Fixed Point Theory, 16(2015), No. 2, 353-370
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A MULTIPARAMETER GLOBAL BIFURCATION THEOREM WITH APPLICATION TO A FEEDBACK CONTROL SYSTEM

NGUYEN VAN LOI*[1,2], VALERI OBUKHOVSKII**, JEN-CHIH YAO***

* [1]Institute for Computational Science (INCOS)
and Faculty of Civil Engineering,
Ton Duc Thang University, Vietnam.
E-mail: nguyenvanloi1@tdt.edu.vn
[2]Faculty of Fundamental Science,
PetroVietNam University, Viet Nam.
E-mail: loinv@pvu.edu.vn

** Faculty of Physics and Mathematics
Voronezh State Pedagogical University, Russia
E-mail: valerio-ob2000@mail.ru

***The corresponding author,
Center for General Education,
Kaohsiung Medical University, Taiwan.
E-mail: yaojc@kmu.edu.tw

Abstract. In this paper, applying the method of integral guiding functions, we consider a multiparameter global bifurcation problem for periodic solutions of first order operator-differential inclusions whose multivalued parts are not necessarily convex-valued. It is shown how the abstract result can be applied to study the global structure of periodic solutions of a feedback control system with a two-dimensional parameter.
Key Words and Phrases: Global bifurcation, guiding function, operator-differential inclusion, periodic solution, fixed point.
2010 Mathematics Subject Classification: 34C23, 34C25, 34A60, 47H04, 47H10.

1. Introduction

The multiparameter global bifurcation problem for inclusions was first studied by Alexander and Fitzpatrick [1]. In that paper, by using the homotopy group theory the authors considered the global bifurcation at $(0,0)$ for solutions of a family of inclusions of the form

$$x \in F(x, \lambda),$$

where $F\colon \mathcal{O} \subset X \times \mathbb{R}^k \multimap X$ is a condensing upper semicontinuous multivalued mapping with convex and compact values, while X is a Banach space. Górniewicz and Kryszewski [15, 16] considered the case when F is a multivalued mapping with acyclic values in finite-dimensional spaces. Recently, Gabor and Kryszewski [10, 11] studied the multiparameter global bifurcation problem for linear Fredholm inclusions of the form

$$Lx \in F(x, \lambda),$$

where $L\colon X \to Y$ is a linear Fredholm operator of nonnegative index and $F\colon \mathcal{O} \subset X \times \mathbb{R}^k \multimap Y$ is a $c-$admissible multivalued mapping with compact values, while X, Y are Banach spaces. A cohomological index of Fuller type for periodic orbits of differential inclusions in finite-dimensional spaces was constructed in [22] and applied to investigate a multiparameter bifurcation of periodic orbits from an equilibrium point.

One frequently occurred in the study of global bifurcation is the problem of evaluation of that called *global bifurcation index* (see, e.g. [10, 21, 24]). In [21] Kryszewski used the method of guiding functions to evaluate the global bifurcation index and to describe the global structure of branches of periodic orbits for families of differential inclusions of the form

$$\begin{cases} u'(t) \in g(t, u(t), \lambda), \\ u(0) = u(T), \end{cases}$$

where $g\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^k \multimap \mathbb{R}^n$ is a upper-Carathéodory multivalued mapping with compact and convex values. Another construction for the evaluation of the global bifurcation index (for one-parameter case) via guiding functions and integral guiding functions was suggested by Loi and Obukhovskii (see [24]).

In the present paper, after necessary preliminaries, by using the method of integral guiding functions we consider in Section 3 the global bifurcation problem at $(0,0)$ for a class of operator-differential inclusions with $CJ-$multivalued perturbations (see the definition of $CJ-$multivalued mappings in Section 2). In Section 4 we demonstrate how the abstract result can be applied to the study of bifurcations of branches of periodic trajectories of a feedback control system.

2. Preliminaries

2.1. **Notation.** Throughout this paper by the symbol $\mathcal{C}$ we denote the space $C([0,T];\mathbb{R}^n)$ of continuous functions and by $\mathcal{L}^p$ $(p \geq 1)$ the space $L^p([0,T];\mathbb{R}^n)$ of $p-$th integrable functions with usual norms:

$$\|x\|_{\mathcal{C}} = \max_{t\in[0,T]} |x(t)| \ \text{ and } \ \|f\|_p = \left(\int_0^T |f(s)|^p \, ds \right)^{\frac{1}{p}}, \ n \geq 1.$$

An *open ball* of radius r centered at 0 in $\mathcal{C}$ $[\mathbb{R}^n]$ is denoted by $B_{\mathcal{C}}(0,r)$ [respectively, $B^n(0,r)$]. The unit open ball [unit sphere] in $\mathbb{R}^n$ are denoted by B^n [resp., S^{n-1}]. Consider the space of all absolutely continuous functions $x : [0,T] \to \mathbb{R}^n$ whose derivatives belong to $\mathcal{L}^p$. It is known (see, e.g. [2]) that this space can be identified with the Sobolev space $W^{1,p}([0,T];\mathbb{R}^n)$ endowed with the norm

$$\|x\|_{\mathcal{W}} = \left(\|x\|_p^p + \|x'\|_p^p \right)^{\frac{1}{p}}.$$

We will denote this space by $\mathcal{W}^{1,p}$. Notice that (see, e.g. [6]), for the case $p = 2$, the embedding $\mathcal{W}^{1,2} \hookrightarrow \mathcal{C}$ is compact. By the symbol $\mathcal{W}_T^{1,p}$ we will denote the subspace of all functions $x \in \mathcal{W}^{1,p}$ such that $x(0) = x(T)$.

2.2. **Multimaps.** Let X, Y be metric spaces. Denote by $P(Y)\,[K(Y)]$ the collections of all nonempty [respectively, nonempty compact] subsets of Y. For the case when Y is a normed space, $Kv(Y)$ denotes the set of all nonempty compact and convex subsets of Y.

Definition 1. *(see, e.g.* [3, 4, 14, 20]*). A multivalued mapping (multimap) $\Sigma\colon X \to P(Y)$ is said to be: (i) upper semicontinuous (u.s.c.) if for every open subset $V \subset Y$ the set*

$$\Sigma_{+}^{-1}(V) = \{x \in X\colon \Sigma(x) \subset V\}$$

is open in X; (ii) closed if its graph $\{(x, y) \in X \times Y : y \in \Sigma(x)\}$ is a closed subset of $X \times Y$; (iii) compact, if the set $\Sigma(X)$ is relatively compact in Y.

Definition 2. *A set $M \in K(Y)$ is said to be* aspheric *(or UV^{∞}, or ∞-*proximally connected*) (see, e.g.* [23, 3, 14, 21]*), if for every $\varepsilon > 0$ there exists $\delta > 0$ such that each continuous map $\phi\colon S^n \to O_\delta(M)$, $n = 0, 1, 2, \cdots$, can be extended to a continuous map $\widetilde{\phi}\colon \overline{B^{n+1}} \to O_\varepsilon(M)$, where $O_\varepsilon(M)$ denotes the $\varepsilon-$neighbourhood of M.*

Definition 3. *(see* [19]*). A nonempty compact space is said to be an R_δ-*set *if it can be represented as the intersection of a decreasing sequence of compact, contractible spaces.*

Definition 4.*(see* [14]*) A u.s.c. multimap $\Sigma : X \to K(Y)$ is said to be a J-multimap ($\Sigma \in J(X, Z)$) if every value $\Sigma(x)$, $x \in X$, is an aspheric set.*

Now let us recall (see, e.g. [5]) that a metric space Z is called *the absolute retract (the AR-space)* [resp., *the absolute neighborhood retract (the ANR-space)*] provided for each homeomorphism h taking it onto a closed subset of a metric space Z', the set $h(Z)$ is the retract of Z' [resp., of its open neighborhood $O(h(Z))$ in Z']. Notice that the class of ANR-spaces is broad enough: in particular, a finite-dimensional compact set is the ANR-space if and only if it is locally contractible. Therefore, it means that compact polyhedrons and compact finite-dimensional manifolds are the ANR-spaces. The union of a finite number of convex closed subsets in a normed space is also the ANR-space.

