Fixed Point Theory, 16(2015), No. 2, 237-260
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

APPROXIMATION METHODS FOR TRIPLE HIERARCHICAL VARIATIONAL INEQUALITIES (II)

L.-C. CENG*, Q.H. ANSARI**, A. PETRUŞEL*** AND J.-C. YAO****

*Department of Mathematics, Shanghai Normal University, Shanghai 200234
and
Scientific Computing Key Laboratory of Shanghai Universities, China
E-mail: zenglc@hotmail.com

**Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India
and
Department of Mathematics and Statistics, King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia
E-mail: qhansari@gmail.com

***Department of Mathematics, Babeş-Bolyai University Cluj-Napoca
Kogălniceanu Street, No.1, 400084 Cluj-Napoca, Romania
E-mail: petrusel@math.ubbcluj.ro

****(Corresponding author) Center for General Education, China Medical University
Taichung 40402, Taiwan
and
Department of Mathematics, King Abdulaziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail: yaojc@mail.cmu.edu.tw

Abstract. In this paper, we consider a triple hierarchical variational inequalities (in short, THVI) with a finite family of nonexpansive mappings. By combining the viscosity approximation method, hybrid steepest-descent method and Mann's iteration method, we propose the hybrid steepest-descent viscosity approximation method for solving the THVI. The strong convergence of this method to a unique solution of the THVI is studied. Under some mild conditions, a strong convergence result (to the unique solution of THVI) for another iterative algorithm is also presented.

Key Words and Phrases: Triple hierarchical variational inequalities, hybrid steepest-descent viscosity approximation method, monotone operators, nonexpansive mappings, fixed point, strong convergence theorem.
2010 Mathematics Subject Classification: 49J40, 47H10, 65J20, 65J25, 49J30, 47H05, 47H09.

1. Introduction and formulations

Let H be a real Hilbert space with its inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. The set of all fixed points of a mapping $T : H \to H$ is denoted by $\mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$. Throughout the paper we follow the usual notation in nonlinear analysis (see [2]).

Let K be a nonempty convex subset of a Hilbert space H and $F : K \to H$ be a monotone mapping, that is, $\langle Fx - Fy, x - y\rangle \geq 0$, $\forall x, y \in K$. The monotone variational inequality problem [7] is to find $x^* \in K$ such that

$$\langle Fx^*, x - x^*\rangle \geq 0, \quad \forall x \in K.$$

The set of solutions of this problem is denoted by $\mathrm{VI}(K, F)$. The following variational inequality problem defined over the set $\mathrm{Fix}(T)$ of fixed points of a mapping $T : H \to H$ is called hierarchical variational inequality problem (in short, HVIP).

Problem 1.1. *Given a monotone, continuous operator $A : H \to H$ and a nonexpansive mapping $T : H \to H$,*

$$\textit{find } x^* \in \mathrm{VI}(\mathrm{Fix}(T), A) := \{x^* \in \mathrm{Fix}(T) : \langle Ax^*, v - x^*\rangle \geq 0,\ \forall v \in \mathrm{Fix}(T)\}.$$

Recently, Iiduka [3, 4] introduced three-stage variational inequality problem, that is, the monotone variational inequality problem over the solution set of HVIP.

Problem 1.2. *Assume that*

(A1) *$A_1 : H \to H$ is α-inverse-strongly monotone;*
(A2) *$A_2 : H \to H$ is η-strongly monotone and κ-Lipschitzian;*
(A3) *$T : H \to H$ is a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$;*
(A4) $\mathrm{VI}(\mathrm{Fix}(T), A_1) \neq \emptyset$.

Then, the purpose is to find $x^* \in \mathrm{VI}(\mathrm{VI}(\mathrm{Fix}(T), A_1), A_2)$, *where*

$$VI\,(VI(\Omega, A_1), A_2) := \left\{x^* \in VI\,(\Omega, A_1) : \langle A_2x^*, v - x^*\rangle \geq 0, \forall v \in VI\,(\Omega, A_1)\right\}.$$

The convergent analysis of the above problem was treated in the first part of this work, see [2].

Very recently, Ceng et al. [1] considered the following monotone variational inequality problem over the solution set of the variational inequality which is defined over the set of common fixed points of N nonexpansive mappings $T_i : H \to H$.

Problem 1.3. *Assume that*

(B1) *$A_1 : H \to H$ is α-inverse-strongly monotone;*
(B2) *$A_2 : H \to H$ is η-strongly monotone and κ-Lipschitzian;*
(B3) *for $i = 1, 2, \ldots, N$, $T_i : H \to H$ is a nonexpansive mapping with $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$;*
(B4) $\mathrm{VI}\left(\bigcap_{i=1}^N \mathrm{Fix}(T_i), A_1\right) \neq \emptyset$.

Then the objective is to find $x^* \in \mathrm{VI}\left(\mathrm{VI}\left(\bigcap_{i=1}^N \mathrm{Fix}(T_i), A_1\right), A_2\right)$.

In [1], the authors also proposed another relaxed hybrid steepest-descent algorithm with variable parameters for computing the approximate solutions of Problem 1.3.

We remark that $T_{[k]} := T_{k \bmod N}$ for integer $k \geq 1$ with the mod function taking values in the set $\{1, 2, \ldots, N\}$, that is, if $k = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_{[k]} = T_N$ if $q = 0$ and $T_{[k]} = T_q$ if $0 < q < N$.

In this paper, by combining the viscosity approximation method hybrid steepest-descent method and Mann's iteration method, using the approach given in the first part of this work for one mapping, we introduce two hybrid steepest-descent viscosity approximation algorithms for computing the appropriate solutions of Problem 1.3.

The strong convergence of the sequences generated by this algorithm is derived under some appropriate conditions. Obviously, whenever $\beta_n = \gamma_n = 0$, $\forall n \geq 0$, this algorithms reduce to Algorithm 2, in [1].

2. Preliminaries

Let H be a real Hilbert space. We denote by $x_n \rightharpoonup x$ (respectively, $x_n \to x$) to indicate that the sequence $\{x_n\}$ converges weakly (respectively, strongly) to x.

Definition 2.1. An operator $A : H \to H$ is called

(a) *strongly monotone* (or more precisely, *α-strongly monotone*) if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in H;$$

(b) *inverse-strongly monotone* (or more precisely, *β-inverse-strongly monotone*) (also called *co-coercive*) if there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in H;$$

(c) *hemicontinuous* if for all $x, y \in H$, the mapping $g : [0, 1] \to H$, defined by $g(t) := A(tx + (1 - t)y)$, is continuous.

Definition 2.2. Let C be a nonempty convex subset of a real Hilbert space H. A function $\varphi : C \to \mathbb{R}$ is said to be

(a) *convex* if for all $x, y \in C$ and all $\lambda \in [0, 1]$,

$$\varphi(\lambda x + (1 - \lambda)y) \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y);$$

(b) *strongly convex* if there exists $\alpha > 0$ such that for all $x, y \in C$ and all $\lambda \in [0, 1]$,

$$\varphi(\lambda x + (1 - \lambda)y) \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y) - \frac{1}{2}\alpha\lambda(1 - \lambda)\|x - y\|^2.$$

The metric projection $P_C : H \to C$ onto the nonempty, closed and convex subset C of H is defined by $P_C x \in C$ and $\|x - P_C x\| = \inf_{x\in C} \|x - y\|$, $\forall x \in H$. The metric projection P_C onto a given nonempty, closed and convex subset C of H is nonexpansive with $\mathrm{Fix}(P_C) = C$.

3. Iterative methods involving a nonexpansive mapping

In this section, we just recall the main result (proved in [2]), concerning the following hybrid steepest-descent viscosity iterative algorithm for solving Problem 1.2. Suppose that the assumptions (A1)–(A4) in Problem 1.2 are satisfied.

Algorithm 3.1.

Step 0. Take $\{\lambda_n\} \subset (0, 2\alpha]$, $\{\mu_n\} \subset \big(0, 2\eta/\kappa^2\big)$, $\{\alpha_n\} \subset (0, 1]$ and $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ with $\beta_n + \gamma_n \leq 1$, $\forall n \geq 0$. Choose $x_0 \in H$ arbitrarily, and let $n := 0$.

Step 1. Given $x_n \in H$, compute $x_{n+1} \in H$ as

$$\begin{aligned} y_n &:= \beta_n x_n + \gamma_n f(x_n) + (1 - \beta_n - \gamma_n)T_n x_n, \\ x_{n+1} &:= y_n - \alpha_n\mu_n A_2 y_n, \quad n \geq 0, \end{aligned}$$

where $T_n := T(I - \lambda_n A_1)$, $\forall n \geq 0$.

Update $n := n + 1$ and go to Step 1.

The main result concerning the convergence of the above algorithm is as follows.

Theorem 3.1. *Assume that the sequence $\{y_n\}$ generated by Algorithm 3.1 is bounded. Let $\{\mu_n\} \subset \left(0, \eta/\kappa^2\right]$, $\{\alpha_n\} \subset (0,1]$, $\{\beta_n\} \subset [0,1]$, $\{\gamma_n\} \subset (0,1]$ and $\{\lambda_n\} \subset (0, 2\alpha]$ be such that the following conditions hold:*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$, $\lim_{n\to\infty} \lambda_n = 0$,

(ii) $\lim_{n\to\infty} (1/\gamma_n) \left|1/\lambda_n - 1/\lambda_{n+1}\right| = 0$,

(iii) $\lim_{n\to\infty} (1/\lambda_{n+1}) \left|1 - \gamma_{n+1}/\gamma_n\right| = 0$,

(iv) $\lim_{n\to\infty} \alpha_n\mu_n/\lambda_n = 0$, $\lim_{n\to\infty} \left(\lambda_n\beta_n + \gamma_n + \lambda_n^2\right)/\alpha_n\mu_n = 0$,

(v) $\sum_{n=1}^{\infty} \left(|\beta_n - \beta_{n-1}| + |\alpha_n\mu_n - \alpha_{n-1}\mu_{n-1}|\right)/\lambda_n < \infty$.

Then, the sequence $\{x_n\}$ generated by Algorithm 3.1 satisfies the following assertions:

(a) *The sequences $\{x_n\}$, $\{A_1x_n\}$ and $\{A_2y_n\}$ are bounded;*
(b) $\lim_{n\to\infty} \|x_{n+1} - x_n\|/\lambda_n = 0$, $\lim_{n\to\infty} \|x_n - y_n\|/\lambda_n = 0$ *and* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$;
(c) *The sequence $\{x_n\}$ converges strongly to a unique solution of Problem 1.2 provided that there exists $r > 0$ such that $\|x - Tx\| \geq r \inf_{y\in\mathrm{Fix(T)}} \|x - y\|$ for all $x \in H$.*

4. Iterative methods involving a finite family of nonexpansive mappings

In this section, we consider a hybrid steepest-descent viscosity iterative algorithm for solving Problem 1.3 in the setting of a real Hilbert space H. We write $T_{[k]} := T_{k \bmod N}$ for integer $k \geq 1$ with the mod function taking values in the set $\{1, 2, \ldots, N\}$, that is, if $k = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_{[k]} = T_N$ if $q = 0$ and $T_{[k]} = T_q$ if $0 < q < N$. For each $i = 1, 2, \ldots, N$, assume that the operator $T_i : H \to H$ is nonexpansive and $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$, and the mappings $A_1, A_2 : H \to H$ satisfy the conditions (B1)–(B4) such that VI $\left(\bigcap_{i=1}^N F_i\left(T_i\right), A_1\right) \neq \emptyset$.

Algorithm 4.1.
Step 0. Take $\{\lambda_n\} \subset (0, 2\alpha]$, $\{\mu_n\} \subset \left(0, 2\eta/\kappa^2\right)$, $\{\alpha_n\} \subset (0,1]$ and $\{\beta_n\}$, $\{\gamma_n\} \subset [0,1]$ with $\beta_n + \gamma_n \leq 1$, $\forall n \geq 0$. Choose $x_0 \in H$ arbitrarily, and let $n := 0$.
Step 1. Given $x_n \in H$, compute $x_{n+1} \in H$ as

$$\begin{aligned} y_n &:= \beta_n x_n + \gamma_n f(x_n) + (1 - \beta_n - \gamma_n)\widetilde{T}_{[n+1]} x_n, \\ x_{n+1} &:= y_n - \alpha_n\mu_n A_2 y_n, \quad n \geq 0, \end{aligned} \tag{4.1}$$

where $\widetilde{T}_{[n+1]} := T_{[n+1]}(I - \lambda_n A_1)$, $\forall n \geq 0$.
Update $n := n + 1$ and go to Step 1.

