that $D(x_n,T(x_n)) \to 0$, as $n \to +\infty$. Denote $M := \{x_n : n \in \{1,2,\cdots\}\}$. Then $T(M) = \bigcup\limits_{n\in\mathbb{N}^*} T(x_n)$. Since $D(x_n,T(x_n)) \to 0$ as $n \to +\infty$, given any $\varepsilon > 0$ the ε-neighborhood $V(T(M);\varepsilon)$ of $T(M)$ contains all except a finite number of elements of M. Then for each $\varepsilon > 0$ we have that

$$\alpha(M) \le \alpha(V(T(M);\varepsilon)) \le \alpha(T(M)) + 2\varepsilon.$$

Hence $\alpha(T(M)) \ge \alpha(M)$. This implies that $\alpha(M) = 0$ and thus M is compact. Using the continuity of the functional $x \mapsto D(x,T(x))$, we obtain that all the limit points of $(x_n)_{n\in\mathbb{N}}$ are fixed points of T. Thus $F_T \neq \emptyset$. For the second conclusion, let $(x_n)_{n\in\mathbb{N}} \in X$ be a sequence such that $D(x_n,T(x_n)) \to 0$ as $n \to +\infty$. As before, we get that $(x_n)_{n\in\mathbb{N}}$ has a subsequence which converges to a fixed point of T. The proof is now complete. □

Another example comes via Kannan nonexpansive multivalued operators. Recall that, if (X,d) is a metric space, then $T : X \to P_{cl}(X)$ is called a Kannan nonexpansive multivalued operator if for each $x,y \in X$ we have

$$H(T(x),T(y)) \le \frac{1}{2} \cdot [D(x,T(x)) + D(y,T(y))].$$

It is obvious that a Kannan nonexpansive multivalued operator is not necessary closed. Nevertheless we have:

Theorem 11.7.5. *Let (X,d) be a complete metric space. If $T : X \to P_{cp}(X)$ is a Kannan nonexpansive multivalued operator such that*

$\inf_{x \in X} D(x, T(x)) = 0$*, then the fixed point problem is well-posed in the generalized sense for* T *with respect to* D*.*

For other results see S. Reich and A.J. Zaslavski R[5], Y.P. Fang, N.J. Huang and J.-C. Yao R[1], E. Llorens-Fuster, A. Petruşel and J.-C. Yao B[1], A. Petruşel and I.A. Rus B[2], etc.

11.8 Other results

For other aspects of the fixed point theory of multivalued operators see also R.P. Agarwal and D. O'Regan R[1]-R[5], J. Andres and L. Górniewicz R[1], Yu.G. Borisovich, B.D. Gelman, A.D. Myškis and V.V. Obukhovskii R[1], B.C. Dhage R[2]-R[5], D. Downing and W.O. Ray R[1], Y. Feng and S. Liu R[2], G. Gabor R[2], L. Górniewicz, S.A. Marano and M. Slosarski R[1], E. Llorens-Fuster R[1], S.A. Marano R[1], B. Ricceri R[1], R[2], R[5], L. Rybinski R[1], T. Wang R[1], H.K. Xu R[1]-R[3], R[5] O. Naselli Ricceri R[1], R[2], J. Saint-Raymond R[2]-R[3], as well as, Chapter 12, Section 12.6., Chapter 10, Section 10.3. and Chapter 14, Section 14.5.

11.9 Applications

For several applications of the fixed point theory of multivalued operators see G. Isac and M.G. Cojocaru B[1], G. Isac, D.H. Hyers and T.M. Rassias B[1], A. Muntean B[1], A. Petruşel B[1]. See also J.-P. Aubin and A. Cellina R[1], J.-P. Aubin and H. Frankowska R[1], H. A. Antosiewicz and A. Cellina R[1], Yu.G. Borisovich, B.D. Gelman, A.D. Myškis and V.V. Obukhovskii R[1], A. Cernea R[4] and R[5], M. Mureşan R[1], A. Petruşel R[1], B. Ricceri and S. Simons (Eds.) R[1], A. Ştefănescu R[1], A. Kristály and Cs. Varga R[1], P. D. Panagiotopoulos, M. Bocea and V. Rădulescu R[1].

Chapter 12

Multivalued generalized contractions on g.m.s.

Guidelines: H. Covitz and S.B. Nadler jr. (1970) ($d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$); C. Avramescu(1970), V.G. Angelov (1998), (2008), M. Frigon (2002), R.P. Agarwal and D. O'Regan (2001) (gauge spaces); S.G. Matthews (1994), S.J. O'Neill (1998) (partial metric spaces); S. Czerwik (1998) (b-metric spaces); S. Priess-Crampe and P. Ribenboim (2000) (ultrametric spaces).
General references: H. Covitz and S.B. Nadler jr. R[1], A. Petruşel, I.A. Rus and M.A. Şerban R[1], C. Avramescu B[5], V.G. Angelov R[2] and R[6], M. Frigon R[3], R.P. Agarwal and D. O'Regan R[1], R.P. Agarwal, D. O'Regan and N.S. Papageorgiou R[1], S.G. Matthews R[1], S.J. O'Neill R[1], S. Czerwik R[2], V. Berinde B[15], M. Boriceanu, A. Petruşel and I.A. Rus R[1], S. Priess-Crampe and P. Ribenboim R[1].

12.0 $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$

Let X be a nonempty set. Recall that a functional $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is said to be a generalized metric in the sense of Luxemburg on X if it satisfies all the well-known axioms of a metric. In this case, the pair (X, d) is called a generalized metric space, see also Chapter 5.

Let us recall first some contractive-type conditions for multivalued operators.

Definition 12.0.1. Let (X, d) be a generalized metric space. Then $T : X \to P_{cl}(X)$ is called a multivalued a-contraction if $a \in [0, 1[$ and

$$H_d(T(x), T(y)) \leq ad(x, y), \text{ for each } x, y \in X, \text{ with } d(x, y) < +\infty.$$

Definition 12.0.2. Let (X, d) be a generalized metric space. If $T : X \to P(X)$ is a multivalued operator, then we consider the following multivalued operators generated by T:

$$\widehat{T} : X \to \mathcal{P}(X),\ \widehat{T}(x) := T(x) \cap X_{i(x)}$$

(where $X_{i(x)}$ denotes the unique element of the canonical decomposition of X where x belongs),

$$\tilde{T}^i : X \to \mathcal{P}(X),\ \tilde{T}^i(x) := T(x) \cap X_i$$

(where X_i denotes an arbitrary element of the canonical decomposition of X).

Then we have:

Lemma 12.0.1. $F_T = F_{\widehat{T}}$.

Lemma 12.0.2. $F_T \neq \emptyset \iff$ *if there exists* $i \in I$ *such that* $F_{\tilde{T}^i} \neq \emptyset$.

Covitz-Nadler fixed point principle (see Theorem 11.1.2.) gave rise to the following concept (see also Definition 11.6.1.)

Definition 12.0.1. (Rus-Petruşel-Sîntămărian B[1]-B[2]) Let $(X, \to)$ be an L-space. Then $T : X \to P(X)$ is a multivalued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

i) $x_0 = x,\ x_1 = y$

ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$

iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

Moreover, a sequence $(x_n)_{n\in\mathbb{N}}$ in X satisfying the conditions (i) and (ii) in the previous definition is called a sequence of successive approximations for T starting from (x, y).

The following result is a straightforward version of Covitz and Nadler (see R[1]) alternative theorem.

Theorem 12.0.1. *Let (X,d) be a generalized complete metric space and $T : X \to P_{cl}(X)$ be a multivalued a-contraction. Suppose that for each $x \in X$ there is $y \in T(x)$ such that $d(x,y) < +\infty$. Then there exists a sequence of successive approximations of T starting from any arbitrary $x \in X$ which converges to a fixed point of T.*

The previous result gives rise to the following open question.

Open question. Let $T : X \to P_{cl}(X)$ be a multivalued a-contraction as in the above Covitz-Nadler fixed point result. Is T a MWP operator ?

Theorem 12.0.2. *Let (X,d) be a generalized complete metric space and $T : X \to P_{cl}(X)$ be a multivalued a-contraction. Suppose there exists $x_0 \in X$ and $x_1 \in T(x_0)$ such that $d(x_0,x_1) < +\infty$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from x_0 which converges to a fixed point of T.*

Proof. Let $X := \bigcup_{i\in I} X_i$ be the canonical decomposition of X into metric spaces. Recall that X is complete if and only if X_i is complete for each $i \in I$. Let $j \in I$ such that $x_0 \in X_j$.

For $x \in X$ we successively have: $D(x,T(x)) < +\infty \Leftrightarrow$ there exists $y \in T(x)$ such that $d(x,y) < +\infty \Leftrightarrow y \in T(x) \cap X_{i(x)}$. Hence

$$D(x,T(x)) < +\infty \Leftrightarrow T(x) \cap X_{i(x)} \neq \emptyset.$$

Consider now the multivalued operator

$$\tilde{T}^j : X \to \mathcal{P}(X),\ \tilde{T}^j(x) := T(x) \cap X_j.$$

We will prove that $\tilde{T}^j_{|X_j} : X_j \to P_{cl}(X_j)$. For this purpose, it is enough to show that

$$D(x,T(x)) < +\infty,\ \text{for each } x \in X_j.$$

For $x \in X_j$ we have:
$D(x,T(x)) \le D(x,T(x_0)) + H(T(x_0),T(x)) \le d(x,x_0) + D(x_0,T(x_0)) + ad(x_0,x) < +\infty$.

Hence $\tilde{T}^j_{|X_j} : X_j \to P_{cl}(X_j)$ is a multivalued a-contraction on the complete metric space $(X_j, d_{|X_j \times X_j})$. The conclusion follows from Lemma 12.0.2. and Theorem 12.0.1. □

An answer to the above problem is the following result.

Theorem 12.0.3. *Let* (X, d) *be a generalized complete metric space and* $T : X \to P_{cl}(X)$ *be a multivalued a-contraction. Suppose that for each* $x \in X$ *and* $y \in T(x)$ *we have* $d(x, y) < +\infty$ *(or equivalently, for each* $x \in X$ *we have* $T(x) \subset X_{i(x)}$*). Then* T *is a MWP operator.*

Proof. From the hypothesis we have that $D(x, T(x)) < +\infty$, for each $x \in X$. Hence, for each $x \in X$ we have that $T : X_{i(x)} \to P_{cl}(X_{i(x)})$. Since $(X_{i(x)}, d_{|X_{i(x)} \times X_{i(x)}})$ is a complete metric space, by Theorem 12.0.1., we conclude that T is a MWP operator. □

We introduce now the following concepts.

Definition 12.0.2. Let $(X, \to)$ be an L-space and $T : X \to P(X)$ be a MWP operator. Define the multivalued operator $T^\infty : Graph(T) \to P(F_T)$ by the formula $T^\infty(x, y) = \{\ z \in F_T \mid$ there exists a sequence of successive approximations of T starting from (x, y) that converges to $z\ \}$.

Definition 12.0.3. Let (X, d) be a generalized metric space and $T : X \to P(X)$ be a MWP operator such that for each $x \in X$ and $y \in T(x)$ we have that $d(x, y) < +\infty$. Then, T is called a c-multivalued weakly Picard operator (briefly c-MWP operator) if there exists a selection t^∞ of T^∞ such that $d(x, t^\infty(x, y)) \le c\ d(x, y)$, for all $(x, y) \in Graph(T)$.

We have:

Theorem 12.0.4. *Let* (X, d) *be a generalized complete metric space and* $T : X \to P_{cl}(X)$ *be a multivalued a-contraction, such that for each* $x \in X$ *and* $y \in T(x)$ *we have* $d(x, y) < +\infty$.

Then T *is a* $\frac{1}{1-a}$*-MWP operator.*

We present now an abstract data dependence theorem for the fixed point set of c-MWP operators on generalized metric spaces.

Theorem 12.0.5. *Let* (X, d) *be a generalized metric space and* $T_1, T_2 : X \to P(X)$ *be two multivalued operators. We suppose that:*

i) T_i *is a* c_i*-MWP operator, for* $i \in \{1, 2\}$

ii) there exists $\eta > 0$ *such that* $H(T_1(x), T_2(x)) \le \eta$*, for all* $x \in X$.

Then $H(F_{T_1}, F_{T_2}) \le \eta\ \max\ \{\ c_1,\ c_2\ \}$.

We also have:

Theorem 12.0.6. *Let* (X, d) *be a generalized complete metric space and*

$T : X \to P_{cl}(X)$ be a multivalued a-contraction. Suppose:

(i) $(SF)_T \neq \emptyset$;

(ii) If $x, y \in F_T$ then $d(x, y) < +\infty$.

Then $F_T = (SF)_T = \{x^*\}$.

Proof. We will prove first that $(SF)_T = \{x^*\}$. Indeed, if $z \in (SF)_T$ with $z \neq x^*$, then $d(z, x^*) < +\infty$ and $d(z, x^*) = H(T(z), T(x^*)) \leq ad(z, x^*)$, a contradiction. Next we will prove that $F_T \subseteq (SF)_T$. Let $y \in F_T$. Then $d(y, x^*) < +\infty$. Thus $d(y, x^*) = D(y, T(x^*)) \leq H(T(y), T(x^*)) \leq ad(y, x^*)$, which implies $y = x^*$. This completes the proof. □

We will discuss now the case of multivalued pseudo-a-contractive operators, introduced by D. Azé and J.-P. Penot in R[1].

Definition 12.0.4. (Azé-Penot R[1]) Let (X, d) be a metric space. A multivalued operator $T : X \to P(X)$ is said to be pseudo-a-Lipschitzian with respect to the subset $U \subset X$ whenever, for all $x, y \in U$, we have

$$\rho_d(T(x) \cap U, T(y)) \leq ad(x, y).$$

Also, the multivalued operator T is called pseudo-a-contractive with respect to U if it is pseudo-a-Lipschitzian with respect to U for some $a \in [0, 1[$.

In Azé-Penot R[1], the fixed point theory for multivalued pseudo-a-contractive operators with respect to the open ball $B_d(x_0, r)$ of a complete metric space (X, d) is studied. Next theorem is a fixed point results for multivalued pseudo-a-contractive operators in the setting of a generalized metric space.

Theorem 12.0.7. *Let (X, d) be a generalized complete metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. Let $X := \bigcup_{i \in I} X_i$ be the canonical decomposition of X. Suppose that there exists $x_0 \in X$ such that $D(x_0, T(x_0)) < +\infty$ and T is pseudo a-contractive with respect to $X_{i(x_0)}$. Then $F_T \neq \emptyset$.*

Proof. Since $D(x_0, T(x_0)) < +\infty$ there exists $b > 0$ and $x_1 \in T(x_0)$ such that $d(x_0, x_1) < b < +\infty$. Then $x_1 \in X_{i(x_0)}$ and thus $x_1 \in T(x_0) \cap X_{i(x_0)}$. Hence we have $D(x_1, T(x_1)) \leq \rho(T(x_0) \cap X_{i(x_0)}, T(x_1)) \leq ad(x_0, x_1) < ab$. Thus there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) < ab < +\infty$. Thus $x_2 \in T(x_1) \cap X_{i(x_0)}$. In a similar way, we have $D(x_2, T(x_2)) \leq \rho(T(x_1) \cap X_{i(x_0)}, T(x_2)) \leq ad(x_1, x_2) < a^2 b < +\infty$.

By induction, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ with the following properties:

(a) $x_{n+1} \in T(x_n) \cap X_{i(x_0)}$, for all $n \in \mathbb{N}$;

(b) $d(x_n, x_{n+1}) < a^n b$, for all $n \in \mathbb{N}$.

From (b) we get that $(x_n)_{n\in\mathbb{N}}$ is Cauchy and hence convergent in $X_{i(x_0)}$. Thus there exists $x^* \in X_{i(x_0)}$ (since $X_{i(x_0)}$ is d-closed), such that $x_n \to x^*$ as $n \to +\infty$. Let us show now that $x^* \in F_T$. We have $D(x^*, T(x^*)) \leq d(x^*, x_{n+1}) + D(x_{n+1}, T(x^*)) \leq d(x^*, x_{n+1}) + \rho(T(x_n) \cap X_{i(x_0)}, T(x^*)) \leq d(x^*, x_{n+1}) + ad(x^*, x_n)) \to 0$ as $n \to +\infty$. Hence $x^* \in T(x^*)$. □

A second answer to the open problem mentioned above is the following:

Theorem 12.0.8. *Let (X, d) be a generalized complete metric space and $T : X \to P_{cl}(X)$ be a multivalued operator such that for each $x \in X$ and $y \in T(x)$ we have $d(x, y) < +\infty$. Let $X := \bigcup_{i\in I} X_i$ be the canonical decomposition of X. Suppose that T is pseudo a-contractive with respect to $X_{i(x)}$, for each $x \in X$. Then T is a MWP operator.*

Proof. Let $x_0 \in X$ and $x_1 \in T(x)$ such that $d(x_0, x_1) < b < +\infty$, for some $b > 0$. Thus $x_1 \in T(x_0) \cap X_{i(x_0)}$. Hence we have $D(x_1, T(x_1)) \leq \rho(T(x_0) \cap X_{i(x_0)}, T(x_1)) \leq ad(x_0, x_1) < ab$. We obtain that there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) < ab < +\infty$. Thus $x_2 \in T(x_1) \cap X_{i(x_0)}$. In a similar way, we have $D(x_2, T(x_2)) \leq \rho(T(x_1) \cap X_{i(x_0)}, T(x_2)) \leq ad(x_1, x_2) < a^2 b < +\infty$.

By induction, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ with the following properties:

(a) $x_{n+1} \in T(x_n) \cap X_{i(x_0)}$, for all $n \in \mathbb{N}$;

(b) $d(x_n, x_{n+1}) < a^n b$, for all $n \in \mathbb{N}$.

From (b) we get that $(x_n)_{n\in\mathbb{N}}$ is Cauchy and hence convergent in $X_{i(x_0)}$ to a certain x^*. As before, we obtain $x^* \in T(x^*)$. Since $x_0 \in X$ and $x_1 \in T(x_0)$ were arbitrarily chosen, we get that T is a MWP operator. □

12.1 $d(x, y) \in \mathbb{R}^m_+$

Let (X, d) be a complete generalized metric space in the sense that $d(x, y) \in \mathbb{R}^m_+$.

For the singlevalued case, the well-known Perov fixed point theorem is the most important basic result. We will present now a Perov type theorem for multivalued operators.

Theorem 12.1.1. (A. Petruşel B[26]) *Let (X, d) be a complete generalized metric space, (i. e. $d(x, y) \in \mathbb{R}^m_+$) and $T : X \to P_{cl}(X)$ be a multivalued A-contraction, i.e. there exists $A \in \mathcal{M}_{mm}(\mathbb{R})$ such that $A^n \to 0$, $n \to \infty$ and for each $x, y \in X$ and each $u \in T(x)$ there exists $v \in T(y)$ such that $d(u, v) \leq Ad(x, y)$.*

Then, T is a MWP operator.

Proof. By standard arguments, we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

$$\begin{cases} x_{n+1} \in T(x_n), \ n \in \mathbb{N} \\ x_0 \in X \end{cases}$$

and $d(x_n, x_{n+1}) \leq A^n d(x_0, x_1)$ for $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} x_n = x^*$. We prove that $x^* \in T(x^*)$. Indeed, for $x_n \in T(x_{n-1})$ there exists $u_n \in T(x^*)$ such that $d(x_n, u_n) \leq Ad(x_{n-1}, x^*)$, for all $n \in \mathbb{N}^*$. On the other side $d(x^*, u_n) \leq d(x^*, x_n) + d(x_n, u_n) \leq d(x^*, x_n) + Ad(x_{n-1}, x^*) \to 0$ as $n \to \infty$. Hence $\lim\limits_{n\to\infty} u_n = x^*$. But $u_n \in T(x^*)$, for $n \in \mathbb{N}^*$ and because $T(x^*)$ is closed, we have that $x^* \in T(x^*)$. The proof is complete. □

12.2 b-metric spaces

Let (X, d) be a b-metric space. We need first some lemmas.

Lemma 12.2.1. *Let (X, d) be a b-metric space and $A, B \in P(X)$. We suppose that there exists $\eta \in \mathbb{R}, \eta > 0$ such that:*

(i) for each $a \in A$ there is $b \in B$ such that $d(a, b) \leq \eta$;

(ii) for each $b \in B$ there is $a \in A$ such that $d(a, b) \leq \eta$.

Then

$$H(A, B) \leq \eta.$$

Notice that, if A is a nonempty subset of a b-metric space X, then we define the set $\bar{A} := \{x \in X \mid \text{there exists a sequence } (x_n)_{n\in\mathbb{N}} \subset A \text{ such that } d(x_n, x) \to 0 \text{ as } n \to +\infty\}$. Also, a subset A of a b-metric space is closed if and only if $A = \bar{A}$.

Lemma 12.2.2. *Let (X, d) be a b-metric space, $A \in P(X)$ and $x \in X$. Then $D(x, A) = 0$ if and only if $x \in \bar{A}$.*

The following results are useful for some of the proofs in the paper.

Lemma 12.2.3. (Czerwik R[1]) *Let (X, d) be a b-metric space. Then*

$$D(x, A) \leq s[d(x, y) + D(y, A)], \text{ for all } x, y \in X, A \subset X.$$

Lemma 12.2.4. (Bakhtin R[1], Czerwik R[1]) *Let (X, d) be a b-metric space and let $\{x_k\}_{k=0}^{n} \subset X$. Then:*

$$d(x_n, x_0) \leq sd(x_0, x_1) + ... + s^{n-1}d(x_{n-2}, x_{n-1}) + s^{n-1}d(x_{n-1}, x_n).$$

Lemma 12.2.5. (Czerwik R[1]) *Let (X, d) be a b-metric space and for all $A, B, C \in X$ we have:*

$$H(A, C) \leq s[H(A, B) + H(B, C)].$$

Lemma 12.2.6. (Czerwik R[1]) *Let (X, d) be a b-metric space and $A, B \in P(X)$. Then, for each $q > 1$ and for all $a \in A$ there exists $b \in B$ such that:*

$$d(a, b) \leq qH(A, B).$$

One of the main result of this section is:

Theorem 12.2.1 *Let (X, d) be a complete b-metric space and let $T : X \to P_{cl}(X)$ be a multivalued operator. Suppose there exists $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $\frac{\alpha+\beta}{1-\gamma} < 1/s$ such that F satisfies the inequality*

$$H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(x, T(x)) + \gamma D(y, T(y)), \text{ for all } x, y \in X.$$

Then $F_T \neq \emptyset$.

Proof. Let $q > 1$ be arbitrary. Take $x_0 \in X$ and for all $x_1 \in T(x_0)$ there exists $x_2 \in T(x_1)$ such that:

$$\begin{aligned} d(x_1, x_2) \leq qH(T(x_0), T(x_1)) &\leq q[\alpha d(x_0, x_1) + \beta D(x_0, T(x_0)) + \gamma D(x_1, T(x_1))] \\ &\leq q[\alpha d(x_0, x_1) + \beta d(x_0, x_1) + \gamma d(x_1, x_2)]. \end{aligned}$$

So we have that

$$d(x_1, x_2) \leq qd(x_0, x_1)(\alpha + \beta)(1 - q\gamma)^{-1}.$$

For $x_2 \in T(x_1)$ there exists $x_3 \in T(x_2)$ such that:

$$d(x_2, x_3) \leq qH(T(x_1), T(x_2)) \leq q[\alpha d(x_1, x_2) + \beta D(x_1, T(x_1)) + \gamma D(x_2, T(x_2))]$$
$$\leq q[q\alpha(\alpha + \beta)(1 - q\gamma)^{-1}d(x_0, x_1) + q\beta(\alpha + \beta)(1 - q\gamma)^{-1}d(x_0, x_1) + \gamma d(x_2, x_3)].$$

So we obtain that

$$d(x_2, x_3) \leq q^2(\alpha + \beta)^2(1 - q\gamma)^{-2}d(x_0, x_1).$$

We can construct by induction a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$d(x_n, x_{n+1}) \leq (q(\alpha + \beta)(1 - q\gamma)^{-1})^n d(x_0, x_1), \text{ for all } n \in \mathbb{N}.$$

We will prove next that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy, by estimating $d(x_n, x_{n+p})$.

Let us denote by

$$A := q(\alpha + \beta)(1 - q\gamma)^{-1}.$$

$$\begin{aligned} d(x_n, x_{n+p}) &\leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + ... + s^{p-2}d(x_{n+p-3}, x_{n+p-2})+ \\ &+ s^{p-1}d(x_{n+p-2}, x_{n+p-1}) + s^{p-1}d(x_{n+p-1}, x_{n+p}) \leq \\ &\leq sA^n d(x_0, x_1) + s^2A^{n+1}d(x_0, x_1) + ... + s^{p-2}A^{n+p-3}d(x_0, x_1)+ \\ &+ s^{p-1}A^{n+p-2}d(x_0, x_1) + s^{p-1}A^{n+p-1}d(x_0, x_1) = \\ &= A^n d(x_0, x_1)[s + s^2A + ... + s^{p-2}A^{p-3} + s^{p-1}A^{p-2} + s^{p-1}A^{p-1} = \\ &= sA^n d(x_0, x_1)[1 + sA + ... + s^{p-3}A^{p-3} + s^{p-2}A^{p-2} + s^{p-2}A^{p-1}] = \\ &= \frac{(sA)^n}{s^{n-1}}d(x_0, x_1)\left[\frac{1 - (sA)^{p-1}}{1 - sA} + (sA)^{p-2}A\right]. \end{aligned}$$

Taking $1 < q < \frac{1}{s(\alpha+\beta)+\gamma}$ we obtain that:

$$d(x_n, x_{n+p}) \leq \frac{(sA)^n}{s^{n-1}}d(x_0, x_1)\left[\frac{1 - (sA)^{p-1}}{1 - sA} + (sA)^{p-2}A\right] \to 0,$$

as $n \to \infty$. So $(x_n)_{n\in\mathbb{N}}$ is Cauchy and $x_n \to x \in X$.

But $x_n \in T(x_{n-1})$ so we have

$$\begin{aligned} D(x,T(x)) &\le sd(x,x_n) + sD(x_n,T(x)) \le sd(x,x_n) + sH(T(x_{n-1}),T(x)) \\ &\le sd(x,x_n) + s[\alpha d(x_{n-1},x) + \beta D(x_{n-1},T(x_{n-1})) + \gamma D(x,T(x))] \\ &\le sd(x,x_n) + s[\alpha d(x_{n-1},x) + \beta d(x_{n-1},x_n) + \gamma D(x,T(x))]. \end{aligned}$$

So we have that

$$D(x,T(x)) \le (sd(x,x_n) + s\alpha d(x_{n-1},x) + s\beta d(x_{n-1},x_n))(1-s\gamma)^{-1} \to 0$$

and hence $D(x,T(x)) = 0$. From Lemma 12.2.2. and the fact that $T(x)$ closed, we obtain that $x \in T(x)$, i.e. T has a fixed point. □

For other results see S. Czerwik R[2], S. Czerwik, K. Dlutek and S.L. Singh R[1], S.L. Singh, C. Bhatnagar and S.N. Mishra R[1], etc.

12.3 Gauge spaces

For the beginning, let us present a result of C. Avramescu.

Let (E,d_α) and (E,p_β) (where $(d_\alpha|\alpha \in A)$ and $(p_\beta|\beta \in B)$ are two families of gauges) two uniform spaces and X a subset of E.

Definition 12.3.1. A multivalued operator $T: X \to P(X)$ is said to be u-contractive in (E,p_β) if there exists a mapping $u: B \to B$ such that $u(u(B)) = u(B)$, for each $\beta \in B$ and a family of real numbers $(q_\beta \subset [0,1[$ such that for each $y_1 \in T(x_1)$ there exists $y_2 \in T(x_2)$ such that $p_\beta(y_1,y_2) \le q_\beta p_{u(\beta)}(x_1,x_2)$, for each $\beta \in B$ and each $x_1,x_2 \in X$.

If in Definition 12.3.1 the inequality holds for each $v_1 \in T(x_1)$ and each $v_2 \in T(x_2)$ the T is called totally u-contractive.

Definition 12.3.2. The uniform space (E,p_β) is more complete than the uniform space (E,d_α) if each fundamental sequence in (E,p_β) is fundamental in (E,d_α).

Definition 12.3.3. The multivalued operator T is said to be almost-contractive in (E,d_α) if there exists an uniform space (E,p_β) more complete than (E,d_α) such that T is u-contractive in (E,p_β).

Also, T is, by definition, totally almost contractive if in Definition 12.3.3., T is totally u-contractive in (E,p_β).

The following result was proved by C. Avramescu:

Theorem 12.3.1. (C. Avramescu B[5]) *Let us suppose:*

i) The uniform space (E, d_α) is complete with respect to countable sequences;

ii) X is a closed subset of E and the multivalued operator $T : X \to P(X)$ has closed graph;

iii) T is almost u-contractive.

Then T has at least a fixed point in X. Moreover, if T is totally almost u-contractive and the space (E, p_β) is separable, then the fixed point is unique.

One of the basic fixed point results for multivalued operators on gauge spaces was proved by M. Frigon in R[3].

In what follows, $E := (\mathbb{E}, \{d_\alpha\}_{\alpha\in\Lambda})$ is a complete gauge space, but we do not assume that Λ is a directed set.

Let X be a nonempty subset of E. A multivalued operator $T : X \to P(E)$ is called an admissible contraction with constant $k := \{k_{alpha}\}_{\alpha\in\Lambda} \in [0, 1[^\Lambda$ if:

(i) for each $\alpha \in \Lambda$ we have $H_\alpha(T(x), T(y)) \le a_\alpha \cdot d_\alpha(x, y)$, for each $x, y \in \mathbb{E}$;

(ii) for every $x \in \mathbb{E}$ and every $\epsilon \in]0, +\infty[^\Lambda$ there exists $y \in T(x)$ such that $d_\alpha(x, y) \le D_\alpha(x, T(x)) + \epsilon_\alpha$, for each $\alpha \in \Lambda$. Notice that, if $\Lambda = \mathbb{N}$, then E is metrizable with some metric d. Nevertheless, a multivalued operator T can be an admissible contraction without being a contraction in the usual sense, with respect to the metric d.

Theorem 12.3.2. (Frigon (2002)) *Let $\mathbb{E}$ be a complete gauge space and let $T : \mathbb{E} \to P_{cl}(\mathbb{E})$ be an admissible multivalued contraction. Then F_T is nonempty.*

Proof. Denote by $k := \{k_{alpha}\}_{\alpha\in\Lambda} \in [0, 1[^\Lambda$ the contraction constant for T. Let $x_0 \in E$ be arbitrary chosen. For every $\alpha \in \Lambda$, let $r_\alpha > 0$ such that $D_\alpha(x_0, T(x_0)) < (1 - k_\alpha) \cdot r_\alpha$. Then there exists $x_1 \in T(x_0)$ such that $d_\alpha(x_0, x_1) < (1 - k_\alpha) \cdot r_\alpha$, for every $\alpha \in \Lambda$. Next, we can chose $x_2 \in T(x_1)$ such that $d_\alpha(x_1, x_2) < D_\alpha(x_1, T(x_1)) + k_\alpha((1-k_\alpha)r_\alpha - d_\alpha(x_0, x_1)) \le H_\alpha(T(x_0), T(x_1)) + k_\alpha((1-k_\alpha)r_\alpha - d_\alpha(x_0, x_1))$. As a consequence, we get that $d_\alpha(x_1, x_2) < k_\alpha(1 - k_\alpha)r_\alpha$.

Inductively, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

1) $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$;

2) $d_\alpha(x_n, x_{n+1}) < k_\alpha^n(1-k_\alpha)r_\alpha$, for each $n \in \mathbb{N}$ and for every $\alpha \in \Lambda$.

By a standard approach we get that the limit of the sequence $(x_n)_{n\in\mathbb{N}}$ is a fixed point for T. □

Remark 12.3.1. For continuation results for contractions and generalized contractions on complete gauge spaces, see Frigon R[3] and Agarwal, Cho and O'Regan R[1].

Following Frigon R[3], we introduce the notion of admissible a_α-contraction, as follows:

Definition 12.3.4. Let $\mathbb{E} = (\mathbb{E}, \{d_\alpha\}_{\alpha\in\Lambda})$ be a gauge space endowed with a complete gauge structure $\{d_\alpha\}_{\alpha\in\Lambda}$. A multivalued operator $T : \mathbb{E} \to P_{cl}(\mathbb{E})$ is called an admissible multivalued a_α-contraction if $a_\alpha \in]0,1[$, for each $\alpha \in \Lambda$ and the following conditions are satisfied:

i) $H_\alpha(T(x), T(y)) \le a_\alpha \cdot d_\alpha(x, y)$, for each $x, y \in \mathbb{E}$ and for each $\alpha \in \Lambda$;

ii) for every $x \in \mathbb{E}$ and every $q \in]1, +\infty[^\Lambda$ there exists $y \in T(x)$ such that $d_\alpha(x, y) \le q_\alpha \cdot D_\alpha(x, T(x))$, for each $\alpha \in \Lambda$.

Following the argument of Frigon R[3], it is easy to prove the following fixed point result:

Theorem 12.3.3. *Let $\mathbb{E}$ be a complete gauge space and let $T : \mathbb{E} \to P_{cl}(\mathbb{E})$ be an admissible multivalued a_α-contraction. Then F_T is nonempty and closed.*

Moreover, a data dependence result holds for the fixed point set of such admissible multivalued a_α-contractions.

Theorem 12.3.4. *Let $\mathbb{E}$ be a complete gauge space. Let $T : \mathbb{E} \to P_{cl}(\mathbb{E})$ be an admissible multivalued a_α-contraction and $G : \mathbb{E} \to P_{cl}(\mathbb{E})$ be an admissible multivalued b_α-contraction.*

Suppose that there exists $\eta \in]0, +\infty[^\Lambda$ such that $H_\alpha(T(x), G(x)) \le \eta_\alpha$, for each $x \in \mathbb{E}$ and for each $\alpha \in \Lambda$.

Then $H_\alpha(F_T, F_G) \le \frac{1}{1-m_\alpha} \cdot \eta_\alpha$, where $m_\alpha = max\{a_\alpha, b_\alpha\}$, $\alpha \in \Lambda$.

Proof. Let $1 < q_\alpha < a_\alpha^{-1}$, $\alpha \in \Lambda$. Consider $x_0 \in F_T$ and $x_1 \in G(x_0)$ such that $d_\alpha(x_0, x_1) \le q_\alpha \cdot \eta_\alpha$, for each $\alpha \in \Lambda$. Then, there exists $x_2 \in G(x_1)$ such that $d_\alpha(x_1, x_2) \le q_\alpha \cdot D_\alpha(x_1, G(x_1))$, for every $\alpha \in \Lambda$. We have successively: $d_\alpha(x_1, x_2) \le q_\alpha \cdot D_\alpha(x_1, G(x_1)) \le q_\alpha \cdot H_\alpha(G(x_0), G(x_1)) \le q_\alpha \cdot a_\alpha \cdot d_\alpha(x_0, x_1)$, for each $\alpha \in \Lambda$. Hence we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for G starting from x_0 satisfying the relation: $d_\alpha(x_n, x_{n+1}) \leq (q_\alpha a_\alpha)^n \cdot d_\alpha(x_0, x_1)$, for each $\alpha \in \Lambda$. By standard methods, we obtain that $(x_n)_{n\in\mathbb{N}}$ converges and its limit is a fixed point (denoted by x_G^*) of G.

Also, we have the following estimate:
$d(x_n, x_G^*) \leq \dfrac{(q_\alpha a_\alpha)^n}{1 - q_\alpha a_\alpha} q_\alpha \cdot \eta_\alpha$, for each $n \in \mathbb{N}$ and for each $\alpha \in \Lambda$.
Taking $n = 0$ in the above relation we obtain:

$$d(x_0, x_G^*) \leq \frac{1}{1 - q_\alpha a_\alpha} \cdot q_\alpha \eta_\alpha, \text{ for each } \alpha \in \Lambda.$$

Using a similar procedure, we can also show that for each $y_0 \in F_G$, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from y_0, which converges to $y_T^* \in F_T$.

Also we have:

$$d(y_0, x_T^*) \leq \frac{1}{1 - q_\alpha b_\alpha} \cdot q_\alpha \eta_\alpha, \text{ for each } \alpha \in \Lambda.$$

In conclusion, we have proved that:

$$H_\alpha(F_T, F_G) \leq \frac{1}{1 - q_\alpha m_\alpha} \cdot q_\alpha \eta_\alpha, \text{ for each } \alpha \in \Lambda.$$

Letting $q_\alpha \searrow 1$ we get the conclusion. □

An important concept is given in the following definition.

Definition 12.3.5. Let $\mathbb{E}$ be a gauge space and $T : X \to P(X)$ an MWP operator. Then T is an admissible c_α-multivalued weakly Picard operator if and only if $c_\alpha \in]0, +\infty[$ for each α and the following assertions are satisfied:

i) there exists a selection t^∞ of T^∞ such that: $d_\alpha(x, t^\infty(x, y)) \leq c_\alpha\, d_\alpha(x, y)$, for all $(x, y) \in Graph(T)$ and for every $\alpha \in \Lambda$.

ii) for every $x \in \mathbb{E}$ and every $q \in]1, +\infty[^\Lambda$ there exists $y \in T(x)$ such that $d_\alpha(x, y) \leq q_\alpha \cdot D_\alpha(x, T(x))$, for every $\alpha \in \Lambda$.

Example 12.3.1. *Let $\mathbb{E}$ be a complete gauge space and $T : X \to P_{cl}(X)$ be an admissible multivalued a_α-contraction. Then T is a c_α-admissible c_α-multivalued weakly Picard operator, where $c_\alpha = (1 - a_\alpha)^{-1}$.*

Our main abstract result on the data dependence problem for admissible multivalued weakly Picard operators is:

Theorem 12.3.6. *Let $\mathbb{E}$ be a complete gauge space. Let $T_1 : \mathbb{E} \to P_{cl}(\mathbb{E})$ be an admissible c^1_α-multivalued weakly Picard operator and $T_2 : \mathbb{E} \to P_{cl}(\mathbb{E})$ be an admissible c^2_α-multivalued weakly Picard operator.*

Suppose that there exists $\eta \in]0, +\infty[^\Lambda$ such that $H_\alpha(T_1(x), T_2(x)) \leq \eta_\alpha$, for each $x \in \mathbb{E}$ and for each $\alpha \in \Lambda$.

Then $H_\alpha(F_{T_1}, F_{T_2}) \leq c_\alpha \cdot \eta_\alpha$, where $c_\alpha = max\{c^1_\alpha, c^2_\alpha\}$, $\alpha \in \Lambda$.

Proof. Let $t_i : X \to X$ be a selection of $T_i,\ i \in \{1, 2\}$. For each $\alpha \in \Lambda$ we have:

$$H_\alpha(F_{T_1}, F_{T_2}) \leq \max\ \{ \sup_{x \in F_{T_2}} d_\alpha(x, t_1^\infty(x, t_1(x))),\ \sup_{x \in F_{T_1}} d_\alpha(x, t_2^\infty(x, t_2(x)))\ \}.$$

Let $q \in]1, +\infty[^\Lambda$. Then we can choose t_i, for $i \in \{1, 2\}$, such that for each $\alpha \in \Lambda$:

$$d_\alpha(x, t_1^\infty(x, t_1(x))) \leq c^1_\alpha\ q_\alpha\ H_\alpha(T_2(x), T_1(x)),\ \text{for all } x \in F_{T_2},$$

and

$$d_\alpha(x, t_2^\infty(x, t_2(x))) \leq c^2_\alpha\ q_\alpha\ H_\alpha(T_1(x), T_2(x)),\ \text{for all } x \in F_{T_1}.$$

Thus, we have

$$H_\alpha(F_{T_1}, F_{T_2}) \leq q_\alpha\ \eta_\alpha\ \max\ \{c^1_\alpha, c^2_\alpha\},\ \text{for each } \alpha \in \Lambda.$$

Letting $q_\alpha \searrow 1$, the proof is complete. □

Remark 12.3.2. Data dependence results for multivalued admissible contractions (Frigon R[3] and Theorem above), multivalued φ-contractions (Angelov-Rus B[1]), multivalued contractions of Bose-Mukherjee type (see Agarwal-O'Regan R[2]) are particular cases of the above theorem.