Proposition 1. *(see* [14]*). Let Z be an ANR-space. In each of the following cases a u.s.c. multimap $\Sigma : X \to K(Z)$ is a J-multimap:*
for each $x \in X$ the value $\Sigma(x)$ is
a) a convex set;
b) a contractible set;
c) an R_δ-set;
d) an AR-space.
In particular, every continuous map $\sigma : X \to Z$ is a J-multimap.

Definition 5. *Let $\mathcal{F} : X \to P(Y)$ be a multimap. For a given $\varepsilon > 0$, a continuous map $f\colon X \to Y$ is called an ε-approximation of the multimap $\mathcal{F}$ if for each $x \in X$ there exists $x' \in X$ such that $\varrho_X(x, x') < \varepsilon$ and*

$$f(x) \in O_\varepsilon(\mathcal{F}(x')), \text{ for all } x \in X.$$

It is easy to see that the ε-approximation may be equivalently defined as the map whose graph belongs to the ε-neighborhood of the graph of the corresponding multimap. The fact that a map $f\colon X \to Y$ is an $\varepsilon-$approximation of a multimap $\mathcal{F}\colon X \to P(Y)$ is written as $f \in a(\mathcal{F}, \varepsilon)$.

Definition 6. *A multimap $\mathcal{F} : X \to K(Y)$ is called approximable if for every $\varepsilon > 0$ it admits a single-valued ε–approximation and, moreover, for every $\varepsilon > 0$ there exists $\delta_0 > 0$ such that for all δ with $0 < \delta < \delta_0$ and any two δ–approximations $f_\delta, \widetilde{f}_\delta : X \to Y$ of the multimap $\mathcal{F}$ there exists a continuous map $h : X \times [0,1] \to Y$ such that*

(*i*) $h(x,0) = f_\delta(x), \;\; h(x,1) = \widetilde{f}_\delta(x)$ *for all* $x \in X$;
(*ii*) $h(\cdot, \lambda) \in a(\mathcal{F}, \varepsilon)$ *for each* $\lambda \in [0,1]$.

The main approximation property of a class of J–multimaps can be expressed by the following assertion (see [23, 13]).

Proposition 2. *Let X be a compact ANR–space, Y a metric space. Then each J–multimap $\mathcal{F} : X \to K(Y)$ is approximable.*

Definition 7.*Let $\mathcal{O} \subseteq X$. By $CJ(\mathcal{O}, X)$ we will denote the collection of all multimaps $F\colon \mathcal{O} \to K(X)$ that may be represented in the form of composition $F = f \circ \Sigma$, where $\Sigma \in J(\mathcal{O}, Y)$ for a certain metric space Y and $f\colon Y \to X$ is a continuous map.*

2.3. **Linear Fredholm operators.** Let X, Y be Banach spaces.

Definition 8.*(see, e.g.* [12]*). A linear bounded operator $L\colon domL \subseteq X \to Y$ is called Fredholm of index q ($q \geq 0$) if*

(1i) *ImL is closed in Y;*
(2i) *$KerL$ and $CokerL = Y/ImL$ have the finite dimension and, moreover,*

$$dimKerL - dimCokerL = q$$

Let $L\colon domL \subseteq X \to Y$ be a Fredholm operator of index q, then there exist projectors $P_L\colon X \to X$ and $Q_L\colon Y \to Y$ such that $ImP_L = KerL$ and $KerQ_L = ImL$. If the operator

$$L_{P_L}\colon domL \cap KerP_L \to ImL$$

is defined as the restriction of L on $domL \cap KerP_L$ then it's clear that L_{P_L} is an algebraic isomorphism and we may define $K_{P_L}\colon ImL \to domL$ as $K_{P_L} = L_{P_L}^{-1}$.

For the case $q = 0$, if we let $\Pi_L\colon Y \to CokerL$ be the canonical surjection:

$$\Pi_L z = z + ImL$$

and $\Lambda_L\colon CokerL \to KerL$ be a one-to-one linear mapping, then the equation

$$Lx = y, \; y \in Y$$

is equivalent to the equation

$$(i - P_L)x = (\Lambda_L \Pi_L + K_L)y,$$

where i denotes the identity operator and $K_L\colon Y \to X$ be defined as

$$K_L = K_{P_L}(i - Q_L).$$

2.4. **Coincidence index.** For readers's convenience, we recall in this section the definition of coincidence index presented in [21] (see, also [7, 8, 9]). Firstly, let us recall the definition of the topological degree of a continuous map between two finite-dimensional spaces (for the notion of the homotopy groups and the cohomotopy sets we refer readers to [18, 25]).

Let $U \subset \mathbb{R}^m$ be an open bounded subset and $f\colon \overline{U} \to \mathbb{R}^n$ a continuous map, where $m \geq n \geq 1$. Assume that $f(x) \neq 0$ for all x belonging to the boundary ∂U of the set U. Therefore, the distance $d(0, f(\partial U))$ from 0 to the set $f(\partial U)$ in $\mathbb{R}^n$ is positive. Taking $\rho = \frac{1}{2} d(0, f(\partial U))$, we obtain $f(\partial U) \subset \mathbb{R}^n \setminus B^n(0,\rho)$. Hence, the map

$$f\colon \big(\overline{U}, \partial U\big) \longrightarrow \big(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)\big)$$

induces a map between cohomotopy sets

$$f^\sharp\colon \pi^n\big(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)\big) \longrightarrow \pi^n\big(\overline{U}, \partial U\big)$$

Consider the following sequence of maps

$$\pi^n\big(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)\big) \xrightarrow{f^\sharp} \pi^n\big(\overline{U}, \partial U\big) \xleftarrow{i_1^\sharp}$$

$$\xleftarrow{i_1^\sharp} \pi^n\big(\mathbb{R}^m, \mathbb{R}^m \setminus U\big) \xrightarrow{i_2^\sharp} \pi^n\big(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,r)\big),$$

where $r > 0$ is such that $U \subset B^m(0,r)$,

$$i_1\colon \big(\overline{U}, \partial U\big) \longrightarrow \big(\mathbb{R}^m, \mathbb{R}^m \setminus U\big),$$

and

$$i_2\colon \big(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,r)\big) \longrightarrow \big(\mathbb{R}^m, \mathbb{R}^m \setminus U\big)$$

are inclusion maps.

The map $i_1^\sharp$ is a bijection (by the excision property), therefore according to the relations $\pi^n(S^n) = \pi^n\big(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)\big)$ and $\pi^n(S^m) = \pi^n\big(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,r)\big)$, the map

$$\omega_f = i_2^\sharp \circ (i_1^\sharp)^{-1} \circ f^\sharp\colon \pi^n(S^n) \longrightarrow \pi^n(S^m)$$

is well-defined.