In Algorithm 4.1, we introduce a sequence $\{\mu_n\}$ of positive parameters that takes into account possible inexact computation. Taking $\mu \in \left(0, 2\eta/\kappa^2\right)$ and $N = 1$ and putting $\mu_n = \mu$ and $\beta_n = \gamma_n = 0$ for all $n \geq 0$, then Algorithm 4.1 reduces to [3, Algorithm 3.1].

We prove that the sequence $\{x_n\}$ generated by Algorithm 4.1 converges strongly to a unique solution of Problem 1.3.

Theorem 4.1. *Assume that the sequence $\{y_n\}$ generated by Algorithm 4.1 is bounded. Suppose that $\{\mu_n\} \subset (0, \eta/\kappa^2]$, $\{\alpha_n\} \subset (0, 1]$, $\{\beta_n\} \subset [0, 1]$, $\{\gamma_n\} \subset (0, 1]$ and $\{\lambda_n\} \subset (0, 2\alpha]$ such that*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\lim_{n\to\infty} (1/\gamma_n) |1/\lambda_n - 1/\lambda_{n+N}| = 0$,

(iii) $\lim_{n\to\infty} (1/\lambda_{n+1}) |1 - \gamma_{n+N}/\gamma_n| = 0$,

(iv) $\lambda_n \downarrow 0$ *as* $n \to \infty$,

(v) $\lim_{n\to\infty} \alpha_n\mu_n/\lambda_n = 0$,

(vi) $\lim_{n\to\infty} \left(\lambda_n\beta_n + \gamma_n + \lambda_n^2\right) /\alpha_n\mu_n = 0$,

(vii) $\sum_{n=1}^{\infty} (|\beta_{n+N-1} - \beta_{n-1}| + |\alpha_{n+N-1}\mu_{n+N-1} - \alpha_{n-1}\mu_{n-1}|) /\lambda_n < \infty$.

Assume, in addition, that

$$\begin{aligned} \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) &= \mathrm{Fix}\left(T_1 \circ T_2 \circ \cdots \circ T_N\right) \\ &= \mathrm{Fix}\left(T_N \circ T_1 \circ \cdots \circ T_{N-1}\right) \\ &\cdots \\ &= \mathrm{Fix}\left(T_2 \circ T_3 \circ \cdots \circ T_N \circ T_1\right). \end{aligned} \tag{4.2}$$

Then, the sequence $\{x_n\}$ generated by Algorithm 4.1 satisfies the following assertions:

(a) *The sequences $\{x_n\}$, $\{A_1x_n\}$ and $\{A_2y_n\}$ are bounded;*

(b) $\lim_{n\to\infty} \|x_{n+N} - x_n\|/\lambda_n = 0$ *and* $\lim_{n\to\infty} \left\|x_n - T_{[n+N]} \circ \cdots \circ T_{[n+1]}x_n\right\| = 0$*;*

(c) *The sequence $\{x_n\}$ converges strongly to a unique solution of Problem 1.3 provided that $\lim_{n\to\infty} \|x_{n+1} - x_n\|/\lambda_n = 0$ and that there $r > 0$ such that*

$$\|x - T_jx\| \geq r \inf_{y\in\bigcap_{i=1}^{N} \mathrm{Fix}(T_i)} \|x - y\|, \quad \forall x \in H, \text{ and } j = 1, 2, \ldots, N.$$

Proof. (a) By using the similar argument as in the proof of Theorem 3.1 (a) in part I, we obtain that the sequences $\{x_n\}$, $\{A_1x_n\}$ and $\{A_2y_n\}$ are bounded.

(b) We prove

$$\lim_{n\to\infty} \|x_{n+N} - x_n\|/\lambda_n = \lim_{n\to\infty} \left\|x_n - T_{[n+N]} \circ \cdots \circ T_{[n+1]}x_n\right\| = 0.$$

Indeed, from assumption (A3), Proposition 2.3, and the condition $\lambda_n \leq 2\alpha$, $\forall n \geq 0$, we have, for all $n \geq 0$,

$$\begin{aligned}
&\|\widetilde{T}_{[n+N+1]}x_{n+N} - \widetilde{T}_{[n+1]}x_n\| = \|T_{[n+N+1]}(I - \lambda_{n+N}A_1)x_{n+N} - T_{[n+1]}(I - \lambda_n A_1)x_n\| \\
&\leq \|(I - \lambda_{n+N}A_1)x_{n+N} - (I - \lambda_n A_1)x_n\| \\
&= \|(I - \lambda_{n+N}A_1)x_{n+N} - (I - \lambda_{n+N}A_1)x_n + (\lambda_n - \lambda_{n+N})A_1x_n\| \\
&\leq \|(I - \lambda_{n+N}A_1)x_{n+N} - (I - \lambda_{n+N}A_1)x_n\| + |\lambda_n - \lambda_{n+N}|\|A_1x_n\| \\
&\leq \|x_{n+N} - x_n\| + |\lambda_n - \lambda_{n+N}|\|A_1x_n\|,
\end{aligned}$$

and hence,

$$\begin{aligned}
\|y_{n+N} - y_n\| &= \|\beta_{n+N}x_{n+N} + \gamma_{n+N}f(x_{n+N}) \\
&+(1 - \beta_{n+N} - \gamma_{n+N})\widetilde{T}_{[n+N+1]}x_{n+N} - \beta_n x_n - \gamma_n f(x_n) - (1 - \beta_n - \gamma_n)\widetilde{T}_{[n+1]}x_n\| \\
&\leq \|\beta_{n+N}x_{n+N} - \beta_n x_n\| + \|\gamma_{n+N}f(x_{n+N}) - \gamma_n f(x_n)\| \\
&+\|(1 - \beta_{n+N} - \gamma_{n+N})\widetilde{T}_{[n+N+1]}x_{n+N} - (1 - \beta_n - \gamma_n)\widetilde{T}_{[n+1]}x_n\| \\
&\leq |\beta_{n+N} - \beta_n|\|x_{n+N}\| + \beta_n\|x_{n+N} - x_n\| + |\gamma_{n+N} - \gamma_n|\|f(x_{n+N})\| \\
&+\gamma_n\|f(x_{n+N}) - f(x_n)\| + |(1 - \beta_{n+N} - \gamma_{n+N}) - (1 - \beta_n - \gamma_n)| \\
&\|\widetilde{T}_{[n+N+1]}x_{n+N}\| + (1 - \beta_n - \gamma_n)\|\widetilde{T}_{[n+N+1]}x_{n+N} - \widetilde{T}_{[n+1]}x_n\| \\
&\leq |\beta_{n+N} - \beta_n|\|x_{n+N}\| + \beta_n\|x_{n+N} - x_n\| + |\gamma_{n+N} - \gamma_n|\|f(x_{n+N})\| \\
&+\gamma_n\rho\|x_{n+N} - x_n\| + (|\beta_{n+N} - \beta_n| + |\gamma_{n+N} - \gamma_n|)\|\widetilde{T}_{[n+N+1]}x_{n+N}\| \\
&+(1 - \beta_n - \gamma_n)[\|x_{n+N} - x_n\| + |\lambda_n - \lambda_{n+N}|\|A_1x_n\|] \\
&\leq (|\beta_{n+N} - \beta_n| + |\gamma_{n+N} - \gamma_n|)\|x_{n+N}\| + \beta_n\|x_{n+N} - x_n\| \\
&+(|\beta_{n+N} - \beta_n| + |\gamma_{n+N} - \gamma_n|)\|f(x_{n+N})\| + \gamma_n\rho\|x_{n+N} - x_n\| \\
&+(|\beta_{n+N} - \beta_n| + |\gamma_{n+N} - \gamma_n|)\|\widetilde{T}_{[n+N+1]}x_{n+N}\| \\
&+(1 - \beta_n - \gamma_n)[\|x_{n+N} - x_n\| + |\lambda_n - \lambda_{n+N}|\|A_1x_n\|] \\
&= [1 - \gamma_n(1 - \rho)]\|x_{n+N} - x_n\| + (1 - \beta_n - \gamma_n)|\lambda_n - \lambda_{n+N}|\|A_1x_n\| \\
&+(|\beta_{n+N} - \beta_n| + |\gamma_{n+N} - \gamma_n|)(\|x_{n+N}\| + \|f(x_{n+N})\| + \|\widetilde{T}_{[n+N+1]}x_{n+N}\|) \\
&\leq [1 - \gamma_n(1 - \rho)]\|x_{n+N} - x_n\| + (|\beta_{n+N} - \beta_n| + |\gamma_{n+N} - \gamma_n| + |\lambda_{n+N} - \lambda_n|)M_1.
\end{aligned}$$

From Proposition 2.4, noticing that

$$M_1 := \sup_{n\geq 0}\{\|x_n\| + \|f(x_n)\| + \|\widetilde{T}_{[n+1]}x_n\| + \|A_1x_n\|\} < \infty,$$

we obtain

$$\begin{aligned}
\|x_{n+N} - x_n\| &= \left\| T^{(\alpha_{n+N-1},\mu_{n+N-1})} y_{n+N-1} - T^{(\alpha_{n-1},\mu_{n-1})} y_{n-1} \right\| \\
&\le \left\| T^{(\alpha_{n+N-1},\mu_{n+N-1})} y_{n+N-1} - T^{(\alpha_{n+N-1},\mu_{n+N-1})} y_{n-1} \right\| \\
&+ \left\| T^{(\alpha_{n+N-1},\mu_{n+N-1})} y_{n-1} - T^{(\alpha_{n-1},\mu_{n-1})} y_{n-1} \right\| \\
&\le (1 - \alpha_{n+N-1}\tau_{n+N-1})\|y_{n+N-1} - y_{n-1}\| \\
&+ |\alpha_{n+N-1}\mu_{n+N-1} - \alpha_{n-1}\mu_{n-1}|\|A_2 y_{n-1}\| \\
&\le (1 - \alpha_{n+N-1}\tau_{n+N-1})\{[1 - \gamma_{n-1}(1-\rho)]\|x_{n+N-1} - x_{n-1}\| \\
&+ (|\beta_{n+N-1} - \beta_{n-1}| + |\gamma_{n+N-1} - \gamma_{n-1}| \\
&+ |\lambda_{n+N-1} - \lambda_{n-1}|)M_1\} + |\alpha_{n+N-1}\mu_{n+N-1} - \alpha_{n-1}\mu_{n-1}|\|A_2 y_{n-1}\| \\
&\le [1 - \gamma_{n-1}(1-\rho)]\|x_{n+N-1} - x_{n-1}\| \\
&+ (|\beta_{n+N-1} - \beta_{n-1}| + |\gamma_{n+N-1} - \gamma_{n-1}| \\
&+ |\lambda_{n+N-1} - \lambda_{n-1}|)M_1 + |\alpha_{n+N-1}\mu_{n+N-1} - \alpha_{n-1}\mu_{n-1}|M_2,
\end{aligned}$$

where $\tau_{n+N-1} := 1 - \sqrt{1 - \mu_{n+N-1}(2\eta - \mu_{n+N-1}\kappa^2)} \in (0,1]$ as in Proposition 2.4 and $M_2 := \sup_{n\ge 0} \|A_2 y_n\| < \infty$.

Note that $\lambda_n \downarrow 0$ as $n \to \infty$, that is, $\{\lambda_n\}$ is a decreasing sequence such that $\lambda_n \to 0$.