For other results see M. Frigon R[1] and R[3], R.P. Agarwal and D. O'Regan R[2], R.P. Agarwal, J. Dshalalow and D. O'Regan R[1], A. Chiş R[2], A. Chiş and R. Precup R[1], etc.

Chapter 13

Compactness, convexity and fixed points

Precursors: H. Knaster, C. Kuratowski and S. Mazurkiewicz (1929), C. Kuratowski (1930).
Guidelines: J. Dugundji (1951), K. Fan (1952), G. Darbo (1955), E. Michael (1959), B.N. Sadovskii (1967), J. Eisenfeld and V. Lakshmikantham (1975), J.-P. Penot (1979), C. Bardaro and R. Ceppitelli (1968), S. Park and H. Kim (1996).
General references: R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina and B.N. Sadovskii R[1], J. Eisenfeld and V. Lakshmikantham R[1], J.M. Ayerbe Toledano, T. Dominguez Benavides and G. Lopez Acedo R[1], J. Banas and K. Goebel R[1], P.K. Lin and Y. Sternfeld R[1], J.-P. Penot R[1], I. Singer R[1], V. Boltyanski, H. Martini and P. Soltan R[1], W.A. Kirk and B. Sims R[1], A. Fryszkowski R[3], Gh. Bocşan B[2], Gh. Constantin and I. Istrăţescu B[1]. V. I. Istrăţescu B[9], I.A. Rus B[32], B[43], B[44] and B[50], H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano and J.V. Llinares R[1], M. Balaj B[8], G.L. Cain and L. González R[1], J.V. Llinares R[1], C. Horvath R[3].

13.0 Introduction

Notions as compactness (J. Dugundji R[2], N. Bourbaki R[3], R. Engelking R[1], J.L. Kelley R[1], Yu.G. Borisovich, N.M. Bliznyakov, Ya.A. Izrailevich and T.N. Fomenko R[1], K. Kuratowski R[1], C.E. Aull and R. Lowen R[1], K. Kunen and J.F. Vaugham (Eds.) R[1], etc.) and convexity (I. Singer R[1], G. Isac R[1], J.-P. Penot, W. Takahashi R[3], V. Boltyanski, H. Martini and P. Soltan R[1], V. Barbu and T. Precupanu R[1], T. Precupanu R[1], A. Fryszkowski R[3], G. Moţ R[1]) are leading parts in Nonlinear Analysis, especially in Fixed Point Theory (R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina and B.N. Sadovskii R[1], J. Appell R[1], J.M. Ayerbe Toledano, T. Dominguez Benavides and G. López Acedo R[1], J. Banas and K. Goebel R[1], M.S. Berger R[2], K. Deimling R[3], A. Granas R[1], A. Granas and J. Dugundji R[1], D. Guo and V. Lakshmikantham R[1], O. Hadžić and E. Pap R[1], S. Hu and N.S. Papageorgiou R[1], G. Isac, D.H. Hyers and T.M. Rassias R[1], M. Kamenskii, V. Obukhovskii, P. Zecca R[1], R.H. Martin R[1], I.A. Rus B[95], S.P. Singh, S. Thomeier and B. Watson (Eds.) R[1], G.X.-Z. Yuan R[1], etc.). These aspects will be discussed in Chapter 17-19. The aim of this chapter is to present some generalizations of the metric fixed point theorems in terms of compactness and convexity. For Darbo's Theorem and Sadovski's Theorem, see Chapter 18 and Chapter 19.

13.1 Abstract measures of non-compactness and fixed points

Definition 13.1.1. (Kuratowski (1930)). Let (X, d) be a complete metric space. A functional $\alpha_K : P_b(X) \to R_+$ defined by $\alpha_K(A) := \inf\{t > 0 |\ A$ can be covered by a finite number of sets of diameter $\leq t\}$, is called the Kuratowski measure of non-compactness.

Definition 13.1.2. A functional $\alpha_H : P_b(X) \to R_+$, defined by $\alpha_H(A) := \inf\{t > 0 |\ A$ can be covered by a finite number of spheres of radius $\leq t\}$, is called the Hausdorff measure of non-compactness.

Definition 13.1.3. A functional $\alpha : P_b(X) \to R_+$ is called an abstract measure of noncompactness on X if:

(i) $\alpha(A) = 0$ implies $\overline{A} \in P_{cp}(X)$;

(ii) $\alpha(A) = \alpha(\overline{A})$, for all $A \in P_b(X)$;

(iii) $A \subset B$ implies $\alpha(A) \le \alpha(B)$;

(iv) If $A_n \in P_{b,cl}(X)$, $A_{n+1} \subset A_n$, $n \in \mathbb{N}$ and $\alpha(A_n)$ tends to 0 as $n \to \infty$, then

$$A_\infty := \bigcap_{n\in\mathbb{N}} A_n \neq \emptyset \text{ and } \alpha(A_\infty) = 0.$$

Definition 13.1.4. A function $\alpha_{DP} : P_b(X) \to R_+$ is called Danes-Pasicki's measure of noncompactness if it satisfies the conditions (i), (ii) and (iii) in Definition 13.1.3. and the condition:

(iv') $\alpha_{DP}(A \cup \{x\}) = \alpha_{DP}$, for all $A \in P_b(X)$ and $x \in X$.

Definition 13.1.5. Let (X, d) be a metric space, $\theta : P_b(X) \to R_+$ a functional and $\varphi : R_+ \to R_+$ be a comparison function. An operator $f : X \to X$ is a (θ, φ)-contraction if:

(i) $A \in P_b(x)$ implies $f(A) \in P_b(X)$;

(ii) $\theta(f(A)) \le \varphi(\theta(A))$, for all $A \in I_b(f)$.

Definition 13.1.6. An operator $f : X \to X$ is θ-densifying if:

(i) $A \in P_b(X)$ implies $f(A) \in P_b(X)$;

(ii) $\theta(f(A)) < \theta(A)$, for all $A \in I_b(f)$, with $\theta(A) > 0$.

We have:

Theorem 13.1.1. (I. A. Rus B[4]). *Let (X, d) be a complete metric space and $\alpha : P_b(X) \to R_+$ be an abstract measure of noncompactness on X. Let $f : X \to X$ be such that:*

(i) f is an (α, φ)-contraction;

(ii) f is contractive;

(iii) $I_b(f) \neq \emptyset$.

Then:

(a) $F_f = \{x^*\}$;

(b) $f^n(x_0)$ converges to x^*, as $n \to \infty$, for all $x_0 \in A \in I_b(f)$.

Proof. (a) From the assumption (ii) we have that $cardF_f \le 1$. Let $Y \in I_b(f)$. By the continuity of f it follows that $\overline{Y} \in I_{b,cl}(f)$. Let $Y_1 :=$

$\overline{f(\overline{Y})}, \cdots, Y_{n+1} := \overline{f(Y_n)}$, $n \in \mathbb{N}^*$. It is obvious that $Y_n \in I_{b,cl}(f)$, $Y_{n+1} \subset Y_n$, for each $n \in \mathbb{N}^*$. By the assumption (i) we have that

$$\alpha(Y_n) \to 0 \text{ as } n \to +\infty.$$

Hence:

$$Y_\infty := \bigcap_{n\in\mathbb{N}} Y_n \neq \emptyset,\ \alpha(Y_\infty) = 0 \text{ and } Y_\infty \in I_{cp}(f).$$

(b) Let $x_0 \in Y \in I_b(f)$. Then $O_f(x_0) := \{x_0, f(x_0), \cdots, f^n(x_0), \cdots\} \in I_b(f)$. From (i) we get that $\alpha(O_f(x_0) = 0$. Thus, there exists a convergent subsequence $(f^{n_i}(x_0))_{i\in\mathbb{N}}$. From (ii) we have that $f^n(x_0) \to x^*$ as $n \to +\infty$. □

Theorem 13.1.2. (I. A. Rus B[4]). *Let (X, d) be a complete metric space and $f : X \to X$ an operator such that:*

(i) f is α_{DP}-densifying;

(ii) f is contractive;

(iii) there exists a regular element $x_0 \in X$ under f (i.e., its orbit $O_f(x_0)$ is bounded).

Then $F_f = \{x^*\}$.

Proof. Let $Y := O_f(x_0)$. By (iii) we have that $Y \in I_b(f)$. By the continuity of f we have that $\overline{Y} \in I_{b,cl}(f)$. Since $Y = f(Y) \cup \{x_0\}$, from (i) we get that $\alpha_{DP}(\overline{Y}) = 0$. Hence, $F_f \neq \emptyset$ and, moreover, $F_f = \{x^*\}$. □

Remark 13.1.1. The above theorems extend some results given by M. Furi and M. Vignoli R[1]. For other results of this type see I.A. Rus B[4], L. Coroian B[2], V.I. Istrăţescu B[3], I.A. Rus B[59]. See also J.M. Ayerbe Toledano, T. Dominguez Benavides and G. Lopez Acedo R[1], M. Furi and M. Martelli R[1], P.L. Papini R[1].

13.2 Abstract measures of nonconvexity and fixed points

Definition 13.2.1. The Eisenfeld-Lakshmikantham measure of nonconvexity on a Banach space X, is the functional $\beta_{EL} : P_b(X) \to R_+$ defined by $\beta_{EL}(A) := H(A, coA)$.

Definition 13.2.2. Let X be a Banach space. A functional $\beta : P_b(X) \to R_+$ is called an abstract measure of nonconvexity if:

(i) $\beta(A) = 0$ implies that $\overline{A}$ is a convex set;

(ii) $\beta(A) = \beta(\overline{A})$, for all $A \in P_b(X)$;

(iii) $\beta : (P_{b,cl}(X), H) \to R_+$ is continuous.

We have:

Theorem 13.2.1. (I.A. Rus B[50], B[43]). *Let X be a Banach space, α be a measure of noncompactness on X and β be a measure of nonconvexity on X. Let $Y \subset X$ be a nonempty closed bounded set and $f : Y \to Y$ be a continuous operator. We suppose that:*

(i) f is an (α, φ_1)-contraction;

(ii) f is a (β, φ_2)-contraction.

Then $F_f = \{x^*\}$.

Remark 13.2.1. If, in Theorem 13.2.1., we take $\alpha = \alpha_K$ and $\beta = \beta_K$ we have a result given by J. Eisenfeld and V. Lakshmikantham in R[1]. For other results see I.A. Rus B[32], B[43], B[44], B[45], B[50] and B[51], J.S. Bae R[1], D. Bugajewski and R. Espínola R[1], etc.

13.3 Convexity and decomposability

Throughout this section $(T, \mathcal{A}, \mu)$ is a complete σ-finite nonatomic measure space and E is a Banach space. Let $L^1(T, E)$ be the Banach space of all measurable functions $u : T \to E$ which are Bochner μ-integrable. We call a set $K \subset L^1(T, E)$ decomposable if for all $u, v \in K$ and each $A \in \mathcal{A}$:

$$u\chi_A + v\chi_{T\setminus A} \in K, \tag{13.1}$$

where χ_A stands for the characteristic function of the set A.

This notion is, somehow, similar to convexity, but there exist also major differences. However, in several cases the decomposability condition is a good substitute for convexity. For important results in this direction: "convexity replaced by decomposability", we refer to C. Olech R[1], A. Fryskowski R[1] and R[2], A. Bressan and G. Colombo R[1], A. Bressan, A. Cellina and A. Fryskowski R[1], M. Kisielewicz R[1], A. Cellina, G. Colombo and A. Fonda

R[1], A. Cellina and C. Mariconda R[1], F. Hiai and H. Umegaki R[1], etc.

Some basic notions are considered in the following definitions:

Definition 13.3.1. Let (X, d) be a metric space. Then $F : X \to P(E)$ is called locally selectionable at $x_0 \in X$ if for all $y_0 \in F(x_0)$ there exist a neighborhood $N(x_0)$ and a continuous function $f : N(x_0) \to E$ such that $f(x_0) = y_0$ and $f(x) \in F(x)$, for each $x \in N(x_0)$.

Definition 13.3.2. Let X be a nonempty set. Let $F : X \to P(E)$ be a multivalued operator. The set defined by $F^{-1}(y) = \{x \in X | y \in F(x)\}$ is said to be the fibre of F at the point $y \in E$.

An important extension of the well-known concept of selection is the notion of selecting family:

Definition 13.3.3. (P. Deguire R[1], P. Deguire and M. Lassonde R[1]) Let X be a topological space and $\{Y_i | i \in I\}$ an arbitrary family of topological spaces.

i) We say that $\{f_i : X \to Y_i | i \in I\}$ is a selecting family for the family of multivalued operators $\{F_i : X \to \mathcal{P}(Y_i) | i \in I\}$ if for each $x \in X$ there exists $i \in I$ such that $f_i(x) \in F_i(x)$.

ii) If $\{Y_i |\ i \in I\}$ is an arbitrary family of convex subsets of a Hausdorff topological vector space then the family $\{F_i : X \to \mathcal{P}(Y_i) | i \in I\}$ is said to be of Ky Fan-type if each F_i has convex values and open fibres and for every $x \in X$ there is $i \in I$ such that $F_i(x) \neq \emptyset$.

Concerning the existence of continuous selections for a locally selectionable multifunction with decomposable values, we have:

Theorem 13.3.1. (A. Petruşel and A. Muntean, B[1]) *Let (X, d) be a separable metric space, $(T, \mathcal{A}, \mu)$ be a complete σ-finite and nonatomic measure space and E be a Banach space. Let $F : X \to \mathcal{P}_{dec}(L^1(T, E))$ be a locally selectionable multivalued operator. Then F has a continuous selection.*

An important result is the following Browder-type selection theorem for a multivalued operator on decomposable sets:

Theorem 13.3.2. (A. Petruşel and A. Muntean, B[1]) *Let E be a Banach space such that $L^1(T, E)$ is separable. Let K be a nonempty, paracompact, decomposable subset of $L^1(T, E)$ and let $F : K \to \mathcal{P}_{dec}(K)$ be a multivalued operator with open fibres. Then F has a continuous selection.*

For the case of Ky Fan-type multivalued operators with decomposable values we have:

Theorem 13.3.3. (A. Petruşel and G. Moţ, R[1])*Let E be a Banach space such that $L^1(T, E)$ is separable. Let I be an arbitrary set of indices, $\{K_i | i \in I\}$ be a family of nonempty, decomposable subsets of $L^1(T, E)$ and X a paracompact space. Let us suppose that the family*
$\{F_i : X \to \mathcal{P}_{dec}(K_i) | i \in I\}$ is of Ky Fan-type. Then there exists a selecting family for $\{F_i\}_{i \in I}$.

Remark 13.3.1. For the convex case see the papers of F.E. Browder R[1], P. Deguire R[1], P. Deguire and M. Lassonde R[1].

For the basic fixed point theory on decomposable sets see A. Fryszkowski R[3].

Chapter 14

Common fixed points

Guidelines: A.A. Markov (1936), S. Kakutani (1938), A.D. Myskis (1954), A. Tarski (1955), J.R. Isbell (1957), M.M. Day (1961), Z. Hedrin (1961), P.C. Baayen (1963), R. De Marr (1964), R. Kannan (1968).
General references: T. van der Walt R[1], I.A. Rus B[90], B[81], B[70] and B[4], D. Butnariu and I. Markowitz B[1], N. Negoescu B[3], T. Kuczumow, S. Reich and D. Shoikhet R[1], K. Merryfield and J.D. Stern R[1], R.P. Pant and V. Pant R[1], P.L. Papini R[1], Dan Butnariu and A.N. Iusem B[1]. See also 18.7, 24.22, 24.23.

14.0 Set-theoretical aspects of the common fixed point theory

Let X be a nonempty set and $f, g : X \to X$ be two operators. An element $x^* \in X$ is a common fixed point for the pair f, g if

$$x^* \in F_f \cap F_g.$$

The following remarks are very useful in the common fixed point theory:

Lemma 14.0.1. *Let X be a nonempty set and $f, g : X \to X$ two commuting operators. Then:*

(i) $f(F_g) \subset F_g$ and $g(F_f) \subset F_f$;

(ii) $f(x), g(x) \in I(f) \cap I(g)$.

Lemma 14.0.2. *Let X be a nonempty set and $f_i, g_i : X \to X$, $i \in \{1, 2\}$. If:*

(i) $F_{f_1} = F_{g_1} = \{x^*\}$;

(ii) $f_1 \circ f_2 = f_2 \circ f_1,\ g_1 \circ g_2 = g_2 \circ g_1$,

then:

$$F_{f_2} \cap F_{g_2} \neq \emptyset.$$

Lemma 14.0.3. *Let $f, g : X \to X$ be two operators. If*

$$F_{f\circ g} = F_{g\circ f} = \{x^*\},$$

then

$$F_f \cap F_g = \{x^*\}.$$

14.1 Order-theoretical aspects of the common fixed point theory

Let $(X, \leq)$ be an ordered set and $f, g : X \to X$ two operators. We continue our considerations with the following remarks:

Lemma 14.1.1. *We have*

$$x \leq f(x),\ x \leq g(x),\ \textit{for all } x \in X \ \Rightarrow\ \mathrm{Max}(X) \subset F_f \cap F_g.$$

For example, if (X, d) is a metric space and $\varphi : X \to R_+$ is a functional, then if we put:

$$x \leq_\varphi y \ \Leftrightarrow\ d(x, y) \leq \varphi(x) - \varphi(y),$$

then we obtain a partial order on X.

From Lemma 14.1.1. we have:

Lemma 14.1.2. *Let (X, d) be a metric space and $\varphi : X \to R_+$ be a functional. If*

$$\mathcal{F} := \{f : X \to X |\ d(x, f(x)) \leq \varphi(x) - \varphi(f(x)),\ \textit{for all } x \in X\},$$

then

$$\operatorname{Max}(X, \leq_{\varphi}) \subset \bigcap_{f \in \mathcal{F}} F_f.$$

14.2 Generalized contraction pairs

Let (X, d) be a metric space and $f, g : X \to X$ be two operators. The following metrical conditions appear in some common fixed theorems:

(1) (R. Kannan (1968)) There exists $a \in \left[0, \frac{1}{2}\right[$ such that:

$$d(f(x), g(y)) \leq a[d(x, f(x)) + d(y, g(y))], \text{ for all } x, y \in X;$$

(2) (S. K. Chatterjea (1972)) There exists $a \in \left[0, \frac{1}{2}\right[$ such that:

$$d(f(x), g(y)) \leq a[d(x, g(y)) + d(y, f(x))], \text{ for all } x, y \in X;$$

(3) (I. A. Rus (1973)) There exist $a, b, c \in R_+$, $a + b + c < 1$, such that:

$$d(f(x), g(y)) \leq ad(x, y) + b[d(x, f(x)) + d(y, g(y))]+$$

$$+c[d(x, g(y)) + d(y, f(x))], \text{ for all } x, y \in X;$$

(4) (L. B. Ćirić (1974)) There exist $a \in [0, 1[$, such that:

$$d(f(x), g(y)) \leq a \max\Big\{d(x, y), d(x, f(x)), d(y, g(y)),$$

$$\frac{1}{2}[d(x, g(y)) + d(y, f(x))]\Big\}, \text{ fora ll } x, y \in X;$$

(5) (K. Iseki (1974)) There exists $a, b \in \mathbb{R}_+$, $a < 1$, $b < 1$ such that:

$$d(f(g(x)), g(y)) \leq ad(x, g(y)),$$

and

$$d(g(f(x)), f(y)) \leq bd(x, f(y)),$$

for all $x, y \in X$;

(6) (N. Negoescu (1982)) There exists a function $\varphi : R_+^5 \to R_+$, increasing in every variable, $\varphi(t,t,2t,0,t) < t$, $\varphi(t,t,0,2t,t) < t$, $\varphi(0,t,t,0,0) < t$, $\varphi(0,0,t,t,t) < t$, such that:

$$d(f(x),g(y)) \le \varphi(d(x,f(x)),d(y,g(y)),d(x,g(y)),d(y,f(x)),d(x,y)),$$

for all $x,y \in X$;

(7) (V. Popa (1984)) There exists $a \in [0,1[$ such that:

$$[d(f(x),g(y))]^2 \le a \max\{d(x,f(x))d(y,g(y)),d(x,g(y))d(y,f(x)),$$

$$d(x,f(x))d(x,g(y)),d(y,f(x))d(y,g(y)),d^2(x,y)\},$$

for all $x,y \in X$.

For other generalized contractions pairs see N. Negoescu B[3], B[18], B[20], B[21] and B[22], V. Popa and G. Puiu B[1], I.A. Rus B[67], B[68], B[70], B[71] and B[80]. See also, J. Dugundji and A. Granas R[1], O. Hadžić R[3], M.R. Tasković R[1].

14.3 Basic problems of the metrical common fixed point theory

We present in what follows the basic problems of the metrical common fixed point theory (see I. A. Rus B[70], B[74], B[67], B[26]):

Problem 14.3.1. Find the generalized contraction pair $f,g : X \to X$ such that:

(i) $F_f \cap F_g \neq \emptyset$;

(ii) $F_f = f_g = \{x^*\}$;

(iii) f and g are Bessaga operators;

(iv) f and g are Janos operators, i. e.,

$$card \bigcap_{n\in\mathbb{N}} f^n(X) = card \bigcap_{n\in\mathbb{N}} g^n(X) = 1.$$

Problem 14.3.2. Let (X,d) be a complete metric space, $f,g : X \to X$ be a generalized contraction pair and $f_n, g_n : X \to X$, $n \in \mathbb{N}$ be such that $F_{f_n} \neq \emptyset$, $F_{g_n} \neq \emptyset$, $n \in \mathbb{N}$ and $F_f = F_g = \{x^*\}$.

If $f_n \overset{unif}{\longrightarrow} f$, $g_n \overset{unif}{\longrightarrow} g$, $x_n \in F_{f_n}$, $y_n \in F_{g_n}$, does

$$x_n \to x^*, \quad y_n \to y^* \ ?$$

Problem 14.3.3. Let (X,d) be a complete metric space, Y be a topological space and $f,g : X \times Y \to X$ be two continuous operators. We assume that:

(a) $f(\cdot,y), g(\cdot,y)$ is a generalized contraction pair;

(b) $F_{f(\cdot,y)} = F_{g(\cdot,y)} = \{x_y^*\}$.

We define the operator $P : Y \to X, \quad y \mapsto x_y^*$.

Does the above conditions imply the continuity of the operator P ?

We have:

Theorem 14.3.1. *Let (X,d) be a complete metric space and $f,g : X \to X$ be two mappings for which there exists $a \in \left[0, \frac{1}{2}\right[$ such that:*

$$d(f(x),g(y)) \le a[d(x,f(x)) + d(y,g(y))], \text{ for all } x,y \in X.$$

Then:

(i) (R. Kannan) $F_f = F_g = \{x^*\}$;

(ii) (I.A. Rus, B[26]) the operators f and g are Picard operators.

Theorem 14.3.2. (I.A. Rus, B[67]). *Let f,g be as in the Theorem 14.3.1. If f_n and g_n are as in the Problem 14.3.2., then:*

$$x_n \to x^*, \ y_n \to y^* \text{ as } n \to \infty.$$

Theorem 14.3.3. (I.A. Rus, B[67]). *Let (X,d) be a complete metric space, Y be a topological space and $f,g : X \times Y \to X$. We suppose that:*

(i) there exists $a \in \left[0, \frac{1}{2}\right[$ such that:

$$d(f(x_1,y),g(x_2,y)) \le a[d(x_1,f(x_1,y)) + d(x_2,g(x_2,y))],$$

for all $x_1,x_2 \in X$ and $y \in Y$;

(ii) $f(x,\cdot)$ and $g(x,\cdot)$ are continuous, for all $x \in X$.

Then, the operator P (see Problem 14.3.3.) is continuous.

Remark 14.3.1. For other results in connection with Problem 14.3.1., Problem 14.3.2. and Problem 14.3.3., see I.A. Rus B[67] and B[70], L.B. Ćirić and J.S. Ume R[1], P.L. Papini R[1], R.A. Rashwan and M.A. Ahmed R[2], R.P. Agarwal, J. Dshalalow and D. O'Regan R[2], etc.

Remark 14.3.2. If we have a commuting pair of operators then there are other types of metric conditions which imply the existence of a common fixed point. For such results see N. Negoescu B[1], B[4], V. Popa B[11], B[13], B[16], B[17], B[18], and B[22], V. Popa, H.K. Pathak and V.V.S.N. Lakshmi B[1], B.C. Dhage R[1], R.P. Pant and V. Pant R[1].

14.4 Almost common fixed points of totally nonexpansive families of operators

Let $(\Omega, \mathcal{A}, \mu)$ be a complete probability space, X a separable reflexive Banach space and Y be a nonempty closed, convex subset of X. Let

$$T_\omega : Y \to Y, \quad \omega \in \Omega$$

a measurable family of operators, i. e. the function

$$T_*(x) : \Omega \to X, \quad T_*(x)(\omega) := T_\omega(x),$$

is measurable, for all $x \in X$.

By definition an element $x^* \in Y$ such that:

$$\mu(\{\omega \in \Omega |\ T_\omega(x^*) = x^*\}) = 1,$$

is called an almost common fixed point of family T_ω, $\omega \in \Omega$. The almost common fixed point set of T_ω, $\omega \in \Omega$ is denoted by $(AF)_{T_*}$.

A lower semicontinuous convex function $f : X \to]-\infty, +\infty[$ is called Bregman function on the set $Y \subset Int(Dom(f))$ if, for each $x \in Y$ the following conditions are satisfied:

(i) f is Gâteaux differentiable and totally convex at x;

(ii) For any $\alpha \geq 0$, the set

$$R^f_\alpha(x; Y) := \{y \in Y |\ Df(x, y) \leq \alpha\}$$

is bounded.

The family T_ω, $\omega \in \Omega$ is by definition totally nonexpansive with respect to the Bregman function f on the set Y, if there exists a point $z \in Y$ such that, for each $x \in Y$

$$Df(z, T_\omega(x)) + Df(T_\omega(x), x) \leq Df(z, x), \ \mu - a.e.$$

A point $z \in Y$, as above, is called a nonexpansive pole with respect to f of the family T_ω, $\omega \in \Omega$. The set of all nonexpansivity poles with respect to f of the family T_ω, $\omega \in \Omega$ is denoted by $Nex_f(T_*)$.

For a sequence of complete probability measure $(\mu_k)_{k\in\mathbb{N}}$ on $(\Omega, \mathcal{A})$ and for a sequence $(\lambda_k)_{k\in\mathbb{N}}$, $\lambda_k \in]0, 1]$, bounded away from zero, we consider the operators

$$T_k : Y \to Y, \quad k \in \mathbb{N}$$

given by

$$T_k(x) := (1 - \lambda_k)x + \lambda_k \int_\Omega T_\omega(x) d\mu_k(\omega).$$

The function f satisfies (see D. Butnariu and I. Markowitz, B[1]) the separability requirement on Y if

$$\left.\begin{array}{r} y_k, z_k \in Y, \quad y_k \to y \text{ as } k \to \infty \\ z_k \to z \text{ as } k \to \infty \end{array}\right\} \Rightarrow \lim_{k\to\infty} \inf |f'(y_k) - f'(z_k), y - z| > 0.$$

We have:

Theorem 14.4.1. (D. Butnariu and I. Markowitz, B[1]) *Let T_ω, $\omega \in \Omega$ be a totally nonexpansive measurable family of operators, with respect to the continuously differentiable Bregman function f on Y and we suppose that, for some $z \in Nex_f(T_*)$, the function $D_f(z, \cdot)$ is convex. Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of real numbers such that for some $\lambda > 0$ we have $\lambda_k \in [\lambda, 1]$, for all $k \in \mathbb{N}$. If for all $k \in N$ the complete probability measure μ_k is absolutely continuous with respect to μ and if*

$$\lim_{k\to\infty} \inf \frac{d\mu_k}{d\mu}(\omega) > 0, \ \textit{for } \mu - \textit{almost all } \omega \in \Omega,$$

then, any orbit $(x_k)_{k\in\mathbb{N}}$ of $(T_k)_{k\in\mathbb{N}}$, has the following properties:

(i) The sequence $(x_k)_{k\in\mathbb{N}}$ is bounded, has weak accumulation points and, for μ-almost all $\omega \in \Omega$, we have

$$\lim_{k\to\infty} \inf Df(T_\omega(x_k), x_k) = 0;$$

(ii) If the function $x \mapsto Df(T_\omega(x), x)$ is sequentially weakly lower semicontinuous, for μ-almost all $\omega \in \Omega$, then

(a) any weak accumulation point of $(x_k)_{k\in\mathbb{N}}$ is contained in $(AF)_{T_*}$;

(b) the sequence $(x_k)_{k\in\mathbb{N}}$ converges weakly to a point in $(AF)_{T_*}$ whenever either $(AF)_{T_*} = \{x^*\}$ or $(AF)_{T_*} = Nex_f(T_*)$ and f satisfies the separability requirement.

For other results of this type see D. Butnariu and A.N. Iusem B[1], B[2], D. Butnariu and I. Markowitz B[1].

14.5 Multivalued operators

Let $T, S : X \multimap X$ be two multivalued operators. Some basic problems of the common fixed point theory are the following: in which conditions we have:

(i) $F_T \cap F_S \neq \emptyset$;

(ii) $(SF)_T \cap (SF)_S \neq \emptyset$;

(iii) $F_T = F_S \neq \emptyset$;

(iv) $(SF)_T = (SF)_S \neq \emptyset$;

(v) $(SF)_T = (SF)_S = \{x^*\}$;

(vi) $F_T = (SF)_T = F_S = (SF)_S = \{x^*\}$;

(vii) $F_T \cup F_S \neq \emptyset$.

There are some metrical results for the above problems. As an example we have:

Theorem 14.5.1. (M. Avram, B[1]). *Let (X, d) be a complete metric space and let S, T be two multivalued operators from X to $P_{b,cl}(X)$, for which there exist positive numbers a, b, and c with $a + 2b + 4c < 1$ such that:*

$$\delta(T(x), S(y)) \leq ad(x, y) + b[\delta(x, T(x)) + \delta(y, S(y))]+$$

$$+c[\delta(x, S(y)) + \delta(y, T(x))], \text{ for all } x, y \in X.$$

Then $F_T = F_S = \{x^*\}$.

Theorem 14.5.2. (A. Sîntămărian, B[4]). *Let (X, d) be a complete metric space and $S, T : X \to P(X)$ be two multivalued operators. We suppose that there exist $a_1, \ldots, a_5 \in \mathbb{R}_+$, with $a_3 + a_4 < 1$ such that for each $x \in X$, any $u_x \in S(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ such that:*

$$d(u_x, u_y) \leq a_1 d(x, y) + a_2 d(x, u_x) + a_3(y, u_y) + a_4(d, u_y) + a_5 d(y, u_x).$$

Then $F_S \subset F_T$.

Theorem 14.5.3. (A. Sîntămărian, B[4]). *Let (X, d) be a complete metric space and $S, T : X \to P_{cl}(X)$ be two multivalued operators. We suppose that there exists $a \in R_+$, with $a < 1$, such that for each $x \in X$, any $u_x \in S(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ such that:*

$$d(u_x, u_y) \leq a \max\left\{d(x, y), d(x, u_x), d(y, u_y), \frac{1}{2}[d(x, u_y) + d(y, u_x)]\right\}.$$

Then $F_S = F_T \in P_{cl}(X)$.

Theorem 14.5.4. (A. Sîntămărian, B[2]). *Let (X, d) be a complete metric space and $S, T : X \to P_{cl}(X)$ be two multivalued operators for which there exists $a \in \left[0, \frac{1}{2}\right[$ such that:*

$$H(S(x), T(y)) \leq a[D(x, S(x)) + d(y, T(y))], \text{ for all } x, y \in X.$$

Then:

(a) $F_T = F_S \in P_{cl}(X)$;

(b) the operators S, T are MWP operators.

For other results see A. Muntean B[1], B[2], N. Negoescu B[3], B[9] and B[23], V. Popa B[27], B[30], B[32] and B[33], I.A. Rus B[18] and B[4], A. Sîntămărian B[2] and B[4], R.P. Agarwal, D. O'Regan and N.S. Papageorgiou R[1], L.B. Ćirić R[4], etc.

Chapter 15

Coincidence point theory

Guidelines: S. Lefschetz (1923), F. Fuller (1954), A. Tarski (1955), H. Schirmer (1955), K. Fan (1961), W. Holsztynski (1964), E. Fadell (1965), R.F. Brown (1968), K. Goebel (1968), F.E. Browder (1968), H. Schirmer (1970), W.A. Horn (1970), J. Mawhin (1972), J. Peetre and I.A. Rus (1973), S. Kasahara (1975), S. Reich (1975), J. Dugundji (1976), J. Mawhin and K. Schmidt (1976), L. Cesari (1977).
General references: R.F. Brown, M. Furi, L. Górniewicz and B. Jiang R[1], L. Gorniewicz and A. Granas R[1], R. Sine R[1], V.G. Angelov R[1], M. Furi, M. Martelli and A. Vignoli R[1], R. Gaines and J. Mawhin R[1], G. Isac B[2], I.A. Rus B[73], B[23], A. Buică B[2], W. Kulpa R[1], Z. Liu, S.J. Ume and M.S. Khan R[1], R.P. Agarwal and D. O'Regan R[1], C. Mortici B[3], A. Petruşel B[20] and B[19], A. Muntean and A. Petruşel B[1], Q H. Ansari, A. Idzik and J.-C. Yao R[1], V. Sadoveanu B[1]. See also Chapter 18.8 and Chapter 17.4.

15.0 $C(f,g)$ and $F_{g_l^{-1}\circ f}$

Let X and Y be two nonempty sets and $f,g : X \to Y$ be two operators. We denote by

$$C(f,g) := \{x \in X|\ f(x) = g(x)\},$$

the coincidence point set of the operators f and g.

We suppose that the operator g is injective. Then g has a left-inverse

$$g_l^{-1} : g(X) \to X.$$

If we suppose that $f(X) \subset g(X)$, then

$$C(f,g) = F_{g_l^{-1} \circ f}.$$

Thus, every fixed point for $g_1^{-1} \circ f$ will be a coincidence point for f and g. For example we have:

Theorem 15.0.1. (I.A. Rus, B[23]). *Let $(X, \leq)$ be a complete lattice, $(Y, \leq)$ be an ordered set and $f, g : X \to Y$ two operators. We suppose that:*

(i) f is increasing operator;

(ii) $f(X) \subset g(X)$;

(iii) g is injective operator;

(iv) $g(x_1) < g(x_2)$ implies $x_1 < x_2$.

Then

$$C(f,g) \neq \emptyset.$$

Theorem 15.0.2. (I.A. Rus, B[23]). *Let (X, d) be a complete metric space, (Y, ρ) be a metric space and $f, g : X \to Y$ be two operators. We suppose that:*

(i) $f(X) \subset g(X)$;

(ii) there exists $a_1 > 0$ such that

$$\rho(g(x_1), g(x_2)) \geq a_1 d(x_1, x_2), \quad \text{for all } \ x_1, x_2 \in X$$

(iii) there exists $a_2 > 0$ such that

$$\rho(f(x_1), f(x_2)) \leq a_2 d(x_1, x_2), \quad \text{for all } \ x_1, x_2 \in X;$$

(iv) $a_2 a_1^{-1} < 1$.

Then

$$C(f,g) = \{x^*\}.$$

For other results of this type see I.A. Rus B[23], A. Buică B[2] and B[4].

15.1 $C(f,g)$ and $F_{f\circ g_r^{-1}}$

Let X and Y be two sets, and $f, g : X \to Y$ two operators. We consider the following multivalued operator

$$T : X \multimap X, \quad x \multimap \{z \in X |\ f(x) = g(z)\}$$

It is clear that

$$C(f,g) = F_T$$

If g is surjective and g_r^{-1} denotes a right-inverse of g then

$$g_r^{-1}(F_{f\circ g_r^{-1}}) \subset C(f,g).$$

Hence, if $F_{f\circ g_r^{-1}} \neq \emptyset$, then $C(f,g) \neq \emptyset$.

For example, we have:

Theorem 15.1.1. (I.A. Rus, B[23]). *Let X be a set and $(Y, \leq)$ be an inductive ordered set. Let $f, g : X \to Y$ be two operators. We suppose that:*

(i) the operator g is surjective;

(ii) $f(x) \geq g(x)$, for all $x \in X$.

Then

$$C(f,g) \neq \emptyset.$$

Theorem 15.1.2. (I.A. Rus, B[23]). *Let X be a set and (Y, ρ) a complete metric space. Let $f, g : X \to Y$ be two operators. We suppose that:*

(i) the operator g is surjective;

(ii) there exists a lower semicontinuous function $\varphi : Y \to R_+$, such that:

$$d(f(x), g(x)) \leq \varphi(g(x)) - \varphi(f(x)), \quad \textit{for all} \ \ x \in X.$$

Then

$$C(f,g) \neq \emptyset.$$

For other results of this type see I. A. Rus B[23], A. Buică B[2].

15.2 Data dependence

Let (f_i, g_i), $i \in \{1,2\}$, $f_i, g_i : (X,d) \to (Y,\rho)$, be such that

$$d(f_i(x), g_i(x)) \leq \eta_i, \text{ for all } x \in X, \ i \in \{1,2\}.$$

If

$$C(f_1, g_1) = \{x^*\} \quad \text{and} \quad y^* \in C(f_2, g_2),$$

the problem is to estimate $d(x^*, y^*)$.

We have:

Theorem 15.2.1. (A. Buică, B[4]). *Let (X,d) be a complete metric space, (Y,ρ) be a metric space and $f_i, g_i : X \to Y$, $i \in \{1,2\}$. We suppose that:*

(i) the pair (f_1, g_1) is as in Theorem 15.0.2.

(ii) $C(f_2, g_2) \neq \emptyset$. Then

$$d(x^*, y^*) \leq \frac{\eta_1 + \eta_2}{a_1 - a_2}.$$

Theorem 15.2.2. (F. Aldea and A. Buică, B[1]). *Let (X,d) be a complete metric space, (Y,ρ) be a metric space and $f_i, g_i : X \to Y$, $i \in \{1,2\}$. We suppose that:*

(i) there exist $0 < \alpha < 1$, $K > 0$ and $\psi : X \to X$ such that

$$d(x, \psi(x)) \leq K\rho(f_1(x), g_1(x)) \text{ and } \rho(f(\psi(x)), g(\psi(x))) \leq$$

$$\leq \alpha\rho(f(x), g(x)), \quad \text{for all } x \in X;$$

(ii) $z^* \in C(f_2, g_2)$;

(iii) f_1, g_1 are continuous operators.

Then:

(a) $C(f_1, g_1) = \{x^*\}$

(b) $d(x^*, z^*) \leq \dfrac{K}{1-\alpha}(\eta_1 + \eta_2)$.

For other results on data dependence of the coincidence points see F. Aldea and A. Buică B[1] and B[2], A. Buică B[2] and B[4].

15.3 Nearness and coincidence

A nice coincidence result was proved by K. Goebel in 1968, as follows.