Definition 9. *The element $\omega_f(\mathbf{1}) \in \pi^n(S^m) = \pi_m(S^n)$ is called the* topological degree *of the map f on $\overline{U}$ and it is denoted by $deg\,(f, \overline{U})$, where $\mathbf{1}$ is the homotopy class of the identity map $id\colon S^n \to S^n$ in $\pi^n(S^n) \cong \mathbb{Z}$.*

Notice that the topological degree $deg(f, \overline{U})$ does not depend on the choice of $r > 0$. For illustration of the above definition let us recall an example presented in [10, Example 4.1]

Example 1. Let $U = B^m$ and $\overline{f} = f_{|_{S^{m-1}}}\colon S^{m-1} \to \mathbb{R}^n \setminus B^n(0,\rho)$, where ρ is taken as above. Consider the diagram

$$\begin{array}{ccccc}
\pi^n(S^n) \cong \pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)) & \xrightarrow{f^\sharp} & \pi^n(\overline{B^m}, S^{m-1}) \cong \pi^n(S^m) & \xrightarrow{=} & \pi_m(S^n) \\
\cong \uparrow \delta_1 & & \uparrow \delta & & \uparrow \Sigma \\
\pi^{n-1}(S^{n-1}) \cong \pi^{n-1}(\mathbb{R}^n \setminus B^n(0,\rho)) & \xrightarrow{\overline{f}^\sharp} & \pi^{n-1}(S^{m-1}) & \xrightarrow{=} & \pi_{m-1}(S^{n-1})
\end{array}$$

where δ, δ_1 are the respective coboundary operators and Σ is the suspension homomorphism. This diagram is commutative. Therefore, if $f\colon (\overline{B^m}, S^{m-1}) \to (\mathbb{R}^n, S^{n-1})$, then $deg(f, \overline{B^m}) = \Sigma([\overline{f}]) \in \pi_m(S^n)$, where $[\overline{f}] \in \pi_{m-1}(S^{n-1})$ is the homotopy class of $\overline{f}\colon S^{m-1} \to S^{n-1}$.

Now let X, Y be Banach spaces; $U \subset X$ an open bounded subset; $L\colon X \to Y$ a linear Fredholm map of index $q \geq 0$ and $F = (f \circ \Sigma) \in CJ(\overline{U}, Y)$ a compact multimap such that $Lx \notin F(x)$ for all $x \in \partial U$. Let $C = \{x \in U\colon Lx \in F(x)\}$ be the set of coincidence points of L and F. Since the restriction $L_{|\overline{U}}$ is proper, the set C is compact. Hence, there exists an open bounded set N such that

(a) $C \subset N \subset \overline{N} \subset U$;
(b) $\overline{N}$ is a compact $ANR-$space.

Let $\delta = \frac{1}{2} dist_Y\big(0, (L - F)(\partial N)\big)$. For $\varepsilon \in (0, \delta]$ let $p_\varepsilon\colon \overline{F(N)} \to Y$ be the Schauder projection of the compact set $\overline{F(N)}$ into a finite-dimensional subspace Z of Y such that $\|p_\varepsilon y - y\| < \varepsilon$ for all $y \in \overline{F(N)}$. Denote by W' the finite-dimensional subspace of $Im\, L$ such that $Z \subset W = W' \oplus Im(Q_L)$. Set $T = L^{-1}(W)$, $N_T = N \cap T$. It is clear that $L_{|T}\colon T \to W$ is Freholm operator of index q and

$$dimT = dimW + q.$$

W.l.o.g. assume that $dimW = n \geq q + 2$. Then the coincidence index $Ind(L, F, \overline{U})$ is defined as

Definition 10.

$$Ind(L, F, \overline{U}) := deg(L - p_\varepsilon \circ f_\kappa, \overline{N_T}) \in \pi^n(S^{n+q}) \cong \Pi_q,$$

where f_κ is an $\kappa-$approximation of F on $\overline{N_T}$ while $\kappa \in (0, \varepsilon)$ is sufficiently small and Π_q *denotes $q-$th stable homotopy group of spheres (see, e.g.* [18]*).*

The given coincidence index has the following properties.

(i) (Existence) If $Ind(L, F, \overline{U}) \neq 0 \in \Pi_q$, then there exists $x \in U$ such that $Lx \in F(x)$.

(ii) (Localization) If $U' \subset U$ is open and
$$C := \{x \in U\colon Lx \in F(x)\} \subset U',$$
then $Ind(L, F, \overline{U}) = Ind(L, F, \overline{U'})$.

(ii) (Additivity) If U_1, U_2 are open bounded disjoint subsets of X and $U = U_1 \cup U_2$, then
$$Ind(L, F, \overline{U}) = Ind(L, F, \overline{U_1}) + Ind(L, F, \overline{U_2}).$$

(iii) (Restriction) If $F(\overline{U})$ belongs to a subspace Y' of Y, then
$$Ind(L, F, \overline{U}) = Ind(L, F, \overline{U_T}),$$
where $U_T = U \cap T$, $T = L^{-1}(Y')$.

(iii) (Homotopy) If there exists a compact $CJ-$multimap $\Phi\colon \overline{U} \times [0, 1] \to K(Y)$ such that $Lx \notin G(x, \lambda)$ for all $(x, \lambda) \in \partial U \times [0, 1]$, then
$$Ind\big(L, \Phi(\cdot, 0), \overline{U}\big) = Ind\big(L, \Phi(\cdot, 1), \overline{U}\big).$$

3. A multiparameter global bifurcation theorem

Consider a family of operator-differential inclusions

$$Lx \in \mathcal{Q}(x, \mu), \tag{1}$$

where $L\colon \mathcal{W}_T^{1,2} \to \mathcal{L}^2$, $Lx = x'$, and $\mathcal{Q}\colon \mathcal{C} \times \mathbb{R}^k \to K\big(\mathcal{L}^2\big)$ be such that

($Q1$) Q is a CJ−multimap and $0 \in \mathcal{Q}(0, \mu)$ for all $\mu \in \mathbb{R}^k$, $k \geq 1$;
($Q2$) for every bounded subset $\Omega \subset \mathcal{C} \times \mathbb{R}^k$ the set $Q(\Omega)$ is bounded in $\mathcal{L}^2$.

By a solution to (1) we mean a pair $(x, \mu) \in \mathcal{W}_T^{1,2} \times \mathbb{R}^k$ that satisfies (1). It is obvious that problem (1) has the trivial solution $(0, \mu)$ for all $\mu \in \mathbb{R}^k$. Denote by $\mathcal{S}$ the set of all non-trivial solutions of (1).

Definition 11. *A point $(0, \mu_0) \in \mathcal{C} \times \mathbb{R}^k$ is said to be a bifurcation point of problem* (1) *if for every open bounded subset $U \subset \mathcal{W}_T^{1,2} \times \mathbb{R}^k$ containing $(0, \mu_0)$, there exists a solution $(x, \mu) \in U$ to problem* (1) *such that $x \neq 0$.*

Definition 12. *A family of continuously differentiable functions $V_\mu\colon \mathbb{R}^n \to \mathbb{R}$, $\mu \in \mathbb{R}^k$, is said to be a family of local integral guiding functions for* (1) *at* $(0,0)$*, if there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ there is a sufficiently small number $\delta_\varepsilon > 0$ (which continuously and nondecreasingly depends on ε) such that for every $x \in \mathcal{C}$, $0 < \|x\|_{\mathcal{C}} \leq \delta_\varepsilon$, the following relation holds*

$$\int_0^T \big\langle \nabla V_\mu(x(s)), f(s) \big\rangle \, ds > 0$$

for all $\mu \in S^{k-1}(0, \varepsilon)$ and all $f \in \mathcal{Q}(x, \mu)$, where ∇V_μ denotes the gradient of V_μ and the symbol $\big\langle \cdot, \cdot \big\rangle$ denotes the inner product in $\mathbb{R}^n$.