If we notice that $M_3 := \sup_{n\ge 0} \|x_{n+N} - x_n\| + M_1 + M_2 < \infty$, we obtain, for all $n \ge 1$, that

$$\begin{aligned}
\frac{\|x_{n+N} - x_n\|}{\lambda_n} &\le [1 - \gamma_{n-1}(1-\rho)] \frac{\|x_{n+N-1} - x_{n-1}\|}{\lambda_n} \\
&+ \frac{|\beta_{n+N-1} - \beta_{n-1}| + |\gamma_{n+N-1} - \gamma_{n-1}| + |\lambda_{n+N-1} - \lambda_{n-1}|}{\lambda_n} M_1 \\
&+ \frac{|\alpha_{n+N-1}\mu_{n+N-1} - \alpha_{n-1}\mu_{n-1}|}{\lambda_n} M_2 = [1 - \gamma_{n-1}(1-\rho)] \frac{\|x_{n+N-1} - x_{n-1}\|}{\lambda_{n-1}} \\
&+ [1 - \gamma_{n-1}(1-\rho)] \left\{ \frac{\|x_{n+N-1} - x_{n-1}\|}{\lambda_n} - \frac{\|x_{n+N-1} - x_{n-1}\|}{\lambda_{n-1}} \right\} \\
&+ \frac{|\beta_{n+N-1} - \beta_{n-1}| + |\gamma_{n+N-1} - \gamma_{n-1}| + |\lambda_{n+N-1} - \lambda_{n-1}|}{\lambda_n} M_1 \\
&+ \frac{|\alpha_{n+N-1}\mu_{n+N-1} - \alpha_{n-1}\mu_{n-1}|}{\lambda_n} M_2 \\
&\le [1 - \gamma_{n-1}(1-\rho)] \frac{\|x_{n+N-1} - x_{n-1}\|}{\lambda_{n-1}} + M_3 \left| \frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}} \right| \\
&+ \frac{|\gamma_{n+N-1} - \gamma_{n-1}| + |\lambda_{n+N-1} - \lambda_{n-1}|}{\lambda_n} M_3 \\
&+ \frac{|\beta_{n+N-1} - \beta_{n-1}| + |\alpha_{n+N-1}\mu_{n+N-1} - \alpha_{n-1}\mu_{n-1}|}{\lambda_n} M_3
\end{aligned}$$

$$
\begin{aligned}
&= [1-\gamma_{n-1}(1-\rho)]\frac{\|x_{n+N-1}-x_{n-1}\|}{\lambda_{n-1}} \\
&+\frac{M_3}{1-\rho}\gamma_{n-1}(1-\rho)\frac{1}{\gamma_{n-1}}\left|\frac{1}{\lambda_n}-\frac{1}{\lambda_{n-1}}\right| + \frac{2\alpha M_3}{1-\rho}\gamma_{n-1}(1-\rho)\frac{1}{\gamma_{n-1}}\frac{|\lambda_{n+N-1}-\lambda_{n-1}|}{2\alpha\lambda_n} \\
&+\frac{M_3}{1-\rho}\gamma_{n-1}(1-\rho)\frac{1}{\gamma_{n-1}}\frac{|\gamma_{n+N-1}-\gamma_{n-1}|}{\lambda_n} \\
&+\frac{|\beta_{n+N-1}-\beta_{n-1}|+|\alpha_{n+N-1}\mu_{n+N-1}-\alpha_{n-1}\mu_{n-1}|}{\lambda_n}M_3 \\
&\le [1-\gamma_{n-1}(1-\rho)]\frac{\|x_{n+N-1}-x_{n-1}\|}{\lambda_{n-1}} \\
&+\frac{M_3}{1-\rho}\gamma_{n-1}(1-\rho)\frac{1}{\gamma_{n-1}}\left|\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_{n+N-1}}\right| \\
&+\frac{2\alpha M_3}{1-\rho}\gamma_{n-1}(1-\rho)\frac{1}{\gamma_{n-1}}\left|\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_{n+N-1}}\right| \\
&+\frac{M_3}{1-\rho}\gamma_{n-1}(1-\rho)\frac{1}{\lambda_n}|1-\frac{\gamma_{n+N-1}}{\gamma_{n-1}}| \\
&+\frac{|\beta_{n+N-1}-\beta_{n-1}|+|\alpha_{n+N-1}\mu_{n+N-1}-\alpha_{n-1}\mu_{n-1}|}{\lambda_n}M_3 \\
&= [1-\gamma_{n-1}(1-\rho)]\frac{\|x_{n+N-1}-x_{n-1}\|}{\lambda_{n-1}} \\
&+\gamma_{n-1}(1-\rho)\cdot\frac{M_3}{1-\rho}\left\{(1+2\alpha)\frac{1}{\gamma_{n-1}}|\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_{n+N-1}}|+\frac{1}{\lambda_n}|1-\frac{\gamma_{n+N-1}}{\gamma_{n-1}}|\right\} \\
&+\frac{|\beta_{n+N-1}-\beta_{n-1}|+|\alpha_{n+N-1}\mu_{n+N-1}-\alpha_{n-1}\mu_{n-1}|}{\lambda_n}M_3.
\end{aligned}
$$

Therefore, by Lemma 2.1 and the conditions (i), (ii), (iii) and (vii), we obtain

$$\lim_{n\to\infty}\frac{\|x_{n+N}-x_n\|}{\lambda_n}=0. \tag{4.3}$$

Since $\|x_{n+1}-y_n\| = \alpha_n\mu_n\|A_2y_n\| \le M_2\alpha_n\mu_n$, by condition (v), we get

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=0.$$

From conditions (v) and (vi), we have

$$\lim_{n\to\infty}\frac{\gamma_n}{\lambda_n}=\lim_{n\to\infty}\frac{\gamma_n}{\alpha_n\mu_n}\cdot\frac{\alpha_n\mu_n}{\lambda_n}=0 \text{ and } \lim_{n\to\infty}\beta_n=\lim_{n\to\infty}\frac{\lambda_n\beta_n}{\alpha_n\mu_n}\cdot\frac{\alpha_n\mu_n}{\lambda_n}=0. \tag{4.4}$$

Since $\lim_{n\to\infty}(\beta_n+\gamma_n)=0$, it follows from (4.1) that:

$$
\begin{aligned}
&\|x_{n+1}-T_{[n+1]}(x_n-\lambda_nA_1x_n)\| \le \|x_{n+1}-y_n\|+\|y_n-T_{[n+1]}(x_n-\lambda_nA_1x_n)\| \\
&= \|x_{n+1}-y_n\|+\|\beta_n(x_n-T_{[n+1]}(x_n-\lambda_nA_1x_n))+\gamma_n(f(x_n)-T_{[n+1]}(x_n-\lambda_nA_1x_n))\| \\
&\le \|x_{n+1}-y_n\|+\beta_n\|x_n-\widetilde{T}_{[n+1]}x_n\|+\gamma_n\|f(x_n)-\widetilde{T}_{[n+1]}x_n\| \\
&\le \|x_{n+1}-y_n\|+\beta_n(\|x_n\|+\|\widetilde{T}_{[n+1]}x_n\|)+\gamma_n(\|f(x_n)\|+\|\widetilde{T}_{[n+1]}x_n\|) \\
&\le \|x_{n+1}-y_n\|+(\beta_n+\gamma_n)M_1 \to 0 \quad \text{as } n\to\infty,
\end{aligned}
$$

that is, $\lim_{n\to\infty}\|x_{n+1}-T_{[n+1]}(x_n-\lambda_n A_1x_n)\|=0$. Since $\lambda_n\to 0$ and each T_i $(i=1,2,...,N)$ is nonexpansive and $\{A_1x_n\}$ is bounded, we obtain

$$\begin{aligned}
&\|x_{n+N}-T_{[n+N]}(x_{n+N-1}-\lambda_{n+N-1}A_1x_{n+N-1})\|\to 0,\\
&\|T_{[n+N]}(x_{n+N-1}-\lambda_{n+N-1}A_1x_{n+N-1})\\
&\quad -T_{[n+N]}T_{[n+N-1]}(x_{n+N-2}-\lambda_{n+N-2}A_1x_{n+N-2})\|\to 0,\\
&\vdots\\
&\|T_{[n+N]}\circ\cdots\circ T_{[n+2]}(x_{n+1}-\lambda_{n+1}A_1x_{n+1})\\
&\quad -T_{[n+N]}\circ\cdots\circ T_{[n+1]}(x_n-\lambda_nA_1x_n)\|\to 0.
\end{aligned}$$

Furthermore, we observe that

$$\begin{aligned}
x_{n+N}-x_n &= x_{n+N}-T_{[n+N]}(x_{n+N-1}-\lambda_{n+N-1}A_1x_{n+N-1})\\
&\quad + T_{[n+N]}(x_{n+N-1}-\lambda_{n+N-1}A_1x_{n+N-1})\\
&\quad - T_{[n+N]}T_{[n+N-1]}(x_{n+N-2}-\lambda_{n+N-2}A_1x_{n+N-2})\\
&\quad +\cdots\\
&\quad + T_{[n+N]}\circ\cdots\circ T_{[n+2]}(x_{n+1}-\lambda_{n+1}A_1x_{n+1})\\
&\quad - T_{[n+N]}\circ\cdots\circ T_{[n+1]}(x_n-\lambda_nA_1x_n)\\
&\quad + T_{[n+N]}\circ\cdots\circ T_{[n+1]}(x_n-\lambda_nA_1x_n)-x_n.
\end{aligned}$$

Consequently, we have $\lim_{n\to\infty}\left\|T_{[n+N]}\circ\cdots\circ T_{[n+1]}(x_n-\lambda_nA_1x_n)-x_n\right\|=0$.

Note that

$$\begin{aligned}
&\|T_{[n+N]}\circ\cdots\circ T_{[n+1]}x_n-x_n\|\\
&\le \|T_{[n+N]}\circ\cdots\circ T_{[n+1]}x_n-T_{[n+N]}\circ\cdots\circ T_{[n+1]}(x_n-\lambda_nA_1x_n)\|\\
&+\|T_{[n+N]}\circ\cdots\circ T_{[n+1]}(x_n-\lambda_nA_1x_n)-x_n\|\\
&\le \lambda_n\|A_1x_n\|+\|T_{[n+N]}\circ\cdots\circ T_{[n+1]}(x_n-\lambda_nA_1x_n)-x_n\|\to 0\quad \text{as } n\to\infty,
\end{aligned}$$

that is,

$$\lim_{n\to\infty}\left\|T_{[n+N]}\circ\cdots\circ T_{[n+1]}x_n-x_n\right\|=0. \tag{4.5}$$

(c) We divide the proof into the following three steps:

(I) We prove $\limsup_{n\to\infty}\langle x^*-x_{n+1},A_2x^*\rangle\le 0$.

We note that

$$\begin{aligned}
\|x_n-y_n\| &\le \|x_{n+1}-x_n\|+\|x_{n+1}-y_n\|=\|x_{n+1}-x_n\|+\alpha_n\mu_n\|A_2y_n\|\\
&\le \|x_{n+1}-x_n\|+\alpha_n\mu_nM_2.
\end{aligned}$$

Hence, by the condition (v) and the assumption $\|x_{n+1}-x_n\|/\lambda_n\to 0$, we get

$$\frac{\|x_n-y_n\|}{\lambda_n}\le\frac{\|x_{n+1}-x_n\|}{\lambda_n}+\frac{\alpha_n\mu_n}{\lambda_n}M_2,$$

which implies

$$\lim_{n\to\infty}\frac{\|x_n-y_n\|}{\lambda_n}=0. \tag{4.6}$$

By the condition (vi), (4.6) and

$$(\lambda_n/\alpha_n\mu_n)\|x_n - y_n\| = (\lambda_n^2/\alpha_n\mu_n)(\|x_n - y_n\|/\lambda_n)\ (\forall n \geq 0),$$

we get $\lim_{n\to\infty}(\lambda_n/\alpha_n\mu_n)\|x_n - y_n\| = 0$.