Theorem 15.3.1. (K. Goebel R[4]) *Let A be an arbitrary set and (X,ρ) be a metric space. Suppose that $S,T : A \to X$ are operators such that $T(A)$ is complete, $S(A) \subset T(A)$ and $\rho(S(x),S(y)) \leq k\rho(T(x),T(y))$ for some constant $k < 1$ and all $x,y \in A$.*

Then:

(1) There exists $\overline{x} \in A$ such that $S(\overline{x}) = T(\overline{x})$;

(2) if $S(\overline{x}) = T(\overline{x}) = T(x)$ then $S(x) = T(x)$;

(3) if $S(\overline{x}) = T(\overline{x})$ and $S(\overline{y}) = T(\overline{y})$ then $T(\overline{x}) = T(\overline{y})$.

Let us notice that the result is obtained by applying the Banach contraction principle to the mapping $H = S \circ T^{-1}$.

Let X be a normed space and Y be a Banach space. By definition (see M. Furi, M. Martelli and A. Vignoli R[1]), a continuous operator $f : X \to Y$ is called:

(1) strong surjection if $C(f,g) \neq \emptyset$ for any continuous $g : X \to Y$, with $\overline{h(X)}$ a compact set;

(2) stable solvable if $C(f,g) \neq \emptyset$ for any completely continuous operator $g : X \to Y$ with quasinorm $|g| = 0$.

Following S. Campanato R[1], we say that $g : X \to Y$ is near $f : X \to Y$ if there exist $\alpha > 0$, $K \in]0,1[$ such that:

$$\|f(x_1) - f(x_2) - \alpha[g(x_1) - g(x_2)]\| \leq K\|f(x_1) - f(x_2)\|$$

for all $x_1, x_2 \in X$.

We have:

Theorem 15.3.2. (A. Buică, R[7]). *Let $g : X \to Y$ be near $f : X \to Y$. If f is a strong surjection, then g is also a strong surjection.*

Theorem 15.3.3. (A. Buică, R[7]). *Let $g : X \to Y$ be near $f : X \to Y$. If f is stable solvable, then g is also stable solvable.*

15.4 Coincidence point theory via Picard operators

Let X be a nonempty set and $(Y, d, \leq)$ be an ordered metric space. Let $f, g : X \to Y$ be two operators. In what follow we consider the coincidence equation

$$f(x) = g(x) \tag{1}$$

and the iterative scheme

$$f(x_{n+1}) = g(x_n). \tag{2}$$

The operator g is Picard w.r.t f, if there exists a unique $y^* \in Y$ with the following properties:
(i) there exists $x^* \in X$ such that $f(x^*) = g(x^*) = y^*$;
(ii) $g(X) \subset f(X)$;
(iii) for every $x_0 \in X$ a sequence defined by (2) is such that

$$f(x_n) \to y^* \text{ as } n \to \infty.$$

We have:
Theorem 15.4.1. (A. Buică, R[6]). *We suppose that:*
(i) $f(x_1) \leq f(x_2),\ x_1, x_2 \in X \ \Rightarrow\ g(x_1) \leq g(x_2)$*;*
(ii) g *is Picard w.r.t.* f*.*
Then:
(a) $f(x_0) \leq g(x_0)$ *implies* $f(x_0) \leq y^*$*;*
(b) $f(x_0) \geq g(x_0)$ *implies* $f(x_0) \geq y^*$*.*
If, in addition, $(X, \leq)$ *is an ordered set and* $f(x_1) \leq f(x_2)$ *implies* $x_1 \leq x_2$*, then*
(c) $f(x_0) \leq g(x_0)$ *implies* $x_0 \leq x^*$*;*
(d) $f(x_0) \geq g(x_0)$ *implies* $x_0 \geq x^*$*.*
Remark 15.4.1. The above theorem extends an abstract Gronwall lemma of I.A. Rus B[14].

Remark 15.4.2. For some applications of Theorem 15.4.1. to monotone iterative technique for coincidence equations, see A. Buică R[6].

15.5 Coincidence point theory on convex cones

Let $(X, \tau, \leq)$ be a locally convex space ordered by a closed convex cone $K \subset X$.

Let $G, \Lambda : X \to X$ be two operators and $Y \subset X$ be a nonempty set.

We have:

Theorem 15.5.1. (G. Isac, B[3]). *Let X be a metrizable locally convex space ordered by a normal closed convex cone $K \subset X$.*

Let $Y \subset X$ be a closed subset and let $f : Y \to Y$ such that:

(i) all sequences of the form

$$f(x_1) \geq \cdots \geq f(x_n) \geq \dots$$

contains a convergent subsequence;

(ii) there exists $x_0 \in Y$ such that $f(x_0) \leq x_0$.

Then

$$F_f \neq \emptyset.$$

From this fixed point theorem we have the following coincidence result:

Theorem 15.5.2. (G. Isac, B[3]). *Let X be a metrizable locally convex space ordered by a normal closed convex cone $K \subset X$. Let $Y \subset X$ be a closed subset, $f : Y \to X$ and $g, \Lambda : X \to X$.*

We suppose that:

(i) there exists $(g + \Lambda)^{-1}$ and is monotone increasing;

(ii) $f + \Lambda$ is monotone increasing;

(iii) $(g + \Lambda)^{-1}(f + \Lambda)(Y) \subset Y$;

(iv) all sequences of the form

$$(g + \Lambda)^{-1}(f + \Lambda)(x_1) \geq \cdots \geq (g + \Lambda)^{-1}(f + \Lambda)(x_n) \geq \dots$$

contains a convergent subsequence;

(v) there exists $x_0 \in Y$ such that $f(x_0) \leq g(x_0)$.

Then

$$C(f, g) \neq \emptyset.$$

For other results see G. Isac B[3] and B[16].

15.6 Coincidence point theory for multivalued operators

Let X, Y be two nonempty set and $A, B : X \multimap Y$ be two multivalued operators. An element $x \in X$ is a coincidence point of the pair A, B if

$$A(x) \cap B(x) \neq \emptyset.$$

We denote by

$$C(A, B) := \{x \in X |\ A(x) \cap B(x) \neq \emptyset\}$$

the coincidence point set of the pair A, B.

One of the basic coincidence point theorem for multivalued operators, with several applications to mathematical economics, is the well-known Ky Fan coincidence theorem, established in 1966. We present here a proof based on K^2M operator technique.

We start this section by presenting the concept of K^2M operator.

Let X a vector space over $\mathbb{R}$. A subset A of X is called a linear subspace if for all $x, y \in A$ $x + y \in A$ and for all $x \in X$ and each $\lambda \in \mathbb{R}$ we have that $\lambda \cdot x \in A$. If A is a nonempty subset of X, then $spanA$ is, by definition, the intersection of all subspaces which contains A, i. e. the smallest linear subspace containing A. We have the following characterization of the span.

$$spanA = \{x \in X | x = \sum_{i=1}^{n} \lambda_i \cdot x_i, \text{ with } x_i \in A, \lambda_i \in \mathbb{R}, n \in \mathbb{N}\}.$$

Also, a k-dimensional flat (or a k-dimensional linear variety) in X is a subset L of X with $dimL = k$ such that for each $x, y \in L$, with $x \neq y$, the whole line joining x and y is included in L, i. e. $(1 - \lambda) \cdot x + \lambda \cdot y \in L$, for each $\lambda \in \mathbb{R}$.

Definition 15.6.1. A subset A of a vector space X is said to be finitely closed if its intersection with any finite-dimensional flat $L \subset X$ is closed in the Euclidean topology of L.

Obviously if X is a vector topological space then any closed subset of X is finitely closed.

Definition 15.6.2. A family $\{A_i|\ i \in I\}$ of sets is said to have the finite intersection property if the intersection of each finite subfamily is not empty.

Definition 15.6.3. Let X be a vector space and Y a nonempty subset of X. The multifunction $G : Y \to P(X)$ is called a Knaster-Kuratowski-Mazurkiewicz operator (briefly K^2M operator) if and only if

$$co\{x_1, \dots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i),$$

for each finite subset $\{x_1, \dots, x_n\} \subset Y$.

The main property of K^2M operators is given in:

K^2M **Abstract Principle.** *Let X be a vector space, Y be a nonempty subset of X and $G : Y \to P(X)$ be a K^2M operator, such that $G(x)$ is finitely closed, for each $x \in Y$. Then, the family $\{G(x)|\ x \in Y\}$ of sets has the finite intersection property.*

Proof. We argue by contradiction: assume that there exist $\{x_1, \dots, x_n\} \subset X$ such that $\bigcap_{i=1}^{n} G(x_i) = \emptyset$. Denote by L the finite dimensional flat spanned by $\{x_1, \dots, x_n\}$, i.e. $L = span\{x_1, \cdots, x_n\}$. Let us denote by d the Euclidean metric in L and by $C := co\{x_1, \dots, x_n\} \subset L$.

Because $L \cap G(x_i)$ is closed in L, for all $i \in \{1, 2, \dots, n\}$ we have that:

$$D_d(x, L \cap G(x_i)) = 0 \ \Leftrightarrow \ x \in L \cap G(x_i), \text{ for all } i \in \{1, \cdots, n\}.$$

Since $\bigcap_{i=1}^{n} [L \cap G(x_i)] = \emptyset$ it follows that the map $\lambda : C \to \mathbb{R}$ given by

$$\lambda(c) = \sum_{i=1}^{n} D_d(c, L \cap G(x_i)) \neq 0, \text{ for each } c \in C.$$

Hence we can define the continuous map $f : C \to C$ by the formula

$$f(c) = \frac{1}{\lambda(c)} \sum_{i=1}^{n} D_d(c, L \cap G(x_i)) x_i.$$

By Brouwer's fixed point theorem there is a fixed point $c_0 \in C$ of f, i. e. $f(c_0) = c_0$. Let

$$I = \{i|\ D_{d_E}(c_0, L \cap G(x_i)) \neq 0\}.$$

Then for $i \in$ we have $c_0 \notin L \cap G(x_i)$ which implies

$$c_0 \notin \bigcup_{i \in I} G(x_i).$$

On the other side:

$$c_0 = f(c_0) \in co\{x_i |\ i \in I\} \subset \bigcup_{i \in I} G(x_i)$$

(last inclusion follows from the K^2M assumption of G). This is a contradiction. □

As an immediate consequence we obtain the following theorem:

Corollary 15.6.1. (Ky Fan) *Let X be a vector topological space, Y a nonempty subset of X and $G : Y \to P_{cl}(X)$ a K^2M operator. If at least one of the sets $G(x)$, $x \in Y$ is compact, then*

$$\bigcap_{x \in Y} G(x) \neq \emptyset.$$

We can present now, the Ky Fan coincidence theorem.

Ky Fan Coincidence Theorem. (Ky Fan R[4]) *Let E, F be vector topological spaces and $X \in P_{cp,cv}(E)$, $Y \in P_{cp,cv}(F)$. Let $A, B : X \to \mathcal{P}(Y)$ two multivalued operators satisfying the following assumptions:*

i) $A(x) \in \mathcal{P}_{op}(Y)$ and $B(x) \in P_{cv}(Y)$, for each $x \in X$;

ii) $A^{-1}(y) \in P_{cv}(X)$ and $B^{-1}(y) \in \mathcal{P}_{op}(X)$, for each $y \in Y$.

Then there exists an element $x_0 \in X$ such that $A(x_0) \bigcap B(x_0) \neq \emptyset$, i. e. $C(A, B) \neq \emptyset$.

Proof. Let $Z = X \times Y$ and $G : X \times Y \to \mathcal{P}(E \times F)$ be given by

$$G(x, y) = Z - (B^{-1}(y) \times A(x)).$$

Because $G(x, y) \in P_{cl}(X \times Y)$ and $X \times Y$ is compact we get that $G(x, y) \in P_{cp}(X \times Y)$.

It is easy to observe that:

$$Z = \bigcup \{B^{-1}(y) \times A(x) |\ (x, y) \in Z\} \qquad (1).$$

Indeed, let $(x_0, y_0) \in Z$ be arbitrarily. Choose an $(x, y) \in A^{-1}(y_0) \times B(x_0) \neq \emptyset$ which is equivalent with $(x_0, y_0) \in B^{-1}(y) \times A(x)$. Thus from (1) we have:

$$\bigcap_{z \in Z} G(z) = \emptyset.$$

From the Corollary of K^2M principle G cannot be a K^2M operator. Hence there exist $z_1, z_2, \ldots, z_n \in Z$ such that

$$co\{z_1, \ldots, z_n\} \not\subset \bigcup_{i=1}^{n} G(z_i),$$

which means that there is a $w \in co\{z_1, \ldots, z_n\}$,

$$w = \sum_{i=1}^{n} \lambda_i z_i$$

with

$$w \notin \bigcup_{i=1}^{n} G(z_i).$$

Because Z is convex and $z_i \in Z$, for each $i \in \{1, \cdots, n\}$ we obtain that $w \in Z$. Hence:

$$w \in Z - \bigcup_{i=1}^{n} G(z_i) = \bigcap_{i=1}^{n} (B^{-1}(y_i) \times A(x_i)).$$

How

$$w = \left(\sum_{i=1}^{n} \lambda_i x_i, \sum_{i=1}^{n} \lambda_i y_i \right)$$

it follows that

$$\sum_{i=1}^{n} \lambda_i x_i \in B^{-1}(y_i)$$

and

$$\sum_{i=1}^{n} \lambda_i y_i \in A(x_i), \text{ for each } i \in \{1, \cdots, n\}.$$

Successively we have:

$$y_i \in B\left(\sum_{i=1}^{n} \lambda_i x_i \right) \text{ and } x_i \in A^{-1}\left(\sum_{i=1}^{n} \lambda_i y_i \right), \text{ for each } i \in \{1, \cdots, n\} \Rightarrow$$

$$\sum_{i=1}^{n} \lambda_i y_i \in B\left(\sum_{i=1}^{n} \lambda_i x_i\right) \text{ and } \sum_{i=1}^{n} \lambda_i x_i \in A^{-1}\left(\sum_{i=1}^{n} \lambda_i y_i\right) \Rightarrow$$
$$\sum_{i=1}^{n} \lambda_i y_i \in B\left(\sum_{i=1}^{n} \lambda_i x_i\right) \text{ and } \sum_{i=1}^{n} \lambda_i y_i \in A\left(\sum_{i=1}^{n} \lambda_i x_i\right).$$

Writing $x_0 = \sum_{i=1}^{n} \lambda_i x_i$ we got that $A(x_0) \cap B(x_0) \neq \emptyset$ and hence $C(A, B) \neq \emptyset$. □

For some related results to Ky Fan Coincidence Theorem, see M. Balaj B[18], H. Ben-El-Mechaiekh and R. Dimand R[1], L. Deng anf M.G. Yang R[1], J.S. Jung R[1], A. Muntean B[7], D. O'Regan R[2], R[4], J. Guillerme R[1], etc.

We present now a result given by A. Petruşel.

Let (X, d) and (Y, ρ) be two metric spaces. Two multivalued operators S, T of X into Y are said to be pM-proximate if there exist increasing functions $\varphi, \psi : R_+ \to R_+$ and $M > 0$ satisfying the following conditions:

(i) $\varphi^n(t) \to 0$ as $n \to \infty$ and $\sum_{i=1}^{\infty} \psi(\varphi^i(t)) < +\infty$;

(ii) there exists $x \in X$ such that

$$D(S(x), T(x)) \leq M;$$

(iii) there exists an operator $p : X \to X$ such that

$$d(x, p(x)) \leq \psi(M) \text{ and } D(S(p(x)), T(p(x))) \leq \varphi(M), \text{ for all } x \in X.$$

Theorem 15.6.1. (A. Petruşel, B[19]). *Let $S, T : X \to P(Y)$ be pM-proximate multivalued operators of a complete metric space (X, d) into a metric space (Y, ρ). If T is u.s.c. and $T(x)$ is compact for each $x \in X$, then there exist $a \in X$, $b \in T(a)$, sequences $(x_n)_{n \in N}$, $(y_n)_{n \in \mathbb{N}}$ such that*

$$x_n \to a,\ y_n \to b \text{ as } n \to \infty, \text{ and } y_n \in S(x_n), \text{ for all } n \in \mathbb{N}.$$

If, in addition, $Graph(S)$ is closed, then $S(a) \cap T(a) \neq \emptyset$.

Remark 15.6.1. The above theorem generalizes a result by J. Peetre and I.A. Rus (see I.A. Rus B[81]).

The following results are in topological vector spaces.

Theorem 15.6.2. (A. Muntean and A. Petruşel, B[2]). *Let X be a nonempty convex subset of a locally convex Hausdorff topological space E, D a nonempty set of a topological vector space Y. If $S : D \to P(X)$ and $T : X \to P(D)$ are such that:*

(i) S is l.s.c.;

(ii) $S(y) \in P_{clc,cv}(X)$;

(iii) $Q(x) := coT(x)$ is a subset of D;

(iv) $S(D) \subset C$, where C is a compact metrizable subset of X;

(v) for each $x \in X$ there exists $y \in D$ such that $x \in intQ^{-1}(y)$.

Then, there exist $\overline{x} \in X$ and $\overline{y} \in D$ such that

$$\overline{x} \in S(\overline{y}) \text{ and } \overline{y} \in Q(\overline{x}).$$

Theorem 15.6.3. (A. Muntean and A. Petruşel, B[2]). *Let X be a nonempty convex compact a metrizable subset of locally convex Hausdorff topological vector space E, D be a nonempty subset of a topological vector space Y and $S, T : D \to P(X)$ be such that:*

(i) S is l.s.c.;

(ii) $S(y) \in P_{cl,cv}(X)$, for all $y \in D$;

(iii) $T^{-1}(x)$ is a nonempty convex subset of D for each $x \in X$;

(iv) $T(y)$ is open in X for each $y \in D$.

Then, there exists $\overline{y} \in D$ such that

$$S(\overline{y}) \cap T(\overline{y}) \neq \emptyset.$$

Remark 15.6.1 Theorem 15.6.2. and Theorem 15.6.3. are in connections with some results given by F.E. Browder (1968), S. Sessa (1988), S. Sessa and G. Mehta (1995) and X. Wu (1997).

A topological coincidence result was proved by H. Schirmer R[3].

Theorem 15.6.4. *Let X be a compact Hausdorff space, Y be a tree (i.e. a continuum in which every pair of distinct points is separated by a third), $T : X \to P(Y)$ be an u.s.c multivalued operator and $S : X \to P(Y)$ be either* *continuous or u.s.c. with connected values. Then,* $C(F,S) \neq \emptyset$*, provided* T *is either open or the set* $T^{-1}(y)$ *is connected for every* $y \in Y$*.*

An interesting approach is based on the following lemma.

Let $S : X \to P(Y)$ and $G : X \to P(Y)$ be two multivalued operators. Suppose that G is onto and consider

$$T : X \times Y \to P(X \times Y), \text{ defined by } T(x,y) := G^{-1}(y) \times S(x),$$

where $G^{-1}(y) := \{x \in A | y \in G(x)\}$. Then, the following lemma holds:

Lemma 15.6.1. *The following statements are equivalent:*

(i) $F_T \neq \emptyset$;

(ii) $F_{S \circ G^{-1}} \neq \emptyset$;

(iii) $F_{G^{-1} \circ S} \neq \emptyset$;

(iv) $C(S,G) \neq \emptyset$.

The following result is well-known, see for example Granas and Dugundji R[1], pp. 543.

Theorem 15.6.5. *Let* X *be a convex subset of a locally convex metrizable space and* $T : X \to P_{ac}(X)$ *an acyclic multivalued operator, such that* $\overline{T(X)}$ *is compact. Then* T *has a fixed point.*

Recall that, a topological space W is said to be acyclic if $\widetilde{H}^n(W) = \{0\}$, for every $n \geq 0$, where $\widetilde{H}^n$ stands for the reduced Čech cohomology with coefficients in $\mathbb{Q}$. Convex or star-shaped sets are simple examples of acyclic sets. For more details about acyclicity see J. Andres and L. Górniewicz R[2] and the references therein.

Let E be a Banach space. Let A and B be nonempty subsets of E. A multivalued operator $T : A \to P(B)$ is said to be acyclic if T is u.s.c. and for each $x \in A$ the values $T(x)$ are acyclic sets in B. If A and B are acyclic sets then $A \times B$ is acyclic too.

A multivalued operator $G : A \to P_{cl}(B)$ is said to be proper if $G^{-}(K) := \{x \in A | G(x) \cap K \neq \emptyset\}$ is a compact set, whenever K is compact. Assume that G is continuous and onto. Denote by $G^{-1} : B \to P(A)$ the inverse of G. Since G is continuous and proper we have that $G(V)$ is a closed set for each closed set $V \subset A$. Hence G^{-1} is u.s.c. on B.

Next we present a coincidence theorem for multivalued operators with non-convex values, see R. Espínola, G. López and A. Petruşel B[1].

Theorem 15.6.6. *Let E be a Banach space and let X and Y be convex subsets of E. Let $S : X \to P_{ac}(Y)$ be u.s.c. and compact and let $G : X \to P_{cl}(Y)$ be a continuous onto and proper multivalued operator. Suppose that $G^{-1}(y)$ is an acyclic set for each $y \in Y$. Then $C(S,G) \neq \emptyset$.*

Proof. Consider the multivalued operator $T : X \times S(X) \to P(X \times S(X))$ given by $T(x,y) := G^{-1}(y) \times S(x)$. Then T is acyclic and compact. Theorem 15.6.5. implies the existence of at least one fixed point $(x^*, y^*) \in X \times S(X)$. From Lemma 15.6.1., the conclusion follows. □

For other results on this topic see A. Granas and F.C. Liu R[1], D. O'Regan R[2]-R[3], K. Włodarczyk, D. Klim R[1], H. Ben-El-Mechaiekh R[1]-R[2].

15.7 Other results

For other results in coincidence point theory, see F.E. Browder (Ed.) R[1], R.F. Brown, M. Furi, L. Górniewicz and B. Jiang R[1], A. Buică B[2], B[7], J. Dugundji R[1], M. Furi, M. Martelli and A. Vignoli R[1], R. Gaines and J. Mawhin R[1], A. Granas and J. Dugundji R[1], J.K. Hale and J. Mawhin R[1], W. Holsztynski R[1], C. Horvath R[1], S. Kasahara R[3], D. O'Regan and N. Shahzad R[1], S. Park R[2], etc.

Chapter 16

Topological degree theory

Precursors: C.F. Gauss (1799), C.F. Sturm (1829), L. Kronecker (1869), H. Poincaré (1892), P. Bohl (1904), J. Hadamard (1910).
Guidelines: L.E.J. Brouwer (1912), J. Leray and J. Schauder (1934), J. Cronin-Scanlon (1950), M. Nagumo (1951), M.A. Krasnoselskii (1956), I. Bernstein and A. Halanay (1956), M.A. Krasnoselskii and A.I. Perov (1958), A. Granas (1959), E. Heinz (1959), M. Hukuhara (1967), F.E. Browder and R. Nussbaum (1968), G.M. Vainikko and B.N. Sadovskii (1968), F.E. Browder and W.V. Petryshyn (1969), A. Cellina and A. Lasota (1969), J. Mawhin (1972), T. O'Neil and J.W. Thomas (1972), I.V. Skrypnik (1973).
References: M.A. Krasnoselskii R[3], M.A. Krasnoselskii, A.I. Perov, A.I. Povolockii and P.P. Zabrejko R[1], T. van der Walt R[1], M.A. Krasnoselskii and P.P. Zabrejko R[1], N.G. Lloyd R[1], J.W. Milnor R[1], R.E. Gaines and J. Mawhin R[1], J. Cronin R[1], S. Sburlan B[1], Gh. Marinescu R[1], W.V. Petryshyn R[2], D. O'Regan, Y.J. Cho and Y.-Q. Chen R[1], A. Granas and J. Dugundji R[1], K. Deimling R[4], R.D. Nussbaum R[2], J.T. Schwartz R[1], M. Berger R[1], P.P. Zabrejko R[2], W. Forster R[1] (article by H.W. Siegberg), H.W. Siegberg R[1], D. Pascali and S. Sburlan R[1], F.E. Browder R[6], I.A. Rus B[73], C. Vladimirescu and C. Avramescu B[1]. For homological theory of topological degree see R.F. Brown, M. Furi, L. Górniewicz and B. Jiang (Eds.) R[1], A. Dold R[2], R.F. Brown R[1] and R[5], L. Górniewicz R[1], R[2] and R[3], S. Eilenberg and N.E. Steenrod R[1], W. Krawcewicz and J. Wu R[1], M.

Efendiev, I. Fonseca and W. Gangbo R[1], A.J. Homburg and W.L. Wendland R[1].

16.0 Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^n$ be in $C^1(\Omega, \mathbb{R}^n)$. We denote by $J_f(x)$ the Jacobian matrix, $\left(\dfrac{\partial f_i(x)}{\partial x_j}\right)_n^n$, of f. By definition a point $x \in \Omega$ is a critical point of f if $\det J_f(x) = 0$, and is regular if $\det J_f(x) \neq 0$. We have

Sard's Lemma. *Let $f \in C^1(\Omega, \mathbb{R}^n)$ and G an open set such that $\overline{G} \subset \Omega$. If*

$$B := \{x \mid \det J_f(x) = 0,\ x \in G\}$$

then $mes(f(B)) = 0$.

Let $y \in \mathbb{R}^n$ be such that:

(a) $y \notin f(\partial\Omega)$;

(b) each $x \in f^{-1}(y)$ is a regular element.

In this case from inverse function theorem the set $f^{-1}(y)$ is finite.

Let $f \in C(\overline{\Omega}, \mathbb{R}^n)$ and $y \in \mathbb{R}^n$ such that $y \notin f(\partial\Omega)$. In this chapter we shall define a functional $\deg(f, \Omega, y) \in Z$ which is "continuous" with respect to f, Ω and y and if $\deg(f, \Omega, y) \neq 0$ then $f^{-1}(y) \neq \emptyset$, i.e., equation $f(x) = y$ has at least a solution.

Let X and Y be two Banach spaces. By definition, a linear operator $f : Dom(f) \subset X \to Y$ (with $Kerf := f^{-1}(0)$ and $Imf := f(Dom(f)))$ is called a Fredholm operator if the following two conditions hold:

(i) $Kerf$ has finite dimension;

(ii) Imf is a closed subset of Y and it has finite codimension, where the codimension $codim(Imf) := dim(Y/Imf)$.

If f is a Fredholm operator, then by definition, the number $dim(Kerf) - codim(Imf)$ is called the index of f and it is denoted by $Ind(f)$.

16.1 Brouwer's degree

We shall define the topological degree in $\mathbb{R}^n$, in three steps.

(i). Let $f \in C^1(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$, $y \notin f(\partial\Omega)$ and $J_f(x) \neq 0$, for all $x \in f^{-1}(y)$. Then by definition the degree of f with respect to y in Ω is

$$\deg(f, \Omega, y) := \sum_{x \in f^{-1}(y)} sign \det J_f(x).$$

(ii). The case in which y is a critical value of f, i.e., there exists $x \in f^{-1}(y)$, such that $\det J_f(x) = 0$.

Let $f \in C^1(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ and $y \notin f(\partial\Omega)$, $m \in \mathbb{N}$, a sequence of regular values of f such that

$$y_m \to y \text{ as } m \to \infty.$$

Then by definition

$$\deg(f, \Omega, y) := \lim_{m \to \infty} d(f, \Omega, y_m).$$

From the definition (i) it follows that the degree does not depend on the sequence $(y_m)_{m \in \mathbb{N}}$ chosen.

(iii). $f \in C(\overline{\Omega}, \mathbb{R}^n)$ and $y \in \mathbb{R}^m$ is such that $y \notin f(\partial\Omega)$. Let $f_m \in C^1(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ be as in the case (ii) and $(f_m)_{m \in \mathbb{N}}$ converges uniformly to f. Then by definition

$$\deg(f, \Omega, y) := \lim_{m \to \infty} d(f_m, \Omega, y).$$

This definition does not depend on the sequence $(f_m)_{m \in \mathbb{N}}$ chosen.

From the above definition we have

(1) $d(1_\Omega, \Omega, y) = \begin{cases} 1 & if \quad y \in \Omega \\ 0 & if \quad y \in \mathbb{R}^n \setminus \overline{\Omega} \end{cases}$

(2) If $\deg(f, \Omega, y) \neq 0$, *then* $f^{-1}(y) \neq \emptyset$.

(3) (Invariance w.r.t. an homotopy).

Let $f, g \in C(\overline{\Omega}, \mathbb{R}^n)$ and $y \in \mathbb{R}^n$ such that $y \notin f(\partial\Omega) \cup g(\partial\Omega)$. If there exists an homotopy $H \in C(\overline{\Omega} \times [0,1], \mathbb{R}^n)$ such that $H(\cdot, 0) = f$, $f(\cdot, 1) = g$ and $y \notin H(\partial\Omega, t)$, for all $t \in [0,1]$, then

$$\deg(f, \Omega, y) = \deg(g, \Omega, y).$$

(4) If Ω_1 *and* Ω_2 *are disjoint open subset in* Ω *such that* $y \notin f(\overline{\Omega}\backslash(\Omega_1 \cup \Omega_2))$, *then*

$$\deg(f, \Omega, y) = \deg(f, \Omega_1, y) + \deg(f, \Omega_2, y).$$

(5) If $f, g \in C(\overline{\Omega}, \mathbb{R}^n)$, $f|_{\partial\overline{\Omega}} = g|_{\partial\overline{\Omega}}$ *and* $d(f, \Omega, y)$, $d(g, \Omega, y)$ *are defined, then*

$$d(f, \Omega, y) = d(g, \Omega, y).$$

Some applications of the Brouwer's degree shall be given in the next chapters.

16.2 Leray-Schauder's degree

In the above section we have defined the topological degree in $\mathbb{R}^n$. In a similar way we can define the topological degree in a finite dimensional Banach space for the class of continuous operators. Now let X be an arbitrary Banach space. In this case we consider the operators of the form $1_X - f$ where f is completely continuous.

We need the following results.

Theorem 16.2.1. *Let* X *and* Y *be two Banach spaces,* $U \subset X$ *a bounded subset of* X *and* $f : U \to Y$ *a completely continuous operator. Given* $\varepsilon > 0$, *there is a continuous operator* f_ε *whose range* $f_\varepsilon(U)$ *is in a finite dimensional subspace of* Y *such that*

$$\|f(u) - f_\varepsilon(u)\| < \varepsilon, \quad \textit{for all} \ \ u \in U.$$

Theorem 16.2.2. *Let* $f \in C(\overline{\Omega}, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^{n+m}$, $y \in \mathbb{R}^n$. *Then*

$$\deg(1_{\mathbb{R}^{n+m}} - f, \Omega, y) = \deg(1_{\mathbb{R}^n} - f, \Omega \cap \mathbb{R}^n, y),$$

where in the first part, $f = (f_1, \ldots, f_n, 0, \ldots, 0)$ *and* $y = (y_1, \ldots, y_n, 0, \ldots, 0)$.

Now, let X be a Banach space, Ω a bounded and open subset of X, $f : \Omega \to X$ a completely continuous operator and $y \in X$ such that $y \notin (1_X - f)(\partial\Omega)$.

By Theorem 16.2.1, there exist $f_n : \Omega \to X_n \subset X$, continuous, X_n linear subspace of X with $\dim X_n < +\infty$, $y_n \in X$, $y_n \notin (1_X - f)(\partial\Omega_n)$, where $\Omega_n := \Omega \cap X_n$, $f_n \overset{unif}{\to} f$, $y_n \to y$ and $\deg(1_{X_n} - f_n, \Omega_n, y_n)$ is defined. So, by definition the Leray-Schauder degree of $1_X - f$ with respect to y in Ω is

$$\deg(1_X - f, \Omega, y) := \lim_{n\to\infty} \deg(1_{X_n} - f_n, \Omega_n, y_n).$$

The Leray-Schauder degree has the following properties:

(1) $d(1_\Omega, \Omega, y) = \begin{cases} 1 & \textit{if} \quad y \in \Omega, \\ 0 & \textit{if} \quad y \in X - \overline{\Omega}. \end{cases}$

(2) If $d(1_X - f, \Omega, y) \neq 0$, *then* $\exists\, x \in \Omega$: $x - f(x) = y$.

(3) (Invariance w.r.t. an homotopy).

Let $f, g : \Omega \to X$ *completely continuous and* $y \in X$ *such that* $y \notin (1_X - f)(\partial\Omega) \cup (1_X - g)(\partial\Omega)$. *If there exists a completely continuous homotopy* $H : \overline{\Omega} \times [0,1] \to X$ *such that* $H(\cdot, 0) = f$, $F(\cdot, 1) = g$ *and* $y \notin (1_X - H(\cdot, t))(\partial\Omega)$, *for all* $t \in [0,1]$, *then*

$$\deg(1_X - f, \Omega, y) = \deg(1_X - g, \Omega, y).$$

(4) If Ω_1 *and* Ω_2 *are disjoint open subset in* Ω *such that* $y \notin (1_X - f)(\overline{\Omega} - (\Omega_1 \cup \Omega_2))$, *then*

$$\deg(1_X - f, \Omega, y) = \deg(1_X - f, \Omega_1, y) + \deg(1_X - f, \Omega_2, y).$$

(5) If $f, g : \overline{\Omega} \to X$ *are completely continuous,* $f|_{\partial\overline{\Omega}} = g|_{\partial\overline{\Omega}}$ *and* $\deg(1_X - f, \Omega, y)$, $\deg(1_X - g, \Omega, y)$ *are defined, then*

$$\deg(1_X - f, \Omega, y) = \deg(1_X - g, \Omega, y).$$

For some applications of the Leray-Schauder degree see H. Brézis R[1], F.E. Browder R[1] and R[6], F.E. Browder and C.P. Gupta R[1], F.E. Browder and R.D. Nussbaum R[1], J. Mawhin R[1] and R[4], S. Sburlan B[1], G. Dincă and P. Jebelean B[1], B[2]. Other applications will be given in the next chapters.

Topological degree theory has been extended to various classes of noncompact operators. The basic references are the following:

- K-set contractions and condensing operators: R.D. Nussbaum R[1] and R[2]; N.G. Lloyd R[1], D. O'Regan, Y.J. Cho and Y.-Q. Chen R[1], K. Deimling R[1];

- A-proper operator: W.V. Petryshyn R[2], F.E. Browder and W.V. Petryshyn R[2], R[3]; D. O'Regan, V.J. Cho and Y.-Q. Chen R[1];

- axiomatic point of view: H. Amann and S. Weiss R[1], F.E. Browder R[1], R[6]; R.D. Nussbaum R[1]; N.G. Lloyd R[1], S. Sburlan B[1];

- multiplicity and topological degree: J. Cronin-Scanlon R[1], R[2], T. O'Neil and J.W. Thomas R[1]

- rotation and topological degree: M.A. Krasnoselskii, A.I. Perov, A.I. Povolockii and P.P. Zabrejko R[1], P.P. Zabrejko R[2].

16.3 Topological degree theory for multivalued operators

Let X be a Banach space and $\Omega \subset X$ be a subset of X. By definition a multivalued operator $T : \Omega \to P(X)$ is completely continuous if it is upper semicontinuous and is compact (i.e., maps bounded sets to relatively compact sets).

Let $\Omega \subset X$ be an open, bounded subset of X and $T : \overline{\Omega} \to P(X)$ a completely continuous operator. Following Cellina and Lasota R[1], in order to define the degree of $1_X - T$ we approximate T by single-valued operators and we define $\deg(1_X - T, \Omega, y)$ in terms of the degrees of these approximant. Thus, we need the following result.

Theorem 16.3.1. (A. Cellina R[2], N.G. Lloyd R[1], pp. 116) *Let X be a Banach space, $Y \subset X$ a closed subset of X and let $T : Y \to P_{cl,cv}(X)$ be upper semicontinuous. Given $\varepsilon > 0$, there exists a continuous singlevalued operator $t_\varepsilon : Y \to X$ such that for each $x \in Y$ there are $y \in Y$ and $z \in T(y)$ with*

$$\|y - x\| < \varepsilon \text{ and } \|z - t_\varepsilon(x)\| < \varepsilon.$$

Moreover, t_ε can be chosen such that $t_\varepsilon(Y) \subset \overline{co}T(Y)$ and t_ε is completely continuous if T is completely continuous.

Now, let $\Omega \subset X$ be an open and bounded subset of X, $T : \overline{\Omega} \to P_{cl,cv}(X)$ a completely continuous operator and $y \in X$ such that $y \notin (1_X - T)(\partial\Omega)$. By Theorem 16.2.1. we choose a sequence t_n of single valued, completely continuous operators, $t_n : \overline{\Omega} \to X$, such that, $t_n \stackrel{H_{\|\cdot\|}}{\to} T$ as $n \to \infty$, $t_n(\overline{\Omega}) \subset \overline{co}T(\overline{\Omega})$, for all $n \in \mathbb{N}$, and $y \notin (1_X - t - n)(\partial\Omega)$.

By definition

$$\deg(1_X - T, \Omega, y) := \lim_{n\to\infty} \deg(1_X - t_n, \Omega, y).$$

From the properties of Leray-Schauder degree it follows that $d(1_X - T, \Omega, y)$ is independent of the particular sequence $(t_n)_{n\in\mathbb{N}}$ chosen.

This degree has the following properties:

(1) If $d(1_X - T, \Omega, y) \neq 0$, then there is $x \in \overline{\Omega}$ such that $x - T(x) \ni y$.

(2) Let $H : \overline{\Omega} \times [0,1] \to P_{cl,cv}(X)$ be a completely continuous operator such that $y \notin (1_X - H(\cdot, t))(\partial\Omega)$, for all $t \in [0,1]$. Then $\deg(H(t,\cdot), \Omega, y)$ is independent of $t \in [0,1]$.

(3) If Ω_1 and Ω_2 are disjoint open subset in Ω such that $y \notin (1_X - T)(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$, then

$$\deg(1_X - T, \Omega, y) = \deg(1 - X - T, \Omega_1, y) + \deg(1_X - T, \Omega_2, y).$$

For more considerations on the degree theory of multivalued operators see M. Hukuhara R[2], A. Cellina and A. Lasota R[1], T.W. Ma R[1], N.G. Lloyd R[1], J. Dugundji and A. Granas R[2], W.V. Petryshyn and P.M. Fitzpatrick R[1], D. O'Regan, Y.J. Cho and Yu.-Q. Chen R[1].

16.4 Coincidence degree theory

Let X and Y be two Banach space, $\Omega \subset X$ a subset of X and $f, g : \Omega \to Y$ two operators. The problem is to study the coincidence point set of the pair (f, g), i.e.,

$$C(f,g) := \{x \in \Omega \mid f(x) = g(x)\},$$

in terms of the degree theory.

For example let Ω be an open and bounded subset of X and we suppose that

(a) $f, g : \overline{\Omega} \to Y$;

(b) f is injective;

(c) $g(\overline{\Omega}) \subset f(\overline{\Omega})$.

Then the coincidence equation

$$f(x) = g(x)$$

is equivalent with the fixed point equation

$$x = (f^{-1} \circ g)(x)$$

with $f^{-1} \circ g : \overline{\Omega} \to X$.

If the $\deg(1_X - f^{-1} \circ g, \Omega, 0)$ is defined, then we define the coincidence degree of the pair (f, g) w.r.t. Ω by

$$\text{codeg}((f, g), \Omega) := \deg(1_X - f^{-1} \circ g, \Omega, 0).$$

The problem is in which conditions on X, Y, f and g the degree of $1_X - f^{-1} \circ g$ w.r.t. 0, in Ω is defined.

More general, let $h : \overline{\Omega} \to Y$ be such that

(1) $f + h$ is injective;

(2) $(g + h)(\overline{\Omega}) \subset (f + h)(\overline{\Omega})$.