Lemma 1. *If V_μ is a family of local integral guiding functions of* (1) *at* $(0,0)$ *then for every $\varepsilon \in (0, \varepsilon_0)$:*

(a) *inclusion* (1) *has only trivial solutions $(0, \mu)$ on $\overline{B_{\mathcal{C}}(0, \delta_\varepsilon)} \times S^{k-1}(0, \varepsilon)$;*
(b) *equation $\nabla V_\mu(w) = 0$ has only trivial solutions $(0, \mu)$ on $\overline{B^n(0, \delta_\varepsilon)} \times S^{k-1}(0, \varepsilon)$.*

Proof. (a) Assume that $(x, \mu) \in \overline{B_{\mathcal{C}}(0, \delta_\varepsilon)} \times \mathbb{R}^k$, $|\mu| = \varepsilon$, is a nontrivial solution to (1). Therefore, there exists $f \in \mathcal{Q}(x, \mu)$ such that $x'(t) = f(t)$ for a.e. $t \in [0, T]$. Since $|\mu| = \varepsilon$ and $\|x\|_{\mathcal{C}} \leq \delta_\varepsilon$ we have

$$\int_0^T \big\langle \nabla V_\mu(x(t)), f(t) \big\rangle dt > 0.$$

On the other hand,

$$\int_0^T \big\langle \nabla V_\mu(x(t)), f(t) \big\rangle dt > 0 = \int_0^T \big\langle \nabla V_\mu(x(t)), x'(t) \big\rangle dt > 0 =$$

$$V_\mu(x(T)) - V_\mu(x(0)) = 0,$$

giving the contradiction.

Similarly we obtain the conclusion (b). ■

For each $\varepsilon \in (0, \varepsilon_0)$, set $O_\varepsilon = B^{n+k}\big(0, \sqrt{\varepsilon^2 + \delta_\varepsilon^2}\,\big)$ and define the map

$$\widetilde{V_\varepsilon} \colon \overline{O_\varepsilon} \to \mathbb{R}^{n+1},$$
$$\widetilde{V_\varepsilon}(w, \mu) = \big\{ -\nabla V_\mu(w), \varepsilon^2 - |\mu|^2 \big\}.$$

From Lemma 1 it follows that $\widetilde{V_\varepsilon}$ has no zeros on the sphere ∂O_ε. Hence, the topological degree

$$deg(\widetilde{V_\varepsilon}, \overline{O_\varepsilon}) = \omega_{\widetilde{V_\varepsilon}}(\mathbf{1}) \in \pi^{n+1}(S^{n+k}) = \pi_{n+k}(S^{n+1})$$

is well-defined.

Let us show that this degree does not depend on the choice of $\varepsilon \in (0, \varepsilon_0)$. In fact, let $\varepsilon_1, \varepsilon_2$, $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$, be arbitrary numbers. For each $\lambda \in [0,1]$, set $\varepsilon_{\lambda+1} = \lambda\varepsilon_2 + (1-\lambda)\varepsilon_1$,

$$O_{\varepsilon_{\lambda+1}} = B^{n+k}\Big(0, \sqrt{\varepsilon_{\lambda+1}^2 + \delta_{\varepsilon_{\lambda+1}}^2}\,\Big),$$

where $\delta_{\varepsilon_{\lambda+1}}$ is the constant from Definition 12, and consider the map

$$V_\lambda^\sharp \colon \overline{O_{\varepsilon_{\lambda+1}}} \to \mathbb{R}^{n+1},$$
$$V_\lambda^\sharp(w, \mu) = \{-\nabla V_\mu(w), \varepsilon_{\lambda+1}^2 - |\mu|^2\}.$$

Assume that there exist $\lambda_* \in [0,1]$ and $(w_*, \mu_*) \in \partial O_{\varepsilon_{\lambda_*+1}}$ such that

$$V_{\lambda_*}^\sharp(w_*, \mu_*) = 0,$$

or equivalently,

$$\begin{cases} \nabla V_{\mu_*}(w_*) = 0, \\ |\mu_*| = \varepsilon_{\lambda_*+1}. \end{cases}$$

From $(w_*, \mu_*) \in \partial O_{\varepsilon_{\lambda_*+1}}$ it follows that $|w_*| = \delta_{\varepsilon_{\lambda_*+1}}$. That contradicts to Lemma 1(b). Hence, the topological degree $deg(\widetilde{V_\varepsilon}, \overline{O_\varepsilon})$ is the same for all $\varepsilon \in (0, \varepsilon_0)$. This element is called *the index of the family of local integral guiding functions* V_μ and is denoted by $indV_\mu$.

Theorem 1. *Let conditions $(Q1)-(Q2)$ hold. In addition, assume that there exists a family of local integral guiding functions V_μ of (1) such that $indV_\mu \neq 0$. Then $(0,0)$ is a bifurcation point of problem (1). Moreover, there is a connected subset $\mathcal{R} \subset \mathcal{S}$ such that $(0,0) \in \overline{\mathcal{R}}$ and either $\mathcal{R}$ is unbounded or $\overline{\mathcal{R}} \ni (0, \mu_*)$ for some $\mu_* \neq 0$.*

Proof. It is clear that L is a linear Fredholm operator of index zero and

$$Ker\, L \cong \mathbb{R}^n \cong Coker\, L.$$

The projection

$$\Pi_L \colon \mathcal{L}^2 \to \mathbb{R}^n,$$

is defined as

$$\Pi_L(f) = \frac{1}{T}\int_0^T f(s)\, ds$$

and the homeomorphism $\Lambda_L\colon \mathbb{R}^n \to \mathbb{R}^n$ is an identity operator.
The space $\mathcal{L}^2$ can be represented as

$$\mathcal{L}^2 = \mathcal{L}_0 \oplus \mathcal{L}_1,$$

where $\mathcal{L}_0 = Coker\, L$ and $\mathcal{L}_1 = Im\, L$. The decomposition of an element $f \in \mathcal{L}^2$ is denoted by

$$f = f_0 + f_1, f_0 \in \mathcal{L}_0, f_1 \in \mathcal{L}_1.$$

Substitute inclusion (1) by the following equivalent inclusion

$$x \in G(x, \mu), \tag{2}$$

where

$$G\colon \mathcal{C} \times \mathbb{R}^k \to K(\mathcal{C}),$$
$$G(x, \mu) = P_L x + (\Pi_L + K_L)\mathcal{Q}(x, \mu).$$

Define the map

$$\ell\colon \mathcal{C} \times \mathbb{R}^k \to \mathcal{C} \times \mathbb{R},\ \ell(x, \mu) = (x, 0).$$

For $r, \varepsilon > 0$ set

$$B_{r,\varepsilon} = \big\{(x, \mu) \in \mathcal{C} \times \mathbb{R}^k\colon \|x\|^2_{\mathcal{C}} + |\mu|^2 \le r^2 + \varepsilon^2\big\},$$

and consider the multimap

$$G_r\colon B_{r,\varepsilon} \to K(\mathcal{C} \times \mathbb{R}),$$
$$G_r(x, \mu) = \{G(x, \mu), r^2 - \|x\|_{\mathcal{C}}\}.$$

It is clear that ℓ is a linear Fredholm map of index $k - 1$.

Step 1. We will show that G_r is a compact $CJ-$multimap. Indeed, from the fact that $\mathcal{Q}$ is a $CJ-$multimap and the operator $\Pi_L + K_L$ is linear and continuous it follows that the multimap $(\Pi_L + K_L) \circ \mathcal{Q}$ is a $CJ-$multimap, hence G_r is a $CJ-$multimap. Further, from $(Q2)$ it follows that the set $(\Pi_L + K_L) \circ \mathcal{Q}(B_{r,\varepsilon})$ is bounded in $\mathcal{W}^{1,2}_T$, and by the Sobolev embedding theorem [6] it is a relatively compact subset in $\mathcal{C}$. Now, our assertion follows from the fact that the operator P_L is continuous and has a finite-dimensional range.