Put $z_n := x_n - \lambda_n A_1 x_n$, $\forall n \geq 0$. Then, we have $\|z_n - x_n\| = \lambda_n\|A_1x_n\| \leq \lambda_n M_1$, and hence, $(\lambda_n/\alpha_n\mu_n)\|z_n - x_n\| \leq (\lambda_n^2/\alpha_n\mu_n)M_1$ for every $n \geq 0$. The condition (vi) implies that $\lim_{n\to\infty}(\lambda_n/\alpha_n\mu_n)\|z_n - x_n\| = 0$. Consequently, we get

$$\lim_{n\to\infty}\frac{\lambda_n\|z_n - y_n\|}{\alpha_n\mu_n} = 0. \tag{4.7}$$

Now, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\limsup_{n\to\infty}\langle x^* - x_n, A_2x^*\rangle = \lim_{i\to\infty}\langle x^* - x_{n_i}, A_2x^*\rangle$. The boundedness of $\{x_{n_i}\}$ implies the existence of a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and a point $\hat{x} \in H$ such that $\{x_{n_{i_j}}\}$ converges weakly to $\hat{x}$. Since $\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|z_n - x_n\| = 0$, we have

$$\lim_{j\to\infty}\langle x_{n_{i_j}+1}, w\rangle e = \lim_{j\to\infty}\langle z_{n_{i_j}}, w\rangle = \lim_{j\to\infty}\langle x_{n_{i_j}}, w\rangle = \langle\hat{x}, w\rangle, \quad \forall w \in H.$$

Without loss of generality, we may assume that $\lim_{i\to\infty}\langle x_{n_i}, w\rangle = \langle\hat{x}, w\rangle, \forall w \in H$. Since the pool of mappings $\{T_i : 1 \leq i \leq N\}$ is finite, we may further assume (passing to a further subsequence if necessary) that, for some integer $k \in \{1, 2, \ldots, N\}$, we have $T_{[n_i]} \equiv T_k, \quad \forall i \geq 1$. Then, it follows by (4.5) that $\left\|x_{n_i} - T_{[i+N]} \circ \cdots \circ T_{[i+1]}x_{n_i}\right\| \to 0$. Hence, by Lemma 2.4, we conclude that $\hat{x} \in \mathrm{Fix}(T_{[i+N]} \circ \cdots \circ T_{[i+1]})$. Together with the condition (4.2), this implies that $\hat{x} \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$.

Let $y \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ be fixed but arbitrarily chosen and put $M_4 := \sup_{n\geq 0}\{\|x_n - y\| + \|y_n - y\| + \|f(x_n) - y\|\} < \infty$. Then, it follows by (A3) and Proposition 2.3 that, for every $n \geq 0$, we have

$$\begin{aligned}
&\|y_n - y\|^2 = \|\beta_n(x_n - y) + \gamma_n(f(x_n) - y) + (1 - \beta_n - \gamma_n)(\widetilde{T}_{[n+1]}x_n - y)\|^2 \\
&\leq \beta_n\|x_n - y\|^2 + \gamma_n\|f(x_n) - y\|^2 + (1 - \beta_n - \gamma_n)\|\widetilde{T}_{[n+1]}x_n - y\|^2 \\
&= \beta_n\|x_n - y\|^2 + \gamma_n\|f(x_n) - y\|^2 \\
&+ (1 - \beta_n - \gamma_n)\|T_{[n+1]}(x_n - \lambda_n A_1x_n) - T_{[n+1]}y\|^2 \\
&\leq \beta_n\|x_n - y\|^2 + \gamma_n\|f(x_n) - y\|^2 + (1 - \beta_n - \gamma_n)\|z_n - y\|^2 \\
&= \beta_n\|x_n - y\|^2 + \gamma_n\|f(x_n) - f(y) + f(y) - y\|^2 \\
&+ (1 - \beta_n - \gamma_n)\|(x_n - \lambda_n A_1x_n) - (y - \lambda_n A_1y) - \lambda_n A_1y\|^2 \\
&\leq \beta_n\|x_n - y\|^2 + \gamma_n[\|f(x_n) - f(y)\|^2 + 2\langle f(y) - y, f(x_n) - y\rangle] \\
&+ (1 - \beta_n - \gamma_n)[\|(x_n - \lambda_n A_1x_n) - (y - \lambda_n A_1y)\|^2 + 2\lambda_n\langle y - z_n, A_1y\rangle] \\
&\leq \beta_n\|x_n - y\|^2 + \gamma_n[\rho^2\|x_n - y\|^2 + 2\|f(y) - y\|\|f(x_n) - y\|] \\
&+ (1 - \beta_n - \gamma_n)[\|x_n - y\|^2 + 2\lambda_n\langle y - z_n, A_1y\rangle] \\
&\leq [1 - \gamma_n(1 - \rho)]\|x_n - y\|^2 + 2\gamma_n\|f(y) - y\|\|f(x_n) - y\| \\
&+ 2(1 - \beta_n - \gamma_n)\lambda_n\langle y - z_n, A_1y\rangle \\
&\leq \|x_n - y\|^2 + 2\gamma_n\|f(y) - y\|\|f(x_n) - y\| + 2(1 - \beta_n - \gamma_n)\lambda_n\langle y - z_n, A_1y\rangle,
\end{aligned} \tag{4.8}$$

which implies that, for every $n \geq 0$, we get

$$\begin{aligned}
0 &\leq \frac{1}{\lambda_n}(\|x_n - y\|^2 - \|y_n - y\|^2) + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|\|f(x_n) - y\| \\
&+ 2(1 - \beta_n - \gamma_n)\langle y - z_n, A_1 y\rangle \leq (\|x_n - y\| + \|y_n - y\|)\frac{\|x_n - y\| - \|y_n - y\|}{\lambda_n} \\
&+ 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|M_4 + 2(1 - \beta_n - \gamma_n)\langle y - z_n, A_1 y\rangle \leq M_4\frac{\|x_n - y\| - \|y_n - y\|}{\lambda_n} \\
&+ 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|M_4 + 2(1 - \beta_n - \gamma_n)\langle y - z_n, A_1 y\rangle \leq \\
&M_4\left(\frac{\|x_n - y_n\|}{\lambda_n} + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|\right) + 2(1 - \beta_n - \gamma_n)\langle y - z_n, A_1 y\rangle.
\end{aligned}$$

By the weak convergence of $\{z_{n_i}\}$ to $\hat{x} \in \bigcap_{i=1}^N \operatorname{Fix}(T_i)$, and (4.4) and (4.6), we get

$$\langle y - \hat{x}, A_1 y\rangle \geq 0, \quad \text{for all } y \in \bigcap_{i=1}^N \operatorname{Fix}(T_i).$$

Assumption (A1) ensures $\langle y - \hat{x}, A_1\hat{x}\rangle \geq 0$, for all $y \in \bigcap_{i=1}^N \operatorname{Fix}(T_i)$, that is, $\hat{x} \in \operatorname{VI}\left(\bigcap_{i=1}^N \operatorname{Fix}(T_i), A_1\right)$. Since $\{x^*\} = \operatorname{VI}\left(\operatorname{VI}\left(\bigcap_{i=1}^N \operatorname{Fix}(T_i), A_1\right), A_2\right)$, we have

$$\limsup_{n\to\infty}\langle x^* - x_{n+1}, A_2 x^*\rangle = \lim_{i\to\infty}\langle x^* - x_{n_i+1}, A_2 x^*\rangle = \langle x^* - \hat{x}, A_2 x^*\rangle \leq 0. \tag{4.9}$$

(II) We next prove $\limsup_{n\to\infty}(\lambda_n/\alpha_n\mu_n)\langle x^* - z_n, A_1 x^*\rangle \leq 0$.

Since $P_{\bigcap_{i=1}^N \operatorname{Fix}(T_i)} z_n \in \bigcap_{i=1}^N \operatorname{Fix}(T_i)$ and $x^* \in \operatorname{VI}\left(\bigcap_{i=1}^N \operatorname{Fix}(T_i), A_1\right)$, we have

$$\begin{aligned}
\langle x^* - z_n, A_1 x^*\rangle &= \langle P_{\bigcap_{i=1}^N \operatorname{Fix}(T_i)} z_n - z_n, A_1 x^*\rangle + \langle x^* - P_{\bigcap_{i=1}^N \operatorname{Fix}(T_i)} z_n, A_1 x^*\rangle \\
&\leq \langle P_{\bigcap_{i=1}^N \operatorname{Fix}(T_i)} z_n - z_n, A_1 x^*\rangle \\
&\leq \|P_{\bigcap_{i=1}^N \operatorname{Fix}(T_i)} z_n - z_n\|\|A_1 x^*\|, \quad \forall n \geq 0.
\end{aligned}$$

The hypothesis of (c) implies that

$$\begin{aligned}
\langle x^* - z_n, A_1 x^*\rangle &\leq \left\|P_{\bigcap_{i=1}^N \operatorname{Fix}(T_i)} z_n - z_n\right\| \|A_1 x^*\| \leq \frac{1}{r}\|z_n - T_{[n+1]} z_n\|\|A_1 x^*\| \\
&\leq \frac{1}{r}[\|z_n - y_n\| + \|y_n - T_{[n+1]} z_n\|]\|A_1 x^*\| \\
&\leq \frac{1}{r}[\|z_n - y_n\| + \beta_n\|x_n - T_{[n+1]} z_n\| + \gamma_n\|f(x_n) - T_{[n+1]} z_n\|]\|A_1 x^*\| \\
&\leq \frac{1}{r}[\|z_n - y_n\| + (\beta_n + \gamma_n)M_5]\|A_1 x^*\|,
\end{aligned}$$

for every $n \geq 0$, where

$$M_5 := \sup_{n\geq 0}\left\{\|x_n - T_{[n+1]} z_n\| + \|f(x_n) - T_{[n+1]} z_n\|\right\} < \infty.$$

So, we obtain

$$\frac{\lambda_n}{\alpha_n\mu_n}\langle x^* - z_n, A_1 x^*\rangle \leq \frac{\|A_1 x^*\|}{r}\left\{\frac{\lambda_n\|z_n - y_n\|}{\alpha_n\mu_n} + \frac{\lambda_n(\beta_n + \gamma_n)}{\alpha_n\mu_n}M_5\right\},$$

for every $n \geq 0$. This relation together with the condition (vi) and (4.7) implies that

$$\limsup_{n\to\infty} \frac{\lambda_n}{\alpha_n\mu_n}\langle x^* - z_n, A_1x^*\rangle \leq 0. \tag{4.10}$$

(III) Finally, we prove $\lim_{n\to\infty} \|x_n - x^*\| = 0$.

Indeed, repeating the same argument as that of $\lim_{n\to\infty} \|x_n - x^*\| = 0$ in the proof of Theorem 3.1 from part I, from (4.8) we have

$$\lim_{n\to\infty} \|x_n - x^*\| = 0.$$

This completes the proof. □

Remark 4.1. We extended [3, Algorithm 3.1] to develop Algorithm 4.1 for solving Problem 1.3. Since Algorithm 4.1 involves a contractive self-mapping f, the pool of nonexpansive self-mappings $\{T_i\}_{i=1}^N$ and several variable parameters, Algorithm 4.1 is more flexible and more subtle than [3, Algorithm 3.1]. However, the proof of Theorem 4.1 is very different from that of Theorem 3.2 in [3] because our argument depends on Lemmas 2.1 and 2.4.

Theorem 4.2. *Assume that the sequence $\{y_n\}$ generated by Algorithm 4.1 is bounded. Suppose that $\{\mu_n\} \subset (0, \eta/\kappa^2]$, $\{\alpha_n\} \subset (0,1]$, $\{\beta_n\} \subset [0,1]$, $\{\gamma_n\} \subset (0,1]$ and $\{\lambda_n\} \subset (0, 2\alpha]$ satisfying*

(i) $\sum_{n=0}^{\infty} \alpha_n\mu_n = \infty$,

(ii) $\lim_{n\to\infty}(1/\alpha_{n+N}\mu_{n+N})|1/\lambda_n - 1/\lambda_{n+N}| = 0$,

(iii) $\lim_{n\to\infty}(1/\lambda_{n+1})|1 - \alpha_n\mu_n/\alpha_{n+N}\mu_{n+N}| = 0$,

(iv) $\lambda_n \downarrow 0$ *as* $n \to \infty$,

(v) $\lim_{n\to\infty} \alpha_n\mu_n/\lambda_n = 0$,

(vi) $\lim_{n\to\infty}(\lambda_n\beta_n + \gamma_n + \lambda_n^2)/\alpha_n\mu_n = 0$,

(vii) $\sum_{n=1}^{\infty}(|\beta_{n+N-1} - \beta_{n-1}| + |\gamma_{n+N-1} - \gamma_{n-1}|)/\lambda_n < \infty$.

Assume, in addition, that (4.2) in Theorem 4.1 holds. Then the sequence $\{x_n\}$ generated by Algorithm 4.1 satisfies the following assertions:

(a) *The sequences $\{x_n\}$, $\{A_1x_n\}$ and $\{A_2y_n\}$ are bounded;*

(b) $\lim_{n\to\infty}\|x_{n+N} - x_n\|/\lambda_n = 0$ *and* $\lim_{n\to\infty}\left\|x_n - T_{[n+N]} \circ \cdots \circ T_{[n+1]}x_n\right\| = 0$;

(c) *The sequence $\{x_n\}$ converges strongly to a unique solution of Problem 1.3 provided $\lim_{n\to\infty}\|x_{n+1} - x_n\|/\lambda_n = 0$ and there is $r > 0$ such that*

$$\|x_n - T_jx_n\| \geq r \inf_{y\in\bigcap_{i=1}^N \mathrm{Fix}(T_i)} \|x_n - y\| (\forall n \geq n_0 \text{ and } j = 1,\ldots,N)$$

for some integer $n_0 \geq 1$.