Then the coincidence equation of the pair (f, g) is equivalent with the coincidence equation of the pair $(f + h, g + h)$ which is equivalent with the fixed point equation

$$x = (f + h)^{-1} \circ (g + h)(x).$$

In this case if the degree of $1_X - (f+h)^{-1} \circ (g+h)$ w.r.t. 0, in Ω is defined then we define the coincidence degree of the pair (f, g) w.r.t. Ω as the degree of the operator $1_X - (f + g)^{-1} \circ (g + h)$ w.r.t. 0, in Ω.

The problem is to choose a class of operators h and the conditions on f and g such that:

• the degree of $1_X - (f + h)^{-1} \circ (g + h)$ w.r.t. 0, in Ω is defined for h in the chosen class;

• the degree of $1_X - (f+h)^{-1} \circ (g+h)$ does not depend upon the choice of h in that class.

In the paper R[3], J. Mawhin (see also R. Gaines and J. Mawhin R[1]) considers the case in which $f : X \to Y$ is a Fredholm linear continuous operator of index 0 and g is completely continuous. Let $j : \mathrm{Ker} f \to \mathrm{coker} f$ be a linear isomorphism and $p : X \to \mathrm{Ker} f$ be a continuous projector. Then, Mawhin take $h := j \circ p$.

For the theory and applications of the coincidence degree see R. Gaines and J. Mawhin R[1]. See also J.K. Hale and J. Mawhin R[1], J. Pejsachowicz and A. Vignoli R[1], S. Sburlan B[2], A. Buică B[1] and B[2].

For homological theory of coincidence degree theory see R.F. Brown, M. Furi, L. Górniewicz and B. Jiang R[1] and the references therein.

Chapter 17

Topological spaces with the fixed point property

Precursors: B. Bolzano (1817), H. Poincaré (1883).
Guidelines: P. Bohl (1904), L.E.J. Brouwer (1909), J. Hadamard (1910), L.E.J. Brouwer (1912), G.D. Birkhoff and O.D. Kellog (1922), B. Knaster, K. Kuratowski and S. Mazurkiewicz (1929), J. Schauder (1930), K. Borsuk (1931), A. Tychonoff (1935), C. Miranda (1940), S. Kakutani (1941), H.F. Bohnenblust and S. Karlin (1950), K. Fan (1952), I.L. Glicksberg (1952), K. Fan (1961), F.E. Browder (1968).
References: T. van der Walt R[1], V.I. Istrăţescu B[5], I.A. Rus B[81], D.R. Smart R[1], M.A. Krasnoselskii and P. Zabrejko R[1], J. Dugundji and A. Granas R[1], A. Granas and J. Dugundji R[1], I.A. Rus B[73], K. Border R[1], R.F. Brown R[1], R.P. Agarwal, M. Meehan and D. O'Regan R[1], E. Zeidler R[1], C. Vladimirescu and C. Avramescu B[1], A. Buică B[2], A. Petruşel B[1], J. Franklin R[1], R.H. Bing R[1], J. Andres R[3], R. Mańka R[2], S. Reich and Y. Sternfeld R[1], N.H. Pavel B[2], B[3], L. Górniewicz R[4], G.L. Cain and L. González R[1], M. Balaj B[8], S. Park R[3], R[7], S. Park in J. Jaworowski, W.A. Kirk and S. Park R[1], K.D. Joshi R[1].

17.0 Topological spaces with the fixed point property

Let (X, τ) be a Hausdorff topological space. By definition X is with the fixed point property (f.p.p.) if

$$f \in C(X, X) \Rightarrow F_f \neq \emptyset.$$

One of the main problem of the topological fixed point theory is the following:

Problem 17.0.1. Which topological spaces have the f.p.p.?

We have

Lemma 17.0.1. *Let (X, τ) and (Y, τ) be two topological spaces. We suppose that*

(i) (X, τ) has the f.p.p.

(ii) there exists a topological isomorphism $\varphi : X \to Y$.

Then (Y, τ) has the f.p.p.

Proof. Let $f \in C(Y, Y)$. By (ii) we have that the operator $\varphi^{-1} \circ f \circ \varphi$: $X \to X$ is continuous. Let $x_0 \in F_{\varphi^{-1} \circ f \circ \varphi}$. We remark that $\varphi(x_0) \in F_f$.

Lemma 17.0.2. *Let (X, τ) be a topological space and $Y \subset X$. We suppose that:*

(i) (X, τ) has the f.p.p.;

(ii) there exists a topological retraction $\varphi : X \to Y$.

Then Y has f.p.p.

Proof. Let $f \in C(Y, Y)$. Then $f \circ \varphi \in C(X, X)$. So, $F_{f \circ \varphi} \neq \emptyset$. But $F_f = F_{f \circ \varphi}$.

Remark 17.0.1. Let X be a Banach space and $x_0 \in X$, $x_0 \neq 0$. The translation operator $t : X \to X$, $x \mapsto x + x_0$ is continuous and $F_t = \emptyset$.

So, $(X, \tau_{\|\cdot\|})$ is not with f.p.p.

This remark give rise to the following problems:

Problem 17.0.2. Which subsets of a Banach space have the f.p.p.?

Problem 17.0.3. Let X be a Banach space and $f \in C(X, X)$. In which conditions there exists $Y \in I(f)$ with f.p.p.?

Example 17.0.1. Any compact interval $[a, b] \subset \mathbb{R}$ is with f.p.p.

Example 17.0.2. Let $f \in C(\mathbb{R}, \mathbb{R})$. If $f(\mathbb{R})$ is a bounded subset of $\mathbb{R}$, then there exists a compact interval in $I(f)$.

On the other hand Lemma 17.0.1 give rise to:

Problem 17.0.4. Let X be a Banach space and $Y, Z \in P(X)$. In which conditions there exists a topological isomorphism $\varphi : Y \to Z$?

Problem 17.0.5. Let X be a Banach space and Y be a nonempty subset of X. In which conditions, Y is a topological retract of X ?

For these problems we have:

Theorem 17.0.1. *If $Y, Z \in P_{b,cl,cv}(X)$ are with nonempty interior, $\overset{\circ}{Y} \neq \emptyset$, $\overset{\circ}{Z} \neq \emptyset$, then there is a topological isomorphism $\varphi : Y \to Z$, i.e., Y and Z are topological isomorphic.*

Theorem 17.0.2. *Every nonempty closed convex subset of a Banach space X is a topological retract of X.*

Remark 17.0.2. For the above results see A. Granas and J. Dugundji R[1], K. Deimling R[3], H. Nikaido R[1], V. Klee R[2], I.A. Rus and P. Pavel R[1], E.G. Begle R[1].

The aim of this chapter is to give examples of topological space with f.p.p.

17.1 Equivalent statements with the f.p.p.

In this section we shall refer to the f.p.p. of the bounded closed convex subset in $\mathbb{R}^n$, with nonempty interior. For this we need some notations:

$\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is the Euclidean metric;

$B^n := \overline{B}(0,1) \subset (\mathbb{R}^n, \rho)$;

$S^n := \partial B^{n+1}$

$I^n := [-1,1]^n := [-1,] \times \cdots \times [-1,1]$;

$I_k^- := \{(x_1, \ldots, x_n) \in I^n \mid x_k = -1\}$, $k = \overline{1,n}$;

$I_k^+ := \{(x_1, \ldots, x_n) \in I^n \mid x_k = 1\}$, $k = \overline{1,n}$;

$\overline{\sigma}_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1, \ldots, x_n \in \mathbb{R}_+,\ x_1 + \cdots + x_n \leq 1\}$.

We have

Theorem 17.1.1. *Let $n \in \mathbb{N}^*$ and X be a real Banach space with* $\dim X = n$*. The following statements are equivalent:*

(i) (Brouwer (1912)). B^n *has the f.p.p.*

(ii) (Bolzano (1817), Poincaré (1883), Miranda (1940)). Let $f = (f_1, \ldots, f_n) \in C(I^n, \mathbb{R}^n)$ *be such that*

(a) $f_k(x) \le 0$, *for all* $x \in I_k^-$, $k = \overline{1,n}$;

(b) $f_k(x) \ge 0$, *for all* $x \in I_k^+$, $k = \overline{1,n}$.

Then, $Z_f \ne \emptyset$.

(iii) (Borsuk (1931)). ∂B^n *is not a topological retract of* B^n.

(iv) (Bohl (1904)). ∂I^n *is not a topological retract of* I^n.

(v) Each $Y \in P_{b,cl,cv}(\mathbb{R}^n)$, $\overset{\circ}{Y} \ne \emptyset$, *has the f.p.p.*

(vi) Each $Y \in P_{b,cl,cv}(X)$, $\overset{\circ}{Y} \ne \emptyset$, *has the f.p.p.*

(vii) There exists $Y \in P_{b,cl,cv}(X)$, $\overset{\circ}{Y} \ne \emptyset$, *with the f.p.p.*

(viii) (Sperner (1928)). Let Σ *be a simplicial subdivision of an* n*-simplex* $P_0P_1 \ldots P_n \subset \mathbb{R}^n$. *To each vertex of* Σ *we assign an integer in such a way that if a vertex* P *of* Σ *lies on a face* $P_{i_0}P_{i_1} \ldots P_{i_k}$ *then the number assigned to* P *is one of the integers* $i_0, i_1, \ldots, i_k$. *Then the total number of* n*-simplexes of* Σ *whose vertices receive all integers* $\{0, 1, \ldots, n\}$ *is odd.*

(ix) (Knaster-Kuratowski-Mazurkiewicz ($\equiv$ KKM *or* K^2M*). Let* $P_0P_1 \ldots P_n \subset \mathbb{R}^n$ *an* n*-simplex and* $K_0, K_1, \ldots, K_n$ *some compact subset of* $\mathbb{R}^n$. *If the inclusion* $P_{i_0}P_{i_1} \ldots P_{i_k} \subset K_{i_0} \cup K_{i_1} \cup \cdots \cup K_{i_k}$ *holds for all faces* $P_{i_0} \ldots P_{i_k}$ *of* $P_0P_1 \ldots P_n$, *then,* $\bigcap_{i=0}^{n} K_i \ne \emptyset$.

For the above equivalences see K. Border R[1], J. Frankin R[1], M. Yoseloff R[1], S. Park in J. Jaworowski, W.A. Kirk and S. Park R[1], R.F. Brown R[1], C. Avramescu B[6], B[7], C. Avramescu and C. Vladimirescu B[1], M. Balaj B[8], M. Yoseloff R[1], etc. See also the next section.

17.2 Brouwer fixed point theorem

The first general example of topological space with the f.p.p. is given by

Brouwer's Theorem. *Every nonempty bounded, closed and convex subset* Y *of a finite dimensional Banach space* X *has the fixed point property.*

First proof. K^2M lemma $\Rightarrow$ Brouwer theorem. Let $\dim X = n$. By Lemma 17.0.1 and Theorem 17.0.1 we consider an n-simplex $\overline{\sigma} = P_0 \ldots P_n$

in $\mathbb{R}^n$. We shall prove that K^2M lemma implies that $\overline{\sigma}$ has the fixed point property. Let $f : \overline{\sigma} \to \overline{\sigma}$ be a continuous function. Let $\overline{\sigma} = P_0P_1 \dots P_n$ and $P \in \overline{\sigma}$. Then $P = \sum_{i=0}^{n} \lambda_i P_i$, $\lambda_i \geq 0$ and $\sum_{i=0}^{n} \lambda_i = 1$. Since $f(P) \in \overline{\sigma}$ we have that

$$f(P) = \sum_{i=0}^{n} f_i(\lambda_0, \dots, \lambda_n)P_i, \quad f_i(\lambda) \geq 0, \quad \sum_{i=0}^{n} f_i(\lambda) = 1.$$

The continuity of f implies that the functions f_i, $i = \overline{0, n}$ are continuous. Now we consider the following sets:

$$K_i := \left\{ \sum_{i=0}^{n} \lambda_j P_j \mid \lambda_j \geq 0, \ \sum_{i=0}^{n} \lambda_j = 1, \ f_i(\lambda) \leq \lambda_i \right\}.$$

These sets are as in K^2M lemma (statement (ix) in Theorem 17.2). Let $x^* \in \bigcap_{i=0}^{n} K_i$, $x^* = \lambda_0^* P_0 + \dots + \lambda_n^* P_n$. We have

$$\lambda_i^* \geq 0, \quad f_i(\lambda^*) \geq 0, \quad \sum_{j=0}^{n} \lambda_j^* P_j = 1, \quad \sum_{j=0}^{n} f_j(\lambda^*) P_j = 1$$

and

$$f_i(\lambda^*) \leq \lambda_i^*, \quad \text{for all} \quad i \in \{0, 1, \dots, n\}.$$

These imply that $f_i(\lambda^*) = \lambda_i^*$, $i = \overline{0, n}$. So, $x^* \in F_f$.

Second proof. By Lemma 17.0.1 it is sufficiently to prove the theorem in the case $Y = B^n$, $n \in \mathbb{N}^*$. If $F_f \cap \partial B^n = \emptyset$, then we consider the homotopy $H(x, t) := x - tf(x)$, $x \in B^n$, $t \in [0, 1]$. Then we have

$$1 = \deg(1_{B^n}, \overset{\circ}{B}{}^n, 0) = \deg(1_{B^n} - f, \overset{\circ}{B}{}^n, 0) \neq 0,$$

where deg stands for the degree of Brouwer. This implies that $F_f \neq 0$.

For other proof of the Brouwer theorem see K. Border R[1], J. Franklin R[1], I.A. Rus B[73], A. Granas and J. Dugundji R[1], S. Park in J. Jaworowski, W.A. Kirk and S. Park R[1].

17.3 Generalizations of the Brouwer fixed point theorem

The generalizations of the Brouwer fixed point theorem are in a deep connections with the generalizations of:

• topological degree

• Knaster-Kuratowski-Mazurkiewicz lemma

• Schauder approximation theorem of completely continuous operators by finite dimensional operators.

We begin with:

Schauder fixed point theorem (first variant). *Let X be a Banach space, $Y \subset X$ a nonempty closed bounded and convex subset of X and $f : Y \to Y$ a completely continuous operator. Then $F_f \neq \emptyset$.*

In a particular case in which Y is a compact convex set we have:

Schauder fixed point theorem (second variant). *Let X be a Banach space, $Y \subset X$ a nonempty compact convex subset of X and $f : Y \to Y$ a continuous operator. Then $F_f \neq \emptyset$.*

Remark 17.2.1. The above variants are equivalent. Indeed, let us prove that "second" $\Rightarrow$ "first". Let $Y \in P_{cl,b,cv}(X)$ and $f : Y \to Y$ completely continuous. Then $\overline{f(Y)} \in P_{cp}(X)$ and $'ov\overline{f(Y)}$ is an invariant subset of f. By a Mazur's lemma we have that $Z := \overline{co}(\overline{f(Y)}) \in I_{cp,cv}(f)$. Now we consider $f|_Z : Z \to Z$ which are in the conditions of the second variant. So, $F_f \neq \emptyset$.

Proof of first variant. By Lemma 17.0.1, we take $Y := \overline{B}(0,1) \subset (X, d_{\|\cdot\|})$. If $F_f \cap \partial\overline{B}(0,1) = \emptyset$, then we consider the homotopy $H(x,t) := x - tf(x)$, $x \in \overline{B}(0,1)$, $t \in [0,1]$. We have

$$1 = \deg(1_{\overline{B}}, B(0,1), 0) = \deg(1_{\overline{B}} - f, B(0,1), 0) \neq 0,$$

where deg stands for the degree of Leray-Schauder. This implies that $F_f \neq \emptyset$.

Another generalization is:

Tychonoff's fixed point theorem. *Let X be a locally convex linear topological space and $Y \subset X$ a nonempty compact convex set and $f : Y \to Y$ a continuous operator. Then, $F_f \neq \emptyset$.*

For to prove this fixed point theorem Ky Fan use the following results (K.

Fan R[3]):

K^2M **lemma of K. Fan.** *Let X be a topological linear space, $Y \subset X$ a nonempty subset of X and $T : Y \to P_{cl}(X)$ an operator. We suppose that:*

(i) $co\{y_1, \ldots, y_m\} \subset \bigcup_{i=1}^{m} T(y_i)$, for any finite subset $\{y_1, \ldots, y_m\} \subset Y$;

(ii) there is $y_0 \in Y$ such that $T(y_0)$ is a compact subset of X.

Then, $\bigcap_{y \in Y} T(y) \neq \emptyset$.

Geometric Lemma of Ky Fan. *Let X be a topological linear space and $Y \subset X$ a compact convex subset of X. Let $Z \subset Y \times Y$ be such that:*

(i) $(y, y) \in Z$, for every $y \in Y$;

(ii) for each $y \in Y$, the set $\{x \in Y \mid (x, y) \notin Z\}$ is convex.

Then there exists at least a point $y_0 \in Y$ such that $Y \times \{y_0\} \subset Z$.

The main open problem of this fixed point theory is the following.

Schauder's conjecture. *Let X be a linear topological space and $Y \subset X$ a nonempty compact convex subset and $f : Y \to Y$ a continuous operator. Then, $F_f \neq \emptyset$.*

For this open problem see T. van der Walt R[1], J. Dugundji and A. Granas R[2], O. Hadžić R[1], S. Park R[7], R. Cauty R[1].

17.4 Multivalued operators

Some basic topological fixed point principles for multivalued operators are now presented.

For the beginning, we define the notion of Kakutani-type multivalued operator:

Definition 17.4.1. Let X, Y be two vector topological spaces. Then $T : X \to P(Y)$ is said to be a Kakutani-type multivalued operator if and only if:

i) $F(x) \in P_{cp,cv}(Y)$, for all $x \in X$

ii) F is u.s.c. on X.

Definition 17.4.2. Let X be a vector topological space and $Y \in P(X)$. Then, by definition, Y has the Kakutani fixed point property (briefly K.f.p.p.) if and only if each Kakutani-type multivalued operator $T : Y \to P(Y)$ has at least a fixed point in Y.

The first topological fixed point result was given by Kakutani in 1941.

Theorem 17.4.1. (Kakutani) *Any compact convex subset K of $\mathbb{R}^n$ has the K.f.p.p.*

For the infinite dimensional case, we have:

Theorem 17.4.2. (Bohnenblust-Karlin) *Any compact convex subset K of a Banach space X has the K.f.p.p.*

Theorem 17.4.3. (Fan-Glicksberg) *Any compact convex subset K of a Hausdorff locally convex topological space X has the K.f.p.p.*

For the infinite dimensional case we also have the following result (see for example Kirk-Sims (Eds.) R[1]) of Bohnenblust-Karlin:

Theorem 17.4.4. (Bohnenblust-Karlin) *Let X be a Banach space and $Y \in P_{b,cl,cv}(X)$. The any upper semicontinuous multivalued operator $T : Y \to P_{cl,cv}(Y)$ with relatively compact range has at least a fixed point in Y.*

Other interesting results were proved by F.E. Browder and C. Himmelberg R[].

Theorem 17.4.5. (Browder R[4]) *Let X be a Hausdorff vector topological space and K be a nonempty compact and convex subset of X. Let $T : K \to P_{cv}(K)$ be a multivalued operator such that $T^{-1}(y)$ is open for each $y \in K$. Then $F_T \neq \emptyset$.*

Theorem 17.4.6. (Himmelberg's Theorem R[1]) *Let K be a convex subset of a locally convex Hausdorff topological vector space X and let Y be a nonempty compact subset of K. If $T : K \to P_{cl,cv}(Y)$ is an u.s.c. multivalued operator then $F_T \neq \emptyset$.*

Another approach is based on the crossed cartesian product of two multivalued operators, see R. Espínola, G. López and A. Petruşel B[1].

Let X and Y be nonempty subsets of the Banach spaces E_1, respectively E_2. Consider $F_1 : Y \to P(X)$ and $F_2 : X \to P(Y)$. Let us define the crossed cartesian product of F_1 and F_2 as follows: $T : X \times Y \multimap X \times Y$ by $T(x, y) := F_1(y) \times F_2(x)$. It is known that if F_1 and F_2 are u.s.c. with compact values then their cartesian product $T : X \times Y \to P_{cp}(X \times Y)$ is u.s.c. too.

The following lemma is quite obvious:

Lemma 17.4.1. *The following statements are equivalent:*

i) $F_T \neq \emptyset$;

ii) $F_{F_2 \circ F_1} \neq \emptyset$;

iii) $F_{F_1 \circ F_2} \neq \emptyset$.

Proof. If $(x, y) \in T(x, y)$, then $x \in F_1(y)$ and $y \in F_2(x) \Leftrightarrow x \in (F_1 \circ F_2)(x)$ and $y \in (F_2 \circ F_1)(y)$. □

It is obvious that the cartesian product of multivalued operators preserves better some properties of their images (such as convexity, for example) than the composition of multivalued operators. Thus, it is of interest to consider the crossed cartesian product technique in nonlinear analysis, in general and for fixed point theory, in particular.

A result by this approach is:

Theorem 17.4.7. *Let X be a Banach space and Y a nonempty closed convex subset of X. Let $S : Y \to P_{cl,cv}(X)$ be u.s.c. with $\overline{S(Y)}$ compact and $G : \overline{co}S(Y) \to P_{cl,cv}(Y)$ be an u.s.c. multivalued operator. Then $F_{G \circ S} \neq \emptyset$.*

Proof. Let us define $T : Y \times \overline{co}S(Y) \to Y \times \overline{co}S(Y)$ by $T(x, y) := G(y) \times S(x)$, for $(x, y) \in Y \times \overline{co}S(Y)$. Then T is u.s.c. with nonempty closed and convex values. Note that $T(Y \times \overline{co}S(Y) \subset G(\overline{co}S(Y)) \times S(Y)$. Hence $T(Y \times \overline{co}S(Y))$ is contained in a compact set. Using Himmelberg's fixed point theorem we obtain that there exists at least one fixed point for T, i. e. $(x^*, y^*) \in T(x^*, y^*)$. From Lemma 17.4.1 the conclusion follows. □

For other results by this approach, see R. Espínola, G. López and A. Petruşel B[1].

For other results see: R.F. Brown, M. Furi, L. Górniewicz and B. Jiang R[1], R.P. Agarwal, M. Meehan and D. O'Regan R[1], J. Andres and L. Górniewicz R[2], J.-P. Aubin and A. Cellina R[1], J.-P. Aubin and H. Frankowska R[1], J. Banas and K. Goebel R[1], L.J. Lin, N.-C. Wong and Z.-T. Yu R[1], J. Andres R[4], F.E. Browder R[4], Yu.G. Borisovich, B.D. Gelman, A.D. Myškis and V.V. Obukhovskii R[1], M. Kamenskii, V. Obukhovskii, P. Zecca R[1], F.H. Clarke, Yu.S. Ledyaev and R.J. Stern R[1], etc.

17.5 Continuity, convexity, compactness and fixed points

There are large classes of generalized continuity, generalized convexity and generalized compactness. Having in mind the results of this chapter, the following problem arises:

Problem 17.5.1. Which of these generalizations are useful and relevant in fixed point theory ?

For the above problem see Chapter 13, 18, 19, 24.6 and the references therein.

For some generalization of the continuity concept see R. Engelking R[1], C.E. Aull and R. Lowen R[1], A.I. Ban and S.G. Gal R[1], L.M. Blumenthal R[1], R.F. Brown, M. Furi, L. Górniewicz and B. Jiang R[1], G. Beer R[1], V. Klee R[2], J. Guillerme R[2], Z. Wu R[1], R.E. Smithson R[4], M. Matejdes R[1], M.C. Anisiu B[4], S. Jafari, T. Noiri, N. Rajesh and M.L. Thivagar R[1], V.N. Akis R[1], D. Miklaszewski R[1], etc.

17.6 Other results

As other examples of topological fixed point theorems, we present here the following results:

Interior Fixed Point Property. (R.F. Brown and R.E. Green R[1]) *Let D be the unit closed disc in $\mathbb{C}$ and S its boundary. Let $f : D \to D$ be a mapping. We suppose:*

(i) f is continuously differentiable;

(ii) there exists an integer $m \geq 2$ such that $f(x) = x^m$, for all $x \in S$.

Then, $F_f \cap int(D) \neq \emptyset$.

Cartwright-Littlewood-Bell's Theorem. (K. Kuperberg R[1]) *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping. Suppose that:*

(i) f is an orientation reversing topological isomorphism;

(ii) there exists a continuum (i.e., a nonempty connected and compact set) $Y \subset \mathbb{R}^2$, such that $f(Y) \subset Y$. Then, $F_f \cap Y \neq \emptyset$.

Poincaré Theorem. *Every area preserving topological isomorphism of an annulus* $A := S^1 \times [a, b]$*, rotating the two boundaries in opposite directions, posses at least two fixed points in the interior.*

For other results for symplectic operators (area preserving mappings in $\mathbb{R}^2$, two-form preserving operators on a symplectic manifold, etc.) see J. Moser R[1], E. Zehnder R[1], C.C. Conley and E. Zehnder R[1], S.M. Boyles R[1], M. Brown and W.D. Neumann R[1], G. Choquet R[1], P.N. Dowling, C.J. Lennard and B. Turett R[1], S.S. Dragomir R[1], etc.

Chapter 18

Fixed point structures

Precursors: C. Kuratowski (1930), G. Darbo (1955), B.N. Sadowski (1967), T. B. Muenzenberger and R. E. Smithson (1973).
Guidelines: I.A. Rus (1986), I.A. Rus (1987), I.A. Rus (1993), I.A. Rus (2006).
General references: I. A. Rus B[95]. See also I. A. Rus B[13], B[15], B[17], B[24], B[27], B[31], B[33], B[35], B[36], B[39], A. Muntean B[7], S. Mureşan B[2], A. Petruşel B[23], M.A. Şerban B[2], B[6], A. Sîntămărian B[7], R. Precup B[27], I.A. Rus, E. Miklos and S. Mureşan B[1].

18.0 Preliminaries

The notion "fixed point structures" is a generalization of some notions such as:

• "ordered set with the fixed point property w.r.t. increasing operators" (B. Knaster, A. Tarski, G. Birkhoff, S. Ginsburg, L.E. Ward, A.C. Davis, S. Abian, A.B. Brown, A. Abian, I. Rival, A. Pelczar, H. Amann, H. Cohen, D. Duffus, Z. Shmuely,...);

• "ordered set with the fixed point property w.r.t. progressive operators" (E. Zermelo, N. Bourbaki, I. Ekeland, A. Brøndsted, W.A. Kirk, J. Caristi, H. Brezis, F.E. Browder, B.S.W. Schröder, M. Turinici,...);

• "metric space with fixed point property w.r.t. contractions" (S. Banach,

R. Caccioppoli, C. Bessaga, P.R. Meyers, E.H. Connell, T.K. Hu, L. Janos, V.I. Opoitsev, P. Amato, L. Leader, W.A. Kirk, S. Park, I.A. Rus, M.C. Anisiu, V. Anisiu, J. Jachymski,...);

• "Menger space with the fixed point property w.r.t. probabilistic contractions" (V.M. Sehgal, A.T. Bharucha-Reid, O. Hadžić, T.L. Hicks, H. Sherwood, Gh. Constantin, V.I. Istrăţescu, E. Pap, V. Radu, R.M. Tardiff, B. Schweizer, D. Miheţ,...);

• "Banach space with the fixed point property w.r.t. nonexpansive operators" (F.E. Browder, D. Göhde, W.A. Kirk, L.A. Karlovitz, T.C. Lim, M. Edelstein, B. Maurey, J.B. Baillon, R.E. Bruck, K. Goebel, M.A. Khamsi, S. Reich, B. Sims, W. Takahashi, T. Dominguez Benavides, R. Espínola, J. Elton, P.K. Lim, E. Odell, S. Szarek, D.S. Jaggi, G. Kassay,...);

• "topological space with the fixed point property with respect to continuous operators" (L.E.J. Brouwer, J. Schauder, A. Tychonoff, B. Knaster, K. Kuratowski, S. Mazurkiewicz, V.L. Klee, E.H. Connel, E. Fadell, H. Schirmer, S. Kakutani, S. Eilenberg, D. Montgomery, H.F. Bohnenblust, S. Karlin, K. Fan, I.L. Glicksberg, R.F. Brown, J. Dugundji, A. Granas, R.H. Bing, R. Mańka,...);

• "operator with the fixed point property on family of sets" (G.S. Jones, F.S. De Blasi,...);

• "object of a category with the fixed point property" (F.W. Lawvere, J. Lambek, I.A. Rus, M. Wand, J. Soto-Andrade, F.J. Varela, M. Barr, C. Wells, A. Baranga,...).

The fixed point structure theory offers a solution for the following problems:

Problem 18.0.1. Let T be a fixed point theorem with respect to a structured set X and some single valued operators from X to X. Let $f : X \to X$ be an operator which does not satisfy the conditions of T. In which conditions the operator f has an invariant subset Y such that the restriction of f to Y, $f|_Y : Y \to Y$ satisfies the conditions of T?

Problem 18.0.2. Let T be a fixed point theorem with respect to a structured set X and some multivalued operators from X to X. Let $F : X \to P(X)$ be an operator which does not satisfy the conditions of T. In which conditions the operator f has an invariant subset Y such that the restriction of F to Y, $F|_Y : Y \to P(Y)$ satisfies the conditions of T?

The aim of this chapter is to present some notions, results and open problem of the fixed point structure theory of singlevalued operators. For the multivalued operators case see the next chapter.

18.1 Fixed point structures. Examples

Let X be a nonempty set. A triple $(X, S(X), M)$ is a fixed point structure on X, if:

(i) $S(X) \subset P(X),\ S(X) \neq \emptyset$;

(ii) $M : P(X) \to \bigcup_{Y \in P(X)} \mathbb{M}(Y),\ Y \mapsto M(Y) \subset \mathbb{M}(Y)$ is an operator such that, if $Z \subset Y$, $Z \neq \emptyset$, then

$$M(Z) \supset \{f|_Z|\ f \in M(Y) \text{ and } f(Z) \subset Z\};$$

(iii) Every $Y \in S(X)$ has the fixed point property with respect to $M(Y)$.

Remark 18.1.1. For the definition of the large fixed point structure see 2.3.

We present now some examples.

Example 18.1.1. X is a nonempty set, $S(X) := \{\{x\}|\ x \in X\}$ and $M(Y) := \mathbb{M}(Y)$.

Example 18.1.2. (A. Tarski). Let $(X, \leq)$ be an ordered set, $S(X) := \{Y \in P(X)|\ (Y, \leq) \text{ is a complete lattice}\}$ and $M(Y) := \{f : Y \to Y|\ f \text{ is an increasing operator}\}$.

Example 18.1.3. (S. Banach, R. Caccioppoli). (X, d) is a metric space, $S(X) := \{Y \in P(X)|\ (Y, d) \text{ is a complete metric space}\}$ and $M(Y) := \{f : Y \to Y|\ f \text{ is contraction}\}$.

Example 18.1.4 (V. Niemytzki and M. Edelstein) (X, d) is a metric space, $S(X) := P_{cp}(X)$ and $M(Y) := \{f : Y \to Y|\ f \text{ is an contractive operator}\}$.

Example 18.1.5. (J. Schauder). Let X be a Banach space, $S(X) := P_{cp,cv}(X)$ and $M(Y) := C(Y, Y)$.

Example 18.1.6. (F. E. Browder). X is a Hilbert space, $S(X) := P_{b,cl,cv}(X)$ and $M(Y) := \{f : Y \to Y |\ f$ is an nonexpansive operator $\}$.

Example 18.1.7. (Girolo). X is a Banach space, $S(X) := P_{cp,cv}(X)$ and $M(Y) := \{f : Y \to Y |\ f$ is connective$\}$. Then, the triple $(X, S(X), M)$ is a large fixed point structure and it is not a fixed point structure.

18.2 Functionals with the intersection property. Examples

Let X be a nonempty set, $Z \subset P(X)$, $Z \neq \emptyset$. By definition a functional $\theta : Z \to R_+$ has the intersection property if $Y_n \in Z$, $Y_{n+1} \subset Y_n$, $n \in \mathbb{N}$ and $\theta(Y_n) \to 0$ as $n \to \infty$, implies

$$Y_\infty := \bigcap_{n\in\mathbb{N}} Y_n \neq \emptyset,\ Y_\infty \in Z \text{ and } \theta(Y_\infty) = 0.$$

Example 18.2.1. Let (X, d) be a complete metric space, $Z := P_{b,cl}(X)$ and $\theta = \delta$ (diameter function).

Example 18.2.2. Let (X, d) be a complete metric space, $Z := P_{b,cl}(X)$ and $\theta = \alpha_K$ (Kuratowski's measure of noncompactness).

Example 18.2.3. Let (X, d) be a complete metric space, $Z = P_{b,cl}(X)$ and $\theta := \alpha_H$ (Hausdorff's measure of noncompactness).

Example 18.2.4. Let (X, d, W) be a convex metric space with the property (c), $Z := P_{b,cl}(X)$ and $\theta := \beta_{EL}$ (Eisenfeld-Lakshmikantham's measure of nonconvexity).

18.3 Compatible pair with a fixed point structure

Let $(X, S(X), M)$ be a fixed point structure, $\theta : Z \to R_+$ ($S(X) \subset M \subset P(X)$) and $\eta : P(X) \to P(X)$. The pair (θ, η) is compatible with the fixed point structure $(X, S(X), M)$ if:

(i) η is a closure operator, $S(X) \subset \eta(Z) \subset Z$, and

$$\theta(\eta(Y)) = \theta(Y),\ \text{for all } Y \in Z$$

(ii) $F_\eta \cap Z_\theta \subset S(X)$.

Example 18.3.1. Let (X,d) be a complete metric space, $S(X) := P_{cp}(X)$, $M(Y) := \{f : Y \to Y |\ f$ is a contractive operator$\}$, $Z = P_b(X)$, $\theta = \alpha_K$, $\eta(A) = \overline{A}$.

Example 18.3.2. Let X be a Banach space, $S(X) := P_{cp,cv}(X)$, $M(Y) := C(Y,Y)$, $Z = P_b(X)$, $\theta = \alpha_K$ and $\eta(A) := \overline{co}A$.

18.4 (θ,φ)-contraction and θ-condensing operators

Let X be a nonempty set, $Z \subset P(X)$, $Z \neq \emptyset$ and $\theta : Z \to \mathbb{R}_+$ a functional. An operator $f : X \to X$ is a strong (θ,φ)-contraction if

(i) φ is a comparison function;

(ii) $A \in Z$ implies that $f(A) \in Z$;

(iii) $\theta(f(A)) \leq \varphi(\theta(A))$, for all $A \in Z$.

An operator $f : X \to X$ is a (θ,φ)-contraction if

(i) φ is a comparison function;

(ii) $A \in Z$ implies that $f(A) \in Z$;

(iii') $\theta(f(A)) \leq \varphi(\theta(A))$, for all $A \in Z \cap I(f)$.

An operator $f : X \to X$ is strong θ-condensing if

(ii) $A \in Z$ implies that $f(A) \in Z$;

(iii") $A \in Z$, $\theta(A) \neq 0$ imply $\theta(f(A)) < \theta(A)$.

An operator $f : X \to X$ is θ-condensing if

(ii) $A \in Z$ implies that $f(A) \in Z$;

(iii"') $A \in Z \cap I(f)$, $\theta(A) \neq 0$ imply $\theta(f(A)) < \theta(A)$.

Example 18.4.1. *Let (X,d) be a metric space and $f : X \to X$ a Ćirić-Reich-Rus operator, i.e., there exist $a,b \in \mathbb{R}_+$, $a+2b<1$, such that*

$$d(f(x),f(y)) \leq ad(x,y) + b[d(x,f(x)) + d(y,f(y))], \quad \textit{for all } x,y \in X.$$

Then f is a (δ,φ)-contraction, where $\varphi(t) := (a+2b)t$, $t \in \mathbb{R}_+$.

Example 18.4.2. *Let (X,d) be a metric space and $f : X \to X$ be a φ-contraction. Then f is a strong (α_K,φ)-contraction, where $\alpha_K : P_b(X) \to \mathbb{R}_+$ denotes the Kuratowski measure of noncompactness of X*

Example 18.4.3. *Let (X,d) be a metric space, $Z := P_b(X)$ and $\theta := \delta$.*

Then, an operator $f : X \to X$ is a strong (δ, φ)-contraction if and only if f is a φ-contraction.

Example 18.4.4. *Let (X, d) be a metric space, $\alpha_K : P_b(X) \to \mathbb{R}_+$ be the Kuratowski measure of noncompactness of X and $f : X \to X$ be a compact operator. Then f is a strong $(\alpha_K, 0)$-contraction.*

Example 18.4.5. *Let X be a Banach space, $f : X \to X$ be a compact operator and $g : X \to X$ be a φ-contraction. Then the operator $h := f + g$ is a strong (α_K, φ)-contraction.*

Example 18.4.6. *The radial retraction ρ on a Banach space X to $\overline{B}(0; 1)$ is l-Lipschitz. Moreover, $l = l(X)$ and we have that $1 \leq l(X) \leq 2$. It is also known that ρ is strong α_K-nonexpansive, i.e.*

$$\alpha_K(\rho(A)) \leq \alpha_K(A), \text{ for each } A \in P_b(X).$$

Example 18.4.7. (D.E. De Figueiredo and L.A. Karlovitz R[1]) *Let X be an infinite dimensional Banach space and $\overline{B}(0; 1) \subset X$. Consider the operator $f : \overline{B}(0; 1) \to \overline{B}(0; 1)$ defined by $f(x) := (1 - \|x\|)x$. Then, f is α_K-condensing and it is not (α_K, l)-contraction, for any $l \in]0, 1[$.*

Remark 18.4.1 For the above definitions and examples see I.A. Rus B[95], pp. 62-68.

18.5 First general fixed point principle

We have:

Theorem 18.5.1. *Let $(X, S(X), M)$ be a f.p.s., (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f \in M(Y)$. We suppose that:*

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) f is a (θ, φ)-contraction.

Then:

(a) $I(f) \cap S(X) \neq \emptyset$;

(b) $F_f \neq \emptyset$;

(c) If $F_f \in Z$, then $\theta(F_f) = 0$.

From this general fixed point principle we have:

Theorem 18.5.2. *Let (X,d) be a bounded and complete metric space and $f : X \to X$ a (δ,φ)-contraction. Then, $F_f = \{x^*\}$.*

Proof. We consider the trivial f.p.s. on X. Let $Z := P(X)$, $\theta := \delta$, $\eta(A) := \overline{A}$ and $Y := X$. We are in the conditions of Theorem 18.5.1. So, we have $F_f \neq \emptyset$ and $\delta(F_f) = 0$, i.e., $F_f = \{x^*\}$.

Theorem 18.5.3. *Let X be a Banach space, α an abstract measure of noncompactness on X, $Y \in P_{b,cl,cv}(X)$ and $f : Y \to Y$ an operator. We suppose that:*

(i) f is a continuous operator;

(ii) f is an (α,φ)-contraction.

Then:

(a) $F_f \neq \emptyset$;

(b) F_f is a compact subset of Y.

Proof. Let $(X, P_{cp,cv}(X), M)$ be the f.p.s. of Schauder. Let $Z := P_b(X)$, $\theta := \alpha$ and $\eta(A) = \overline{co}A$. From Theorem 18.5.1 we have that $F_f \neq \emptyset$ and $\alpha(F_f) = 0$, i.e., F_f is compact.□

Theorem 18.5.4. (G. Darbo (1955)) *Let X be a Banach space, $l \in]0,1[$, $Y \in P_{b,cl,cv}(X)$ and $f : Y \to Y$. We suppose that:*

(i) f is a continuous operator;

(ii) f is an (α_K, l)-contraction.

Then:

(a) $F_f \neq \emptyset$;

(b) F_f is a compact subset of Y.