Step 2. Choosing arbitrarily $\varepsilon \in (0, \varepsilon_0)$ and sufficiently small $r \in (0, \delta_\varepsilon)$, where ε_0 and δ_ε are the constants from Definition 12, we will show that $\ell(x, \mu) \notin G_r(x, \mu)$ for all $(x, \mu) \in \partial B_{r,\varepsilon}$.

Indeed, assume to the contrary that there is $(x, \mu) \in \partial B_{r,\varepsilon}$ such that $\ell(x, \mu) \in G_r(x, \mu)$. Then,

$$x \in G(x, \mu), \tag{3}$$

and

$$\|x\|_{\mathcal{C}} = r. \tag{4}$$

From (3) it follows that there is $f \in \mathcal{Q}(x, \mu)$ such that $x'(t) = f(t)$ for a.e. $t \in [0, T]$. Applying (4), we obtain $|\mu| = \varepsilon$. Moreover, from $\|x\|_{\mathcal{C}} = r < \delta_\varepsilon$ we have

$$\int_0^T \big\langle \nabla V_\mu(x(s)), f(s)\big\rangle ds > 0$$

for all $\mu \in S^{k-1}(0,\varepsilon)$.
On the other hand,

$$\int_0^T \big\langle \nabla V_\mu(x(s)), f(s)\big\rangle ds = \int_0^T \big\langle \nabla V_\mu(x(s)), x'(s)\big\rangle ds = 0,$$

giving the contradiction. Therefore, the coincidence index $Ind(\ell, G_r, B_{r,\varepsilon})$ is well-defined for each $\varepsilon \in (0,\varepsilon_0)$ and $r \in (0,\delta_\varepsilon)$

STEP 3. Fix $\varepsilon \in (0,\varepsilon_0)$ and $r \in (0,\delta_\varepsilon)$. Let us evaluate $Ind(\ell, G_r, B_{r,\varepsilon})$. Towards this goal, consider the multimap $\Sigma\colon B_{r,\varepsilon} \times [0,1] \to K(\mathcal{C} \times \mathbb{R})$,

$$\Sigma(x,\mu,\lambda) = \Big\{P_L x + (\Pi_L + K_L) \circ \alpha\big(\mathcal{Q}(x,\mu),\lambda\big), \tau\Big\},$$

where

$$\tau = \lambda(r^2 - \|x\|_{\mathcal{C}}^2) + (1-\lambda)(|\mu|^2 - \varepsilon^2),$$

and $\alpha\colon \mathcal{L}^2 \times [0,1] \to \mathcal{L}^2$,

$$\alpha(f,\lambda) = f_0 + \lambda f_1, f_0 \in \mathcal{L}_0, f_1 \in \mathcal{L}_1, f = f_0 + f_1.$$

Following Step 1 we can easily prove that Σ is a compact $CJ-$multimap.

Assume $(x^*,\mu^*,\lambda_*) \in \partial B_{r,\varepsilon} \times [0,1]$ is such that $\ell(x^*,\mu^*) \in \Sigma(x^*,\mu^*,\lambda^*)$. Then

$$\lambda^*(r^2 - \|x^*\|_{\mathcal{C}}^2) + (1-\lambda^*)(|\mu^*|^2 - \varepsilon^2) = 0 \tag{5}$$

and there is a function $f^* \in \mathcal{Q}(x^*,\mu^*)$ such that

$$x^* = P_L x^* + (\Pi_L + K_L) \circ \alpha(f^*,\lambda^*)$$

or equivalently,

$$\begin{cases} (x^*)' = \lambda^* f_1^* \\ 0 = f_0^*, \end{cases}$$

where $f_0^* + f_1^* = f^*$, $f_0^* \in \mathcal{L}_0$ and $f_1^* \in \mathcal{L}_1$.
From $(x^*,\mu^*) \in \partial B_{r,\varepsilon}$ it follows that

$$r^2 - \|x^*\|_{\mathcal{C}}^2 = |\mu^*|^2 - \varepsilon^2.$$

Hence, from (5) we obtain

$$\|x^*\|_{\mathcal{C}} = r \ \text{ and } \ |\mu^*| = \varepsilon.$$

From the choice of r it follows that

$$\int_0^T \big\langle \nabla V_{\mu^*}(x^*(s)), f(s)\big\rangle ds > 0 \text{ for all } f \in \mathcal{Q}(x^*,\mu^*).$$

If $\lambda^* \neq 0$: then

$$\int_0^T \big\langle \nabla V_{\mu^*}(x^*(s)), f^*(s)\big\rangle\, ds = \int_0^T \big\langle \nabla V_{\mu^*}(x^*(s)), \frac{1}{\lambda^*} x^{*\prime}(s)\big\rangle\, ds =$$
$$= \frac{1}{\lambda^*}\Big(V_{\mu^*}\big(x^*(T)\big) - V_{\mu^*}\big(x^*(0)\big)\Big) = 0,$$

giving the contradiction.

If $\lambda^* = 0$ then $(x^*)' = 0$. Therefore $x^* \equiv a$ for some $a \in \mathbb{R}^n$, $|a| = r$. According to Definition 12, for every $f \in \mathcal{Q}(a, \mu^*)$ we have

$$\int_0^T \big\langle \nabla V_{\mu^*}(a), f(s) \big\rangle \, ds = \Big\langle \nabla V_{\mu^*}(a), \int_0^T f(s)\, ds \Big\rangle = T \big\langle \nabla V_{\mu^*}(a), \Pi_L f \big\rangle > 0. \tag{6}$$

Consequently, $\Pi_L f \neq 0$ for all $f \in \mathcal{Q}(a, \mu^*)$, in particular, $\Pi_L f^* \neq 0$. But $\Pi_L f^* = \Pi_L f_0^* = 0$. That is the contradiction.

Thus multimap Σ is a homotopy on $\partial B_{r,\varepsilon}$ connecting the multimaps $\Sigma(x, \mu, 1) = G_r(x, \mu)$ and

$$\Sigma(x, \mu, 0) = \big\{ P_L x + \Pi_L \mathcal{Q}(x, \mu), |\mu|^2 - \varepsilon^2 \big\}.$$

By virtue of the homotopy invariance of the coincidence index we obtain

$$Ind(\ell, G_r, B_{r,\varepsilon}) = Ind(\ell, \Sigma(\cdot, \cdot, 0), B_{r,\varepsilon}).$$

The multimap $P_L + \Pi_L \mathcal{Q}$ takes values in $\mathbb{R}^n$, so applying the restriction property of the coincidence degree we have

$$Ind(\ell, \Sigma(\cdot, \cdot, 0), B_{r,\varepsilon}) = Ind(\ell, \Sigma(\cdot, \cdot, 0), \overline{U}_{r,\varepsilon}),$$

where $\overline{U}_{r,\varepsilon} = B_{r,\varepsilon} \cap \mathbb{R}^{n+k}$.
In the space $\mathbb{R}^{n+1}$ the vector field $\ell - \Sigma(\cdot, \cdot, 0)$ has the form

$$\ell(y, \mu) - \Sigma(y, \mu, 0) = \big\{ -\Pi_L Q(y, \mu), \varepsilon^2 - |\mu|^2 \big\}, \ \forall (y, \mu) \in \overline{U}_{r,\varepsilon} \subset \mathbb{R}^{n+k}.$$

Consider now the multimap: $\Gamma \colon \overline{U}_{r,\varepsilon} \times [0, 1] \to K(\mathbb{R}^{n+1})$ defined as

$$\Gamma(y, \mu, \lambda) = \big\{ -\lambda \Pi_L \mathcal{Q}(y, \mu) + (\lambda - 1) \nabla V_\mu(y), \varepsilon^2 - |\mu|^2 \big\}.$$