Proof. (a) By using the similar arguments as in the proof of Theorem 4.1 (a), we obtain the sequences $\{x_n\}$, $\{A_1x_n\}$ and $\{A_2y_n\}$ are bounded.

(b) We prove that

$$\lim_{n\to\infty} \|x_{n+N} - x_n\|/\lambda_n = \lim_{n\to\infty} \left\|x_n - T_{[n+N]} \circ \cdots \circ T_{[n+1]}x_n\right\| = 0.$$

As in the proof of Theorem 4.1 (b), from assumption (A3), Proposition 2.3, and the condition $\lambda_n \le 2\alpha$ ($\forall n \ge 0$), we obtain, for all $n \ge 0$,

$$\left\|\widetilde{T}_{[n+N+1]}x_{n+N} - \widetilde{T}_{[n+1]}x_n\right\| \le \|x_{n+N} - x_n\| + |\lambda_n - \lambda_{n+N}|\|A_1x_n\|,$$

and hence,

$$\begin{aligned}
\|y_{n+N} - y_n\| &= \|\beta_{n+N}x_{n+N} + \gamma_{n+N}f(x_{n+N}) + (1-\beta_{n+N}-\gamma_{n+N})\widetilde{T}_{[n+N+1]}x_{n+N} \\
&\quad -\beta_n x_n - \gamma_n f(x_n) - (1-\beta_n-\gamma_n)\widetilde{T}_{[n+1]}x_n\| \\
&\le \|\beta_{n+N}x_{n+N} - \beta_n x_n\| + \|\gamma_{n+N}f(x_{n+N}) - \gamma_n f(x_n)\| \\
&\quad +\|(1-\beta_{n+N}-\gamma_{n+N})\widetilde{T}_{[n+N+1]}x_{n+N} - (1-\beta_n-\gamma_n)\widetilde{T}_{[n+1]}x_n\| \\
&\le |\beta_{n+N}-\beta_n|\|x_{n+N}\| + \beta_n\|x_{n+N}-x_n\| + |\gamma_{n+N}-\gamma_n|\|f(x_{n+N})\| \\
&\quad +\gamma_n\|f(x_{n+N}) - f(x_n)\| + |(1-\beta_{n+N}-\gamma_{n+N}) - (1-\beta_n-\gamma_n)|\|\widetilde{T}_{[n+N+1]}x_{n+N}\| \\
&\quad +(1-\beta_n-\gamma_n)\|\widetilde{T}_{[n+N+1]}x_{n+N} - \widetilde{T}_{[n+1]}x_n\| \\
&\le |\beta_{n+N}-\beta_n|\|x_{n+N}\| + \beta_n\|x_{n+N}-x_n\| + |\gamma_{n+N}-\gamma_n|\|f(x_{n+N})\| \\
&\quad +\gamma_n\rho\|x_{n+N}-x_n\| + (|\beta_{n+N}-\beta_n| + |\gamma_{n+N}-\gamma_n|)\|\widetilde{T}_{[n+N+1]}x_{n+N}\| \\
&\quad +(1-\beta_n-\gamma_n)[\|x_{n+N}-x_n\| + |\lambda_n-\lambda_{n+N}|\|A_1x_n\|] \\
&\le (|\beta_{n+N}-\beta_n| + |\gamma_{n+N}-\gamma_n|)\|x_{n+N}\| + \beta_n\|x_{n+N}-x_n\| \\
&\quad +(|\beta_{n+N}-\beta_n| + |\gamma_{n+N}-\gamma_n|)\|f(x_{n+N})\| + \gamma_n\rho\|x_{n+N}-x_n\| \\
&\quad +(|\beta_{n+N}-\beta_n| + |\gamma_{n+N}-\gamma_n|)\|\widetilde{T}_{[n+N+1]}x_{n+N}\| \\
&\quad +(1-\beta_n-\gamma_n)[\|x_{n+N}-x_n\| + |\lambda_n-\lambda_{n+N}|\|A_1x_n\|] \\
&= [1-\gamma_n(1-\rho)]\|x_{n+N}-x_n\| + (1-\beta_n-\gamma_n)|\lambda_n-\lambda_{n+N}|\|A_1x_n\| \\
&\quad +(|\beta_{n+N}-\beta_n| + |\gamma_{n+N}-\gamma_n|)(\|x_{n+N}\| + \|f(x_{n+N})\| + \|\widetilde{T}_{[n+N+1]}x_{n+N}\|) \\
&\le [1-\gamma_n(1-\rho)]\|x_{n+N}-x_n\| + (|\beta_{n+N}-\beta_n| + |\gamma_{n+N}-\gamma_n| + |\lambda_{n+N}-\lambda_n|)M_1 \\
&\le \|x_{n+N}-x_n\| + (|\beta_{n+N}-\beta_n| + |\gamma_{n+N}-\gamma_n| + |\lambda_{n+N}-\lambda_n|)M_1,
\end{aligned}$$

where $M_1 := \sup_{n\geq 0}\{\|x_n\| + \|f(x_n)\| + \|\widetilde{T}_{[n+1]}x_n\| + \|A_1x_n\|\} < \infty$.
From Proposition 2.4, we get

$$\begin{aligned}
\|x_{n+N} - x_n\| &= \|T^{(\alpha_{n+N-1},\mu_{n+N-1})}y_{n+N-1} - T^{(\alpha_{n-1},\mu_{n-1})}y_{n-1}\| \\
&\leq \|T^{(\alpha_{n+N-1},\mu_{n+N-1})}y_{n+N-1} - T^{(\alpha_{n+N-1},\mu_{n+N-1})}y_{n-1}\| \\
&+ \|T^{(\alpha_{n+N-1},\mu_{n+N-1})}y_{n-1} - T^{(\alpha_{n-1},\mu_{n-1})}y_{n-1}\| \\
&\leq (1 - \alpha_{n+N-1}\tau_{n+N-1})\|y_{n+N-1} - y_{n-1}\| \\
&+ |\alpha_{n+N-1}\mu_{n+N-1} - \alpha_{n-1}\mu_{n-1}|\|A_2y_{n-1}\| \\
&\leq (1 - \alpha_{n+N-1}\tau_{n+N-1})\{\|x_{n+N-1} - x_{n-1}\| \\
&+ (|\beta_{n+N-1} - \beta_{n-1}| + |\gamma_{n+N-1} - \gamma_{n-1}| \\
&+ |\lambda_{n+N-1} - \lambda_{n-1}|)M_1\} \\
&+ |\alpha_{n+N-1}\mu_{n+N-1} - \alpha_{n-1}\mu_{n-1}|\|A_2y_{n-1}\| \\
&\leq (1 - \alpha_{n+N-1}\tau_{n+N-1})\|x_{n+N-1} - x_{n-1}\| + (|\beta_{n+N-1} - \beta_{n-1}| \\
&+ |\gamma_{n+N-1} - \gamma_{n-1}| \\
&+ |\lambda_{n+N-1} - \lambda_{n-1}|)M_1 + |\alpha_{n+N-1}\mu_{n+N-1} - \alpha_{n-1}\mu_{n-1}|M_2,
\end{aligned}$$

where $\tau_{n+N-1} := 1 - \sqrt{1 - \mu_{n+N-1}(2\eta - \mu_{n+N-1}\kappa^2)} \in (0, 1]$ as in Proposition 2.4 and $M_2 := \sup_{n\geq 0}\|A_2y_n\| < \infty$. At the same time, observe that for all $n \geq 0$,

$$\sqrt{1 - \mu_n(2\eta - \mu_n\kappa^2)} \leq \sqrt{1 - \mu_n\eta} \leq 1 - \frac{1}{2}\mu_n\eta,$$

and hence,

$$\tau_n = 1 - \sqrt{1 - \mu_n(2\eta - \mu_n\kappa^2)} \geq 1 - \left(1 - \frac{1}{2}\mu_n\eta\right) = \frac{1}{2}\mu_n\eta,$$

where $0 < \mu_n \leq \eta/\kappa^2$ for all $n \geq 0$. Also, note that $\lambda_n \downarrow 0$ as $n \to \infty$, that is, $\{\lambda_n\}$ is a decreasing sequence such that $\lambda_n \to 0$. Thus, by a similar argument as in the proof of Theorem 4.1, we obtain for all $n \geq 1$,

$$\begin{aligned}
\frac{\|x_{n+N} - x_n\|}{\lambda_n} &\leq \left(1 - \frac{1}{2}\alpha_{n+N-1}\mu_{n+N-1}\eta\right)\frac{\|x_{n+N-1} - x_{n-1}\|}{\lambda_{n-1}} \\
&+ \frac{1}{2}\alpha_{n+N-1}\mu_{n+N-1}\eta \cdot \frac{2M_3}{\eta}\Big\{(1 + 2\alpha)\frac{1}{\alpha_{n+N-1}\mu_{n+N-1}}\left|\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_{n+N-1}}\right| \\
&+ \frac{1}{\lambda_n}\left|1 - \frac{\alpha_{n-1}\mu_{n-1}}{\alpha_{n+N-1}\mu_{n+N-1}}\right|\Big\} + \frac{|\beta_{n+N-1} - \beta_{n-1}| + |\gamma_{n+N-1} - \gamma_{n-1}|}{\lambda_n}M_3,
\end{aligned}$$

where $M_3 := \sup_{n\geq 0}\|x_{n+N} - x_n\| + M_1 + M_2 < \infty$. Therefore, by Lemma 2.1 and the conditions (i), (ii), (iii) and (vii), we have

$$\lim_{n\to\infty}\frac{\|x_{n+N} - x_n\|}{\lambda_n} = 0. \tag{4.11}$$

From $\|x_{n+1} - y_n\| = \alpha_n \mu_n \|A_2 y_n\| \le M_2 \alpha_n \mu_n$ and the condition (v), we get

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0.$$

In addition, note that the conditions (v) and (vi) imply

$$\lim_{n\to\infty} \frac{\gamma_n}{\lambda_n} = \lim_{n\to\infty} \frac{\gamma_n}{\alpha_n\mu_n} \cdot \frac{\alpha_n\mu_n}{\lambda_n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \frac{\lambda_n\beta_n}{\alpha_n\mu_n} \cdot \frac{\alpha_n\mu_n}{\lambda_n} = 0. \quad (4.12)$$

Since $\lim_{n\to\infty} (\beta_n + \gamma_n) = 0$, as in the proof of Theorem 4.1, it follows from (4.1) that

$$\|x_{n+1} - T_{[n+1]}(x_n - \lambda_n A_1 x_n)\| \le \|x_{n+1} - y_n\| + (\beta_n + \gamma_n) M_1 \to 0 \quad \text{as } n \to \infty,$$

that is,

$$\lim_{n\to\infty} \|x_{n+1} - T_{[n+1]}(x_n - \lambda_n A_1 x_n)\| = 0.$$

Since $\lambda_n \to 0$, each T_i $(i = 1, 2, \ldots, N)$ is nonexpansive and $\{A_1 x_n\}$ is boundedness, we have

$$\begin{cases} \|x_{n+N} - T_{[n+N]}(x_{n+N-1} - \lambda_{n+N-1} A_1 x_{n+N-1})\| \to 0, \\ \|T_{[n+N]}(x_{n+N-1} - \lambda_{n+N-1} A_1 x_{n+N-1}) \\ \quad - T_{[n+N]} T_{[n+N-1]}(x_{n+N-2} - \lambda_{n+N-2} A_1 x_{n+N-2})\| \to 0, \\ \vdots \\ \|T_{[n+N]} \circ \cdots \circ T_{[n+2]}(x_{n+1} - \lambda_{n+1} A_1 x_{n+1}) \\ \quad - T_{[n+N]} \circ \cdots \circ T_{[n+1]}(x_n - \lambda_n A_1 x_n)\| \to 0. \end{cases}$$