Proof. We take in Theorem 18.5.3., $\alpha := \alpha_K$. □

Theorem 18.5.5. (M.A. Krasnoselskii (1958)) *Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$ and $f, g : Y \to Y$ two operators. We suppose that:*

(i) f is a completely continuous operator;

(ii) g is an l-contraction;

(iii) $f(x) + g(x) \in Y$, for all $x \in Y$.

Then the operator $f + g$ has at least a fixed point.

Proof. We remark that $f + g$ is an (α_K, l)-contraction. The proof follows from Darbo's theorem. □

Remark 18.5.1. From Theorem 18.5.1 we have some results given by: J.M. Ayerbe Toledano, T. Dominguez Benavides and G. López Acedo R[1], J. Appell R[1], J. Banas and K. Goebel R[1], V. Berinde B[7]. See also I.A. Rus [95], S. Czerwik R[1], O. Hadžić R[1] and R[4].

Remark 18.5.2. For Krasnoselskii's Theorem, see K. Deimling R[3], I.A. Rus B[73], B[95], V.I. Istrăţescu B[3], D.R. Smart R[1], T.A. Burton and C. Kirk R[1], C. Avramescu and C. Vladimirescu B[4], Y. Liu and Z. Li R[1], M. Boriceanu B[1], I. Muntean B[2], A. PetruşelB[14], B[15], B[17], G.L. Cain and M.Z. Nashed R[1], B.C. Dhage R[5], W.V. Petryshyn R[4], J. Reinermann R[2], etc.

18.6 Second general fixed point principle

We have:

Theorem 18.6.1. *Let $(X, S(X), M)$ be a f.p.s., (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f \in M(Y)$. We suppose that:*

(i) $A \in Z$, $x \in Y$ imply that $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) f is θ-condensing.

Then:

(a) $I(f) \cap S(X) \neq \emptyset$;

(b) $F_f \neq \emptyset$;

(c) if $F_f \in Z$, then $\theta(F_f) = 0$.

From Theorem 18.6.1. we have:

Theorem 18.6.2. *Let X be a Banach space, $\alpha_{DP} : P_b(X) \to \mathbb{R}_+$ a measure of noncompactness of Danes-Pasicki, $Y \in P_{b,cl,cv}(X)$ and $f : Y \to Y$ a continuous α_{DP}-condensing operator. Then:*

(a) $F_f \neq \emptyset$;

(b) $\alpha_{DP}(F_f) = 0$.

Proof. We consider $S(X) := P_{cp,cv}(X)$, $M(Y) := C(Y, Y)$, $\theta := \alpha_{DP}$ and $\eta(A) := \overline{co}A$. Now, we are in the conditions of Theorem 18.6.1. □

Theorem 18.6.3. (B.N. Sadovskii (1967)) *Let X be a Banach space, α_H*

the Hausdorff measure of noncompactness on X*,* $Y \in P_{b,cl,cv}(X)$ *and* $f : Y \to Y$ *a continuous* α_H*-condensing operator. Then* F_f *is a nonempty compact subset of* Y*.*

Proof. We take in Theorem 18.6.2., $\alpha_{DP} := \alpha_H$.

Theorem 18.6.4. *Let* X *be a Banach space,* $\omega : P_b(X) \to \mathbb{R}_+$ *an abstract measure of weak noncompactness on* X*,* $Y \in P_{b,cl,cv}(X)$*, and* $f : Y \to Y$ *an operator. We suppose that:*

(i) f *is weakly continuous;*

(ii) f *is* ω*-condensing operator.*

Then, F_f *is a nonempty and weak compact subset of* Y*.*

Proof. We consider in Theorem 18.6.1., the fixed point structure of Tychonoff, i.e., $Z := P_b(X)$, $S(X) := P_{wcp,cv}(X)$, $M(Y) := \{g : Y \to Y \mid g$ is weakly continuous$\}$, $\theta := \omega$ and $\eta(A) := \overline{co}^W A$.

Theorem 18.6.5. (G. Emmanuele (1981)) *Let* X *be a Banach space,* ω_D *the De Blasi weak measure of noncompactness on* X *(see De Blasi R[2]),* $Y \in P_{b,wcl,cv}(X)$ *and* $f : Y \to Y$ *an operator. We suppose that:*

(i) f *is weakly continuous;*

(ii) f *is* ω_D*-condensing operator.*

Then, F_f *is a nonempty and weak compact subset of* Y*.*

For more considerations on θ-condensing operators see L. Pasicki R[1], R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina and B.N. Sadovskii R[1], J. Appell R[1], I.A. Rus B[95], etc.

18.7 Fixed point structures with the common fixed point property

A f.p.s. $(X, S(X), M)$ is with the common fixed point property if

$$Y \in S(X), \quad f, g \in M(Y), \quad f \circ g = g \circ f \Rightarrow F_f \cap F_g \neq \emptyset.$$

In this section we shall consider the following

Problem 18.7.1. Which fixed point structures have the common fixed point property?

Example 18.7.1. The Tarski f.p.s. is with the common fixed point property. Indeed, let $(X, \leq)$ be an ordered set and $Y \subset X$ a complete lattice. Let $f, g : X \to X$ be increasing operators such that $f \circ g = g \circ f$. By Tarski's fixed point theorem $F_f \neq \emptyset$, $F_g \neq \emptyset$ and $(F_f, \leq)$, $(F_g, \leq)$ are complete lattices. From $f \circ g = g \circ f$ it follows that $F_f \in I(g)$ and $F_g \in I(f)$. By Tarski's fixed point theorem the operator, $g|_{F_f} : F_f \to F_f$ has at least a fixed point. So, $F_f \cap F_g \neq \emptyset$.

Example 18.7.2. The fixed point structure of contractions has the common fixed point property. More general, if $(X, S(X), M)$ is a f.p.s. such that

$$Y \in S(X), \quad f \in M(Y) \Rightarrow F_f \in S(X),$$

then, $(X, S(X), M)$ has the common fixed point property.

Example 18.7.3. (J.P. Huneke and H.H. Glover R[1]). The Brouwer fixed point structure on $\mathbb{R}$ has not the common fixed point property.

For other counterexample on $\mathbb{R}$ see J.R. Jachymski R[7].

These counterexample give rise to

Problem 18.7.2. Let $(X, S(X), M)$ be a f.p.s. Let $Y \in S(X)$ and $f, g \in M(Y)$ be such that $f \circ g = g \circ f$. In which conditions we have that $F_f \cap F_g \neq \emptyset$?

We have

Theorem 18.7.1. *Let $(X, S(X), M)$ be a f.p.s. having the common fixed point property and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f, g \in M(Y)$.*

We suppose that:

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) $f \circ g = g \circ f$;

(iii) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that:

$$\theta(f(A) \cup g(A)) \leq \varphi(\theta(A)), \quad \text{for all} \quad A \in I(f) \cap I(g) \cap Z.$$

Then:

(a) $F_f \cap F_g \neq \emptyset$

(b) if $F_f \cap F_g \in Z$, then $\theta(F_f \cap F_g) = 0$.

For other results on the Problem 18.37.1. and 18.7.2. see I.A. Rus B[95].

For the common fixed point theory, see Chapters 14, 24.22, 24.23.

18.8 Fixed point structures with the coincidence property

A f.p.s. $(X, S(X), M)$ is with the coincidence property if

$$Y \in S(X), \quad f, g \in M(Y), \quad f \circ g = g \circ f \Rightarrow C(f, g) \neq \emptyset.$$

Example 18.8.1. Each f.p.s. with the common fixed point property is a f.p.s. with the coincidence property.

Example 18.8.2. (W.A. Horn R[1]). Let $X := \mathbb{R}$, $S(X) := P_{cp,cv}(\mathbb{R})$, and $M(Y) := C(Y, Y)$. Then $(\mathbb{R}, P_{cp,cv}(\mathbb{R}), M)$ is a f.p.s. with the coincidence property.

The following problem is a very hard one.

Problem 18.8.3. Which are the f.p.s. with the coincidence property?

This problem has the following well known particular cases.

Horn's conjecture. *The Schauder f.p.s. $(X, P_{cp,cv}(X), M)$ has the coincidence property.*

Schauder's conjecture. *Let X be a Banach space and $Y \in P_{cl,cv}(X)$. If $f : Y \to Y$ is a continuous operator such that $\overline{f^n(Y)}$ is compact for some $n \in \mathbb{N}^*$, then f has at least a fixed point.*

It is clear that if Horn's conjecture is a theorem then Schauder's conjecture is a theorem.

For the above problems see F.E. Browder (Ed.) R[2], R. Sine R[1], I.A. Rus B[95], R.D. Nussbaum R[2], F.E. Browder R[7], K. Deimling R[2], W.A. Horn R[1], V. Seda R[2], J. Eells and G. Fournier R[1].

For the coincidence point theory, see Chapters 15, 16 and the following papers: L. Cesari R[1], A. Granas and J. Dugundji R[1], E.U. Tarafdar and M.S.R. Chowdhury R[1], R.F. Brown, M. Furi, L. Górniewicz and B. Jiang R[1] and the references therein.

18.9 Other results

For other aspects of the fixed point structure theory see S. Budişan B[1], A. Horvat-Marc B[1], E. Miklos B[1], S. Mureşan B[2], A. Sîntuamărian B[7], M.A. Şerban B[2] and B[6], I.A. Rus B[95], etc.

For other results for (θ, φ)-contractions and θ-condensing operators see J. Appell R[3], J.S. Bae R[1], A.I. Ban and S.G. Gal R[1], S. Czerwik R[1], J. Danes R[1], F.S. De Blasi R[2], T. Dominguez Benavides R[1], J. Eells and G. Fournier R[2], J. Eisenfeld and V. Lakshmikantham R[1], G. Emmanuele R[1], M. Furi and M. Martelli R[1], M. Furi and M. Vignoli R[1], O. Hadžić R[4], J.K. Hale R[1], R.H. Martin R[1], R. Precup R[13], etc.

Chapter 19

Fixed point structures for multivalued operators

For **Precursors, Guidelines and General references** see Chapter 18.

19.0 Notations

Let X and Y be two sets. Then we denote by $\mathbb{M}^0(X,Y)$ the set of all multivalued operators $T : X \multimap Y$. If $T : X \multimap X$ is a multivalued operator then:

$F_T := \{x \in X \mid x \in T(x)\}$, the fixed point set of T,

$(SF)_T := \{x \in X \mid T(x) = \{x\}\}$, the strict fixed point set of T,

$P_T := \bigcup\limits_{n\in\mathbb{N}^*} F_{T^n}$, the periodic point set of T,

$(SP)_T := \bigcup\limits_{n\in\mathbb{N}^*} (SF)_{T^n}$, the strict periodic point set of T.

19.1 Examples of fixed point structures for multivalued operators

Let X be a nonempty set. By definition:

- A triple $(X, S(X), M^0)$ is a fixed point structure on X (f.p.s.) if:

(i) $S(X) \subset P(X)$ and $S(X) \neq \emptyset$;

(ii) $M^0 : P(X) \multimap \bigcup_{Y \in P(X)} \mathbb{M}^0(Y)$, $Y \multimap M^0(Y) \subset \mathbb{M}^0(Y)$ is an operator such that if $Z \subset Y$, $Z \neq \emptyset$, then

$$M^0(Z) \supset \{T|_Z \mid T \in M^0(Y) \text{ and } Z \in I(T)\};$$

(iii) every $Y \in S(X)$ has the fixed point property with respect to $M^0(Y)$.

• A triple $(X, S(X), M^0)$ which satisfies (i) and (iii), in the above definition, and the condition

(ii') $M : P(X) \multimap \cup \mathbb{M}^0(Y)$, $Y \multimap M^0(Y) \subset \mathbb{M}(Y)$ is an operator;
is called a large fixed point structure (l.f.p.s.).

Example 19.0.1. (The trivial f.p.s.) X is a nonempty set, $S(X) := \{\{x\} \mid x \in X\}$ and $M^0(Y) := \mathbb{M}^0(Y)$.

Example 19.0.2.(The fixed point structure of contractions (Avramescu-Markin-Nadler) (X, d) is a complete metric space, $S(X) := P_{cl}(X)$ and $M^0(Y) := \{T : Y \to P_{cl}(Y) \mid T \text{ is a contraction}\}$.

Example 19.0.3. (The f.p.s. of graphic contraction (I.A. Rus (1975))) (X, d) is a complete metric space, $S(X) := P_{cl}(X)$ and $M^0(Y) := \{T : Y \to P_{cl}(Y) \mid$ there exist $\alpha, \beta \in \mathbb{R}_+$, $\alpha + \beta < 1$, such that $H(T(x), T(y)) \leq \alpha d(x, y) + \beta D(y, T(y))$, for every $x \in X$ and every $y \in T(x)$, and T is a closed operator$\}$.

Example 19.0.4. (The f.p.s. of nonexpansive operators (T.C. Lim (1974)) X is a uniformly convex Banach space, $S(X) := P_{b,cl,cv}(X)$ and $M^0(Y) := \{T : Y \to P_{cp}(Y) \mid T \text{ is nonexpansive}\}$.

Example 19.0.5. (The f.p.s. of contractive operators (R.E. Smithson (1971))) (X, d) is a complete metric space, $S(X) := P_{cp}(X)$ and $M^0(Y) := \{T : Y \to P_{cp}(Y) \mid T \text{ is a contractive operator}\}$.

Example 19.0.6. (The f.p.s. of S. Kakutani (1941)) $X = \mathbb{R}^n$, $S(X) := P_{cp,cv}(X)$ and $M^0(Y) := \{T : Y \to P_{cp,cv}(Y) \mid T \text{ is a u.s.c. operator}\}$.

Example 19.0.7. (The f.p.s. of Fan-Glicksberg (1952)) X is a Hausdorff locally convex topological space, $S(X) := P_{cp,cv}(X)$ and $M^0(Y) := \{T : Y \to P_{cp,cv}(Y) \mid T \text{ is a u.s.c. operator}\}$. If X is a Banach space then this f.p.s. is called the f.p.s. of Bohnenblust-Karlin (1950).

Example 19.0.8. (The f.p.s. of F.E. Browder (1968)) X is a Hausdorff topological linear space, $S(X) := P_{cp,cv}(X)$ and $M^0(Y) := \{T : Y \to P_{cv}(Y) \mid T^{-1}(y) \text{ is an open subset in } Y, \text{ for all } y \in Y\}$.

From the above examples, it is clear that for any fixed point theorem we have an example of a f.p.s. or of a l.f.p.s.

The following notion is fundamental in the f.p.s. theory of multivalued operators.

Let $(X, S(X), M^0)$ be a f.p.s., $S(X) \subset Z \subset P(X)$, $\theta : Z \to \mathbb{R}_+$ and $\eta : P(X) \to P(X)$. The pair (θ, η) is a compatible pair with $(X, S(X), M^0)$ if:

(i) η is a closure operator, $S(X) \subset \eta(Z) \subset Z$ and $\theta(\eta(Y)) = \theta(Y)$, for all $Y \in Z$;

(ii) $F_\eta \cap Z_\theta \subset S(X)$.

Example 19.0.9. Let $(X, S(X), M^0)$ be the f.p.s. of nonexpansive operators, $Z := P_b(X)$, $\theta := \alpha_K$ or α_H and $\eta(A) := \overline{A}$. Then the pairs (α_K, η) and (α_H, η) are compatible with $(X, S(X), M^0)$.

Example 19.0.10. Let $(X, S(X), M^0)$ be the f.p.s. of Kakutani, $Z := P_b(X)$, $\theta := \alpha_K$ or α_H and $\eta(A) := \overline{co}A$. Then the pairs (α_K, η), (α_H, η) are compatible with $(X, S(X), M^0)$.

19.2 Examples of strict fixed point structures

Let X be a nonempty set. By definition:

• A triple $(X, S(X), M^0)$ is a strict fixed point structure on X (s.f.p.s.) if:

(i) $S(X) \subset P(X)$ and $S(X) \neq \emptyset$;

(ii) $M^0 : P(X) \multimap \bigcup_{Y \in P(X)} M^0(Y)$, $Y \multimap M^0(Y) \subset \mathbb{M}^0(Y)$ is an operator such that if $Z \subset Y$, $Z \neq \emptyset$, then

$$M^0(Z) \supset \{T|_Z \mid T \in M^0(Y) \text{ and } Z \in I(T)\};$$

(iii) every $Y \in S(X)$ has the strict fixed point property w.r.t. $M^0(Y)$, i.e.,

$$Y \in S(X) \text{ and } T \in M^0(Y) \Rightarrow \exists\, x \in Y : \ T(x) = \{x\}.$$

• A triple $(X, S(X), M^0)$ which satisfies (i) and (iii), in the above definition, and the condition

(ii') $M^0 : P(X) \multimap \bigcap_{Y \in P(X)} \mathbb{M}^0(Y)$, $Y \multimap M^0(Y) \subset \mathbb{M}^0(Y)$;

is called a large strict fixed point structure (l.s.f.p.s.).

Example 19.1.1. The trivial f.p.s. is a s.f.p.s.

Example 19.1.2. (The s.f.p.s. of S. Reich (1972)) (X, d) is a complete metric space, $S(X) := P_{cl}(X)$ and $M^0(Y) := \{T : Y \to P_b(Y) \mid$ there exist $a, b, c \in \mathbb{R}_+,\ a + b + c < 1$ such that $\delta(T(x), T(y)) \leq ad(x, y) + b\delta(x, T(x)) + c\delta(y, T(y))$, for all $x, y \in Y\}$.

Example 19.1.3. (The s.f.p.s. of H.W. Corley (1986)) (X, d) is a metric space, $S(X) := P_{cp}(X)$ and $M^0(Y) := \{T : Y \to P_{cl}(Y) \mid T$ is reflexive, antisymmetric and transitive$\}$.

Let $(X, S(X), M^0)$ be a s.f.p.s., $S(X) \subset Z \subset P(X)$, $\theta : Z \to \mathbb{R}_+$ and $\eta : P(X) \to P(X)$. The pair (θ, η) is a compatible pair with $(X, S(X), M^0)$ if:

(i) η is a closure operator, $S(X) \subset \eta(Z) \subset Z$ and $\theta(\eta(Y)) = \theta(Y)$, for all $Y \in Z$;

(ii) $F_\eta \cap Z_\theta \subset S(X)$.

19.3 (θ, φ)-contractions and θ-condensing operators

Let X be a nonempty set, $Z \subset P(X)$, $Y \subset X$, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a comparison function, $\theta : Z \to \mathbb{R}_+$ a functional. By definition:

• An operator $T : Y \multimap X$ is called a strong (θ, φ)-contraction if:

(i) $A \in P(Y) \cap Z$ implies $T(A) \in Z$;

(ii) $\theta(T(A)) \leq \varphi(\theta(A))$, for all $A \in P(Y) \cap Z$.

• An operator $T : Y \multimap Y$ is called a (θ, φ)-contraction if:

(i) $A \in P(Y) \cap Z$ implies $T(A) \in Z$;

(ii') $\theta(T(A)) \leq \varphi(\theta(A))$, for all $A \in P(Y) \cap Z \cap I(T)$.

For some examples of strong (θ, φ)-contractions and of (θ, φ)-contractions see I.A. Rus B[95], pp. 153-156.

Let X be a nonempty set $Z \subset P(X)$, $Y \subset X$ and $\theta : Z \to \mathbb{R}_+$ a functional. By definition:

• An operator $T : Y \multimap X$ is called strong θ-condensing if:

(i) $A \in P(Y) \cap Z$ implies $T(A) \in Z$;

(ii) $A \in P(Y) \cap Z$, $\theta(A) \neq 0$ implies $\theta(T(A)) < \theta(A)$.

• An operator $T : Y \multimap Y$ is called θ-condensing if:

(i) $A \in P(Y) \cap Z$ implies $T(A) \in Z$;

(ii') $A \in I(T) \cap Z$, $\theta(A) \neq 0$, implies $\theta(T(A)) < \theta(A)$.

For some examples of strong θ-condensing and of θ-condensing operators see I.A. Rus B[95], pp. 156-158.

19.4 First general fixed point principle for multivalued operators

Let X be a nonempty set. We have

Theorem 19.2.1. *Let $(X, S(X), M^0)$ be a f.p.s. on the set X and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ be a compatible pair with $(X, S(X), M^0)$. Let $Y \in \eta(Z)$ and $T \in M^0(Y)$. We suppose that:*

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) T is a (θ, φ)-contraction.

Then:

(a) $I(T) \cap S(X) \neq \emptyset$;

(b) $F_T \neq \emptyset$;

(c) if $F_T \in Z$ and $T(F_T) = F_T$, then $\theta(F_T) = 0$;

(d) if:

(1) T is a strong (θ, φ)-contraction;

(2) $A, B \in Z$, $A \subset B \Rightarrow \theta(A) \leq \theta(B)$;

(3) $F_T \in Z$;

then, $\theta(F_T) = 0$.

Theorem 19.2.1'. *Let $(X, S(X), M^0)$ be a s.f.p.s. and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M^0)$. Let $Y \in \eta(Z)$ and $T \in M^0(Y)$. We suppose that:*

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) T is a (θ, φ)-contraction.

Then:

(a) $I(T) \cap S(X) \neq \emptyset$;

(b) $(SF)_T \neq 0$;

(c) if $(SF)_T \in Z$, then $\theta((SF)_T) = 0$.

From these general fixed point principle we have:

Theorem 19.2.2. *Let (X, d) be a bounded and complete metric space,*

$\alpha : P(X) \to \mathbb{R}_+$ an abstract measure of noncompactness on X and $T : X \to P_{cl}(X)$. We suppose that:

(i) T is an (α, φ)-contraction;

(ii) T is a contractive operator.

Then:

(a) $F_T \neq \emptyset$;

(b) if $T(F_T) = F_T$, then $\alpha(F_T) = 0$;

(c) if T is a strong (α, φ)-contraction, then $\alpha(F_T) = 0$.

Proof. We take in Theorem 19.2.1, the f.p.s. of Smithson, $Z := P(X)$, $\theta := \alpha$ and $\eta(A) = \overline{A}$.

Theorem 19.2.3. *Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$ and α an abstract measure of noncompactness on the Banach space X. Let $T : Y \to P_{cp,cv}(Y)$ a multivalued operator. We suppose that:*

(i) T is u.s.c.;

(ii) T is an (α, φ)-contraction.

Then:

(a) $F_T \neq \emptyset$;

(b) if T is a strong (α, φ)-contraction, then $\alpha(F_T) = 0$.

Proof. We consider in Theorem 19.2.1, the f.p.s. of Bohnenblust-Karlin, $Z := P_b(X)$, $\theta := \alpha$ and $\eta(A) := \overline{co}A$.

Theorem 19.2.4. (S. Czerwik R[1] (1980)) *Let X be a Banach space, $Y \in P_{b,clc,cv}(X)$ and $T : Y \to P_{cp,cv}(Y)$ an operator. We suppose that:*

(i) T is u.s.c.;

(ii) T is an (α_H, φ)-contraction.

Then:

(a) $F_T \neq \emptyset$;

(b) if $T(F_T) = F_T$, then $\alpha_H(F_T) = 0$;

(c) if T is a strong (α_H, φ)-contraction, then $\alpha_H(F_T) = 0$.

Proof. We take $\alpha := \alpha_H$ in Theorem 19.2.3.

Theorem 19.2.5. (W.V. Petryshyn and Fitzpatrick R[1] (1974)) *Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$, $T : X \to P_{cp,cv}(X)$ and $S : Y \to P_{cp,cv}(Y)$. We suppose that:*

(i) T is a strong (α_H, l)-contraction ($l \in]0, 1[$);

(ii) $Y \in I(T)$*;*

(iii) T *and* S *are u.s.c.;*

(iv) S *is a compact operator;*

(v) $T(x) + S(x) \in Y$*, for all* $x \in Y$*.*

Then, $F_{T+S} \neq \emptyset$*.*

Proof. We take $\alpha := \alpha_H$ in Theorem 19.2.3 and we remark that $T + S : Y \multimap Y$ is a strong (α_H, l)-contraction.

Theorem 19.2.6. *Let* X *be a Banach space,* $\omega : P_b(X) \to \mathbb{R}_+$ *an abstract measure of weak noncompactness on* X*,* $Y \in P_{b,wcl,cv}(X)$ *and* $T : Y \to P_{wcp,cv}(Y)$*. We suppose that:*

(i) T *is weakly u.s.c.;*

(ii) T *is an* (ω, φ)*-contraction.*

Then:

(a) $F_T \neq \emptyset$*;*

(b) if $F_T \in I(T)$*, then* $\omega(F_T) = 0$*;*

(c) if T *is a strong* (ω, φ)*-contraction, then* $\omega(F_T) = 0$*.*

Proof. Let $S(X) := P_{wcp,cv}(X)$ and $M^0(A) := \{T : A \to P_{wcp,cv}(A) \mid T$ is weakly u.s.c.$\}$. Then by a theorem of J. Ewert R[1], $(X, S(X), M^0)$ is a f.p.s. We name it the fixed point structure of Ewert. Now, we take in Theorem 19.2.1, $Z := P_b(X)$, $\theta := \omega$, $\eta(A) := \overline{co}^W A$.

If we take in Theorem 19.2.6, $\omega = \omega_D$, $\varphi(t) = lt$, $0 < l < 1$, then we have

Theorem 19.2.7. (J. Ewert R[1] (1986)). *Let* X *be a Banach space,* ω_D *the De Blasi measure of weak noncompactness on* X*,* $Y \in P_{b,wcl,cv}(X)$ *and* $T : Y \to P_{wcp,cv}(Y)$*. We suppose that:*

(i) T *is weakly u.s.c.;*

(ii) T *is an* (ω_D, l)*-contraction.*

Then:

(a) $F_T \neq \emptyset$*;*

(b) if $F_T \in I(T)$*, then* $\omega_D(F_T) = 0$*;*

(c) if T *is a strong* (ω_D, l)*-contraction, then* $\omega_D(F_T) = 0$*.*

Theorem 19.2.8. *Let* (X, d) *be a bounded and complete metric space and* $T : X \to P(X)$ *a* (δ, φ)*-contraction. Then:*

(a) $(SF)_T = \{x^*\}$*;*

(b) $F_T = (SF)_T$.

19.5 Second general fixed point principle for multivalued operators

The main tresult of this section are:

Theorem 19.3.1. *Let* X *be a nonempty set and* $(X, S(X), M^0)$ *be a f.p.s. and* (θ, η) *a compatible pair with* $(X, S(X), M^0)$*. Let* $Y \in \eta(Z)$ *and* $T \in M^0(Y)$*. We suppose that:*

(i) $A \in Z$, $x \in Y$ *imply* $A \cup \{x\} \in Z$ *and* $\theta(A \cup \{x\}) = \theta(A)$;

(ii) T *is* θ*-condensing.*

Then:

(a) $F_T \neq \emptyset$;

(b) if $F_T \in Z$ *and* $T(F_T) = F_T$*, then* $\theta(F_T) = 0$.

Theorem 19.3.2'. *Let* X *be a nonempty set,* $(X, S(X), M^0)$ *a s.f.p.s. on* X *and* (θ, η) *a compatible pair with* $(X, S(X), M^0)$*. Let* $Y \in \eta(Z)$ *and* $T \in M^0(Y)$*. We suppose that:*

(i) $A \in Z$, $x \in Y$ *imply* $A \cup \{x\} \in Z$ *and* $\theta(A \cup \{x\}) = \theta(A)$;

(ii) T *is* θ*-condensing.*

Then:

(a) $(SF)_T \neq \emptyset$;

(b) if $(SF)_T \in Z$*, then* $\theta((SF)_T) = 0$.

For these general results and for some applications see I.A. Rus B[95]. See also J. Appell, E. De Pascale, H.T. Nguyen and P.P. Zabrejko R[1], Yu.G. Borisovich, B.D. Gelman, A.D. Myskis and V.V. Obukhovskii R[1]. S. Czerwik R[1], O. Hadzić R[1], M. Kamenskii, V. Obukhovskii and P. Zecca R[1] and the references therein.

19.6 Other results

For the fixed point structures with the common fixed point property see I.A. Rus B[95] pp. 181-190, while for the fixed point structure with the coincidence property see pp. 191-197.

For other results for multivalued (θ, φ)-contractions and θ-condensing operators, see J. Appell, E. De Pascale H.T. Nguyen and P.P. Zabrejko R[1], C.C. Bui R[1], K. Deimling R[1], J. Ewert R[1], P.M. Fitzpatrick and W.V. Petryshyn R[1], L. Górniewicz R[1], C.J. Himmelberg, J.R. Porter and F.S. Van Vleck R[1], C. Horvath R[2], M. Kamenskii, V. Obukhovskii and P. Zecca R[1], W.V. Petryshyn and P.M. Fitzpatrik R[1], etc.

Chapter 20

Fixed point theory for operators on product spaces

Precursors: P. Bohl (1904), L.E.J. Brouwer (1910).
Guidelines: K. Kuratowski (1930), W.L. Strother (1953), S. Ginsburg (1954), E. Dyer (1956), E. Connell (1959), V. Klee (1960), S.B. Prešič (1965), F.E. Browder (1966), R. Fiorenza (1966), P. Zecca (1968), S.B. Nadler jr. (1968), R.H. Bing (1969), C. Avramescu (1970), M.W. Hirsch and C.C. Pugh (1970), R.B. Thompson (1970), I.A. Rus (1972), H. Cohen (1973), O. Hadžić (1973), J. Matkowski (1973), A. Dold (1974).
General references: R.F. Brown R[3] and R[4], A. Granas R[1], M.A. Şerban B[2], R. Manka R[2], T. van der Walt R[1], I.A. Rus B[73], R. Espínola and W.A. Kirk R[3], J. Guillerme R[2], B. Rzepecki R[4].

20.0 Basic problems

Let (X, τ) be a topological space. By definition, a subset Y of X is called:

(a) arcwise connected if every pair of its points constitutes end points of an arc contained in Y;

(b) a continuum if it is arcwise connected and compact;

(c) a manifold if it is compact and every point of it has a neighborhood homeomorphic to $\mathbb{R}^n$ or, in the case of boundary points, to $\mathbb{R}^{n-1} \times \mathbb{R}_+$.

We will present now some basic problems of the fixed point theory for operators on product spaces.

Kuratowski's Problem. If X and Y are locally connected metric continuum with the topological fixed point property, does $X \times Y$ have the topological fixed point property ?

Ia a more general setting, we have:

Brown's Problem. If the manifolds X and Y are having the topological fixed point property, does $X \times Y$ have the topological fixed point property ?

Browder's Problem. Let X be a real Banach space, Y a closed, bounded, convex subset of X. In which conditions on the operator $f : X \times X \to Y$, the operator $g : Y \to Y$, $g(x) := f(x, x)$ has at least a fixed point ?

In the multivalued case, we have:

Strother's Problem. Let X be a topological space with the fixed point property with respect to continuous multivalued operators $T : X \to P(X)$. In which conditions on X, the product space $X \times X$ has the fixed point property with respect to continuous multivalued operators ?

For other details on these basic problems of the fixed point theory for operators on product spaces, see the above General References. Other problems of this topic are treated in the next sections of this chapter.

20.1 $f : X \times Y \to X \times Y$

Let X and Y be two sets. The problem is to give fixed point theorems for operators $f : X \times Y \to X \times Y$.

We have:

Theorem 20.1.1. (M. A. Şerban, B[6]). *Let (U, S_1, M_1) and (V, S_2, M_2) be two fixed point structures. Let $X \in S_1$, $Y \in S_2$ and $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$, such that:*

(i) $f_1(\cdot, y) \in M_1(X)$, for all $y \in Y$

(ii) the operator $f_2(x, \cdot) \in M_2(Y)$, for all $x \in X$

(iii) the operator $P \circ Q : X \multimap X$ has at least a fixed point, or the operator

$Q \circ P : Y \multimap Y$ has at least a fixed point, where

$$P : Y \multimap X, \quad P(y) := \{x \in X |\ x = f_1(x, y)\}$$

$$Q : X \multimap Y, \quad Q(x) := \{y \in Y |\ y = f_2(x, y)\}.$$

Then the operator f has at least a fixed point.

From this abstract result we have the following consequences:

Theorem 20.1.2. (M. A. Şerban, B[6]). *Let $(X, \leq)$ be a complete lattice, $(Y, \leq)$ be a right inductively ordered set and $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$. We suppose that:*

(i) $f_1(\cdot, y)$ is monotone increasing, for each $y \in Y$

(ii) $y \leq f_2(x, y)$, for each $x \in X$, $y \in Y$

(iii) for each $y \in Y$ and $x \in F_{f_1(\cdot,y)}$ there exists $z \in F_{f_2(x,\cdot)}$, such that $y \leq z$.

Then, $F_f \neq \emptyset$.

Theorem 20.1.3. (M. A. Şerban, B[6]). *Let (X, d) and (Y, ρ) two complete metric space and $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$. We suppose that:*

(i) $f_1(\cdot, y) : X \to X$ is a_1-contraction, for each $y \in Y$

(ii) $f_2(x, \cdot) : Y \to Y$ is a_2-contraction, for each $x \in X$

(iii) $f_1(x, \cdot) : Y \to X$ is L_1-Lipschitz, for each $x \in X$

(iv) $f_2(\cdot, y) : X \to Y$ is L_2-Lipschitz, for each $y \in Y$

(v) $\dfrac{L_1 L_2}{(1 - a_1)(1 - a_2)} < 1$.

Then, $F_f \neq \emptyset$.

Remark 20.1.1. For other consequences of Theorem 20.1.1., see M. A. Şerban B[6].

20.2 $f : X^k \to X$

Let X be a set and $f : X^k \to X$ an operator. We consider the following operators:

$$\widetilde{f} : X \to X, \quad \widetilde{f}(x) := f(x, \ldots, x),$$

and

$$A_f : X^k \to X^k, \quad A_f(u_1, \ldots, u_k) := (u_2, \ldots, u_k, f(u_1, \ldots, u_k)).$$

We have:

Theorem 20.2.1. (M. A. Şerban, B[7]). *Let* (X, d) *be a complete metric space and* $f : X^k \to X$*. Suppose that there exists* $\varphi : R_+^k \to R$ *such that:*

(i) φ *is a (c)-comparison function*

(ii) $d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq \varphi(d(x_0, x_1), \ldots, d(x_{k-1}, x_k))$*, for all* $x_0, \ldots, x_k \in X$

(iii) $\varphi(r, 0, \ldots, 0) + \varphi(0, r, 0, \ldots, 0) + \cdots + \varphi(0, \ldots, 0, r) \leq \varphi(r, \ldots, r)$*.*

Then:

(a) the operator $\widetilde{f}$ *is PO*

(b) the operator A_f *is PO*

(c) if ψ*,* $\psi(r) := \varphi(r, \ldots, r)$ *is positive semihomogeneous, then* A_f *is c-PO with*

$$c = k \sum_{i=0}^{\infty} \psi^i(1)$$

(d) if ψ *is positive semihomogeneous, then* $\widetilde{f}$ *is c-Po with*

$$c = \sum_{i=0}^{\infty} \psi^i(1).$$

Remark 20.2.1. For other results of the above type see: M.A. Şerban B[7], G. Caius B[1], I.A. Rus B[56].

20.3 Other results

For other results see: R.F. Brown R[3], R[4], H. Cohen R[1], S. Czerwik R[1], A. Granas R[1], O. Hadžić R[5], R. Mańka R[1], T.B. McLean and S.B. Nadler jr. R[1], F. Robert R[2], M. Albu B[1], S. András B[4], C. Avramescu B[4], G. Dezsö B[2], V. Mureşan B[1], M. Nicolescu B[1], A. Petruşel B[22], I.A. Rus B[70], B[72], B[81], B[83] and B[84], M. Turinici B[12], M.A. Şerban B[2], P. Bassanini and M. Galaverni R[1], S.B. Persić R[1], etc.

For set-to-set operators on cartesian product see B. Breckner B[1].

For the fixed point theory of triangular operators see M.W. Hirsch and C.C. Pugh R[1], S. András B[1], C. Bacoţiu B[1], I.A. Rus B[6], B[8] and B[9], M.A. Şerban B[1], B[2] and B[5].

Chapter 21

Fixed point theory for nonself operators

Precursors: H. Poincaré (1884), P. Bohl (1904), S. Bernstein (1912).
Guidelines: B. Knaster, C. Kuratowski and S. Mazurkiewicz (1929), K. Borsuk (1931), J. Leray and J. Schauder (1934), E. Sperner (1934), E.H. Rothe (1938), S. Eilenberg and D. Montgomery (1946), O.H. Hamilton (1948), H.H. Schaefer (1955), A. Granas (1959), M.A. Krasnoselskii (1960), K. Fan (1961), F.E. Browder (1967), R.L. Frum-Ketkov (1967), W.V. Petryshyn (1967), F.E. Browder (1968), K. Fan (1969), R.D. Nussbaum (1969), S. Reich (1971), A.J.B. Potter (1972), J.A. Gatica and W.A. Kirk (1974), R.H. Martin (1976), J. Caristi (1976), R.F. Brown (1984), L. Pasicki (1985), I.A. Rus (1986), R. Precup (1991), A. Petruşel (1993), I.A. Rus (1993), M. Frigon (1996), A. Jiménez-Melado and C.H. Morales (2006), V. Berinde (2007), S. Reich and A.J. Zaslavski (2008).
General references: D. O'Regan and R. Precup B[2], T. van der Walt R[1], K. Deimling R[3], T.E. Williams R[1] and R[2], A. Granas and J. Dugundji R[1], J. Jaworowski, W.A. Kirk and S. Park R[1], M. Frigon R[1], R.F. Brown R[2] and R[7], I.A. Rus B[95], A. Jiménez-Melado C.H. Morales R[1], A. Chiş and R. Precup B[1], M.K. Kwong R[1], D. Azé and J.-N. Corvellec R[1], M. Kamenskii and M. Quincampoix R[1].

21.0 Basic fixed point principles for nonself operators

We begin our considerations on fixed point theory of nonself operators with the following classical results.

Theorem 21.0.1. *Let* (X, d) *a complete metric space,* $x_0 \in X$ *and* $r > 0$. *If* $f : B(x_0; r) \to X$ *is an* a*-contraction and* $d(x_0, f(x_0)) < (1 - a)r$, *then* f *has a unique fixed point.*

Theorem 21.0.2. (Leray-Schauder's Continuation Principle.) *Let* X *be a Banach space,* $Y \subset X$ *a bounded open subset and* $H : \overline{Y} \times [0, 1] \to X$ *be a completely continuous operator. We suppose that:*

(i) $H(x, \lambda) \neq x$ *for all* $x \in \partial Y$ *and* $\lambda \in [0, 1]$;

(ii) $\deg(1_X - H(\cdot, 0), Y, 0) \neq 0$.

Then, $F_{H(\cdot,1)} \neq \emptyset$.

Theorem 21.0.3. (Granas's Topological Transversality Principle.) *Let* X *be a Banach space,* $Y \subset X$ *a bounded open subset and* $H : \overline{Y} \times [0, 1] \to X$ *be a completely continuous operator. We suppose that:*

(i) $H(x, \lambda) \neq x$ *for all* $x \in \partial Y$ *and* $\lambda \in [0, 1]$;

(ii) $H(\cdot, 0)$ *is essential, i.e., each completely continuous extension to* $\overline{Y}$ *of* $H(\cdot, 0)|_{\partial Y}$ *has a fixed point.*

Then, $F_{H(\cdot,1)} \neq \emptyset$.

Theorem 21.0.4. (Hamilton's Theorem.) *Let* $Y \subset \mathbb{R}^2$ *be a two-cell and* $f : Y \to \mathbb{R}^2$ *a function. We suppose that:*

(i) f *is continuous;*

(ii) f *is open;*

(iii) $Y \subset f(Y)$.

Then $F_f \neq \emptyset$.