It is clear that Γ is a compact $CJ-$multimap. Assume that there exists $(y, \mu, \lambda) \in \partial U_{r,\varepsilon} \times [0, 1]$ such that $0 \in \Gamma(y, \mu, \lambda)$. Then we obtain

$$\begin{cases} |\mu| = \varepsilon \\ (\lambda - 1) \nabla V_\mu(y) \in \lambda \Pi_L \mathcal{Q}(y, \mu), \end{cases}$$

and by virtue of (6) we get the contradiction. So, Γ is a homotopy connecting $\ell - \Sigma(\cdot, \cdot, 0)$ and $\widetilde{V_\varepsilon}$, therefore, by assumption

$$Ind\big(\ell, \Sigma(\cdot, \cdot, 0), \overline{U}_{r,\varepsilon}\big) = deg\big(\widetilde{V_\varepsilon}, \overline{U}_{r,\varepsilon}\big) = ind V_\mu \neq 0. \tag{7}$$

Step 4. Let $\mathcal{O} \subset \mathcal{C} \times \mathbb{R}^k$ be an open set defined as

$$\mathcal{O} = \big(\mathcal{C} \times \mathbb{R}^k\big) \setminus \big(\{0\} \times (\mathbb{R}^k \setminus B^k(0, \varepsilon_0))\big).$$

From $Ind\,(\ell, G_r, B_{r,\varepsilon}) \neq 0$ for every $\varepsilon \in (0, \varepsilon_0)$ and for all $r \in (0, \delta_\varepsilon)$ it follows that there exists $(x, \mu) \in B_{r,\varepsilon}$ such that $\ell(x, \mu) \in G_r(x, \mu)$, or equivalently

$$\begin{cases} x \in G(x, \mu), \\ \|x\|_{\mathcal{C}} = r, \end{cases}$$

i.e., $(x, \mu) \in B_{r,\varepsilon}$ is a nontrivial solution to problem (2). Therefore, $(0, 0)$ is a bifurcation point of problem (2), and hence, it is a bifurcation point of problem (1). Denote by $\mathcal{R} \subset \mathcal{S} \cup \{(0, 0)\} \subset \mathcal{O}$ the connected component of $(0, 0)$. Let us demonstrate that

$\mathcal{R}$ is a non-compact component. Assume to the contrary that $\mathcal{R}$ is compact. Then there exists an open bounded subset $U \subset \mathcal{O}$ such that

$$\overline{U} \subset \mathcal{O},\ \mathcal{R} \subset U \text{ and } \partial U \cap \mathcal{S} = \emptyset.$$

Hence, for each $r > 0$

$$\ell(x,\mu) \notin G_r(x,\mu),\ \forall (x,\mu) \in \partial U.$$

Further, for any $0 < r < R$, the compact $CJ-$multimaps G_r and G_R on $\overline{U}$ can be joined by the homotopy $G_{\lambda r+(1-\lambda)R}$. For sufficiently large R,

$$\ell(x,\mu) \notin G_R(x,\mu),\ \forall (x,\mu) \in \overline{U},$$

so, $Ind\,(\ell, G_R, \overline{U}) = 0$. Therefore, $Ind\,(\ell, G_r, \overline{U}) = 0$ for all $r > 0$.
Now, let $\Lambda = \{\mu \in \mathbb{R}^k \colon (0,\mu) \in \overline{U}\}$. From $\overline{U} \subset \mathcal{O}$ it follows that

$$\Lambda \subset B^k(0, \varepsilon_0). \tag{8}$$

From Lemma 1(a) and the continuous dependence of the number δ_ε on ε it follows that we can choose $0 < \varepsilon < \varepsilon_0$ and $0 < r < \delta_\varepsilon$ such that $B_{r,\varepsilon} \subset U$ and inclusion

$$x \in G(x,\mu)$$

has only trivial solutions in the ball $\overline{B_{\mathcal{C}}(0,r)}$ for all $\mu \in \mathbb{R}^k \colon \varepsilon \leq |\mu| < \varepsilon_0$.
From (8) and the choice of r, ε (we can take r, ε sufficiently small) we have

$$Coin(\ell, G_r, \overline{U}) := \{(x,\mu) \in \overline{U} \colon \ell(x,\mu) \in G_r(x,\mu)\} \subset B_{r,\varepsilon}.$$

So, we obtain

$$0 = Ind\,(\ell, G_r, \overline{U}) = Ind\,(\ell, G_r, B_{r,\varepsilon}) \neq 0,$$

that is the contradiction. Thus, $\mathcal{R}$ is a non-compact component, i.e., either $\mathcal{R}$ is unbounded or $\overline{\mathcal{R}} \cap \partial\overline{\mathcal{O}} \neq \emptyset$.■

4. Application to a feedback control system

Consider the following parametrized control system

$$\begin{cases} x'(t) = A(\mu)x(t) + f\big(t, x(t), y(t), \mu\big), \text{ for a.e. } t \in [0,T], \\ y'(t) \in G\big(t, x(t), y(t), \mu\big), \text{ for a.e. } t \in [0,T], \\ y(0) = 0;\ x(0) = x(T), \end{cases} \tag{9}$$

where $f \colon [0,T] \times \mathbb{R}^2 \times \mathbb{R}^m \times \mathbb{R}^2 \to \mathbb{R}^2$ and $G \colon [0,T] \times \mathbb{R}^2 \times \mathbb{R}^m \times \mathbb{R}^2 \to Kv(\mathbb{R}^m)$ $(m \geq 1)$ are given map and multimap, respectively; $\mu \in \mathbb{R}^2$ is the parameter and $A(\mu)$ is a family of $(2\times 2)-$matricies defined as

$$A(\mu) = \begin{pmatrix} 2\mu_1 & 2\mu_2 \\ 2\mu_2 & -2\mu_1 \end{pmatrix},\ \mu = (\mu_1, \mu_2). \tag{10}$$

Here $x \colon [0,T] \to \mathbb{R}^2$ is a trajectory of the system, $y \colon [0,T] \to \mathbb{R}^m$ is a control function. The first equation describes the dynamics of the system and the differential inclusion represents the feedback law.

The use of the family of $(2\times 2)-$matricies $A(\mu)$ of the form (10) comes from [10]. We will use the symbols $\mathcal{W}_T^{1,2}, \mathcal{W}^{1,1}, \mathcal{C}, \mathcal{L}^1, \mathcal{L}^2$ introduced in Section 2 with a remark that in the present section $n = k = 2$.

By a solution to the system we mean a pair $(x, \mu) \in \mathcal{W}_T^{1,2} \times \mathbb{R}^2$ for which there is a function $y \in \mathcal{W}^{1,1}$ such that the triplet (x, y, μ) satisfies (9).

Assume that

($f1$) for every $(u, v, \mu) \in \mathbb{R}^2 \times \mathbb{R}^m \times \mathbb{R}^2$ the function $f(\cdot, u, v, \mu)\colon [0,T] \to \mathbb{R}^2$ is measurable;

($f2$) for a.e. $t \in [0,T]$ the map $f(t, \cdot, \cdot, \cdot)\colon \mathbb{R}^2 \times \mathbb{R}^m \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous;

($f3$) there is a number $c > 0$ such that

$$|f(t, u, v, \mu)| \leq c|u|(|v| + |\mu|),$$

for a.e. $t \in [0,T]$ and for all $(u, v, \mu) \in \mathbb{R}^2 \times \mathbb{R}^m \times \mathbb{R}^2$;

($G1$) for every $(u, v, \mu) \in \mathbb{R}^2 \times \mathbb{R}^m \times \mathbb{R}^2$ the multifunction $G(\cdot, u, v, \mu)\colon [0,T] \to Kv(\mathbb{R}^m)$ has a measurable selection;

($G2$) for a.e. $t \in [0,T]$ the multimap $G(t, \cdot, \cdot, \cdot)\colon \mathbb{R}^2 \times \mathbb{R}^m \times \mathbb{R}^2 \to Kv(\mathbb{R}^m)$ is u.s.c.;

($G3$) there is a number $d > 0$ such that

$$\|G(t, u, v, \mu)\| := \max\{|z| \colon z \in G(t, u, v, \mu)\} \leq d(|u| + |v| + |\mu|),$$

for a.e. $t \in [0,T]$ and for all $(u, v, \mu) \in \mathbb{R}^2 \times \mathbb{R}^m \times \mathbb{R}^2$.