Furthermore, observe that

$$\begin{aligned} x_{n+N} - x_n = {} & x_{n+N} - T_{[n+N]}(x_{n+N-1} - \lambda_{n+N-1} A_1 x_{n+N-1}) \\ & + T_{[n+N]}(x_{n+N-1} - \lambda_{n+N-1} A_1 x_{n+N-1}) \\ & - T_{[n+N]} T_{[n+N-1]}(x_{n+N-2} - \lambda_{n+N-2} A_1 x_{n+N-2}) \\ & + \cdots \\ & + T_{[n+N]} \circ \cdots \circ T_{[n+2]}(x_{n+1} - \lambda_{n+1} A_1 x_{n+1}) \\ & - T_{[n+N]} \circ \cdots \circ T_{[n+1]}(x_n - \lambda_n A_1 x_n) \\ & + T_{[n+N]} \circ \cdots \circ T_{[n+1]}(x_n - \lambda_n A_1 x_n) - x_n. \end{aligned}$$

Consequently, we have

$$\lim_{n\to\infty} \big\|T_{[n+N]} \circ \cdots \circ T_{[n+1]}(x_n - \lambda_n A_1 x_n) - x_n\big\| = 0.$$

Note that

$$\begin{aligned} & \|T_{[n+N]} \circ \cdots \circ T_{[n+1]} x_n - x_n\| \\ & \le \|T_{[n+N]} \circ \cdots \circ T_{[n+1]} x_n - T_{[n+N]} \circ \cdots \circ T_{[n+1]}(x_n - \lambda_n A_1 x_n)\| \\ & + \|T_{[n+N]} \circ \cdots \circ T_{[n+1]}(x_n - \lambda_n A_1 x_n) - x_n\| \\ & \le \lambda_n \|A_1 x_n\| + \|T_{[n+N]} \circ \cdots \circ T_{[n+1]}(x_n - \lambda_n A_1 x_n) - x_n\| \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

that is,

$$\lim_{n\to\infty} \|T_{[n+N]} \circ \cdots \circ T_{[n+1]} x_n - x_n\| = 0. \quad (4.13)$$

(c) We divide the proof into the following three steps:

(I) As in the proof of Theorem 4.1, we have $\limsup_{n\to\infty}\langle x^* - x_{n+1}, A_2x^*\rangle \le 0$.

(II) We next prove $\limsup_{n\to\infty}(\lambda_n/\alpha_n\mu_n)\langle x^* - z_n, A_1x^*\rangle \le 0$.

We may assume, without loss of generality, that, for some $r > 0$, one have

$$\|x_n - T_jx_n\| \ge r \inf_{y\in\bigcap_{i=1}^N \mathrm{Fix}(T_i)} \|x_n - y\| \ (\forall n \ge 0 \text{ and } j = 1, \dots, N).$$

Since $P_{\bigcap_{i=1}^N \mathrm{Fix}(T_i)} z_n \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ and $x^* \in \mathrm{VI}(\bigcap_{i=1}^N \mathrm{Fix}(T_i), A_1)$, we have

$$\begin{aligned}
&\langle x^* - z_n, A_1x^*\rangle \\
&= \langle P_{\bigcap_{i=1}^N \mathrm{Fix}(T_i)} z_n - z_n, A_1x^*\rangle + \langle x^* - P_{\bigcap_{i=1}^N \mathrm{Fix}(T_i)} z_n, A_1x^*\rangle \\
&\le \langle P_{\bigcap_{i=1}^N \mathrm{Fix}(T_i)} z_n - z_n, A_1x^*\rangle \\
&\le \|P_{\bigcap_{i=1}^N \mathrm{Fix}(T_i)} z_n - z_n\|\|A_1x^*\| \\
&\le [\|P_{\bigcap_{i=1}^N \mathrm{Fix}(T_i)} z_n - P_{\bigcap_{i=1}^N \mathrm{Fix}(T_i)} x_n\| + \|P_{\bigcap_{i=1}^N \mathrm{Fix}(T_i)} x_n - x_n\| + \|x_n - z_n\|]\|A_1x^*\| \\
&\le [2\|x_n - z_n\| + \|P_{\bigcap_{i=1}^N \mathrm{Fix}(T_i)} x_n - x_n\|]\|A_1x^*\|,
\end{aligned}$$

for every $n \ge 0$. This together with the hypothesis of (c) implies that

$$\begin{aligned}
&\langle x^* - z_n, A_1x^*\rangle \\
&\le \|P_{\bigcap_{i=1}^N \mathrm{Fix}(T_i)} x_n - x_n\|\|A_1x^*\| + 2\|x_n - z_n\|\|A_1x^*\| \\
&\le \frac{1}{r}\|x_n - T_{[n+1]}x_n\|\|A_1x^*\| + 2\|x_n - z_n\|\|A_1x^*\| \\
&\le \frac{1}{r}[\|x_n - y_n\| + \|y_n - T_{[n+1]}x_n\|]\|A_1x^*\| + 2\|x_n - z_n\|\|A_1x^*\| \\
&\le \frac{1}{r}[\|x_n - y_n\| + \beta_n\|x_n - T_{[n+1]}x_n\| + \gamma_n\|f(x_n) - T_{[n+1]}x_n\| \\
&+ (1 - \beta_n - \gamma_n)\|T_{[n+1]}z_n - T_{[n+1]}x_n\|]\|A_1x^*\| + 2\|x_n - z_n\|\|A_1x^*\| \\
&\le \frac{1}{r}[\|x_n - y_n\| + \beta_n\|x_n - T_{[n+1]}x_n\| + \gamma_n\|f(x_n) - T_{[n+1]}x_n\| + \|z_n - x_n\|]\|A_1x^*\| \\
&+ 2\|x_n - z_n\|\|A_1x^*\| \\
&= \frac{1}{r}[\|x_n - y_n\| + \beta_n\|x_n - T_{[n+1]}x_n\| + \gamma_n\|f(x_n) - T_{[n+1]}x_n\|]\|A_1x^*\| \\
&+ (\frac{1}{r} + 2)\|x_n - z_n\|\|A_1x^*\| \\
&\le \frac{1}{r}[\|x_n - y_n\| + \beta_nM_5 + \gamma_nM_5]\|A_1x^*\| + (\frac{1}{r} + 2)\|x_n - z_n\|\|A_1x^*\| \\
&= \frac{1}{r}[\|x_n - y_n\| + (\beta_n + \gamma_n)M_5]\|A_1x^*\| + (\frac{1}{r} + 2)\|x_n - z_n\|\|A_1x^*\|,
\end{aligned}$$

for every $n \ge 0$, where

$$M_5 := \sup_{n\ge 0}\left\{\|x_n - T_{[n+1]}x_n\| + \|f(x_n) - T_{[n+1]}x_n\|\right\} < \infty.$$

So, we obtain

$$\begin{aligned}\frac{\lambda_n}{\alpha_n\mu_n}\langle x^*-z_n, A_1x^*\rangle &\leq \frac{\|A_1x^*\|}{r}\left\{\frac{\lambda_n\|x_n-y_n\|}{\alpha_n\mu_n}+\frac{\lambda_n(\beta_n+\gamma_n)}{\alpha_n\mu_n}M_5\right\}\\ &+\left(\frac{1}{r}+2\right)\frac{\lambda_n\|x_n-z_n\|}{\alpha_n\mu_n}\|A_1x^*\|\\ &=\frac{\|A_1x^*\|}{r}\left\{\frac{\lambda_n^2}{\alpha_n\mu_n}\cdot\frac{\|x_n-y_n\|}{\lambda_n}+\frac{\lambda_n(\beta_n+\gamma_n)}{\alpha_n\mu_n}M_5\right\}\\ &+\left(\frac{1}{r}+2\right)\frac{\lambda_n\|x_n-z_n\|}{\alpha_n\mu_n}\|A_1x^*\|, \quad \forall n\geq 0.\end{aligned}$$

This relation together with the condition (vi) implies that

$$\limsup_{n\to\infty}\frac{\lambda_n}{\alpha_n\mu_n}\langle x^*-z_n, A_1x^*\rangle \leq 0. \tag{4.14}$$

(III) Finally, we prove $\lim_{n\to\infty}\|x_n-x^*\|=0$.

Indeed, repeating the same argument as that of $\lim_{n\to\infty}\|x_n-x^*\|=0$ in the proof of Theorem 3.1 of the first part, we get that $\lim_{n\to\infty}\|x_n-x^*\|=0$. This completes the proof. □

Remark 4.2. If $N=1$ in Theorems 4.1 and 4.2, then the conclusion $\lim_{n\to\infty}\|x_{n+N}-x_n\|/\lambda_n=0$ reduces to $\lim_{n\to\infty}\|x_{n+1}-x_n\|/\lambda_n=0$. In this case, we have

$$\frac{\|x_n-y_n\|}{\lambda_n}\leq\frac{\|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|}{\lambda_n}=\frac{\|x_n-x_{n+1}\|}{\lambda_n}+\frac{\alpha_n\mu_n}{\lambda_n}\|A_2y_n\|,$$

which together with the condition (v) implies $\lim_{n\to\infty}\|x_n-y_n\|/\lambda_n=0$. Thus, the condition $\lim_{n\to\infty}\|x_{n+1}-x_n\|/\lambda_n=0$ in the conclusion (c) can be deleted.

The following result is derived under some mild conditions, which are very different from those in Theorems 4.1 and 4.2.

Theorem 4.3. *Assume that the sequence $\{y_n\}$ generated by Algorithm 4.1 is bounded. Suppose that $\{\mu_n\}\subset(0,\eta/\kappa^2]$, $\{\alpha_n\}\subset(0,1]$, $\{\beta_n\}$, $\{\gamma_n\}\subset[0,1]$ and $\{\lambda_n\}\subset(0,2\alpha]$ satisfy the following conditions:*

(i) $\sum_{n=0}^{\infty}\gamma_n=\infty$ *and* $\lim_{n\to\infty}(\beta_n+\gamma_n)=0$;

(ii) $\sum_{n=0}^{\infty}(|\beta_{n+N}-\beta_n|+|\gamma_{n+N}-\gamma_n|+|\lambda_{n+N}-\lambda_n|)<\infty$;

(iii) $\sum_{n=0}^{\infty}|\alpha_{n+N}\mu_{n+N}-\alpha_n\mu_n|<\infty$ *and* $\lim_{n\to\infty}\alpha_n\mu_n=0$;

(iv) $\gamma_n=o(\lambda_n)$ *and* $\lambda_n\leq\alpha_n\mu_n$, $\forall n\geq 0$.

In addition, assume that the condition (4.2) in Theorem 4.1 also holds.
Then the sequence $\{x_n\}$ generated by Algorithm 4.1 satisfies the following assertions:

(a) *The sequences* $\{x_n\}$, $\{A_1x_n\}$ *and* $\{A_2y_n\}$ *are bounded;*
(b) $\lim_{n\to\infty}\|x_{n+N}-x_n\|=0$ *and* $\lim_{n\to\infty}\left\|x_n-T_{[n+N]}\circ\cdots\circ T_{[n+1]}x_n\right\|=0$*;*
(c) *The sequence* $\{x_n\}$ *converges strongly to a unique solution of Problem 1.3 provided* $\|x_n-y_n\|=o(\lambda_n)$*.*

Proof. We divide the proof into several steps.
Step 1. By using the same argument as that in the proof of Theorem 4.1 (a), we see that the sequences $\{x_n\}$, $\{A_1x_n\}$ and $\{A_2y_n\}$ are bounded.
Step 2. $\lim_{n\to\infty}\|x_{n+N}-x_n\|=\lim_{n\to\infty}\left\|x_n-T_{[n+N]}\circ\cdots\circ T_{[n+1]}x_n\right\|=0.$