Theorem 21.0.5. (Reich's Theorem) (Reich R[11]) *Let* X *be a Banach space,* $Y \subset X$ *a closed convex subset and* $T : Y \to P_{cp,cv}(X)$ *a multivalued operator. We suppose that:*

(i) T *is a contraction;*

(ii) T *is weakly inward, i.e.*

$$T(x) \subset \overline{I_Y(x)} := cl\{y \in X \mid y = x + t(z - x) \text{ for some } z \in Y,\ t \geq 0\}.$$

Then, $F_T \neq \emptyset$*.*

Theorem 21.0.6. (Browder's Theorem.) (F.E. Browder R[7]) *Let* X *be a Hausdorff locally convex topological space. The algebraic boundary* $\partial_a Y$ *of a convex subset* $Y \subset X$ *is by definition*

$$\partial_a Y := \{y \in Y \mid \exists\ z \in X :\ y + \lambda z \notin Y \text{ for all } \lambda > 0\}$$

Let $f : Y \to X$ *be an operator. We suppose that:*

(i) Y *is a nonempty compact convex subset of* X*;*

(ii) f *is continuous;*

(iii) for each $y_0 \in \partial_a(Y)$ *there exist a point* z_0 *in* Y *and a real number* $\lambda > 0$ *such that* $f(y_0) - y_0 = \lambda(z_0 - y_0)$*.*

Then, $F_f \neq \emptyset$*.*

Theorem 21.0.7. (Browder's Theorem.) (F.E. Browder R[4]) *Let* X *be a Hausdorff locally convex topological space. Let* Y *be a nonempty compact convex subset of* X*. Let* $T : Y \to P_{cl,cv}(X)$ *be a multivalued operator. We suppose that:*

(i) T *is u.s.c.*

(ii) for each $y_0 \in \partial_a Y$ *there exist* $z_0 \in Y$*,* $y_0 \in T(y_0)$ *and* $\lambda > 0$ *such that* $y_0 - z_0 = \lambda(u_0 - z_0)$*.*

Then, $F_T \neq \emptyset$*.*

Theorem 21.0.8. (Krasnoselskii's Theorem.) (Krasnoselskii R[4]) *Let* $(X, \|\cdot\|)$ *be a Banach space and* $K \subset X$ *be a closed convex cone in* X*. Let* $0 < a < b$*,* $a, b \in \mathbb{R}$*. Let us denote:*

$$K_a := \{x \in K \mid \|x\| = a\},$$
$$K_b := \{x \in K \mid \|x\| = b\},$$
$$K_{a,b} := \{x \in K \mid a \leq \|x\| \leq b\}.$$

Let $f : K_{a,b} \to K$ *be an operator. We suppose that*

(i) f *is completely continuous;*

(ii) $\|f(x)\| \geq \|x\|$*, for all* $x \in K_a$*,*

$\|f(x)\| \leq \|x\|$*, for all* $x \in K_b$*.*

Then, $F_f \neq \emptyset$*.*

In this theorem instead of condition (ii) we can put the following condition:

(ii') $\|f(x)\| \leq \|x\|$, for all $x \in K_a$,
$\|f(x)\| \geq \|x\|$, for all $x \in K_b$.

Theorem 21.0.9. (Nonlinear Alternative of Leray-Schauder type.) *Let X be a locally convex Hausdorff topological space and $Y \subset X$ a subset of X. Let $U \subset Y$ and $f : \overline{U} \to Y$. We suppose that:*

(i) Y is a convex set;

(ii) U is open in Y with $0 \in U$;

(iii) f is completely continuous.

Then, either

(i) $F_f \neq \emptyset$

or

(ii) there is a point $x \in \partial U$ and $\lambda \in]0,1[$ with $x = \lambda f(u)$.

For the above results see the General References.

For boundary conditions and inwardness conditions see T.E. Williams R[1], R[2], and W.A Kirk and C.H. Morales R[1]. For interior conditions see J.-M. Antonio and C.H. Morales. For the condition $Y \subset f(Y)$ see, for example, T.L. Hicks and L.M. Saliga R[1].

For retraction principle in the fixed point theory of nonself operators see R.F. Brown R[2], R[6] and I.A. Rus B[95]. See also Chapter 1 and 2.

For other results and applications see R.P. Agarwal, D. O'Regan and R. Precup B[1], J. Danes and J. Kolomy R[1], D.R. Anderson and R.I. Avery R[1], S.P. Singh, M. Singh and B. Watson R[1], etc.

21.1 Continuation principles for generalized contractions

Let (X, d) be a complete metric space and $Y \subset X$ a domain of X. Let $f, g : \overline{Y} \to X$ be two generalized contractions. We say that they are homotopic if there exists continuous $H : \overline{Y} \times [0,1] \to X$ such that:

(i) $H(\cdot, 0) = f$, $H(\cdot, 1) = g$

(ii) $H(x,t) \neq x$ for all $x \in \partial Y$ and $t \in [0,1]$

(iii) $H(\cdot, t)$ is a generalized contraction for all $t \in [0,1]$.

In what follows we give some results for the following problem:

Is the fixed point property invariant by homotopy for generalized contractions?

Theorem 21.1.1. (R. Precup, B[2]) *Let X be a nonempty set and d and ρ two metrics on X. Let $D \subset X$ be ρ-closed and U a d-open set of X with $U \subset D$. Let $H : D \times [0,1] \to X$. We suppose that:*

(i) (X, ρ) is a complete metric space;

(ii) $H(\cdot, \lambda)$ is an a-contraction, for all $\lambda \in [0,1]$;

(iii) $H(x, \lambda) \neq x$ for all $x \in D \setminus U$ and $\lambda \in [0,1]$

(iv) $F_{H(\cdot,0)} \neq \emptyset$;

(v) H is uniformly (d, ρ)-continuous;

(vi) H is (ρ, ρ)-continuous;

(vii) $H(x, \cdot)$ is d-continuous, uniformly for $x \in U$.

Then:

(a) $F_{H(\cdot,\lambda)} = \{x^*(\lambda)\}$, for all $\lambda \in [0,1]$;

(b) $x^* : [0,1] \to (X, d)$ is continuous.

Theorem 21.1.2. (R. Precup, B[2]) *Let be a Hilbert space, U a bounded open set of H with $0 \in U$ and $f : \overline{U} \to H$ a nonexpansive operator. If*

$$x \neq \lambda f(x)$$

for all $x \in \partial U$, $\lambda \in]0,1[$, then f has at least one fixed point in $\overline{U}$.

For other results see R. Precup B[1], B[2] and B[3].

21.2 A general continuation principle

Let X and Y be two sets, $A \subset X$, $B \subset Y$ proper subsets, $H : X \times [0,1] \to Y$ an operator. Let $\mathcal{A} \subset \mathbb{M}(X, [0,1])$ with $a \in \mathcal{A}$ implies $a|_A$ is a constant function and $0 \in \mathcal{A}$, $1 \in \mathcal{A}$. Let ν be an operator defined at least on the following family of subsets of X,

$$\{H(\cdot, a(\cdot))^{-1}(B) |\ a \in \mathcal{A}\} \cup \{\emptyset\}.$$

Let

$$S := \{x \in X |\ H(x, \lambda) \in B \text{ for some } \lambda \in [0,1]\},$$

and

$$H_\lambda := H(\cdot, \lambda), \quad \lambda \in [0,1].$$

We have:

Theorem 21.2.1. (R. Precup, B[8]) *We suppose that:*

(i) for each $a \in \mathcal{A}$, there exists $a^* \in \mathcal{A}$ such that

$$a^*(x) := \begin{cases} a(x) & \text{for} \quad x \in S, \\ 0 & \text{for} \quad x \in A \end{cases}$$

(ii) the operator H_0 satisfies

$$\nu(H(\cdot, a(\cdot)))^{-1}(B) = \nu(H_0^{-1}(B)) \neq \nu(\emptyset),$$

for any $a \in \mathcal{A}$ with

$$H(\cdot, a(\cdot))|_A = H_0|_A.$$

Then there exists at least one $x \in X \setminus A$ such that $H_1(x) \in B$. Moreover H_1 also satisfies

$$\nu(H(\cdot, a(\cdot)))^{-1}(B) = \nu(H_1^{-1}(B)) \neq \nu(\emptyset)$$

for any $a \in \mathcal{A}$ with $H(\cdot, a(\cdot))|_A = H_1|_A$, and

$$\nu(H_1^{-1}(B)) = \nu(H_0^{-1}(B)).$$

Remark 21.2.1. For some particular cases of this general result, see R. Precup B[8], B[14], B[16] and B[13].

Remark 21.2.2. For continuation theorems for coincidences see R. Precup B[8] and B[6].

Remark 21.2.3. For the coincidence degree theory see A. Buică B[1].

Remark 21.2.4. For other contributions to continuation principles see D. O'Regan and R. Precup B[1] and B[2], S. Sburlan B[2].

21.3 Retractible operators

Let X be a nonempty set and $Y \subset X$ a subset of X. By definition an operator $\rho : X \to Y$ is a set-retraction if $\rho|_Y = 1_Y$. An operator $f : Y \to X$ is retractible with respect to a retraction $\rho : X \to Y$, if $F_{\rho \circ f} = F_f$.

Retraction Principle. *Let $(X, S(X), M)$ be a large fixed point structure (l.f.p.s.). Let $Y \in S(X)$, $\rho : X \to Y$ a set retraction and $f : Y \to X$ an operator. We suppose that:*

(i) $\rho \circ f \in M(Y)$;

(ii) f is retractible w.r.t. ρ.

Then, $F_f \neq \emptyset$.

From this general principle we have (see I.A. Rus B[95]):

Theorem 21.3.1. *Let $(X, S(X), M)$ be a f.p.s. and (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$, $f : Y \to X$ an operator and $\rho : X \to Y$ a set-retraction. We suppose that:*

(i) $\theta|_{\eta(Z)}$ is with the intersection property;

(ii) f is retractible w.r.t. ρ and $\rho \circ f \in M(Y)$;

(iii) ρ is (θ, l)-Lipschitz with $l \in \mathbb{R}_+$;

(iv) f is a strong (θ, φ)-contraction;

(v) the function $l\varphi$ is a comparison function.

Then, $F_f \neq \emptyset$ and if $F_f \in Z$, then $\theta(F_f) = 0$.

Proof. Conditions (iii), (iv) and (v) imply that $\rho \circ f : Y \to Y$ is a strong (θ, l_φ)-contraction. By the First general fixed point principle, of the fixed point structure theory, we have that $F_{\rho\circ f} \neq \emptyset$. From the condition (ii) it follows that, $F_f \neq \emptyset$. From $f(F_f) = F_f$ we have that $\theta(F_f) = 0$.

Theorem 21.3.2. *Let $(X, S(X), M)$ be a f.p.s. on a set X and (θ, η) a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$, $f : Y \to X$ an operator and $\rho : X \to Y$ a set-retraction. We suppose that:*

(i) $A \in Z,\ x \in Y$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) f is retractible w.r.t. ρ and $\rho \circ f \in M(Y)$;

(iii) ρ is $(\theta, 1)$-Lipschitz;

(iv) f is strong θ-condensing.

Then, $F_f \neq \emptyset$ and if $F_f \in Z$, then $\theta(F_f) = 0$.

Proof. Conditions (iii) and (iv) imply that $\rho \circ f : Y \to Y$ is strong θ-condensing. By the Second general fixed point principle, of the fixed point structure theory, we have that, $F_{\rho\circ f} \neq \emptyset$. Condition (ii) implies that $F_f \neq \emptyset$. From $F_f \in Z$, $f(F_f) = F_f$ and the condition (iv) we have that $\theta(F_f) = 0$.

From these general results we have:

Theorem 21.3.3. *Let X be a Banach space, α_K the Kuratowski measure of noncompactness on X and $f : \overline{B}(0, R) \to X$ a continuous operator. We suppose that:*

(i) f is a strong (α_K, φ)-contraction;

(ii) f is retractible w.r.t. the radial retraction.

Then $F_f \neq \emptyset$ and F_f is a compact subset.

Theorem 21.3.4. *Let X be a Banach space and $f : \overline{B}(0, R) \to X$ a continuous operator. We suppose that:*

(i) f is strong α_K-condensing;

(ii) f is retractible w.r.t. the radial retraction.

Then, $F_f \neq \emptyset$ and is a compact subset.

Remark 21.3.1. Each of the following conditions implies the condition (ii) in Theorem 21.3.3. and 21.3.4.

(1) (Leray-Schauder) $x \in \partial B(0, R)$, $f(x) = \lambda x \Rightarrow \lambda \leq 1$.

(2) (E. Rothe) $f(\partial B(0, R)) \subset \overline{B}(0, R)$.

(3) (M. Altman) $\|x - f(x)\|^2 \geq \|f(x)\|^2 - \|x\|^2$, for all $x \in \partial B(0, R)$.

(4) (Martelli-Vignoli) There exists $m \geq 2$, such that,

$$\|f(x) - x\|^m \geq \|f(x)\|^m - \|x\|^m, \quad \text{for all } x \in \partial B(0, R).$$

21.4 Basic fixed point principles for multivalued nonself operators

The first local version of Nadler's contraction principle was proved by M. Frigon and A. Granas R[1], as follows.

Theorem 21.4.1. *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T : \widetilde{B}(x_0; r) \to P_{cl}(X)$ be an a-contraction such that $D(x_0, T(x_0)) < (1 - a)r$. Then $F_T \neq \emptyset$.*

Proof. Let $x_0 \in X$ and $x_1 \in T(x_0)$, with $d(x_0, x_1) < (1 - a)r$. Then $H(T(x_0), T(x_1)) \leq a \cdot d(x_0, x_1) < a(1 - a)d(x_0, x_1)$. Then there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) < a(1-a)r$. Moreover we have $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) < (1 - a)r + a(1 - a)r = (1 - a^2)r$. Thus $x_2 \in \widetilde{B}(x_0; r)$. We can construct inductively a sequence $(x_n)_{n \in \mathbb{N}}$ in $\widetilde{B}(x_0; r)$ having the properties:

(i) $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$;

(ii) $d(x_n, x_{n+1}) \leq a^n \cdot (1-a)r$.

From (ii) the sequence is Cauchy and hence it converges to an element $x^* \in \widetilde{B}(x_0; r)$. From (i) by taking account that T has closed graph, we obtain that $x^* \in T(x^*)$. □

An extension of this result is based on the concept of multivalued graphic contraction introduced by I.A. Rus in B[77].

Definition 21.4.1. Let (X, d) be a complete metric space. A multivalued operator $T : X \to P_{cl}(X)$ is said to be a multivalued graphic contraction if its graph is closed and the following condition is satisfied: there exist $\alpha \in \mathbb{R}_+$, $\alpha < 1$ such that: $H(T(x), T(y)) \leq \alpha d(x, y)$, for every $x \in X$ and every $y \in T(x)$.

Theorem 21.4.2. (A. Petruşel, B[2]) *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T : \tilde{B}(x_0; r) \to P_{cl}(X)$ satisfying:*

i) T is a multivalued graphic-contraction;

ii) T has closed graph;

iii) $D(x_0, T(x_0)) < (1-\alpha)r$.

Then $F_T \neq \emptyset$.

A local result for φ-contractions was recently proved by T. Lazăr, A. Petruşel and N. Shahzad B[1].

Theorem 21.4.3. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strong comparison function such that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ is strictly increasing, continuous in r and $\sum_{n=1}^{\infty} \varphi^n(\psi(s)) \leq \varphi(s)$, for each $s \in]0, r[$. Let $T : B(x_0; r) \to P_{cl}(X)$ be a multivalued φ-contraction such that $D(x_0, T(x_0)) < r - \varphi(r)$. Then $F_T \neq \emptyset$.*

Proof. Let $0 < s < r$ such that $\tilde{B}(x_0; s) \subset B(x_0; r)$ and $D(x_0, T(x_0)) < s - \varphi(s) < r - \varphi(r)$. Let $x_1 \in T(x_0)$ such that $d(x_0, x_1) < s - \varphi(s)$. Then $x_1 \in \tilde{B}(x_0; s)$ and we have $H(T(x_0), T(x_1)) \leq \varphi(d(x_0, x_1)) \leq \varphi(\psi(s))$. Then there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) \leq \varphi(\psi(s))$. Hence $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) \leq s - \varphi(s) + \varphi(\psi(s))$. So $x_2 \in \tilde{B}(x_0; s)$. Inductively, we can obtain a sequence $(x_n)_{n \in \mathbb{N}}$ with the following properties:

(i) $d(x_n, x_{n+1}) \leq \varphi^n(\psi(s))$, for each $n \in \mathbb{N}$;

(ii) $x_n \in \tilde{B}(x_0; s)$, for each $n \in \mathbb{N}$;

From (i) we get that $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Denote by $x^* \in \tilde{B}(x_0; s)$ the limit of this sequence. We get successively: $D(x^*, T(x^*)) \leq d(x^*, x_{n+1}) + H(T(x_n), T(x^*)) \leq d(x^*, x_{n+1}) + \varphi(d(x^*, x_n)) < d(x^*, x_{n+1}) + d(x^*, x_n)$, Since T has closed values, we get, when $n \to +\infty$ that $x^* \in F_T$. □

A similar result is the following theorem.

Theorem 21.4.4. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $T : \widetilde{B}(x_0; r) \to P_{cl}(X)$ be a multivalued φ-contraction such that $\delta(x_0, T(x_0)) < r - \varphi(r)$. Then, $F_T \cap B(x_0; r) \neq \emptyset$.*

Proof. From the condition $\delta(x_0, T(x_0)) < r - \varphi(r)$ we get that $\tilde{B}(x_0; r)$ is invariant with respect to T. Hence, by Węgrzyk's theorem, there exists $x^* \in F_T$. Let us prove now that $x^* \in B(x_0; r)$. If $d(x_0, x^*) = r$, then we have $r = d(x^*, x_0) \leq D(x^*, T(x_0)) + \delta(T(x_0), x_0) \leq H(T(x^*), T(x_0)) + +\delta(T(x_0), x_0) < \varphi(r) + r - \varphi(r) = r$, which is a contradiction. Thus, $x^* \in B(x_0; r)$. □

We will consider now the case of multivalued Meir-Keeler operators, see T. Lazăr, A. Petruşel and N. Shahzad B[1].

Recall first the concept of multivalued Meir-Keeler operator.

Definition 21.4.2. Let (X, d) be a metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. Then, T is called a Meir-Keeler type operator, if for each $\epsilon > 0$ there exists $\eta = \eta(\epsilon) > 0$ such that for $x, y \in X$ with $\epsilon \leq d(x, y) < \epsilon + \eta$ we have $H(T(x), T(y)) < \epsilon$.

Theorem 21.4.5. *Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$. Let $T : X \to P_{cp}(X)$ be a multivalued Meir-Keeler operator. Suppose that $\delta(x_0, T(x_0)) \leq \eta(r)$, where $\eta(r)$ denotes the positive number corresponding to $r > 0$ by Definition 21.4.1.*

Then:

(i) $B(x_0, r + \eta(r)) \in I(T)$;

(ii) If (X, d) is complete, then there exists $x^* \in F_T \cap \widetilde{B}(x_0, r + \eta(r))$.

Proof. (i) Let $y \in B(x_0, r + \eta(r))$ be arbitrary. We will prove that $T(y) \subset B(x_0, r + \eta(r))$. For this purpose, let us consider the following two cases:

a) $0 < d(x_0, y) < r$

Then, since T is contractive we get that $H(T(x_0), T(y)) < d(x_0, y) < r$.

b) $d(x_0, y) \geq r$

Since $y \in B(x_0, r + \eta(r))$, the Meir-Keeler condition implies $H(T(x_0), T(y)) < r$.

Thus, in both cases we get $H(T(x_0), T(y)) < r$.

Let $u \in T(y)$ arbitrary. We have $d(x_0, u) \leq d(x_0, v) + d(v, u) \leq \delta(x_0, T(x_0)) + d(v, u)$, for each $v \in T(x_0)$. Then, $d(x_0, u) \leq \delta(x_0, T(x_0)) + D(u, T(x_0)) \leq \delta(x_0, T(x_0)) + H(T(y), T(x_0)) < \eta(r) + r$. Thus, $T(y) \subset B(x_0, r + \eta(r))$.

(ii) Let us remark first that T maps $\widetilde{B}(x_0, r + \eta(r))$ to itself, since the operator $T : (X, d) \to (P_{cp}(X), H)$ is continuous. Because (X, d) is complete, we can apply for $T|_{\widetilde{B}(x_0, r+\eta(r))}$ the fixed point theorem for multivalued Meir-Keeler operators given by S. Reich see Chapter 11.1. $\square$

We consider now some new concepts.

Definition 21.4.3. Let X be a real Banach space, $Y \in P_{cl}(X)$ and $x \in Y$. We denote:

$$J_Y(x) = \left\{ y \in X | \liminf_{h \to 0_+} D(x + hy, Y)h^{-1} = 0 \right\}$$

$$\tilde{I}_Y(x) := x + J_Y(x)$$

$$I_Y(x) = \{x + \lambda(y - x) | \ \lambda \geq 0, \ y \in Y\}, \quad \text{for} \quad Y \in P_{cl,cv}(X).$$

The set $I_Y(x)$ is called the inward set at x. Notice that $\tilde{I}_Y(x) = I_Y(x)$ for convex subset Y of X.

Definition 21.4.4. Let X be a real Banach space, $Y \in P_{cl}(X)$ and $T : Y \to P(X)$. Then:

i) T is called weakly inward if $T(x) \subset \tilde{I}_Y(x)$, for each $x \in Y$;

iii) T is called inward if $T(x) \cap \tilde{I}_Y(x) \neq \emptyset$, for each $x \in Y$.

Definition 21.4.5. Let (X, d) be a metric space. A multi-valued operator $T : X \to P_{cl}(X)$ is called:

i) γ-condensing if and only if $\gamma(T(A)) < \gamma(A))$, for each $A \in P_b(X)$, with $\gamma(A) > 0$.

ii) (γ, a)-contraction if and only if $a \in [0, 1[$ and $\gamma(T(A)) \leq a\gamma(A)$, for each $A \in P_b(X)$.

(where γ is α_K or α_H-the Kuratowski and respectively the Hausdorff measure of noncompactness). Moreover, γ could be also an abstract measure of noncompactness, see for example Ayerbe Toledano, Dominguez Benavides, López Acedo R[1].

The following results were given by Deimling R[1], R[3].

Theorem 21.4.6. *Let X be a Banach space and $Y \in P_{b,cl,cv}(X)$. Let $T : Y \to P_{cl,cv}(X)$ be upper semicontinuous, γ-condensing and inward. Then $F_T \neq \emptyset$.*

As a consequence of the degree theory for multivalued operators one can prove:

Theorem 21.4.7. *Let X be a Banach space, $Y \in P_b(X)$ and $T : \overline{Y} \to P_{cl,cv}(X)$ be upper semicontinuous and (γ, a)-contraction. Suppose that one of the following conditions holds:*

i) Y is open and there exists $x_0 \in Y$ such that $x_0 + \lambda(x - x_0) \notin T(x)$, for each $x \in \partial Y$ and each $\lambda > 1$

ii) Y is closed, convex and $T(Y) \subset Y$

Then $F_T \neq \emptyset$.

21.5 Continuation principles for multivalued operators

Frigon and Granas have proved some continuation results for multivalued operators on complete metric spaces.

Definition 21.5.1. If X, Y are metric spaces and $G_t : X \to P_{cl}(Y)$ is a family of multivalued operators depending on a parameter $t \in [0, 1]$ then, by definition, $(G_t)_{t\in[0,1]}$ is said to be a family of k-contractions if:

i) G_t is a k-contraction, for each $t \in [0, 1]$.

ii) $H(G_t(x), G_s(x)) \leq |\phi(t) - \phi(s)|$, for each $t, s \in [0, 1]$ and each $x \in X$, where $\phi : [0, 1] \to \mathbb{R}$ is a continuous and strictly increasing function.

If (X, d) is a complete metric space and U is an open connected subset of X, then we will denote by $K(\overline{U}, X)$ the set of all k-contractions $G : \overline{U} \to P_{cl}(X)$.

Also, denote by $\mathcal{K}_0(\overline{U}, X) = \{G \in \mathcal{K}(\overline{U}, X) | x \notin G(x),\ \text{for each}\ x \in \partial U\}$.

Definition 21.5.2. $G \in \mathcal{K}_0(\overline{U}, X)$ is called essential if and only if $F_G \neq \emptyset$. Otherwise G is said to be inessential.

Definition 21.5.3. A family of k-contractions $(G_t)_{t\in[0,1]}$ is called a homotopy of contractions if and only if $G_t \in \mathcal{K}_0(\overline{U}, X)$, for each $t \in [0,1]$. The multifunctions S and T are said to be homotopic if there exists a homotopy of contractions $(F_t)_{t\in[0,1]}$ such that $G_0 = S$ and $G_1 = T$.

The topological transversality theorem is as follows:

Theorem 21.5.1. (Frigon-Granas R[1]) *Let $S, T \in \mathcal{K}_0(\overline{U}, X)$ two homotopic multifunctions. Then S is essential if and only if T is essential.*

The non-linear alternative for multivalued contractions was proved by Frigon and Granas:

Theorem 21.5.2. (Frigon-Granas R[1]) *Let X be a Banach space and $U \in P_{op}(X)$ such that $0 \in U$. If $T : \overline{U} \to P_{cl}(X)$ is a multivalued k-contraction such that $T(\overline{U})$ is bounded, then either:*

i) there exists $x \in \overline{U}$ such that $x \in T(x)$.

or

ii) there exists $y \in \partial U$ and $\lambda \in]0,1[$ such that $y \in \lambda T(y)$.

Let us present now the Leray-Schauder principle for multivalued contractions:

Theorem 21.5.3. (Frigon-Granas R[1]) *Let X be a Banach space and $T : X \to P_{cl}(X)$ such that for each $r > 0$ the multifunction $T|_{\widetilde{B}(0,r)}$ is a k-contraction. Denote by $\mathcal{E}_T := \{x \in X | x \in \lambda T(x),\ \text{for some}\ \lambda \in]0,1[\}$. Then at least one of the following assertions hold:*

i) $\mathcal{E}_T$ is unbounded

ii) $F_T \neq \emptyset$.

For other results see P.S. Milojevic and W.V. Petryshyn R[1], J. Andres R[2], D. O'Regan R[1], A. Chiş R[2], R.P. Agarwal, J. Dshalalow and D. O'Regan R[1], T. Lazăr, D. O'Regan and A. Petruşel R[1], etc.

21.6 Retractible multivalued operators

Let X be a nonempty set and $Y \subset X$ a nonempty subset of X. By definition an operator $T : Y \to P(X)$ is retractible w.r.t. a set-retraction $\rho : X \to Y$ if $F_{\rho\circ T} = F_T$.

Then, we have the following retraction theorem.

Retraction principle for multivalued operators. *Let $(X, S(X), M^0)$ be a l.f.p.s. for multivalued operators on a set X. Let $Y \in S(X)$, $\rho : X \to Y$ a set retraction and $T : X \to P(X)$ an operator. We suppose that:*

(i) $\rho \circ T \in M^0(Y)$;

(ii) T is retractible w.r.t. ρ.

Then, $F_T \neq \emptyset$.

From this general fixed point principle we have (see I.A. Rus B[95]):

Theorem 21.6.1. *Let $(X, S(X), M^0)$ be a f.p.s. on X and (θ, η) ($\theta : Z \to \mathbb{R}_+$) a compatible pair with $(X, S(X), M^0)$. Let $Y \in \eta(Z)$, $T : Y \to P(X)$ a multivalued operator and $\rho : X \to Y$ a set retraction. We suppose that:*

(i) $\theta|_{\eta(Z)}$ is with the intersection property:

(ii) T is retractible w.r.t. ρ and $\rho \circ T \in M(Y)$;

(iii) ρ is (θ, l)-Lipschitz with $l \in \mathbb{R}_+$;

(iv) T is strong (θ, φ)-contraction;

(v) the function l_φ is a comparison.

Then, $F_T \neq \emptyset$.

Proof. Conditions (iii), (iv) and (v) imply that the operator $\rho \circ T : Y \multimap Y$ is a strong (θ, l_φ)-contraction. By First general fixed point principle for multivalued operators it follows that $F_{\rho\circ T} \neq \emptyset$. Now condition (ii) implies that $F_T \neq \emptyset$.

Theorem 21.6.2. *Let $(X, S(X), M^0)$ be a f.p.s. on X and (θ, η) a compatible pair with $(X, S(X), M^0)$. Let $Y \in \eta(Z)$, $T : Y \to P(X)$ a multivalued operator and $\rho : X \to Y$ a set retraction. We suppose that:*

(i) $A \in Z$, $x \in Y$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) T is retractible w.r.t. ρ and $\rho \circ T \in M(Y)$;

(iii) ρ is $(\theta, 1)$-Lipschitz;

(iv) T is strong θ-condensing.

Then, $F_T \neq \emptyset$.

Proof. Conditions (iii) and (iv) imply that the operator $\rho \circ T : Y \multimap Y$ is strong θ-condensing. By Second general fixed point principle of the fixed point structure theory of multivalued operator, we have that $F_{\rho\circ T} \neq \emptyset$. Now, condition (ii) implies that $F_T \neq \emptyset$.

21.7 The case of the strict fixed point structures

Let X be a nonempty set and $(X, S(X), M^0)$ be a large strict fixed point structure on X. Let $Y \in S(X)$, $\rho : X \to Y$ a set retraction and $T : Y \to P(X)$ a multivalued operator. If $\rho \circ T \in M^0(Y)$ and $(SF)_{\rho\circ T} = (SF)_T$ then, $(SF)_T \neq \emptyset$. From this remark and the general strict fixed point principle for self-multivalued operator we have (see I.A. Rus B[95]):

Theorem 21.7.1. *Let X be a nonempty set and $(X, S(X), M^0)$ a s.f.p.s. on X. Let (θ, η) $(\theta : Z \to \mathbb{R}_+)$ a compatible pair with $(X, S(X), M^0)$, $Y \in \eta(Z)$, $T : Y \to P(X)$ a multivalued operator and $\rho : X \to Y$ a set retraction. We suppose that:*

(i) $\theta|_{\eta(Z)}$ is with the intersection property;

(ii) $(SF)_{\rho\circ T} = (SF)_T$ and $\rho \circ T \in M(Y)$;

(iii) ρ is (θ, l)-Lipschitz with $l \in \mathbb{R}_+$;

(iv) T is a strong (θ, φ)-contraction;

(v) the function l_φ is a comparison function.

Then, $(SF)_T \neq \emptyset$ and if $(SF)_T \in Z$, then $\theta((SF)_T) = 0$.

Theorem 21.7.2. *Let X be a nonempty set and $(X, S(X), M^0)$ a s.f.p.s. on X. Let (θ, η) a compatible pair with $(X, S(X), M^0)$, $Y \in \eta(Z)$, $T : Y \to P(X)$ a multivalued operator and $\rho : X \to Y$ a set retraction. We suppose that:*

(i) $A \in Z$, $x \in Y$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) $(SF)_{\rho\circ T} = (SF)_T$ and $\rho \circ T \in M(Y)$;

(iii) ρ is $(\theta, 1)$-Lipschitz;

(iv) T is strong θ-condensing.

Then, $(SF)_T \neq \emptyset$.

For some applications of the above general fixed point theorem for nonself multivalued operators see I.A. Rus B[95], A. Petruşel B[16] and T. Lazăr, A. Petruşel and N. Shahzad B[1].

Chapter 22

A generic view on the fixed point theory

Precursors: R. Baire (1899), M.K. Fort (1951), V. Klee (1959), R.F. Brown (1971).
Guidelines: A. Lasota and J.A. Yorke (1973), G. Vidossich (1974), G. Butler (1974), F.S. De Blasi (1979), F.S. De Blasi and J. Myjak (1989), T. Dominguez Benavides (1985), J. Baillon and N. Rallis (1988), T. Zamfirescu (1993), S. Reich and A.J. Zaslavski (2001).
General references: W.A. Kirk and B. Sims (Eds.) R[1], S. Reich and J. Zaslavski R[2], F.S. De Blasi R[1], T. Zamfirescu B[1], G. Isac and G.X.-Z. Yuan B[1], S. Reich and A.J. Zaslavski R[11], R[13].

22.0 Preliminaries

In this section, we shall present some notions which are needed in the next sections.

Let (X, d) be a metric space and $Y \subset X$. let $\alpha, s \in \mathbb{R}_+^*$. Then, by definition the Hausdorff dimension of Y is defined by:

$$dim_H(Y) := \inf\{\alpha > 0| \lim_{s\to 0} \inf \sum_{B\in\mathcal{B}} (\delta(B))^\alpha = 0\},$$

where $\inf \sum_{B\in\mathcal{B}} (\delta(B))^\alpha$ is taken over all covers $\mathcal{B}$ of Y, by closed balls of diameter at most s.

By definition, a subset Y of X is called porous if there exists $\epsilon \in]0,1[$ and $r_0 > 0$ such that, for each $r \in]0, r_0[$ and each $x \in X$ there exists $y \in X$ such that

$$\tilde{B}(y, \epsilon r) \subset \tilde{B}(x, r).$$

The subset Y of X is called σ-porous if it is a countable union of porous subsets of X.

Let $r_\epsilon(x)$ be the radius of the largest open ball with center in $\tilde{B}(x,\epsilon)$ which is disjoint from Y. By definition, Y is strongly porous if

$$\rho_Y := \inf_{x\in Y} \limsup_{z\to 0} \frac{r_\epsilon(x)}{\epsilon} = 1.$$

The number ρ_Y is called the porosity of Y. The set Y is porous if $\rho_Y > 0$.

For more considerations on the above concepts see J. Heinonen R[1], R.L. Devaney and L. Keen (Eds.) R[1] (107-126 pp.), F.S. De Blasi and J. Myjak R[2], T. Zamfirescu B[1], L. Zajicek R[1] and S.J. Agronsky and A.M. Bruckner R[1].

22.1 Generic aspects on Schauder's theorem

We begin our considerations with some notions.

Let (X, d) be a metric space. A subset $Y \subset X$ is of first category in X if it can be expressed as the union of a countable collection of sets each of which is nowhere dense in X. A subset $Z \subset X$ is of second category in X if it is not of first category in X. X is called a Bairespace if all its open sets are of second category. A set in a Baire space is called residual if its complement is of first category. Most means all except those in a first category set. By definition, a property is generic if it is shared by most elements.

Let X be a Banach space and $Y \in P_{cp,cv}(X)$. Consider the Banach space $(C(Y,Y), +, R, \|\cdot\|_C)$.

We have:

Theorem 22.1.1. (T. Zamfirescu, B[1]). *For most operators in* $C(Y,Y)$ *the set of fixed points is homeomorphic to the Cantor set.*

Theorem 22.1.2. (T. Zamfirescu, B[1]). *Let* $Y \subset R^n$ *be compact, convex and with nonempty interior. Most functions in* $C(Y,Y)$ *admit a set of fixed points which is strongly and totaly porous.*

Theorem 22.1.3. (T. Zamfirescu, B[1]). *Let* $Y \subset R^n$ *be compact convex and with nonempty interior. Most functions in* $C(Y,Y)$ *admit a set of fixed points which has Hausdorff dimension.*

22.2 Generic aspects on Fan-Glicksberg's theorem

A metric space (X,d) is said to be hyperconvex metric space (N. Aronszajn and P. Panitchpakdi (1956)) if for any collection of points $\{x_i |\ i \in I\}$ of X and any collection $\{r_i |\ i \in I\}$ of nonnegative real numbers with $d(x_i, x_j) \le r_i + r_j$, we have

$$\bigcap_{i \in I} \overline{B}(x_i, r_i) \neq \emptyset.$$

We have the following result by G. Isac and G.X.-Z. Yuan

Theorem 22.2.1 (G. Isac and G. X.-Z. Yuan B[1]). *Let* X *be a compact hyperconvex metric space and* M *be the space consisting of all upper semicontinuous multivalued operators from* X *to itself with nonempty closed and acyclic values. Then there exists a dense residual subset* M_1 *of* M *such that:*

(i) the fixed point set of each operator in M_1 *is essential (M. K. Fort)*

(ii) for any given $T \in M$ *and* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for each* $S \in M_1$ *with*

$$\sup_{x \in X} H(T(x), S(x)) < \delta,$$

we have that

$$H(F_T, F_S) < \varepsilon.$$

22.3 Other results

For other results on the generic view in fixed point theory see W.A. Kirk and B. Sims (Eds.) R[1], G. Isac and G.X.-Z. Yuan B[1], S. Reich and A.J. Zaslavski R[2], R[8], R[10].

Chapter 23

Iterated function (operator) systems

Precursors: B. Knaster (1928), S.B. Nadler jr. (1969).
Guidelines: B. Mandelbrot (1975), J. Hutchinson (1981), M.F. Barnsley and S. Demko (1985), S. Demko, L. Hodges and B. Naylor (1985), D.P. Hardin and P. Massopust (1986), L.M. Anderson (1992), J. Andres (2004), V. Glăvan and V. Guţu (2004), J. Fišer (2004).
General references: B. Mandelbrot R[1], R. L. Devaney and L. Keen (ed.) R[1], J.E. Hutchinson R[1], M. Yamaguti, P. Hata and J. Kigami R[1], J.E. Hutchinson and L. Rüschendorf R[1], I.A. Rus and B. Rus B[2], A. Petruşel and I. A. Rus B[1], A. Soós B[1], B. Breckner B[1], J. Andres, J. Fišer, G. Gabor and K. Lesniak R[1], J. Fišer R[1], J. Andres R[2], V. Glăvan and V. Guţu B[1], B[2] and B[3], J. Jachymski, L. Gajek and P. Pokarowski R[1], M. Hegedüs R[2], F.S. De Blasi R[3].

23.0 Set-to-set operators

Let X be a nonempty set and $Z \subset \mathcal{P}(X)$ a nonempty subset of $\mathcal{P}(X)$. By definition, a singlevalued operator $\Psi : Z \to Z$ is called a set-to-set operator from X to X and it is denoted by $\Psi : X \hookrightarrow X$. Notice that $F_\Psi \subset Z$.

Example 23.0.1. Let (X, τ) be a topological space. The following operators are set-to-set operators from X to X:

(i) $^- : \mathcal{P}(X) \to \mathcal{P}(X) \quad A \mapsto \overline{A}$;

(ii) $int : \mathcal{P}(X) \to \mathcal{P}(X) \quad A \mapsto int(A)$;

(iii) $' : \mathcal{P}(X) \to \mathcal{P}(X) \quad A \mapsto A'$;

(iv) $\partial : \mathcal{P}(X) \to \mathcal{P}(X) \quad A \mapsto \partial A$.

Example 23.0.2. Let $(X, +, \mathbb{R})$ be a linear space. Then the operator:

$$co : \mathcal{P}(X) \to \mathcal{P}(X)\ A \mapsto co(A)$$

is a set-to-set operator from X to X.

Some basic problem of the fixed point theory for set-to-set operators are the following:

Problem 23.0.1. In which conditions there exists $A \in Z$ such that $\Psi(A) = A$?

Problem 23.0.2. In which conditions there exists $A \in Z$ such that $A \subset \Psi(A)$?

Problem 23.0.3. In which conditions there exists $A \in Z$ such that $A \supset \Psi(A)$?

Problem 23.0.4. In which conditions there exists $A \in Z$ such that $A \cap \Psi(A) \neq \emptyset$?

Some examples related to the above problems are:

Knaster's Example (1928). *Let (X, d) be a nonempty set and $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$ be a set-to-set operator from X to X. If $\Psi(A) \subseteq \Psi(B)$, for all $A, B \subset X$, with $A \subseteq B$, then $F_\Psi \neq \emptyset$, i.e., there exists $A \subset X$ such that $\Psi(A) = A$.*

Let us remark that Tarski's fixed point theorem (see Chapter 2) is a generalization of the previous result.