From ($f3$) it follows that $(0, \mu)$ is the solution to (9) for all $\mu \in \mathbb{R}^2$. These solutions are trivial. Denote by $\mathcal{S}$ the set of all nontrivial solutions to (9).

In the sequel we need the following assertions.

Lemma 2 *(Gronwall's Lemma, see, e.g.* [17]*). Let $u, v\colon [a,b] \to \mathbb{R}$ be continuous nonnegative functions; $C \geq 0$ a constant and*

$$v(t) \leq C + \int_a^t u(s)v(s)ds, \quad a \leq t \leq b.$$

Then

$$v(t) \leq Ce^{\int_a^t u(s)ds}, \quad a \leq t \leq b.$$

Lemma 3 *(see Theorem 70.6* [14]*). Let $\mathcal{F}\colon [0,T] \times \mathbb{R}^n \to Kv(\mathbb{R}^n)$ be a multimap such that*

($\mathcal{F}1$) *multifunction $\mathcal{F}(\cdot, w)\colon [0,T] \to Kv(\mathbb{R}^n)$ has a measurable selection for every $w \in \mathbb{R}^n$;*

($\mathcal{F}2$) *multimap $\mathcal{F}(t, \cdot)\colon \mathbb{R}^n \to Kv(\mathbb{R}^n)$ is u.s.c. for a.e. $t \in [0,T]$;*

($\mathcal{F}3$) *there is $b > 0$ such that*

$$\|\mathcal{F}(t, w)\| \leq b(1 + |w|)$$

for all $(t, w) \in [0,T] \times \mathbb{R}^n$.

Then the solution set of the Cauchy problem

$$\begin{cases} u'(t) \in \mathcal{F}(t, u(t)), \ \textit{for a.e. } t \in [0,T], \\ u(0) = u_0, \end{cases}$$

is an R_δ−set in $C([0,T]; \mathbb{R}^n)$.

Lemma 4 *(see Theorem 5.2.5* [20]*). Let $\mathcal{E}$ be a separable Banach space, Λ a metric space, and $\mathcal{F}\colon [0,T] \times \mathcal{E} \times \Lambda \to Kv(\mathcal{E})$ a multimap satisfying the following conditions:*

($\mathcal{F}1$) *multifunction* $\mathcal{F}(\cdot, w, \lambda)\colon [0,T] \to Kv(\mathcal{E})$ *has a measurable selection for every* $(w,\lambda) \in \mathcal{E} \times \Lambda$*;*

($\mathcal{F}2$) *multimap* $\mathcal{F}(t, \cdot, \cdot)\colon \mathcal{E} \times \Lambda \to Kv(\mathcal{E})$ *is u.s.c. for a.e.* $t \in [0,T]$*;*

($\mathcal{F}3$) *there is* $k > 0$ *such that*

$$\|\mathcal{F}(t,w,\lambda)\|_{\mathcal{E}} \le k(1 + \|w\|_{\mathcal{E}} + \|\lambda\|_{\Lambda})$$

for all $(w,\lambda) \in \mathcal{E} \times \Lambda$ *and for a.e.* $t \in [0,T]$*;*

($\mathcal{F}4$) *there exists a function* $\gamma \in L^1_+[0,T]$ *such that*

$$\chi\big(\mathcal{F}(t,\Omega,\Lambda)\big) \le \gamma(t)\,\chi(\Omega)$$

for every nonempty bounded subset $\Omega \subset \mathcal{E}$*, where* χ *denotes the Hausdoff measuare of noncompactness.*

For each $\lambda \in \Lambda$ *denote by* $\Sigma_{u_0}^{\mathcal{F}(\cdot,\cdot,\lambda)}$ *the solution set of the Cauchy problem*

$$\begin{cases} u'(t) \in \mathcal{F}(t,u(t),\lambda) \text{ for a.e. } t \in [0,T], \\ u(0) = u_0 \in \mathcal{E}. \end{cases}$$

Then the multimap $\lambda \to \Sigma_{u_0}^{\mathcal{F}(\cdot,\cdot,\lambda)}$ *is u.s.c. .*

Now let us describe the global structure of the solution set of problem (9).

Theorem 2. *Let conditions* $(f1)-(f3)$ *and* $(G1)-(G3)$ *hold. In addition, assume that*

$$c(1 + Tde^{Td}) < 2.$$

Then $(0,0) \in \mathcal{W}^{1,2}_T \times \mathbb{R}^2$ *is a bifurcation point for solutions of* (9) *and, moreover, there is a unbounded subset* $\mathcal{R} \subset \mathcal{S}$ *such that* $(0,0) \in \overline{\mathcal{R}}$*.*

Proof. For a fixed $(x,\mu) \in \mathcal{C} \times \mathbb{R}^2$ consider the following multimap

$$G_{(x,\mu)}\colon [0,T] \times \mathbb{R}^m \to Kv(\mathbb{R}^m),\ G_{(x,\mu)}(t,z) = G(t,x(t),z,\mu).$$

It is easy to verify that $G_{(x,\mu)}$ satisfies all conditions of Lemma 3. So, for every $(x,\mu) \in \mathcal{C} \times \mathbb{R}^2$ the set $\Psi_{(x,\mu)}$ of all solutions to the problem

$$\begin{cases} y'(t) \in G(t,x(t),y(t),\mu) \text{ for a.e. } t \in [0,T] \\ y(0) = 0 \end{cases}$$

is an $R_\delta-$set in $C\big([0,T];\mathbb{R}^m\big)$.

Moreover, for each $r > 0$ take $\Lambda = B_{\mathcal{C}}(0,r) \times B^2(0,r)$, and consider the multimap

$$\Pi\colon [0,T] \times \mathbb{R}^m \times \Lambda \to Kv(\mathbb{R}^m),\ \Pi(t,w,x,\mu) = G_{(x,\mu)}(t,w).$$

It is easy to verify that multimap Π satisfies conditions $(\mathcal{F}1) - (\mathcal{F}4)$ in Lemma 4. Therefore, the multimap $(x,\mu) \to \Psi_{(x,\mu)}$ is u.s.c. at all points $(x,\mu) \in \Lambda$. Since we can choose arbitrarily $r > 0$, so if we define the multimap

$$\Psi\colon \mathcal{C} \times \mathbb{R}^2 \to K\big(C([0,T];\mathbb{R}^m)\big),\ \Psi(x,\mu) = \Psi_{(x,\mu)},$$

then it is u.s.c., too.