Assumption (A3), Proposition 2.3, and the condition $\lambda_n\le 2\alpha$ $(n\ge 0)$ imply that, for all $n\ge 0$, we have:

$$\begin{aligned}
&\left\|\widetilde{T}_{[n+N+1]}x_{n+N}-\widetilde{T}_{[n+1]}x_n\right\|\\
&=\left\|T_{[n+N+1]}(I-\lambda_{n+N}A_1)x_{n+N}-T_{[n+1]}(I-\lambda_nA_1)x_n\right\|\\
&\le\|(I-\lambda_{n+N}A_1)x_{n+N}-(I-\lambda_nA_1)x_n\|\\
&=\|(I-\lambda_{n+N}A_1)x_{n+N}-(I-\lambda_{n+N}A_1)x_n+(\lambda_n-\lambda_{n+N})A_1x_n\|\\
&\le\|(I-\lambda_{n+N}A_1)x_{n+N}-(I-\lambda_{n+N}A_1)x_n\|+|\lambda_n-\lambda_{n+N}|\,\|A_1x_n\|\\
&\le\|x_{n+N}-x_n\|+|\lambda_n-\lambda_{n+N}|\|A_1x_n\|.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|y_{n+N}-y_n\|=\|\beta_{n+N}x_{n+N}+\gamma_{n+N}f(x_{n+N})\\
&+(1-\beta_{n+N}-\gamma_{n+N})\widetilde{T}_{[n+N+1]}x_{n+N}-\beta_nx_n-\gamma_nf(x_n)-(1-\beta_n-\gamma_n)\widetilde{T}_{[n+1]}x_n\|\\
&\le\|\beta_{n+N}x_{n+N}-\beta_nx_n\|+\|\gamma_{n+N}f(x_{n+N})-\gamma_nf(x_n)\|\\
&+\|(1-\beta_{n+N}-\gamma_{n+N})\widetilde{T}_{[n+N+1]}x_{n+N}-(1-\beta_n-\gamma_n)\widetilde{T}_{[n+1]}x_n\|\\
&\le|\beta_{n+N}-\beta_n|\|x_{n+N}\|+\beta_n\|x_{n+N}-x_n\|+|\gamma_{n+N}-\gamma_n|\|f(x_{n+N})\|\\
&+\gamma_n\|f(x_{n+N})-f(x_n)\|+|(1-\beta_{n+N}-\gamma_{n+N})-(1-\beta_n-\gamma_n)|\|\widetilde{T}_{[n+N+1]}x_{n+N}\|\\
&+(1-\beta_n-\gamma_n)\|\widetilde{T}_{[n+N+1]}x_{n+N}-\widetilde{T}_{[n+1]}x_n\|\\
&\le|\beta_{n+N}-\beta_n|\|x_{n+N}\|+\beta_n\|x_{n+N}-x_n\|+|\gamma_{n+N}-\gamma_n|\|f(x_{n+N})\|\\
&+\gamma_n\rho\|x_{n+N}-x_n\|+(|\beta_{n+N}-\beta_n|+|\gamma_{n+N}-\gamma_n|)\|\widetilde{T}_{[n+N+1]}x_{n+N}\|\\
&+(1-\beta_n-\gamma_n)[\|x_{n+N}-x_n\|+|\lambda_n-\lambda_{n+N}|\|A_1x_n\|]\\
&\le(|\beta_{n+N}-\beta_n|+|\gamma_{n+N}-\gamma_n|)\|x_{n+N}\|+\beta_n\|x_{n+N}-x_n\|\\
&+(|\beta_{n+N}-\beta_n|+|\gamma_{n+N}-\gamma_n|)\|f(x_{n+N})\|+\gamma_n\rho\|x_{n+N}-x_n\|\\
&+(|\beta_{n+N}-\beta_n|+|\gamma_{n+N}-\gamma_n|)\|\widetilde{T}_{[n+N+1]}x_{n+N}\|\\
&+(1-\beta_n-\gamma_n)[\|x_{n+N}-x_n\|+|\lambda_n-\lambda_{n+N}|\|A_1x_n\|]\\
&=[1-\gamma_n(1-\rho)]\|x_{n+N}-x_n\|+(1-\beta_n-\gamma_n)|\lambda_n-\lambda_{n+N}|\|A_1x_n\|\\
&+(|\beta_{n+N}-\beta_n|+|\gamma_{n+N}-\gamma_n|)(\|x_{n+N}\|+\|f(x_{n+N})\|+\|\widetilde{T}_{[n+N+1]}x_{n+N}\|)\\
&\le[1-\gamma_n(1-\rho)]\|x_{n+N}-x_n\|+(|\beta_{n+N}-\beta_n|+|\gamma_{n+N}-\gamma_n|+|\lambda_{n+N}-\lambda_n|)M_1,
\end{aligned}$$

where $M_1 := \sup_{n\geq 0}\left\{\|x_n\| + \|f(x_n)\| + \|\widetilde{T}_{[n+1]}x_n\| + \|A_1 x_n\|\right\} < \infty$.

From Proposition 2.4, we have

$$\begin{aligned}
\|x_{n+N} - x_n\| &= \left\|T^{(\alpha_{n+N-1},\mu_{n+N-1})}y_{n+N-1} - T^{(\alpha_{n-1},\mu_{n-1})}y_{n-1}\right\| \\
&\leq \left\|T^{(\alpha_{n+N-1},\mu_{n+N-1})}y_{n+N-1} - T^{(\alpha_{n+N-1},\mu_{n+N-1})}y_{n-1}\right\| \\
&+ \left\|T^{(\alpha_{n+N-1},\mu_{n+N-1})}y_{n-1} - T^{(\alpha_{n-1},\mu_{n-1})}y_{n-1}\right\| \\
&\leq (1-\alpha_{n+N-1}\tau_{n+N-1})\|y_{n+N-1} - y_{n-1}\| \\
&+ |\alpha_{n+N-1}\mu_{n+N-1} - \alpha_{n-1}\mu_{n-1}|\|A_2 y_{n-1}\| \\
&\leq (1-\alpha_{n+N-1}\tau_{n+N-1})\Big\{[1-\gamma_{n-1}(1-\rho)]\|x_{n+N-1} - x_{n-1}\| \\
&+ (|\beta_{n+N-1} - \beta_{n-1}| + |\gamma_{n+N-1} - \gamma_{n-1}| \\
&+ |\lambda_{n+N-1} - \lambda_{n-1}|)M_1\Big\} + |\alpha_{n+N-1}\mu_{n+N-1} - \alpha_{n-1}\mu_{n-1}|\|A_2 y_{n-1}\| \\
&\leq [1-\gamma_{n-1}(1-\rho)]\|x_{n+N-1} - x_{n-1}\| \\
&+ (|\beta_{n+N-1} - \beta_{n-1}| + |\gamma_{n+N-1} - \gamma_{n-1}| \\
&+ |\lambda_{n+N-1} - \lambda_{n-1}|)M_1 + |\alpha_{n+N-1}\mu_{n+N-1} - \alpha_{n-1}\mu_{n-1}|M_2,
\end{aligned}$$

where $\tau_{n+N-1} := 1 - \sqrt{1-\mu_{n+N-1}(2\eta - \mu_{n+N-1}\kappa^2)} \in (0,1]$ as in Proposition 2.4 and $M_2 := \sup_{n\geq 0}\|A_2 y_n\| < \infty$. Hence, for all $n, m \geq 0$,

$$\begin{aligned}
\|x_{n+m+N} - x_{n+m}\| &\leq [1-\gamma_{n+m-1}(1-\rho)]\|x_{n+m+N-1} - x_{n+m-1}\| \\
&+(|\beta_{n+m+N-1} - \beta_{n+m-1}| + |\gamma_{n+m+N-1} - \gamma_{n+m-1}| \\
&+|\lambda_{n+m+N-1} - \lambda_{n+m-1}|)M_1 + |\alpha_{n+m+N-1}\mu_{n+m+N-1} - \alpha_{n+m-1}\mu_{n+m-1}|M_2 \\
&\leq [1-\gamma_{n+m-1}(1-\rho)]\{[1-\gamma_{n+m-2}(1-\rho)]\|x_{n+m+N-2} - x_{n+m-2}\| \\
&+(|\beta_{n+m+N-2} - \beta_{n+m-2}| + |\gamma_{n+m+N-2} - \gamma_{n+m-2}| + |\lambda_{n+m+N-2} - \lambda_{n+m-2}|)M_1 \\
&+|\alpha_{n+m+N-2}\mu_{n+m+N-2} - \alpha_{n+m-2}\mu_{n+m-2}|M_2\} \\
&+M_1(|\beta_{n+m+N-1} - \beta_{n+m-1}| + |\gamma_{n+m+N-1} - \gamma_{n+m-1}| + |\lambda_{n+m+N-1} - \lambda_{n+m-1}|) \\
&+M_2|\alpha_{n+m+N-1}\mu_{n+m+N-1} - \alpha_{n+m-1}\mu_{n+m-1}| \\
&\leq \prod_{k=m}^{n+m-1}[1-\gamma_k(1-\rho)]\|x_{m+N} - x_m\| \\
&+M_1\sum_{k=m}^{n+m-1}(|\beta_{k+N} - \beta_k| + |\gamma_{k+N} - \gamma_k| + |\lambda_{k+N} - \lambda_k|) \\
&+M_2\sum_{k=m}^{n+m-1}|\alpha_{k+N}\mu_{k+N} - \alpha_k\mu_k|.
\end{aligned}$$

Since the condition (i) implies $\prod_{k=m}^{\infty}[1-\gamma_k(1-\rho)] = 0$ $(\forall m \geq 0)$, we have for all $m \geq 0$, that

$$\limsup_{n\to\infty}\|x_{n+N} - x_n\|^2 = \limsup_{n\to\infty}\|x_{n+m+N} - x_{n+m}\|^2$$

$$\leq M_1 \sum_{k=m}^{\infty} (|\beta_{k+N} - \beta_k| + |\gamma_{k+N} - \gamma_k| + |\lambda_{k+N} - \lambda_k|) + M_2 \sum_{k=m}^{\infty} |\alpha_{k+N}\mu_{k+N} - \alpha_k \mu_k|.$$

This together with the conditions (ii) and (iii) ensures that

$$\limsup_{n\to\infty} \|x_{n+N} - x_n\| \leq 0, \text{ that is } \lim_{n\to\infty} \|x_{n+N} - x_n\| = 0. \tag{4.15}$$

From $\|x_{n+1} - y_n\| = \alpha_n \mu_n \|A_2 y_n\| \leq M_2 \alpha_n \mu_n$ and the condition (iii), we get $\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0$. Since $\lim_{n\to\infty}(\beta_n + \gamma_n) = 0$, it follows from (4.1) that

$$\begin{aligned}
\|x_{n+1} - T_{[n+1]}(x_n - \lambda_n A_1 x_n)\| &\leq \|x_{n+1} - y_n\| + \|y_n - T_{[n+1]}(x_n - \lambda_n A_1 x_n)\| \\
= \|x_{n+1} - y_n\| + \|\beta_n(x_n - T_{[n+1]}(x_n - \lambda_n A_1 x_n)) &+ \gamma_n(f(x_n) - T_{[n+1]}(x_n - \lambda_n A_1 x_n))\| \\
&\leq \|x_{n+1} - y_n\| + \beta_n \|x_n - \widetilde{T}_{[n+1]} x_n\| + \gamma_n \|f(x_n) - \widetilde{T}_{[n+1]} x_n\| \\
&\leq \|x_{n+1} - y_n\| + \beta_n \left(\|x_n\| + \left\|\widetilde{T}_{[n+1]} x_n\right\|\right) + \gamma_n \left(\|f(x_n)\| + \|\widetilde{T}_{[n+1]} x_n\|\right) \\
&\leq \|x_{n+1} - y_n\| + (\beta_n + \gamma_n) M_1 \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

that is, $\lim_{n\to\infty} \left\|x_{n+1} - T_{[n+1]}(x_n - \lambda_n A_1 x_n)\right\| = 0$.