Nadler's Example (1969). *Let (X, d) be a complete metric space and $T : X \to P_{cp}(X)$ be an α-Lipschitz multivalued operator, i.e. $\alpha > 0$ and*

$$H_d(T(x), T(y)) \leq \alpha d(x, y), \text{ for each } x, y \in X.$$

The point-to-set operator T induces a set-to-set operator $\hat{T} : P_{cp}(X) \to$

$P_{cp}(X)$, by the relation:

$$\hat{T}(Y) := \bigcup_{x\in Y} T(x), \text{ for each } Y \in P_{cp}(X).$$

Then, $\hat{T}$ is α-Lipschitz. Moreover, if $\alpha < 1$, then $\hat{T}$ is an α-contraction and, by the Contraction Principle, we have that $F_{\hat{T}} = \{A\}$.

Hutchinson's Example (1981). *Let (X, d) be a complete metric space and $f_i : X \to X$ be α-contractions, for $i \in \{1, \ldots, m\}$. This finite system generates a set-to-set operator $T_f : P_{cp}(X) \to P_{cp}(X)$, by the relation:*

$$T_f(Y) := \bigcup_{i=1}^{m} f_i(Y), \text{ for each } Y \in P_{cp}(X).$$

Then, T_f is an α-contraction with respect to the Pompeiu-Hausdorff metric H_d. Thus $F_{T_f} = \{A\}$.

A similar construction can be made for a finite family of multivalued α-contractions $F_i : X \to P_{cp}(X)$, for $i \in \{1, \ldots, m\}$.

Remark 23.0.1. For other examples of set-to-set operators, see J. Andres and L. Górniewicz R[1], R[2], J. Andres, J. Fišer R[1], J. Andres, J. Fišer, G. Gabor and K. Leśniak R[1], B. Breckner B[1].

Remark 23.0.2. For the fixed point theory of set-to-set operators, see M. Hegedüs R[2], F.S. De Blasi R[3] and the references therein.

Remark 23.0.3. For applications to set differential and integral equations see V. Lakshmikantham, T. Gnana Bhaskar and J. Vasundhara Devi R[1] and I. Tişe R[1].

The aim of this chapter is to study the fixed points of the Hutchinson's set-to-set operator, i.e., to study the iterated (multivalued) operators systems.

23.1 Iterated Picard operator systems

Let X be a nonempty set and $f_1, \ldots, f_m : X \to X$ some operators. These operator generates the following operator on $P(X)$

$$T_f : P(X) \to P(X), \quad T_f(A) := f_1(A) \cup \cdots \cup f_m(A).$$

The problem is to study the operator T_f depending on the properties of the operators $f_1, \ldots, f_m$.

In the case of a metric space (X, d) we have:

Theorem 23.1.1. (I. A. Rus, B[11]) *If the operator $f_1, \ldots, f_m : (X, d) \to (X, d)$ are φ-contractions, then the operator $T_f : (P_{cp}(X), H_d) \to (P_{cp}(X), H_d)$ is a φ-contraction.*

Theorem 23.1.2. (A. Petruşel, B[24], B[25]) *Let (X, d) be a complete metric space and $f_i : X \to X$, for $i \in \{1, 2, \ldots, m\}$ are Meir-Keeler type operators. Then the operator $T_f : (P_{cp}(X), H) \to (P_{cp}(X), H)$ defined below is a Meir-Keller type operator and thus $F_{T_f} = \{A^*\}$.*

Proof. We shall prove that for each $\eta > 0$ there is $\delta > 0$ such that the following implication holds

$$\eta \leq H(A, B) < \eta + \delta \text{ we have } H(T_f(A), T_f(B)) < \eta.$$

Let us consider $A, B \in P_{cp}(X)$ such that $\eta \leq H(A, B) < \eta + \delta$.

If $u \in T_f(A)$ then there exists $j \in \{1, \ldots, m\}$ and $x \in A$ such that $u = f_j(x)$.

For $x \in A$ we can choose $y \in B$ such that $d(x, y) \leq H(A, B) < \eta + \delta$. We have the following alternative:

If $d(x, y) \geq \eta$ then $\eta \leq d(x, y) < \eta + \delta$ implies $d(f_j(x), f_j(y)) < \eta$. Hence $D(u, T_f(B)) \leq d(u, f_j(y)) < \eta$.

On the other hand, if $d(x, y) < \eta$ then from the Meir-Keeler assumption, we have $d(f_j(x), f_j(y)) < d(x, y) < \eta$ and again the conclusion $D(u, T_f(B)) < \eta$.

Because $T_f(A)$ is compact we have that $\rho(T_f(A), T_f(B)) < \eta$.

Interchanging the roles of $T_f(A)$ and $T_f(B)$ we obtain $\rho(T_f(B), T_f(A)) < \eta$ and hence $H(T_f(A), T_f(B)) < \eta$, showing the fact that T_f is a Meir-Keeler-type operator. By the Meir-Keeler fixed point theorem, we obtain that there exists an unique $A^* \in P_{cp}(X)$ such that $T_f(A^*) = A^*$. □

Theorem 23.1.3. (I.A. Rus and B. Rus, B[2]). *Let (X, d) be a compact metric space and $f : X \to X$ be a continuous Janos operator. Then the operator $T_f : P_{cp}(X) \to P_{cp}(X)$ is a Janos operator.*

Theorem 23.1.4. (I.A. Rus B[11]). *Let (X, d) be a complete metric space. Let $\mathcal{F} := \{f_1, \ldots, f_m\}$ and $G := \{g_1, \ldots, g_m\}$ be two systems of strict φ-*

contraction on X. Let A^* be the attractor of $\mathcal{F}$ and B^* the attractor of G. We suppose that there exists $\eta > 0$ such that

$d(f_i(x), g_i(x)) \leq \eta$, *for all* $x \in X$, $i \in \{1, \cdots, n\}$.

Then

$H(A^*, B^*) \leq T_\eta := \sup\{t \in R_+ |\ t - \varphi(t) \leq \eta\}$.

We recall that if T_f is a Picard operator then the unique fixed point of T_f is by definition the attractor of the system f.

Remark 23.1.1. Theorem 23.1.4. generalizes a result by J. Jachymski (1996).

Remark 23.1.2. Theorem 23.1.1. generalizes a result by S.B. Nadler R[1].

Theorem 23.1.5. (I.A. Rus and B. Rus, B[2]). *Let X be a nonempty set and $f : X \to X$ a Bessaga operator. Then there exists $Y \subset P(X)$ such that:*

(a) $T_f(Y) \subset Y$;

(b) $T_f : Y \to Y$ is Bessaga operator;

(c) If $cardX > 1$, then there exists $Y \subset P(X)$ such that $cardY > 1$.

For other results on this topic see J. Andres R[2], J. Andres, J. Fišer R[1], J. Andres and L. Górniewicz R[1], J. Andres, J. Fišer, G. Gabor and K. Leśniak R[1], E. De Amo, I. Chiţescu, M.D. Carrillo and N.A. Secelean R[1], K.R. Wicks R[1], etc.

23.2 Iterated multivalued operator systems

Let $F_1, \ldots, F_m : X \to P_{cp}(X)$ be a finite family of u.s.c. multivalued operators. By definition, the fractal operator (or the Hutchinson operator) generated by $F = \{F_1, \ldots, F_m\}$ is

$$T_F : P_{cp}(X) \to P_{cp}(X), \quad T_F(Y) := \bigcup_{i=1}^{m} F_i(Y).$$

With respect to the existence of a fixed point of the fractal operator, we have:

Theorem 23.2.1. (A. Petruşel and I.A. Rus, B[1]). *Let (X, d) be a complete metric space and $F_1, \ldots, F_m, G_1, \cdots, G_m : X \to P_{cl}(X)$ be φ-contractions where φ is a strict comparison function. Then:*

(a) $F_{T_F} = \{A^*\}$ *and* $F_{T_G} = \{B^*\}$

(b) If $H(F_i(x), G_i(x)) \leq \eta$, *for all* $x \in X$ *and* $i \in \{1, \ldots, m\}$, *then* $H(A^*, B^*) \leq t_\eta$, *where* $t_\eta := \sup\{t \in \mathbb{R}_+ : t - \varphi(t) \leq \eta\}$.

We consider now the following problem.

Problem 23.2.1. If the fixed point problem is well-posed for the finite family of continuous operators $f_i : X \to X$ (respectively for the finite family of u.s.c. multivalued operators $F_i : X \to P_{cl}(X)$), then is the fixed point problem well-posed for the Hutchinson operator $T_f : (P_{cp}(X), H) \to (P_{cp}(X), H)$, $T_f(Y) = \bigcup_{i=1}^{m} f_i(Y)$ (respectively for $T_F : (P_{cp}(X), H) \to (P_{cp}(X), H)$, $T_F(Y) = \bigcup_{i=1}^{m} F_i(Y)$) ? If the answer is affirmative, then we say that the self-similar problem is well-posed for the iterated function system $f = (f_1, f_2, \ldots, f_m)$ (respectively for $F = (F_1, F_2, \ldots, F_m)$).

Recall now that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a strict comparison function if:

(i) φ is a comparison function;

(ii) $\lim_{t\to\infty} (t - \varphi(t)) = \infty$.

In this setting, the operator $F : X \to P_{cl}(X)$ is called a multivalued φ-strict contraction if φ is a continuous strict comparison function and for each $x, y \in X$ we have $H(F(x), F(y)) \leq \varphi(d(x, y))$.

An answer to the above problem is the following theorem.

Theorem 23.2.2. (A. Petruşel, I.A. Rus and J.-C. Yao B[1]) *Let* (X, d) *be a complete metric space and* $F_i : X \to P_{cl}(X)$ *be a finite family of multivalued strict* φ_i*-contractions for each* $i \in \{1, \ldots, m\}$. *Then the self-similar problem for the iterated function system* $F = (F_1, F_2, \ldots, F_m)$ *is well-posed.*

Proof. We will prove that $F_{T_F} = \{X^*\}$ and if $(X_n)_{n\in\mathbb{N}} \in P_{cp}(X)$ is such that $H(X_n, T_F(X_n)) \to 0$, then $X_n \overset{H}{\to} X^*$ as $n \to +\infty$. Since $F_i : X \to P_{cl}(X)$ is a strict φ_i-contraction for each $i \in \{1, \ldots, m\}$ then, T_F is a strict $\max\{\varphi_1, \cdots, \varphi_m\}$-contraction, having a unique fixed point $X^* \in P_{cp}(X)$.

Denote by $\varphi := \max\{\varphi_1, \cdots, \varphi_m\}$ and by $\psi(t) := t - \varphi(t)$, for $t \in \mathbb{R}_+$. Obvious ψ is a continuous bijection on $\mathbb{R}_+$ and $\psi^{-1}(\eta) \searrow 0$ as $\eta \searrow 0$.

Next we have $H(X_n, X^*) \leq H(X_n, T_F(X_n)) + H(T_F(X_n), T_F(X^*)) \leq H(X_n, T_F(X_n)) + \varphi(H(X_n, X^*))$. Hence $H(X_n, X^*) \leq \psi^{-1}(H(X_n, T_F(X_n)))$.

The conclusion follows now from the properties of ψ. □

In particular, if F_i are multivalued a_i-contractions we have the following result.

Corollary 23.2.1 *Let* (X, d) *be a complete metric space and* $F_i : X \to P_{cl}(X)$ *be a finite family of multivalued* a_i*-contractions for each* $i \in \{1, \ldots, m\}$. *Then the self-similar problem for the iterated function system* $F = (F_1, F_2, \ldots, F_m)$ *is well-posed.*

Remark 23.2.1. For other results of this type see A. Petruşel and I.A. Rus B[1], B. Breckner B[1], E. Llorens-Fuster, A. Petruşel and J.-C. Yao B[1], S.L. Singh, B. Prasad and A. Kumar R[1], etc.

Remark 23.2.2. For iterated Picard operator systems in probabilistic metric space see A. Soós B[1], J. Kolumbán and A. Soós B[1].

Remark 23.2.3. For shadowing phenomena of iterated multivalued operator systems, see V. Glăvan and V. Guţu B[2].

Remark 23.2.4. For continuation principles see J. Andres R[2] and numerical aspects of iterated multivalued operator systems, see J. Fišer R[1].

Chapter 24

Other results

24.1 Ultra-methods in metric fixed point theory

• **Guidelines:** B. Maurey (1982), B. Sims (1982), J.M. Borwein and B. Sims (1984), P.K. Lin (1985).

• **Results:**

Let $(X, \|\cdot)\|$ be a Banach space and $\mathcal{U}$ be an ultrafilter over an index set I. We consider the following sets:

$$l_\infty(X) := \{(x_i)_{i\in I} \ : \ x_i \in X \text{ and } \sup_{i\in I} \|x_i\| < \infty\},$$

$$N_\mathcal{U}(X) := \{(x_i)_{i\in I} \in l_\infty(X) \ : \ \lim \mathcal{U} \|x_i\| = 0\}.$$

Then, $(l_\infty(X), +, \mathbb{R}, \|\cdot\|_\infty)$ with $\|(x_i)_{i\in I}\|_\infty := \sup\limits_{i\in I} \|x_i\|$ is a Banach space and $N_\mathcal{U}(X)$ is a closed linear subspace of it. By definition, the Banach space $l_\infty(X)/N_\mathcal{U}(X)$ is the ultrapower of X over $\mathcal{U}$ and it is denoted by $(X)_\mathcal{U}$.

If $x \in X$, then $(x)_{i\in I} \in l_\infty(X)$ and we will denote by $\tilde{x}$ the corresponding element in $(X)_\mathcal{U}$. The operator $e : X \to (X)_\mathcal{U}$ defined by $e(x) = \tilde{x}$ is an isometric embedding of X into $(X)_\mathcal{U}$.

The theory of ultrapower Banach spaces is a useful technique in the fixed point theory. The following theorems are some basic results on this line.

Maurey's Theorem. *Let $Y \subset L^1[0,1]$ be a reflexive subspace. If $Z \in P_{b,cl,cv}(Y)$ and $f : Z \to Z$ is a nonexpansive operator, then $F_f \neq \emptyset$.*

Maurey's Theorem. *Let X be a superreflexive Banach space and $Y \in P_{b,cl,cv}(X)$. If $f : Y \to Y$ is an isometry, then $F_f \neq \emptyset$.*

Lin's Theorem. *Let X be a Banach space with a 1-unconditional basis and $Y \subset X$ a nonempty weakly compact convex subset of X. If $f : Y \to Y$ is nonexpansive, then $F_f \neq \emptyset$.*

• **General references:** A. Aksoy and M.A. Khamsi R[1], M.A. Khamsi and W.A. Kirk R[1], M.A. Khamsi and B. Sims in W.A. Kirk and B. Sims (Eds.), pp. 177-199.

24.2 Fixed point theorems in Kasahara spaces

• **Guidelines:** S. Kasahara (1975), K. Iseki (1975), T.L. Hicks (1992), T.L. Hicks and B.E. Rhoades (1992).

• **Results:**

Let X a nonempty set and $d : X \times X \to \mathbb{R}_+$ a functional. An L-space $(X, \to)$ is called d-complete, if any sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < +\infty$ converges in $(X, \to)$ to a point of X. By definition a d-complete L-space is a Kasahara space.

Kasahara's Theorem (1976). *Let $(X, \to, d)$ be a Kasahara space with d a premetric. Let $T, S : X \to P_{cl}(X, d)$ be two multivalued operators. We suppose that:*

(i) d is continuous on $(X, \to) \times (X, \to)$;

(ii) there exist $\varphi, \psi : \mathbb{R}^5_+ \to \mathbb{R}_+$ such that:

(a) there is $\alpha \in]0, 1[$ with $\varphi(t, t, t, 0, 2t) \leq \alpha t$ and $\psi(t, t, t, 0, 2t) \leq \alpha t$ for all $t \in \mathbb{R}^*_+$;

(b) φ and ψ are increasing;

(c) $\rho_d(S(x), T(y)) \leq \varphi(d(x, y), d(s(x), x), d(T(y), y), d(T(y), x), d(s(x), y))$ and $\rho_d(T(x), S(y)) \leq \psi(d(x, y), d(T(x), x), d(S(y), y), d(S(y), x), d(T(x), y))$, for all $x, y \in X$;

(iii) φ or ψ is u.s.c.

Then, $F_S = F_T \neq \emptyset$.

For the definition of ρ_d see Chapter 11.0.

Saliga's Theorem (1996). *Let (X,τ) be a Hausdorff topological space and $(X,\xrightarrow{\tau},d)$ a Kasahara space. Let $Y\in P_{cl}(X,\tau)$ and $f:(Y,\tau_Y)\to(X,\tau)$ an open operator. We suppose that:*

(i) $Y\subset f(Y)$;

(ii) there exists $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ an increasing function with $\varphi(0)=0$ such that $\varphi(d(f(x),f(y)))\geq d(x,y)$, for each $x,y\in X$;

(iii) there exists $x_0\in Y$ with $\sum\limits_{n\in\mathbb{N}}\varphi^n(d(f(x_0),x_0))<+\infty$.

Then $F_f\neq\emptyset$.

• **General references:** S. Kasahara R[3], K. Iseki R[6] and R[7], V. Popa B[42], L.M. Saliga R[1].

24.3 Iterative test of Edelstein

• The following result of M. Edelstein is well-known.

Edelstein's Theorem. *Let (X,d) be a metric space and $f:X\to X$ a contractive operator. We suppose that for some $x_0\in X$, the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ has a convergent subsequence.*

Then:

(i) $F_f=\{x^*\}$;

(ii) $f^n(x_0)\to x^*$ as $n\to\infty$.

Having in mind this result, S.B. Nadler jr. gives the following definition:

Definition 24.3.1. The iterative test, for contractive operators is conclusive for (X,d) if and only if the following implication is valid:

($f:X\to X$ contractive and exists $x_0\in X$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ does not converges) $\Rightarrow F_f=\emptyset$.

• **Results:**

Nadler's Theorem. *If (X,d) is a locally compact and connected metric space, then the iterative test is conclusive for (X,d).*

• References: M. Edelstein R[2], S.B. Nadler jr. R[5], J. Bryant and L.F. Guseman R[2], J. Bryant and T.F. McCabe R[1], D.N. Cheban, J. Duan and A. Gherco R[1].

24.4 Fixed point theorems in 2-metric spaces

• Let X be a nonempty set and $d : X \times X \times X \to \mathbb{R}_+$ a functional.

The S. Gähler axioms for a 2-metric are the following:

(i) for each pair of point, $(x, y) \in X \times X$, with $x \neq y$ there exists a point z such that $d(x, y, z) \neq 0$;

(ii) $d(x, y, z) = 0$, whenever at least two of the point x, y, z are equal;

(iii) $d(x, y, z) = d(y, z, x) = d(z, x, y) = \dots,$ for all $x, y, z \in X$;

(iv) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z),$ for all $x, y, z, u \in X$.

B.C. Dhage axioms are: (iii)+(iv) and

(ii') $d(x, y, z) = 0$ if and only if $x = y = z$.

• For the fixed point theory in 2-metric spaces (D-metric spaces !) see I.A. Rus, A. Petruşel and G. Petruşel B[1] and the references therein (T. Zamfirescu (1971), D.E. Daykin and J.K. Gugdale (1974), K. Iseki (1975), B.E. Rhoades (1978, 1979), G. Dezsö and V. Mureşan (1981), M.S. Khan and M. Imdad (1982),...), B.C. Dhage R[1], I. Beg, F. Ali and T.Y. Minhas R[1], A. Froda R[1], Z. Mustafa and B. Sims R[1], I. Goleţ R[1],...

• For the convergence of the sequences in 2-metric space see K. Iseki R[4], R[5] and Z. Mustafa and B. Sims R[1].

• For area contraction in $\mathbb{R}^2$ see T. Zamfirescu B[10].

24.5 Y-contractions

• Let (X, d) be a metric space, $f : X \to X$ an operator and $Y \subset X \times X$ a subset. If f is a generalized contraction, then the metric condition is satisfied for all $(x, y) \in X \times X$. For a generalized Y-contraction the metric condition is satisfied for all $(x, y) \in Y$.

The basic examples of generalized Y-contractions are:

(1) graphic contractions: I.A. Rus B[86], S. Kasahara R[3], T.L. Hicks and B.E. Rhoades R[1], J. Jachymski R[1],...

(2) contractions outside a bounded set: I.A. Rus B[38] and the references therein (S. Weigram (1969),...)

(3) cyclical generalized contractions: W.A. Kirk, P.S. Srinivasan and P.

Veeramani R[1], I.A. Rus [105], G. Petruşel B[3],...

(4) Y-contractions in ordered metric spaces: M.A. Krasnoselskii and P. Zabrejko R[1], A.C.M. Ran and M.C.B. Reurings R[1], A. Petruşel and I.A. Rus B[4], J.J. Nieto and R. Rodríguez-López R[1], R[2], D. O'Regan and A. Petruşel B[1], J.J. Nieto, R.L. Pouso and R. Rodríguez-López R[1], R.P. Agarwal, M.A. El-Gebeily and D. O'Regan R[1],...

For the fixed point theory of multivalued Y-contractions see I.A. Rus, A. Petruşel and G. Petruşel B[1] and the references therein.

For example we have the following results:

Theorem 24.5.1. *Let (X, d) be a complete metric space and $T : X \to P_b(X)$ be a multivalued operator with closed graph. Suppose that there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that*

$$\delta(T(x), T(y)) \leq a \cdot d(x, y) + b \cdot \delta(x, T(x)) + c \cdot \delta(y, T(y)), \text{ for each } (x, y) \in G(T).$$

Then $F_T = (SF)T \neq \emptyset$.

Proof. Let $q > 1$ and $x_0 \in X$ be arbitrary. Then there exists $x_1 \in T(x_0)$ such that $\delta(x_0, T(x_0)) \leq q \cdot d(x_0, x_1)$. We have $\delta(x_1, T(x_1)) \leq \delta(T(x_0), T(x_1)) \leq \cdot d(x_0, x_1) + b \cdot \delta(x_0, T(x_0)) + c \cdot \delta(x_1, T(x_1)) \leq ad(x_0, x_1) + bqd(x_0, x_1) + c \cdot \delta(x_1, T(x_1))$. Hence $\delta(x_1, T(x_1)) \leq \frac{a+bq}{1-c} \cdot d(x_0, x_1)$. By this procedure, we can obtain the sequence $(x_n)_{n\in\mathbb{N}}$ having the property $d(x_n, x_{n+1}) \leq (\frac{a+bq}{1-c})^n \cdot d(x_0, x_1)$, for each $n \in \mathbb{N}$. If we choose $q > \frac{b}{1-a-c}$ then we get that $\frac{a+bq}{1-c} < 1$. Hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space (X, d). Denote by x^* the limit of the sequence $(x_n)_{n\in\mathbb{N}}$. Since the graph of T is a closed set in $X \times X$ we obtain that $x^* \in T(x^*)$.

Let us establish now the relation $F_T = (SF)_T$. It's enough to prove that $F_T \subset (SF)_T$. For, let $x \in F_T$ be arbitrary. Then, using the hypothesis (with $y = x \in T(x)$) we get successively: $\delta(T(x)) \leq (b + c) \cdot \delta(x, T(x)) \leq (b + c) \cdot \delta(T(x))$. Suppose, by absurdum, that $cardT(x) > 1$. Then $\delta(T(x)) > 0$ and using the above relation we get that $1 \leq b + c$, a contradiction. Hence $\delta(T(x)) = 0$ and so $\{x\} = T(x)$. □

Theorem 24.5.2. *Let X be a Banach space, $Z \in P_b(X)$ and $T : X \to P_{cp,cv}(X)$. Suppose that the following assertions hold:*

(i) T is u. s. c. and compact (i.e., T sends bounded sets into relatively

compact sets);

(ii) there exists $a \in]0,1[$ *such that*

$$H(T(x_1), T(x_2)) \leq a \cdot \|x_1 - x_2\|, \text{ for each } (x_1, x_2) \in (X \setminus Z) \times (X \setminus Z).$$

Then $F_T \neq \emptyset$.

Proof. From (ii) we get that the operator T is quasibounded with the quasinorm $|T| = a < 1$. For, consider first $x \in Z$ Then we have that $\|T(x)\| := \sup\limits_{y \in T(x)} \|y\| \leq \|T(Z)\| < +\infty$. If $x \in X \setminus Z$ then consider an arbitrary but fixed $x_0 \in X \setminus Z$. We have $\|T(x)\| = H(T(x), \{0\}) \leq H(T(x), T(x_0)) + H(T(x_0), \{0\}) \leq a \cdot \|x - x_0\| + \|T(x_0)\| \leq a \cdot \|x\| + a \cdot \|x_0\| + \|T(x_0)\|$. Hence, for all $x \in X$ we get $\|T(x)\| \leq a \cdot \|x\| + \max(\cdot\|x_0\| + \|T(x_0)\|, \|T(Z)\|)$. Then, there exists $R > 0$ such that $T(\widetilde{B}(0, R)) \subset \widetilde{B}(0, R))$, (see M. Martelli, A. Vignoli R[1], Lemma 2.1) Taking into account that $T(\widetilde{B}(0, R))$ is an invariant subset for T, the proof follows now from the Bohnenblust-Karlin fixed point theorem. □

Theorem 24.5.3. *Let* $(X, S(X), M^0)$ *be a fixed point structure, on the L-space* $(X, \to)$. *Let* $A_i \in P_{cl}(X)$, *for* $i \in \{1, 2, \cdots, m\}$. *Define* $Y := \bigcup_{i=1}^{m} A_i$ *and consider* $T : Y \to P(Y)$. *Suppose that:*

(i) $Y := \bigcup_{i=1}^{m} A_i$ *is a cyclic representation of* Y *with respect to* T;

(ii) there exists a convergent sequence $(x_n)_{n \in \mathbb{N}}$, *where* $x_n \in X$, $x_{n+1} \in T(x_n)$, *for each* $n \in \mathbb{N}$;

(iii) If $A := \bigcap_{i=1}^{m} A_i \neq \emptyset$ *then* $A \in S(X)$ *and* $T_{|A} \in M^0(A)$.

Then $F_T \neq \emptyset$.

24.6 Fixed point theorems for Darboux functions

• **Guidelines:** J. Nash (1956), O.H. Hamilton (1957), J. Stallings (1959), P.D. Humke, R.E. Svetic and C.E. Weil (2000).

• **Results:**

Muntean's Theorem. *If $f : [a, b] \to [a, b]$ is a Darboux function in the first class of Baire, then f has at least a fixed point.*

• **General references:** J. Stallings R[1], I. Muntean B[4], P.D. Humke, R.E. Svetic and C.E. Weil R[1], M. Csörnyei, C.T. O'Neil and D. Preiss R[1], V. Berinde B[22], A. Bruckner R[1]. See also Chapter 11 in I.A. Rus, A. Petruşel and G. Petruşel B[2].

24.7 Iterated functions on $\mathbb{R}$

• **Results:**

Hillam's Theorem. *A continuous function $f : [0, 1] \to [0, 1]$ is a weakly Picard function if and only if f is an asymptotically regular function.*

Bellen-Volčič's Theorem. *If $f : [0, 1] \to [0, 1]$ is non-cyclic continuous function, then the following properties are equivalent:*

(i) F_f is connected;

(ii) $F_f = \bigcap_{n \in \mathbb{N}} f^n([0, 1])$;

(iii) f is weakly Picard function and f^∞ is continuous;

(iv) f is weakly Picard function w.r.t. uniform convergence.

Sarkovskii's Theorem. *If a continuous function $f : \mathbb{R} \to \mathbb{R}$ has a periodic point of period 3, then f has periodic points of all periods.*

We consider on $P_{cp}([0, 1])$ the Pompeiu-Hausdorff metric. Let $\omega_f(x)$ be the ω-limit set of x under $f \in C[0, 1]$. By ω_f we denote the multivalued function

$$\omega_f : [0, 1] \to P_{cp}([0, 1]), \quad x \multimap \omega_f(x).$$

Bruckner-Ceder's Theorem. *If $f \in C[0, 1]$ then the following statements are equivalent:*

(i) ω_f is continuous;

(ii) $(f^n)_{n \in \mathbb{N}}$ is equicontinuous;

(iii) $F_{f^2} = \bigcap_{n \in \mathbb{N}} f^n([0, 1])$;

(iv) ω_f is lower semi-continuous;

(v) ω_f is upper semi-continuous.

Let $f \in C^1(\mathbb{R})$ and $x^* \in F_f$. By definition, x^* is called a hyperbolic fixed point of if $|f'(x^*) \neq 1|$. We have:

Theorem 24.7.1. *Let $x^* \in F_f$ be a hyperbolic fixed point with $|f'(x^*) < 1|$. Then, there exists an open neighborhood V of x^* such that*

$$f^n(x^*) \to x^* \text{ as } n \to +\infty,$$

i.e., x^* is an attracting fixed point of f.

Theorem 24.7.2. *Let $x^* \in F_f$ be a hyperbolic fixed point with $|f'(x^*) > 1|$. Then, there exists an open neighborhood V of x^* such that, if $x \in V \setminus \{x\}$, then there exists $m \in \mathbb{N}^*$ such that $f^m(x) \notin V$, i.e., x is a repelling fixed point.*

• **References:** P. Collet and J.-P. Eckmann R[1], R.L. Devaney and L. Keen (Eds.) R[1], J. Milnor and W. Thurston R[1], C. Preston R[1], G. Targonski R[1], A.M. Bruckner and J. Ceder R[1], A. Bellen and A. Volčič R[1], D.R. Smart R[3], M. Kuczma, R. Ger and B. Choczewski R[1], etc.

24.8 Iterated functions on $\mathbb{C}$

• **Guidelines:** G. Julia (1918), M.P. Fatou (1919), J. Ritt (1920), A. Denjoy (1926), P. Montel (1927), B. Mandelbrot (1980), I.N. Baker (1984), P. Blanchard (1984), A. Douady and J.H. Hubbard (1984), D. Sullivan (1985), B. Branner and J.H. Hubbard (1988), A.F. Beardon (1990).

• **Notions and results**

Let $f : \mathbb{C} \cup \{+\infty\} \to \mathbb{C} \cup \{+\infty\}$ be a nonconstant rational function. By definition:

(i) the Fatou set of f, denoted by $F(f)$, is the maximal open subset of $\mathbb{C} \cup \{+\infty\}$ on which $(f^n)_{n\in\mathbb{N}}$ is equicontinuous;

(ii) the Julia set, denoted by $J(f)$, is the complement of $F(f)$ in $\mathbb{C} \cup \{+\infty\}$.

Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function and $x^* \in F_f$. Then, by definition, x^* is (A.F. Beardon R[1]) called:

(a) super-attracting if and only if $f'(x^*) = 0$;
(b) attracting if and only if $0 < |f'(x^*)| < 1$;
(c) repelling if and only if $|f'(x^*)| > 1$;
(d) rationally indifferent if and only if $f'(x^*)$ is a root of unity;

(e) irrationally indifferent if and only if $|f'(x^*)| = 1$, but $f'(x^*)$ is not a root of unity.

Denote $\mathbb{C}_\infty := \mathbb{C} \cup \{+\infty\}$.

Then, we have:

Theorem 24.8.1. *Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational function. Then $F(f)$ and $J(f)$ are completely invariant under f.*

Theorem 24.8.2. *If $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a polynomial function of degree at least 2, then ∞ is in $F(f)$ and the component of $F(f)$ containing ∞ is completely invariant under f.*

Theorem 24.8.3. *Let $f := \frac{f_1}{f_2}$ be a rational function such that $deg(f) := \max\{deg(f_1), deg(f_2)\} \geq 2$. Then:*

(i) $J(f)$ is infinite;

(ii) $J(f)$ is contained in the closure of the set of periodic points of f.

• **References:** A.F. Beardon R[1], R.L. Devaney R[1], R.L. Devaney and L. Kean (Eds.) R[1], P. Blanchard R[1], T. Kuczumow, S. Reich and D. Shoikhet R[1], R.D. Mauldin and M. Urbański R[1], R[2], I. Bârza and D. Ghişa B[1]-B[2] and the references therein.

24.9 Fixed point theory in $\mathbb{C}^n$ and in a complex Banach space

• **Results:**

Whiltington's Theorem. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ two sequences in $\mathbb{C}$. If the sequence $(a_n)_{n\in\mathbb{N}}$ does not have finite limit, then there exists an entire function $f : \mathbb{C} \to \mathbb{C}$ such that:*

(1) $a_n \in F_f$, for all $n \in \mathbb{N}$;

(2) $f'(a_n) = b_n$, for all $n \in \mathbb{N}$.

Earle-Hamilton's Theorem. *Let X be a complex Banach space, $Y \subset X$ a bounded domain and $f : Y \to Y$ an operator. We suppose that:*

(i) f is holomorphic;

(ii) $f(Y)$ lies strictly inside Y.

Then, f is a Picard operator.

Hayden-Suffridge's Theorem. *Let X be a complex Hilbert space, $f : B(0,1) \to B(0,1)$ an operator. We suppose that:*

(i) f is surjective;

(ii) f is biholomorphic.

Then:

(a) h is biholomorphic in a large region and map $\overline{B}(0;1)$ onto $\overline{B}(0;1)$;

(b) f has a fixed point in $\overline{B}(0;1)$.

Shields' Theorem. *Let Y be a commuting family of continuous functions from $\overline{B}(0;1) \subset \mathbb{C}$ to $\overline{B}(0;1)$. If the elements of Y are holomorphic in $B(0;1)$, then $\bigcap\limits_{f \in Y} F_f \neq \emptyset$.*

• **References:** S.G. Krantz R[1], T. Kuczumow, S. Reich and D. Shoikhet R[1], I.A. Rus B[73] (pp. 107-108), I.N. Baker R[1], T.L. Hayden and T.J. Suffridge R[1], N. Suita R[1], A.L. Shields R[1], D. Abts and J. Reinermann R[1], S. Reich and D. Shoikhet R[1], K. Włodarczyk R[1], L.A. Harris R[1].

24.10 Fixed point theory in ordered linear spaces

• **Guidelines:** R. Cristescu R[1] and B[1], G. Isac B[1], V. Berinde B[7], F. Voicu B[2], B[5] and B[8].

• **General references:** E. Zeidler R[1], I.A. Rus B[90], H. Amann R[3], K. Deimling R[3], M.A. Krasnoselskii R[1], D. Guo and V. Lakshmikantham R[1], P. Hess R[1], O. Hadžić R[2]. For σ-complete vector lattice see Chapter 1.3.

24.11 Minimal displacement of points under operators

• Let (X,d) be a metric space and $f : X \to X$ an operator. By definition (K. Goebel (1973)), the minimal displacement of f is the following number

$$(MD)_f := \inf_{x \in X} d(x, f(x)).$$

A point $x_0 \in X$ is called the best almost fixed point of f if and only if $(MD)_f = d(x_0, f(x_0))$.

• **Results:**

Sternfeld-Lin's Theorem. *Let X be a Banach space and $Y \subset X$ a closed bounded convex but noncompact subset of X. Then for any $K > 1$, there exists an operator $f : Y \to Y$ satisfying the Lipschitz condition,*

$$\|f(x) - f(y)\| \le K\|x - y\|, \quad \textit{for all} \quad x, y \in Y$$

and such that $(MD)_f > 0$.

• **References:** K. Goebel R[3], K. Balibok and K. Goebel R[1], M. Furi and M. Martelli R[1], T. Kuczumow, S. Reich and A. Stachura R[1], M. Angrisani and M. Clavelli R[1], A.I. Ban and S.G. Gal B[1].

24.12 Almost and approximate fixed point property

• **Notions:**

Let X be a nonempty set, α a covering of X and $f : X \to X$ an operator. A point $x \in X$ is an α-fixed point of f if there exists $U \in \alpha$ such that x and $f(x)$ belong to U. Let $\mathcal{C}(X)$ be a family of coverings of X and $M(X) \subset \mathbb{M}(X)$ a family of operators. By definition, X has the almost fixed point property with respect to $\mathcal{C}(X)$ and $M(X)$ if, for every $f \in M(X)$ and every $\alpha \in \mathcal{C}(X)$ there exists an α-fixed point of f.

Let (X, d) a metric space, $f : X \to X$ an operator and ε a positive number. A point $x \in X$ is an ε-fixed point of f if, $d(x, f(x)) \le \varepsilon$. By definition, X has the approximate fixed point property with respect to a family $M(X) \subset \mathbb{M}(X)$ if, for every $f \in M(X)$ and every $\varepsilon > 0$, there exists an ε-fixed point of f, i.e., if minimal displacement of f, $(MD)_f = 0$. A convex subset Y of a Banach space X has the approximate fixed point property if the metric space $(Y, d_{\|\cdot\|})$ has the approximate fixed point property w.r.t. the family of all nonexpansive operator $f : Y \to Y$.

Let (X, d) be a metric space and $f : X \to X$ be an operator. A sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

$$d(x_n, f(x_n)) \to 0 \text{ as } n \to +\infty$$

is by definition an approximate fixed point sequence of f.

• **Results:**

Goebel-Karlovitz's Lemma. *Let Y be a weakly compact convex subset of a Banach space X and $f : Y \to Y$ be a nonexpansive operator. We suppose:*

(i) Y is a minimal invariant subset for f;

(ii) $(x_n)_{n\in\mathbb{N}}$ is n approximate fixed point sequence of f.

Then, $\lim\limits_{n\to+\infty} \|y - x_n\| = \delta(Y)$, for all $y \in Y$.

de Groot-de Vries-van der Walt's Theorem. *The Euclidean plane has the almost fixed point property with respect to orientation preserving topological isometries and finite coverings by arcwise connected sets.*

Fort's Theorem. *The open Euclidean sphere $B(0;r) \subset \mathbb{R}^n$ has the approximate fixed point property with respect to the family of all continuous functions $f : B(0,r) \to B(0,r)$.*

Shafrir's Theorem. *A convex subset Y of a Banach space X has the approximate fixed point property if and only if Y is directionally bounded.*

Bruck's Theorem. *Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$ and $f : Y \to Y$ a nonexpansive operator. Then the ε-fixed point set of f is pathwise connected.*

• **General references:** T. van der Walt R[1], W.A. Kirk and B. Sims (Eds.) R[1] (see the chapters by K. Goebel and W.A. Kirk; K. Goebel; T. Kuczumow, S. Reich and D. Shoikhet), A. Granas and J. Dugundji R[1], J. Jaworowski, W.A. Kirk and S. Park R[1], D. Butnariu and A.N. Iusem B[1] and B[2], J.-B. Baillon and S. Simons R[1], Gh. Constantin B[4], Gh. Constantin and V. Radu B[1], S. Park R[4], D.R. Smart R[2], M. Păcurar B[1], M. Păcurar and V. Berinde B[1].

24.13 Periodic points

• **Notions:**

Let X be a nonempty set and $f : X \to X$ an operator. Then by definition:

(1) a point $x \in X$ is called a periodic point of f if $f^n(x) = x$ for some $n \geq 1$ and the minimal such n is called the period of x.

(2) f is called a periodic operator if $f^n = 1_X$ for some $n \geq 1$ and the

minimal such n is called the period of f.

(3) a periodic operator of period 2 is called an involution.

Let X be a topological space and $f : X \to X$ be an operator. An element $x \in X$ is called a recurrent point of f if and only if

$$x \in \omega_f(x) := \{y \in X : \text{ there is } n_k \to +\infty \text{ such that } f^{n_k}(x) \to y \text{ as } n_k \to +\infty\}.$$

• **Results:**

Theorem 24.13.1. (A. Granas and J. Dugundji R[1]) *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a function. We suppose that:*

(i) f is a topological isomorphism;

(ii) f is an involution.

Then, $F_f \neq \emptyset$.