This together with $\lambda_n \to 0$ and the nonexpansiveness of each T_i $(i = 1, 2, \ldots, N)$ and boundedness of $\{A_1 x_n\}$ implies that

$$\begin{cases}
\left\|x_{n+N} - T_{[n+N]}(x_{n+N-1} - \lambda_{n+N-1} A_1 x_{n+N-1})\right\| \to 0, \\
\|T_{[n+N]}(x_{n+N-1} - \lambda_{n+N-1} A_1 x_{n+N-1}) \\
\quad - T_{[n+N]} T_{[n+N-1]}(x_{n+N-2} - \lambda_{n+N-2} A_1 x_{n+N-2})\| \to 0, \\
\cdots \\
\|T_{[n+N]} \circ \cdots \circ T_{[n+2]}(x_{n+1} - \lambda_{n+1} A_1 x_{n+1}) \\
\quad - T_{[n+N]} \circ \cdots \circ T_{[n+1]}(x_n - \lambda_n A_1 x_n)\| \to 0.
\end{cases}$$

Furthermore, observe that

$$\begin{aligned}
x_{n+N} - x_n &= x_{n+N} - T_{[n+N]}(x_{n+N-1} - \lambda_{n+N-1} A_1 x_{n+N-1}) \\
&+ T_{[n+N]}(x_{n+N-1} - \lambda_{n+N-1} A_1 x_{n+N-1}) \\
&- T_{[n+N]} T_{[n+N-1]}(x_{n+N-2} - \lambda_{n+N-2} A_1 x_{n+N-2}) + \cdots \\
&+ T_{[n+N]} \circ \cdots \circ T_{[n+2]}(x_{n+1} - \lambda_{n+1} A_1 x_{n+1}) - T_{[n+N]} \circ \cdots \circ T_{[n+1]}(x_n - \lambda_n A_1 x_n) \\
&+ T_{[n+N]} \circ \cdots \circ T_{[n+1]}(x_n - \lambda_n A_1 x_n) - x_n.
\end{aligned}$$

Consequently, we have that $\lim_{n\to\infty} \|T_{[n+N]} \circ \cdots \circ T_{[n+1]}(x_n - \lambda_n A_1 x_n) - x_n\| = 0$. Note that

$$\begin{aligned}
&\|T_{[n+N]} \circ \cdots \circ T_{[n+1]} x_n - x_n\| \\
&\leq \|T_{[n+N]} \circ \cdots \circ T_{[n+1]} x_n - T_{[n+N]} \circ \cdots \circ T_{[n+1]}(x_n - \lambda_n A_1 x_n)\| \\
&+ \|T_{[n+N]} \circ \cdots \circ T_{[n+1]}(x_n - \lambda_n A_1 x_n) - x_n\| \\
&\leq \lambda_n \|A_1 x_n\| + \|T_{[n+N]} \circ \cdots \circ T_{[n+1]}(x_n - \lambda_n A_1 x_n) - x_n\| \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

that is,

$$\lim_{n\to\infty} \|T_{[n+N]} \circ \cdots \circ T_{[n+1]} x_n - x_n\| = 0. \tag{4.16}$$

Step 3. $\limsup_{n\to\infty}\langle A_1x^*, x^* - x_n\rangle \le 0$.

Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\limsup_{n\to\infty}\langle A_1x^*, x^* - x_n\rangle = \lim_{i\to\infty}\langle A_1x^*, x^* - x_{n_i}\rangle$. The boundedness of $\{x_{n_i}\}$ implies the existence of a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and a point $\hat{x} \in H$ such that $x_{n_{i_j}} \rightharpoonup \hat{x}$. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup \hat{x}$, that is, $\lim_{i\to\infty}\langle w, x_{n_i}\rangle = \langle w, \hat{x}\rangle$, $\forall w \in H$.

Since the pool of mappings $\{T_i : 1 \le i \le N\}$ is finite, we may further assume (passing to a further subsequence if necessary) that, for some integer $k \in \{1, 2, \ldots, N\}$, $T_{[n_i]} \equiv T_k$, $\forall i \ge 1$. Then, it follows from (4.16) that $\left\|x_{n_i} - T_{[i+N]} \circ \cdots \circ T_{[i+1]}x_{n_i}\right\| \to 0$. Hence, by Lemma 2.4, we conclude that $\hat{x} \in \mathrm{Fix}\left(T_{[i+N]} \circ \cdots \circ T_{[i+1]}\right)$. Together with the condition (4.2), this implies that $\hat{x} \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$. Now, since $x^* \in \mathrm{VI}\left(\bigcap_{i=1}^N \mathrm{Fix}(T_i), A_1\right)$, we obtain

$$\limsup_{n\to\infty}\langle A_1x^*, x^* - x_n\rangle = \lim_{i\to\infty}\langle A_1x^*, x^* - x_{n_i}\rangle = \langle A_1x^*, x^* - \hat{x}\rangle \le 0. \tag{4.17}$$

Step 4. $\limsup_{n\to\infty}\langle A_2x^*, x^* - x_n\rangle \le 0$.

Choose a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $\limsup_{n\to\infty}\langle A_2x^*, x^* - x_n\rangle = \lim_{k\to\infty}\langle A_2x^*, x^* - x_{n_k}\rangle$. The boundedness of $\{x_{n_k}\}$ implies that there is a subsequence of $\{x_{n_k}\}$ which converges weakly to a point $\bar{x} \in H$. Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \bar{x}$. Repeating the same argument as in the proof of $\hat{x} \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$, we have $\bar{x} \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$.

Let $y \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ be an arbitrary fixed point. Then, it follows from the nonexpansiveness of each T_i $(i = 1, 2, \ldots, N)$ and monotonicity of A_1 that for all $n \ge 0$, we have:

$$\begin{aligned}
&\|y_n - y\|^2 = \|\beta_n(x_n - y) + \gamma_n(f(x_n) - y) + (1 - \beta_n - \gamma_n)(\widetilde{T}_{[n+1]}x_n - y)\|^2\\
&\le \beta_n\|x_n - y\|^2 + \gamma_n\|f(x_n) - y\|^2 + (1 - \beta_n - \gamma_n)\|\widetilde{T}_{[n+1]}x_n - y\|^2\\
&= \beta_n\|x_n - y\|^2 + \gamma_n\|f(x_n) - f(y) + f(y) - y\|^2\\
&+ (1 - \beta_n - \gamma_n)\|T_{[n+1]}(x_n - \lambda_nA_1x_n) - T_{[n+1]}y\|^2\\
&\le \beta_n\|x_n - y\|^2 + \gamma_n[\|f(x_n) - f(y)\|^2 + 2\langle f(y) - y, f(x_n) - y\rangle]\\
&+ (1 - \beta_n - \gamma_n)\|(x_n - y) - \lambda_nA_1x_n\|^2\\
&\le \beta_n\|x_n - y\|^2 + \gamma_n[\rho^2\|x_n - y\|^2 + 2\langle f(y) - y, f(x_n) - y\rangle]\\
&+ (1 - \beta_n - \gamma_n)[\|x_n - y\|^2 + 2\lambda_n\langle A_1x_n, y - x_n\rangle + \lambda_n^2\|A_1x_n\|^2]\\
&\le [1 - \gamma_n(1 - \rho)]\|x_n - y\|^2 + 2\gamma_n\|f(y) - y\|\|f(x_n) - y\|\\
&+ 2(1 - \beta_n - \gamma_n)\lambda_n\langle A_1x_n, y - x_n\rangle + \lambda_n^2\|A_1x_n\|^2\\
&\le \|x_n - y\|^2 + 2\gamma_n\|f(y) - y\|\|f(x_n) - y\| + 2(1 - \beta_n - \gamma_n)\lambda_n\langle A_1x_n, y - x_n\rangle\\
&+ \lambda_n^2\|A_1x_n\|^2\\
&\le \|x_n - y\|^2 + 2\gamma_n\|f(y) - y\|(M_1 + \|y\|) + 2(1 - \beta_n - \gamma_n)\lambda_n\langle A_1y, y - x_n\rangle\\
&+ \lambda_n^2M_1^2.
\end{aligned} \tag{4.18}$$

The above relation implies that, for all $n \geq 0$, one have:

$$\begin{aligned}0 &\leq \frac{1}{\lambda_n}(\|x_n - y\|^2 - \|y_n - y\|^2) + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|(M_1 + \|y\|)\\ &+ 2(1 - \beta_n - \gamma_n)\langle A_1 y, y - x_n\rangle + \lambda_n M_1^2\\ &= (\|x_n - y\| + \|y_n - y\|)\frac{\|x_n - y\| - \|y_n - y\|}{\lambda_n} + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|(M_1 + \|y\|)\\ &+ 2(1 - \beta_n - \gamma_n)\langle A_1 y, y - x_n\rangle + \lambda_n M_1^2\\ &\leq M_3\frac{\|x_n - y_n\|}{\lambda_n} + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|(M_1 + \|y\|)\\ &+ 2\langle A_1 y, y - x_n\rangle + 2(\beta_n + \gamma_n)\|A_1 y\|M_3 + \lambda_n M_1^2,\end{aligned}$$

where $M_3 := \sup_{n\geq 0}(\|x_n - y\| + \|y_n - y\|) < \infty$.

From $\|x_n - y_n\| = o(\lambda_n)$, $\gamma_n = o(\lambda_n)$ and $\lim_{n\to\infty}(\beta_n + \gamma_n) = 0$, we deduce that for any $\varepsilon > 0$, there exists an integer $m_0 \geq 0$ such that

$$M_3\frac{\|x_n - y_n\|}{\lambda_n} + 2\frac{\gamma_n}{\lambda_n}\|f(y) - y\|(M_1 + \|y\|) + 2(\beta_n + \gamma_n)\|A_1 y\|M_3 + \lambda_n M_1^2 \leq 2\varepsilon$$

for all $n \geq m_0$. Hence, $0 \leq 2\varepsilon + 2\langle A_1 y, y - x_n\rangle$ for all $n \geq m_0$. Putting $n := n_k$, we derive $2\varepsilon + 2\langle A_1 y, y - \bar{x}\rangle \geq 0$ as $k \to \infty$, from $x_{n_k} \rightharpoonup \bar{x} \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$. Since $\varepsilon > 0$ is arbitrary, it is clear that $\langle A_1 y, y - \bar{x}\rangle \geq 0$ for all $y \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$. By Proposition 2.2 (i), we obtain from the α-inverse strong monotonicity of A_1 that $\bar{x} \in \mathrm{VI}\left(\bigcap_{i=1}^N \mathrm{Fix}(T_i), A_1\right)$. Therefore, from $\{x^*\} = \mathrm{VI}\left(\mathrm{VI}\left(\bigcap_{i=1}^N \mathrm{Fix}(T_i), A_1\right), A_2\right)$, we have

$$\limsup_{n\to\infty}\langle A_2 x^*, x^* - x_n\rangle = \lim_{k\to\infty}\langle A_2 x^*, x^* - x_{n_k}\rangle = \langle A_2 x^*, x^* - \bar{x}\rangle \leq 0. \tag{4.19}$$

Step 5. $\lim_{n\to\infty}\|x_n - x^*\| = 0$.

Indeed, repeating the same argument as in Step 5 of the proof of Theorem 3.3, from (4.18), we have $\lim_{n\to\infty}\|x_n - x^*\| = 0$. This completes the proof. □

Acknowledgments. In this research, the first author was partially supported by the National Science Foundation of China (11071169), Innovation Program of Shanghai Municipal Education Commission (09ZZ133) and Leading Academic Discipline Project of Shanghai Normal University (DZL707). The second author was supported by the D.S.T. Research Project No. SR/S4/MS:719/11. The third author express his sincere appreciation to the Deanship of Scientific Research at King Saud University for the funding of this research through the Visiting Professor Program 2015-2016. The fourth author was partially supported by the grant NSC 99-2221-E-037-007-MY3.

References

[1] L.C. Ceng, Q.H. Ansari, J.C. Yao, *Relaxed hybrid steepest-descent methods with variable parameters for triple hierarchical variational inequalities*, Appl. Anal., **91**(2012), no. 10, 1793-1810.

[2] L.-C. Ceng, Q.H. Ansari, A. Petruşel, J.-C. Yao, *Approximation methods for triple hierarchical variational inequalities (I)*, Fixed Point Theory, **16**(2015), no. 1, 67-90.

[3] H. Iiduka, *Strong convergence for an iterative method for the triple-hierarchical constrained optimization problem*, Nonlinear Anal., **71**(2009), 1292–1297.

[4] H. Iiduka, *Iterative algorithm for solving triple-hierarchical constrained optimization problem*, J. Optim. Theory Appl., **148**(2011), 580–592.

[5] H. Iiduka, I. Yamada, *A use of conjugate gradient direction for the convex optimization problem over the fixed point set of a nonexpansive mapping*, SIAM J. Optim., **19**(2009), 1881–1893.

[6] A.F. Izmaelov, M.V. Solodov, *An active set Newton method for mathematical program with complementarity constraints*, SIAM J. Optim., **19**(2008), 1003–1027.

[7] D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and their Applications*, Academic Press, New York, 2008.

[8] I. Yamada, *The hybrid steepest-descent method for the variational inequality problem over the intersection of fixed-point sets of nonexpansive mappings*, in Inherently Parallel Algorithms in Feasibility and Optimization and their Applications, (D. Butnariu, Y. Censor, S. Reich- Eds.), Kluwer Academic Publ., Dordrecht, 2001, pp. 473-504.

Received: October 12, 2012; Accepted: June 21, 2013.