Goebel-Kaczor's Theorem. *Let X be a Banach space and $Y \in P_{cp,cv}(X)$. Then, any continuous operator $f : Y \to Y$ has a periodic point with period n, for any $n \in \mathbb{N}^*$.*

Klee's Theorem. *Let X be an infinite dimensional Hilbert space and $Y \in P_{cp}(X)$. Then there exists a periodic topological isomorphism $f : X \to X$ whose fixed point set is Y.*

Roux-Zanco's Theorem. *Let X be a topological space, $x \in X$ and $f : X \to X$ an operator. We suppose that:*

(i) f is continuous;

(ii) $\omega_f(x)$ is a nonempty compact subset of X.

Then, there exists in $\omega_f(x)$ at least one recurrent point of f.

• **General references:** T. van der Walt R[1], A. Granas and J. Dugundji R[1], W.A. Kirk and B. Sims R[1], P.E. Conner and F.E. Floyd R[1], R.F. Brown, M. Furi, L. Górniewicz and B. Jiang R[1], M.A. Krasnoselskii and P. Zabrejko R[1], D. Roux and C. Zanco R[1], A. Bege B[1], A.C. Donescu B[1], I.A. Rus B[4] and B[73], G. Crăciun, P. Harja, M. Prunescu and T. Zamfirescu B[1], S. López de Medrano R[1].

24.14 Invariability of the fixed point set of a multivalued operator

• **Problem 24.14.1.** Let X be a set and $T : X \to P(X)$ a multivalued operator. In which conditions we have that $T(F_T) = F_T$?

Problem 24.12.2. In which conditions we have $F_T = (SF)_T$?
• **General references:** I.A. Rus B[17], B[18], B[42], B[95], M.C. Anisiu B[6], A.S. Mureşan B[6].

24.15 Stability of the fixed point property

There are many sort of stability about the fixed point property:

- Stability of the fixed point set for a continuous operator
  - stability of a fixed point of a continuous operator
  - stability of a component of the fixed point set of a continuous operator

• **General references:** T. van der Walt R[1], M.K. Fort R[1], A. Granas and J. Dugundji R[1], I.A. Rus B[73], R.F. Brown, M. Furi, L. Górniewicz and B. Jiang R[1],

- Stability of the fixed point property for a nonexpansive operator
  - stability in terms of Banach-Mazur metric
  - stability in terms of equivalent norms

  • **General references:** J. Garcia-Falset, A. Jiménez-Melado and E. Llorens-Fuster, pp. 201-238, in W.A. Kirk and B. Sims (Eds.) R[1], K. Goebel and W.A. Kirk R[1], S. Reich and A. Zaslavski R[13], P.K. Lin R[1].

  - data dependence of the fixed point set.

• **General references:** D.R. Smart R[1], M.A. Krasnoselskii and P. Zabrejko R[1], E. Zeidler R[1], W.A. Kirk and B. Sims (Eds.) R[1], I.A. Rus B[4], B[70], B[73], V. Berinde B[7], T.A. Burton R[5], J.K. Hale R[1], T.C. Lim R[4], I.A. Rus, A. Petruşel and A. Sîntămărian B[1], B[2], L.C. Becker and T.A. Burton R[1].

• For other aspects of the stability in the fixed point theory see: J.T. Markin R[1] and R[2], J.K. Hale R[1], T. Wang R[1], J. Jachymski R[4], S. Czerwik R[1], T.A. Burton and D.P. Dwiggins R[2], I.U. Bronšteĭn, V.A. Glăvan and V.F. Černik R[1], I.U. Bronšteĭn and V.A. Glăvan R[1] and R[2], G. Kassay and I. Kolumban B[1], Şt. Măruşter B[1], O. Cira and Şt. Măruşter R[1], A. Petruşel and E. Kirr B[1] and B[2], V.G. Angelov and I.A. Rus B[1], I.A. Rus, A. Petruşel and A. Sîntămărian B[1], B[2], A. Sîntămărian B[3], M.A. Serban B[1], B[2], B[5], A. Constantin B[8], I.A. Rus and S. Mureşan B[1] and B[2], A. Petruşel B[26], I.A. Rus B[102],...

24.16 Relative fixed point property

• Let X and Y be two topological space. By definition X has the fixed point property with respect to Y if for each pair of continuous operator, $f : X \to X$, $g : X \to Y$ there exists some $x \in X$ such that $g(x) = g(f(x))$.

We have:

Jerrard's Theorem. (R. Jerrard R[1]) *Let X be a topological space. The following statements are equivalent:*

(i) X has the topological fixed point property;

(ii) X has the fixed point property with respect to all nonempty topological spaces;

(iii) X has the fixed point property with respect to X.

• **References:** R. Jerrard R[1], D.K. Bayen R[1], I.A. Rus B[46].

24.17 Antipodal points

• **Guidelines:** L.A. Lusternik and L. Schnirelmann (1930), K. Borsuk (1933), M.A. Krasnoselskii and S.G. Krein (1949), M.A. Krasnoselskii (1950), J.W.

Jaworowski (1956), M. Altman (1958), A. Granas (1962), J. Dugundji (1965), P. Bacon (1966), A. Dold (1983).

• **Results:**

Antipodal Borsuk-Ulam Theorem. *If $f : S^n \to \mathbb{R}^m$ with $m < n$ is a continuous function, then there exist $x \in S^n$ such that $f(x) = f(-x)$.*

Surjectivity Borsuk-Ulam Theorem. *Let $f : S^n \to S^n$ be a continuous function such that $f(x) \neq f(-x)$ for each $x \in S^n$. Then f is surjective.*

Lusternik-Schnirelmann-Borsuk Theorem. *Let $\{Y_1, \ldots, Y_{n+1}\}$ be a closed covering of S^n. Then at least one set Y_i must contain a pair of antipodal points.*

• **General references:** T. van der Walt R[1], A. Granas and J. Dugundji R[1], N.G. Lloyd R[1], K. Deimling R[3] and R[4], J. Jaworowski, W.A. Kirk and S. Park R[1], H. Steinlein R[2], E. Zeidler R[1], M.C. Anisiu B[9]. For the fact that Borsuk-Ulam theorem implies Brouwer theorem see A.Yu. Volovikov R[1].

24.18 Classification of fixed points

• **Precursors:** H. Poincaré.

• **Guidelines:** M.K. Fort (1950), F.E. Browder (1965), A.N. Sharkowskii (1965), N. Levinson R[1].

• **General references:** T. van der Walt R[1], E. Akin R[1], R.L. Devaney R[1], S.Yu. Pilyugin R[1], A.F. Beardon R[1], G.R. Belitskii and Yu.L. Lyubich R[1], V. Barbuti and S. Guerra R[1], M.K. Fort R[1], G. Gabor R[1], Z. Balogh and A. Volberg B[1], I. Del Prete, M. Di Iorio and S. Naimpally R[1], D. Chiorean, B. Rus, I.A. Rus and D. Trif B[1], S. Mureşan B[1], B. Rus, I.A. Rus and D. Trif B[1]. See also 24.7 and 24.8.

24.19 Fixed point theory for fuzzy operators

• **General references:** J. Heilpern R[1], D. Butnariu B[1], B[2], B[4] and B[5], S.G. Gal B[1], V. Radu B[26], A.I. Ban and S.G. Gal B[1].

24.20 Fixed point theory in algebraic structures

• **General references:** D. Gorenstein R[1], I.A. Rus B[90] and the references therein (J.D. Dixon (1967), S. Dubuc (1969), W. Feit and J.G. Thompson (1963), B. Fisher (1966), G. Glauberman (1964), D. Gorenstein and I.N. Herstein (1961), F. Gross (1968), G. Higman (1957), E. Shult (1965), J.G. Thompson (1959), N. Jacobson (1956), A.G. Kuros (1951), E.J. Taft (1968), K. Iseki (1964)), G. Glauberman R[1], D.J. Hemmer R[1], G. Ercan and I.Ş. Güloglu R[1], K. Denecke R[1], C. Smorynski R[1], etc.

For the fixed point theory in ordered sets, see Chapter 2.

24.21 Fixed point theory in algebraic topology

• **General references:** R.F. Brown R[5], R.F. Brown, M. Furi, L. Górniewicz and B. Jiang R[1], A. Dold R[1], L. Górniewicz R[1]-R[3], J. Andres an L. Górniewicz R[2], R.D. Nussbaum R[2], M. Furi, M.P. Pera and M. Spadini, R.P. Agarwal and D. O'Regan R[7], M. Balaj, Y.J. Cho and D. O'Regan R[1], D.L. Ferrario R[1], A. Nestke R[1], V.P. Okhezin R[1], F. Hirzebruch R[1], R. Geoghegan R[1], P. Wong R[1], etc.

For the Algebraic Topology see J. Dieudonné R[1], A. Dold R[2], S. Eilenberg and N. Steenrod R[1], R. Miron and I. Pop R[1], Yu.G. Borisovich, N.M. Bliznyakov, Ya.A. Izrailevich and T.N. Fomenko R[1], etc.

Main topics of this field are:

- Fixed Point Index Theory
- Lefschetz fixed point theorem
- Nielsen theory

As a Romanian contribution in this field, we mention:

Deleanu's Theorem. *Let X be a compact, absolute neighborhood retract and $f : X \to X$ be a continuous operator. If $\bigcap_{n\in\mathbb{N}} f^n(X)$ is homologically trivial in X, then the Lefschetz number of f is equal to* 1.

24.22 Finite commutative family of operators

• **General references:** Y. Kijima and W. Takahashi R[1], T.C. Lim R[1], C. Bonatti R[1], Y. Derriennic R[1], T. Shimizu and W. Takahashi R[1], A. Wiśnicki R[1], G.A. Isaev and A.S. Fainstein R[1].

See also Chapter 18, Section 4, for the fixed point structures with the common fixed point property.

Some basic results are:

Lim's Theorem. *Let Y be a bounded convex subset of a linear space and let $f_1, f_2, \cdots, f_m : Y \to Y$ be a finite commutative family of affine operators. Then*

$$F_{\sum_{i=1}^{m} \lambda_i \cdot f_i} = \bigcap_{i=1}^{m} F_{f_i}.$$

Bonatti's Theorem. *Let $f_1, f_2, \cdots, f_m : S^2 \to S^2$ be a finite commutative family of diffeomorphisms C^1-close to the identity. Then, the family has a common fixed point.*

24.23 Common fixed points for commuting families of operators

• **General references:** A.T.M. Lau and W. Takahashi R[1], T. Kuczumow, S. Reich and D. Shoikhet R[1], A. Kaewcharoen and W.A. Kirk R[1], R. Espínola and W.A. Kirk R[1], T. Hu and W. S. Heng R[1], T. van der Walt R[1], A. Granas and J. Dugundji R[1], J.R. Jachymski R[8], M.M. day R[1].

Two important results are:

Markov-Kakutani's Theorem. *Let X be a Hausdorff topological vector space and Y a nonempty compact convex subset of X. Then, every commuting family of continuous affine operators $f : Y \to Y$ has a common fixed point.*

Lau-Takahashi's Theorem. *Let S be a semitopological semigroup, let Y be a nonempty weakly compact convex subset of a Banach space X which has normal structure and let $\mathcal{S} := \{f_S \| \ s \in S\}$ be a continuous representation of*

S as nonexpansive self operators on Y. We suppose that the set RUC, of all right uniformly continuous functions on S, has a left subinvariant submean. Then, $\mathcal{S}$ has a common fixed point.

24.24 Asymptotic fixed point theory

• **General references:** W.A. Kirk and B. Sims (Eds.) R[1], K. Goebel and W.A. Kirk R[1], K. Deimling R[3], A. Granas and J. Dugundji R[1], R.F. Brown, M. Furi, L. Górniewicz and B. Jiang (Eds.) R[1], S.Yu. Pilyugin R[1], T.A. Burton and D.P. Dwiggins R[1], F.E. Browder R[1], A. Bellen and A. Volčič R[1], T.K. Hu R[1], R.D. Nussbaum R[3], E. Zeidler R[1], V. Seda R[2], J.R. Jachymski and J.D. Stein R[1], W.A. Horn R[1], J. Eells and G. Fournier R[1]-R[2], G. Fournier and H.-O. Peitgen R[1], D.P. Dwiggins R[1], V.I. Istrăţescu B[1], B[2] and B[4], I.A. Rus B[4], B[49], B[70], B[73] and B[95], A. Bege B[1], D. Miheţ B[12], D. Miheţ and V. Radu B[1], T. Baranyai R[1].

- **Basic concepts in terms of the iterates:**
  - asymptotic regular operator
  - Picard operator
  - weakly Picard operator
  - ψ-weakly Picard operator
  - asymptotic center
  - asymptotically nonexpansive operator
  - k-strictly asymptotically pseudocontractive operator
  - uniformly k Lipschitz operator ($k > 1$)
  - compact dissipative operator
  - orbit of a point under an operator
  - ω-limit point
  - attractor
  - shadowing property of an operator
  - shadowing property of a dynamical system
  - rotative operator
  - eventual compact operator
  - asymptotic compact operator

- asymptotic Nielsen number
- ergodic theorem
- asymptotic fixed point theorem
- asymptotic Schauder conjecture

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- Some results and problems:

Istrăţescu's Theorem. (V.I. Istrăţescu B[4]) *Let* (X,d) *be a complete metric space and* $f : X \to X$ *be an operator. We suppose that there exist* $a_1, a_2 \in \mathbb{R}_+$ *with* $a_1 + a_2 < 1$ *such that*

$$d(f^2(x), f^2(y)) \leq a_1 d(x,y) + a_2 d(f(x), f(y)), \text{ for all } x, y \in X.$$

Then, $F_f = \{x^*\}$.

Horn's Theorem. (W.A. Horn R[1]) *Let* X *be a Banach space and* $f : X \to X$ *be an operator. We suppose:*

(i) f *is completely continuous;*

(ii) there exists a bounded subset Y *of* X*, such that for each* $x \in X$ *there exists* $m(x) \in \mathbb{N}^*$ *with* $f^{m(x)}(x) \in Y$.

Then, $F_f \neq \emptyset$.

Browder's Theorem. (F.E. Browder R[10]) *Let* X *be a Banach space,* $Y_0 \subset Y_1 \subset Y \subset X$ *and* $f : Y \to X$ *be an operator. We suppose:*

(i) Y_0 *is closed and* Y_1 *and* Y *are open;*

(ii) f *is a completely continuous operator;*

(iii) there exists $m \in \mathbb{N}^*$ *such that* $f^i(Y_0) \subset Y_1$*, for* $i \in \{1, 2, \cdots m\}$ *and* $f^m(Y_1) \subset Y_0$.

Then, $F_f \cap Y_0 \neq \emptyset$.

Schauder's Conjecture. (F.E. Browder (Ed.) R[2]) *Let* Y *be a bounded closed convex subset of a Banach space* X *and let* $f : Y \to Y$ *be an operator. We suppose:*

(i) f *is continuous;*

(ii) there exists $m \in \mathbb{N}^*$ *such that* f^m *is compact.*

Then, $F_f \neq \emptyset$.

Browder-Nussbaum's Conjecture. (R.D. Nussbaum R[1]) *Let Y be a closed convex subset of a Banach space X and $f : Y \to Y$ be an operator. We suppose that:*

(i) f is continuous;

(ii) there exists $K \subset Y$ an attractor for compact sets under f, i.e.,

(a) K is nonempty and compact;

(b) $f(K) \subset K$;

(c) for each compact subset A of K and each neighborhood V of K, there exists $n(A,V) \in \mathbb{N}^*$ such that $f^m(A) \subset V$ for all $m \geq n(A,V)$.

Then, $F_f \neq \emptyset$.

24.25 Fixed point theory in categories

• **General references:** F. Lawvere R[1], D. Scott R[1], M. Wand R[1], J. Soto-Andrade and F.J. Varela R[1], M. Barr and C. Wells R[1], J. Adámek, V. Koubek and J. Reiterman R[1], J. Lambek R[1], I.A. Rus B[64], B[85], B[90] and B[95], W. Forster R[1], A. Baranga B[1], C. Bănică and N. Popescu B[1], M. Szilagyi B[1], R. Ceterchi B[1].

• **Examples of categories:**

The category SET. The class of objects in the class of all sets. If $A, B \in$ Ob SET, then $\text{Mor}(A,B) := \mathbf{M}(A,B)$.

The category SELF-OP. The objects of this category are self operators. Let $f : A \to A$ and $g : B \to B$ be two objects of this category. A morphism from f to g is an operator $h : A \to B$ such that $h \circ f = g \circ h$.

The category POSET. The class of objects is, in this case, the class of all partially ordered sets and $\text{Mor}(A,B) := \{f : A \to B|$ f is increasing $\}$.

The category TOP. The class of objects is, in this case, the class of all topological spaces and $\text{Mor}(A,B) := \{f : A \to B|$ f is continuous $\}$.

• **Basic notions and results:**

Let $\mathcal{C}$ be a category and $A \in Ob(\mathcal{C})$. A morphism $f \in \text{Mor}(A,A)$ has the fixed point property if and only if there exists $B \in Ob(\mathcal{C})$ and $g \in \text{Mor}(B,A)$

such that $f \circ g = g$. An object $A \in Ob(\mathcal{C})$ has the fixed point property if each morphism $f \in \text{Mor}(A, A)$ has the fixed point property.

Let $\mathcal{C}$ be a category of sets with structure (Set, POSET, TOP, etc.). We have:

J. Soto-Andrade and F.J. Varela Theorem. *Suppose that the category $\mathcal{C}$ has finite products and powers. Given an object A of $\mathcal{C}$, if there exists some other object B of $\mathcal{C}$ and a surjective morphism $f : B \to Mor(B, A)$, then A has the fixed point property.*

Isomorphism Theorem. (I.A. Rus B[95]) *Let $\mathcal{C}$ be a category and $A, B \in Ob(\mathcal{C})$ be two objects of $\mathcal{C}$. Suppose that:*

(i) the object A has the fixed point property;

(ii) there exists an isomorphism $\varphi \in Mor(A, B)$.

Then, B is an object with the fixed point property.

Retraction Theorem. (J. Soto-Andrade and F.J. Varela R[1]) *Let $\mathcal{C}$ be a category with final object and $A, B \in Ob(\mathcal{C})$. We suppose that:*

(i) the object A has the fixed point property;

(ii) the object B is a weak retract of A.

Then, B is an object with the fixed point property.

Problem of J. Adámek, V. Koubek and J. Reiterman. *Characterize categories whose all indecomposable representations have the fixed point property.*

Recall that, by definition, a representation $F : \mathcal{C} \to Set$ has the fixed point property if for each endomorphism $\tau : F \to F$ there exists an object A in $\mathcal{C}$ and a point $a \in F(A)$ with $\tau_A(a) = a$.

24.26 Maximal fixed point structures

• **General references:** I.A. Rus B[95] (pp. 32-36 and pp. 143-144 and the references therein: E.H. Connell (1959), P.K. Lin and Y. Sternfeld (1985), A.C. Davis (1955), T.K. Hu (1967), W.A. Kirk (1976), S. Park (1984), P.P. Subrahmanyam (1975), M.C. Anisiu and V. Anisiu (1977), I.A. Rus, S. Mureşan and E. Miklos (2003), I.A. Rus (2006).

Let $(X, S(X), M)$ be a fixed point structure (briefly f.p.s.) and $S_1(X) \subset P(X)$ such that $S_1(X) \supset S(X)$.

Definition 24.26.1. (I.A. Rus (1996)) The f.p.s. $(X, S(X), M)$ is maximal in $S_1(X)$ if we have

$$S(X) = \{A \in S_1(X) | \ f \in M(A) \ \Rightarrow \ F_f \neq \emptyset\}.$$

Problem 24.26.1. Establish if a given f.p.s. is or isn't maximal.

For example, in some concrete structured sets, the above problem has the following form:

- Characterize the ordered sets with fixed point property with respect to increasing operators.
- Characterize the topological space with f.p.p. with respect to contractions.
- Characterize the metric space with f.p.p. with respect to continuous operators.
- Characterize the metric space with f.p.p. with respect to contractions.
- Characterize the Banach spaces X with the following property:

  $$Y \in P_{b,cl,cv}(X), \ f : Y \to Y \text{ is nonexpansive } \ \Rightarrow \ F_f \neq \emptyset.$$

- Characterize the Banach spaces X with the following property:

  $$Y \in P_{wcp,cv}(X), \ f : Y \to Y \text{ nonexpansive } \ \Rightarrow \ F_f \neq \emptyset.$$

- Does the following implication hold:

  If X is a Banach space and $Y \in P_{cl,cv}(X)$ has the fixed point property with respect to continuous operators $\Rightarrow Y \in P_{cp}(X)$?

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24.27 The computation of fixed points

• **Sequences of operators and fixed points:** F.F. Bonsal (1962), F.E. Browder (1967), S.B. Nadler jr. (1968), R.B. Fraser and S.B. Nadler jr. (1969), M. Furi and M. Martelli (1969), W. Russel and S.P. Singh (1969), G. Vidossich (1971), I.A. Rus (1979), etc.

• **General references:** I.A. Rus B[98].

• **Iterative approximation of fixed points:** W.R. Mann (1953), M.A. Krasnoselskii (1955), H. Schaefer (1975), W.V. Petryshyn (1966), W.G. Dotson (1970), S. Ishikawa (1974), C.E. Chidume (1981), B.E. Rhoades (1990), V. Berinde (2002), Y. Alber S. Reich and J.-C. Yao R[1], Şt. Măruşter and Cristina Popìrlan R[1], N. Castaneda R[1], C.E. Chidume and B. Ali R[1] and R[2], L.-C. Ceng, A. Petruşel and J.-C. Yao R[1]-R[3], etc.

• **General references:** V. Berinde B[37].

• **Fixed point algorithms:** H. Scarf (1967), D.I.A. Cohen (1967), H.W. Kuhn (1968), B.C. Eaves (1971), M.J. Todd (1976), R.B. Kellog, T.Y. Li and J. Yorke (1976), S.-N. Chow, J. Mallet-Paret and J.A. Yorke (1976), H. Tuy (1976), L. Filus (1977), W. Forster (1980), etc.

• **General references:** M.J. Todd R[1], W. Forster R[1], M.-L. Su and X.-R. Lü R[1]-R[2].

• **Other topics:** J.M. Ortega and W.C. Rheinboldt R[1], D. Butanariu B[1], D. Butnariu and A.N. Iusem B[1], D. Butnariu and I. Markowitz B[1], I. Dziţac B[1], B. Finta R[1], D. Trif B[4] and B[5], Şt. Soltuz R[1], Şt. Măruşter R[1], I. Păvăloiu R[1], S. Karamardian R[1], K. Schilling R[1], F. Robert R[1], B.C. Eaves R[1], B.S.W. Schröder R[3], R.B. Kellog, T.Y. Li and J. Yorke R[1], E. Allgower and K. Georg R[1], D. Butnariu and E. Resmerita R[1], etc.

• **Successive approximations of fixed points:** see Chapters 3-11, 14, 15, 17.

24.28 Bifurcation theory

• **Basic concepts.** Let X be a Banach space and $f : \mathbb{R} \times X \to X$ be an operator. We consider the following family of fixed point equations:

$$x = f(\lambda, x), \lambda \in \mathbb{R}.$$

We suppose that $f(\lambda, 0) = 0$, for all $\lambda \in \mathbb{R}$. Let

$$S := \{(\lambda, x) \in \mathbb{R} \times X | x = f(\lambda, x), x \neq 0_X\}.$$

By definition, an element $(\lambda_0, 0_X) \in \mathbb{R} \times X$ is called a bifurcation fixed point for f (or a bifurcation solution) if $(\lambda_0, 0_X) \in \overline{S}$.

Actually, if we have an infinite family of problems $(P_\lambda)_{\lambda \in \mathbb{R}}$, a bifurcation point exists if we have a "change" in the structure of the solutions set of the problems $(P_\lambda)_{\lambda \in \mathbb{R}}$, when the parameter λ varies. Thus, we will have: bifurcation fixed point, bifurcation periodic point, bifurcation zero point, bifurcation coincidence point, bifurcation of eigenvalue, bifurcation equilibrium point, bifurcation critical point, bifurcation limit cycle, etc.

• **General references:** M.A. Krasnoselskii R[3], M.A. Krasnoselskii and P. Zabrejko R[1], K. Deimling R[3] (C., J.M. Lasry and M. Schatzman (1980), M.S. Berger (1969), G. Iooss and D.D. Joseph (1980), J. Ize (1976), J.E. Marsden and M. McCracken (1976), R.D. Nussbaum (1975), P.H. Rabinowicz (1977), M.M. Vainberg and V.A. Trenogin (1974), etc.) A. Granas and J. Dugundji R[1], R.F. Brown, M. Furi, L. Górniewicz and B. Jiang R[1] (Eds.), Y.A. Kuznetsov R[1] (V.I. Arnold (1972), A. Bazykin, Y. Kuznetsov and A. Khibnik (1989), S.N. Chow and J. Hale (1982), B. Fiedler (1988), M. Golubitsky, I. Stewart and D. Schaeffer (1988), J. Guckenheimer and P. Holmes (1983), J. Hale and H. Kocak (1991), S. Wiggins (1988), etc.), L. Nirenberg R[1], D. Pascali B[1], S. Sburlan B[1] and B[2], S. Codreanu and M. László B[1], A. Buică and J. Llibre B[1] and B[2], A. Buică, J.-P. Francoise and J. Llibre B[1], E.U. Tarafdar and M.S.R. Chowdhury R[1], J.S.W. Lamb, I. Melbourne R[1], etc.

24.29 Surjectivity, injectivity, invariance of domain and fixed points

• **General references:** G.J. Minty r[1], S.I. Pohožaev R[1], F.E. Browder R[9] and R[12], P.Q. Khanh R[1] M.S. Berger R[2], L. Nirenberg R[2], K. Deimling R[3], M. Furi, M. Martelli and A. Vignoli R[1], A. Granas and J. Dugundji R[1], M.A. Krasnoselskii R[1], J.T. Schwartz R[1], E. Zeidler R[2], D. Pascali and S. Sburlan B[1], I.A. Rus B[70], B[73], B[81], and B[95], V. Barbu and A. Cellina R[1], M. Nagumo R[1], S. Fucik R[1] and R[2], S. Kasahara R[4], W.O. Ray and A.M. Walker R[1], M. Altman R[7], R.T. Rockafellar R[1], S. Simons R[1], V. Seda R[1], J. Mawhin R[6], W. Kulpa and M. Turzański R[1], J. Gevirtz R[1], L. Janos R[3], J. Danes and J. Kolomy R[1], B. Ricceri R[5], F. Aldea B[1] and B[2], A. Leonte and A. Duma B[1], A.S. Mureşan B[7], V. Mureşan B[1] and B[3], A. Petruşel B[20], T. Lazăr, A. Petruşel and N. Shahzad B[1], F. Voicu B[3], M.C. Anisiu R[2], V. Berinde R[5], R.P. Agarwal, D. O'Regan and R. Precup B[1], D. Reem R[1]., B. Ricceri R[3], A. Pietsch R[1], T.R. Hamlett and L.L. Herrington R[1], C.S. Kubrusly R[1], S. Sawyer R[1], J.M. Soriano and V.G. Angelov R[1], etc. See also Chapter 3.4.

• **Basic results:**

Domain Invariance Theorem. *Let X be a Banach space, $Y \subset X$ an open subset of X and $f : Y \to X$ be a completely continuous operator. If $1_X - f$ is injective, then $1_X - f$ is open.*

Minty's Theorem. *Let $(X, \prec \cdot, \cdot \succ)$ be a complex Hilbert space and $f : X \to X$ be an operator. We suppose that:*

(i) f is continuous;

(ii) there exists $c > 0$ such that

$$Re \prec f(x) - f(y), x - y \succ \geq c\|x - y\|^2, \text{ for all } x.y \in X.$$

Then, f is a topological isomorphism.

Browder-Ray-Walker's Theorem. *Let X and Y be two Banach spaces, $f : X \to Y$ be an open operator with closed graph and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous decreasing function with the property $\int_0^{+\infty} \varphi(s)ds = +\infty$.*

Suppose that, for each $x \in X$ there exists $\epsilon > 0$ such that the following implication holds:

$$x, y \in X,\ \|x - y\| \leq \epsilon \text{ implies } \varphi(\|x\| \cdot \|x - y\| \leq \|f(x) - f(y)\|.$$

Then, $f(X) = Y$.

Rockafellar's Theorem. *Let X be a reflexive Banach space, J be the duality operator and $T : X \to \mathcal{P}(X^*)$ be a maximal monotone operator. Then:*

(1) $T + J$ is injective;
(2) $\overline{T(X)}$ is convex;
(3) $int(T(X))$ is convex.

24.30 Implicit operators and fixed points

• **General references:** L. Nirenberg R[2], K. Deimling R[3], J.T. Schwartz R[1], E. Zeidler R[2], R.S. Hamilton R[1], I.A. Rus B[81], J. Appell, A. Vignoli and P.P. Zabrejko R[1], J. Appell R[4], L. Nirenberg R[1], M. Altman R[1], A. Deleanu and Gh. Marinescu B[1], A. Domokos B[1], W. Alt and I. Kolumban R[1], S.M. Robinson R[1].

• **Basic results:**

Inverse Operator Theorem. *Let X and Y be two Banach spaces, $x_0 \in X$ and $y_0 \in Y$. Let $U \subset X$ and $V \subset Y$ be neighborhoods of x_0 and y_0 respectively. Let $f : U \to V$ be an operator. We suppose that:*

(i) $f(x_0) = y_0$;
(ii) $f \in C^1(U, V)$;
(iii) there exists $(Df(x_0))^{-1} : Y \to X$ and it is continuous.

Then, there exist a neighborhood $U' \subset U$ of x_0 and a neighborhood $V' \subset V$ of y_0 such that $f : U' \to V'$ is a bijection and $f^{-1} \in C^1(V', U')$.

Implicit Operator Theorem. *Let X, Y and Z be three Banach spaces, $x_0 \in X$ and $y_0 \in Y$. Let $U \subset X$ and $V \subset Y$ be neighborhoods of x_0 and y_0 respectively. Let $f : U \times V \to Z$ be an operator. We suppose:*

(i) $f(x_0, y_0) = 0_Z$;

(ii) f is continuous;

)iii) there exist $\frac{\partial f(x,y)}{\partial y} : Y \to X$ and $\frac{\partial f(x_0,y_0)}{\partial y}^{-1}$ and they are continuous.

Then, there exist a neighborhood $U' \subset U$ of x_0, a neighborhood $V' \subset V$ of y_0 and a unique operator $g : U' \to V'$ such that:

(a) $g(x_0) = y_0$;

(b) $f(x, g(x)) = 0$, for all $x \in V'$;

(c) $g \in C(U', V')$.

24.31 Caristi selections for multivalued operators

Caristi's fixed point theorem states that each operator f from a complete metric space (X, d) into itself satisfying the condition:

there exists a lower semi-continuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that:

$$(1) \qquad d(x, f(x)) + \varphi(f(x)) \leq \varphi(x), \text{ for each } x \in X,$$

has at least a fixed point $x^* \in X$, i. e. $x^* = f(x^*)$.

For the multivalued case, if T is an operator of the complete metric space X into the space of all nonempty subsets of X and there exists a lower semi-continuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that:

$$(2) \text{ for each } x \in X, \text{ there is } y \in T(x) \text{ so that } d(x, y) + \varphi(y) \leq \varphi(x),$$

then the multivalued map T has at least a fixed point $x^* \in X$ (see Mizoguchi and Takahashi R[1]).

Moreover, if T satisfies the stronger condition:

$$(3) \text{ for each } x \in X \text{ and each } y \in T(x) \text{ we have } d(x, y) + \varphi(y) \leq \varphi(x),$$

then the multivalued map T has at least a strict fixed point $x^* \in X$, i. e. $\{x^*\} = T(x^*)$. (see Aubin and Siegel R[1]).

Theorem 24.31.1. (A. Petruşel, A. Sîntămărian, B[1]) *Let* (X,d) *be a metric space and* $F : X \to P_{cl}(X)$ *be a Reich-type multivalued map. Then, there exists* $f : X \to X$ *a selection of* F *satisfying the Caristi-type condition (1).*

Let (X,d) be a metric space and $T : X \to P(X)$ be a multivalued map.

Definition 24.31.1. (J.-P. Aubin, J. Siegel R[1]) A function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ is called:

(i) a weak entropy of T if the condition (2) holds.

(ii) an entropy of T if the condition (3) holds.

Moreover, the map $T : X \to P(X)$ is said to be weakly dissipative if there exists a weak entropy of T and it is said to be dissipative if there is an entropy of it.

Theorem 24.31.2. (A. Petruşel, A. Sîntămărian, B[1]) *Let* (X,d) *be a metric space and* $T : X \to P_{cp}(X)$ *be an* α*-contraction. Then,* T *is weakly dissipative with a weak entropy given by the formula* $\varphi(x) = (1-a)^{-1}\ D(x, T(x))$*, for each* $x \in X$*.*

Theorem 24.31.3. (A. Petruşel, A. Sîntămărian, B[1]) *Let* (X,d) *be a metric space and* $T : X \to P_{b,cl}(X)$ *be a multivalued operator, such that there exist* $a, b, c \in \mathbb{R}_+$*, with* $a + b + c < 1$ *such that*

$$\delta(T(x), T(y)) \leq ad(x,y) + bD(x, T(x)) + cD(y, T(y)),\ \textit{for each}\ x, y \in X.$$

Then, the multivalued operator T *is dissipative.*

Remark 24.31.1. For some results in connection with the theory of multivalued dynamical systems, see J.-P. Aubin and J. Siegel R[1], V. Barbu and A. Cellina R[1], D.N. Chebanm and D.S. Fakeeh R[1], K. Włodarczyk, D. Klim, R. Plebaniak R[1], etc.

24.32 Applications of the fixed point theory

24.32.1 Applications to functional equations

• In 2003, V. Radu B[26] gives a new proof of thr Hyers-Rassias-Gajda stability theorem for the Cauchy functional equation in a Banach space which is based on Luxemburg's fixed point theorem. Other results by the same method are given in L. Cădariu and V. Radu B[1]-B[6], L. Cădariu B[1], S.-M. Jung R[1]. See also V. Berinde R[1], etc.

24.32.2 Applications to differential equations

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24.32.3 Applications to integral equations

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List of Symbols

- Let X, Y be two nonempty sets and $f : X \to X$ an operator.

$$\mathbb{N} := \{0, 1, 2, \cdots\} \text{ and } \mathbb{N}^* := \mathbb{N} \setminus \{0\}$$

$$\mathbb{R} \text{ denotes the set of all real numbers and } \mathbb{R}^* := \mathbb{R} \setminus \{0\}$$

$$\mathcal{P}(X) := \{A | A \subseteq X\}$$

$$P(X) := \{A \subset X | A \neq \emptyset\}$$

$$Card X - \text{ the cardinal number of X}$$

$$1_X - \text{ the identity operator}$$

$$F_f := \{x \in X | f(x) = x\} - \text{ the fixed point set of f}$$

$$f^0 := 1_X,\ f^1 := f, ..., f^n := f \circ f^{n-1} - \text{ the iterates of f } n \in \mathbb{N}^*$$

$$O_f(x) := \{x, f(x), ..., f^n(x), ...\}, x \in X - \text{ the orbit of f with respect to x}$$

$$I(f) := \{A \in P(X) | f(A) \subset A\}$$

$$\mathbf{M}(X, Y) := \{f : X \to Y | \text{ f is an operator } \} \text{ and } \mathbf{M}(Y) := \mathbf{M}(Y, Y).$$

$$s(X) := \{(x_n)_{n \in \mathbb{N}^*} \mid x_n \in X,\ n \in \mathbb{N}^*\}$$

and

$$M(X) := \{(x_{ij})_1^\infty \mid x_{ij} \in X,\ i, j \in \mathbb{N}^*\}$$

where

$$(x_{ij})_1^\infty := \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ x_{31} & x_{32} & x_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

is an infinite matrix.

- Let (X, d) be a metric space.

$$\widetilde{B}(x, R) := \{y \in X | d(x, y) \leq R\}, \text{ where } x \in X, R > 0.$$

$$B(x, R) := \{y \in X | d(x, y) < R\}, \text{ where } x \in X, R > 0.$$

$$int(Y) \text{ denotes the interior of the set } Y \subset X.$$

$$\overline{Y} \text{ denotes the closure of the set } Y \subset X.$$

$$\text{If } Y \subset X, \text{ then } \delta(Y) \text{ denotes the diameter of Y.}$$

$$P_b(X) := \{Y \in P(X) | \delta(Y) < +\infty\}.$$

$$P_{cl}(X) := \{Y \in P(X) | Y = \overline{Y}\}.$$

$$P_{cp}(X) := \{Y \in P(X) | Y \text{ is compact }\}.$$

$D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$

$$D(A, B) = \begin{cases} \inf\{d(a, b)| \ a \in A, \ b \in B\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset = B \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

D is called the gap functional between A and B.

In particular, $D(x_0, B) = D(\{x_0\}, B)$ (where $x_0 \in X$) is called the distance from the point x_0 to the set B.

$\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\delta(A, B) = \begin{cases} \sup\{d(a, b)| \ a \in A, \ b \in B\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{otherwise} \end{cases}$$

In particular, $\delta(A) := \delta(A, A)$ is the diameter of the set A.

$\rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\rho(A,B) = \begin{cases} \sup\{D(a,B) \mid a \in A\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset \\ +\infty, & \text{if } B = \emptyset \neq A \end{cases}$$

ρ is called the excess functional of A over B.

$H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$H(A,B) = \begin{cases} \max\{\rho(A,B), \rho(B,A)\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset = B \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

H is called the generalized Pompeiu-Hausdorff functional of A and B.

$$V_r(Y) := \{x \in X \mid D(x,Y) < r\}$$

- Let X be a Banach space.

coA denotes the convex hull of the set $A \subset X$

$\overline{co}A$ denotes the closed convex hull of the set $A \subset X$

$$P_{cv}(X) := \{Y \in P(X) | Y \text{ is convex }\}$$

$$P_{cp,cv}(X) := \{Y \in P(X) | Y \text{ is compact and convex }\}$$

- If X, Z are nonempty sets, then the symbol $T : X \multimap Z$ or $T : X \to \mathcal{P}(Z)$ denotes the multivalued operator from X to Z.

$DomT := \{x \in X \mid T(x) \neq \emptyset\}$
$T(Y) := \bigcup_{x \in Y} T(x)$, for $Y \in P(X)$
$I(T) := \{A \in P(X) | T(A) \subset A\}$
$I_b(T) := \{Y \in I(T) | \delta(Y) < +\infty\}$
$I_{b,cl}(T) := \{Y \in I(T) | \delta(Y) < +\infty, Y = \overline{Y}\}$
$I_{cp}(T) := \{Y \in I(T) | Y \text{ is compact }\}$
$T^{-1}(z) := \{x \in X \mid z \in T(x)\}$

$Graph(T) := \{(x, z) \in X \times Z |\ z \in T(x)\}$

The sequence $(x_n)_{n\in\mathbb{N}} \subset X$, of successive approximations for T starting from $x \in X$ is defined by $x_0 = x,\ x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$.

If $T : X \to P(X)$, then for $x \in X$ we denote by

$$T^0(x) = x,\ T^1(x) = T(x), \cdots T^{n+1}(x) = T(T^n(x))$$

the iterates of T.

Also, an element $x \in X$ is a fixed point (respectively a strict fixed point) for T if $x \in T(x)$ (respectively $\{x\} = T(x)$). We denote by F_T (respectively by $(SF)_T$) the fixed point set (respectively the strict fixed point set) of T.

If X, Y are nonempty sets, then we denote:
$\mathbf{M}^0(X, Y) := \{T \mid T : X \to \mathcal{P}(Y)\}$ and $\mathbf{M}^0(Y) := \mathbf{M}^0(Y, Y)$.

Index of Terms

• C.

Authors Index