Now define the following maps and multimaps

$$\widetilde{\Psi}\colon \mathcal{C} \times \mathbb{R}^2 \to K\big(\mathcal{C} \times C([0,T];\mathbb{R}^m) \times \mathbb{R}^2\big),\ \widetilde{\Psi}(x) = \{x\} \times \Psi(x,\mu) \times \{\mu\},$$

$$B\colon \mathcal{C}\times C([0,T];\mathbb{R}^m)\times\mathbb{R}^2\to\mathcal{L}^2,$$
$$B(x,y,\mu)(t)=A(\mu)x(t)+f(t,x(t),y(t),\mu),\ t\in[0,T].$$

Then problem (9) can be written in the form

$$Lx\in\mathcal{Q}(x,\mu), \tag{11}$$

where $L\colon\mathcal{W}_T^{1,2}\to\mathcal{L}^2$ is the differentiation operator and

$$\mathcal{Q}\colon\mathcal{C}\times\mathbb{R}^2\to K(\mathcal{L}^2),\ \mathcal{Q}(x,\mu)=B\circ\widetilde{\Psi}(x,\mu).$$

It is clear that $\mathcal{Q}$ is a $CJ-$multimap, and hence, condition $(Q1)$ is fulfilled.

Let $\Omega\subset\mathcal{C}\times\mathbb{R}^2$ be a bounded subset and (x,μ) an arbitrary point in Ω. For $g\in\mathcal{Q}(x,\mu)$ there is $y\in\Psi(x,\mu)$ such that

$$g(t)=A(\mu)x(t)+f\big(t,x(t),y(t),\mu\big)\ \text{for a.e. } t\in[0,T]. \tag{12}$$

Since $y\in\Psi(x,\mu)$, there exists $h\in\mathcal{L}^1$ such that $h(t)\in G\big(t,x(t),y(t),\mu\big)$ for a.e. $t\in[0,T]$ and

$$y(t)=\int_0^t h(s)ds,\ \text{for all } t\in[0,T].$$

From $(G3)$ it follows that

$$\begin{aligned}|y(t)|&\le\int_0^t|h(s)|ds\le d\int_0^t\big(|x(s)|+|y(s)|+|\mu|\big)ds\\&\le dT(\|x\|_{\mathcal{C}}+|\mu|)+\int_0^t d|y(s)|ds.\end{aligned}$$

According to Lemma 2 we obtain

$$|y(t)|\le Td(\|x\|_{\mathcal{C}}+|\mu|)e^{Td}\ \text{for all } t\in[0,T]. \tag{13}$$

Now, the boundedness of the set $\mathcal{Q}(\Omega)$ follows from (12)-(13) and $(f3)$. So, condition $(Q2)$ also holds true.

Let us show that the family of functions $V_\mu\colon\mathbb{R}^2\to\mathbb{R}$,

$$V_\mu(w)=\mu_1w_1^2+2\mu_2w_1w_2-\mu_1w_2^2,\quad w=(w_1,w_2),\ \mu=(\mu_1,\mu_2),$$

is a family of local integral guiding functions for (11) at $(0,0)$. In fact, let $\varepsilon>0$ and $\mu\in S^1(0,\varepsilon)$ be arbitrary. For $x\in\mathcal{C}$ take any $g\in\mathcal{Q}(x,\mu)$. Then there is $y\in\Psi(x,\mu)$ such that

$$g(t)=A(\mu)x(t)+f\big(t,x(t),y(t),\mu\big)\ \text{for a.e. } t\in[0,T].$$

By virtue of $(f3)$ and (13) we have

$$\int_0^T\big\langle\nabla V_\mu(x(t)),g(t)\big\rangle dt=\int_0^T\big\langle A(\mu)x(t),A(\mu)x(t)+f\big(t,x(t),y(t),\mu\big)\big\rangle dt$$
$$\ge\int_0^T|A(\mu)x(t)|^2dt-\int_0^T|A(\mu)x(t)||f(t,x(t),y(t),\mu)|dt$$
$$\ge 4|\mu|^2\|x\|_2^2-2|\mu|\int_0^T|x(t)|\,c\,|x(t)|\big(|y(t)|+|\mu|\big)dt$$

$$\begin{aligned}
&\geq 4|\mu|^2\|x\|_2^2 - 2c|\mu| \int_0^T |x(t)|^2 \big(Td(\|x\|_{\mathcal{C}} + |\mu|)e^{Td} + |\mu|\big)dt = \\
&= 4|\mu|^2\|x\|_2^2 - 2c|\mu|\,\|x\|_2^2\,\big(Tde^{Td}\|x\|_{\mathcal{C}} + Tde^{Td}|\mu| + |\mu|\big) \\
&= 2|\mu|^2\|x\|_2^2\big(2 - c(1 + Tde^{Td})\big) - 2cTde^{Td}|\mu|\,\|x\|_2^2\,\|x\|_{\mathcal{C}} \\
&= 2|\mu|\,\|x\|_2^2\Big(|\mu|\big(2 - c(1 + Tde^{Td})\big) - cTde^{Td}\|x\|_{\mathcal{C}}\Big).
\end{aligned} \quad ,$$

Therefore,

$$\int_0^T \big\langle \nabla V_\mu(x(t)), g(t)\big\rangle dt > 0$$

provided

$$0 < \|x\|_{\mathcal{C}} < \frac{|\mu|\big(2 - c(1 + Tde^{Td})\big)}{cTde^{Td}} = \frac{\varepsilon\big(2 - c(1 + Tde^{Td})\big)}{cTde^{Td}}. \tag{14}$$

Thus, V_μ is a family of local integral guiding functions for problem (11) at $(0,0)$.

For each $\varepsilon > 0$, choose δ_ε such that

$$\delta_\varepsilon = \frac{1}{2}\frac{\varepsilon\big(2 - c(1 + Tde^{Td})\big)}{cTde^{Td}}.$$

Set $O_\varepsilon = B^4\big(0, \sqrt{\varepsilon^2 + \delta_\varepsilon^2}\,\big)$ and consider the map

$$\widetilde{V_\varepsilon} \colon \overline{O_\varepsilon} \to \mathbb{R}^3,$$

$$\begin{aligned}
\widetilde{V_\varepsilon}(w,\mu) &= \big\{ -\nabla V_\mu(w), \varepsilon^2 - |\mu|^2 \big\} \\
&= \big\{ -(2\mu_1 w_1 + 2\mu_2 w_2), -(2\mu_2 w_1 - 2\mu_1 w_2), \varepsilon^2 - |\mu|^2 \big\},
\end{aligned}$$

Let us show that $indV_\mu \neq 0$. Toward this goal, consider the following continuous map

$$H \colon \overline{O_\varepsilon} \times [0,1] \to \mathbb{R}^3,$$

$$H(w,\mu,\lambda) = \{-\nabla V_\mu(w), \lambda|w|^2 + (1-\lambda)\varepsilon^2 - |\mu|^2\}.$$

Assume that there exists $(w,\mu,\lambda) \in \partial O_\varepsilon \times [0,1]$ such that $H(w,\mu,\lambda) = 0$, then we have

$$\begin{cases}
-\nabla V_\mu(w) = 0, \\
\lambda|w|^2 - |\mu|^2 = (\lambda - 1)\varepsilon^2, \\
|w|^2 + |\mu|^2 = \varepsilon^2 + \delta_\varepsilon^2.
\end{cases}$$

From the second and third equations of the above system it follows that

$$|w|^2 = \frac{\lambda\varepsilon^2 + \delta_\varepsilon^2}{1+\lambda} \quad \text{and} \quad |\mu|^2 = \frac{\varepsilon^2 + \lambda\delta_\varepsilon^2}{1+\lambda}.$$

Therefore, w and μ are non-zero elements in $\mathbb{R}^2$. That contradicts to the first equation of the system. Thus, H is a homotopy connecting the maps $\widetilde{V_\varepsilon} = H(\cdot,\cdot,0)$ and $H(\cdot,\cdot,1)$. By the homotopy invariance property of the topological degree we obtain

$$deg(\widetilde{V_\varepsilon}, \overline{O_\varepsilon}) = deg\big(H(\cdot, \cdot, 1), \overline{O_\varepsilon}\big).$$

On the other hand, the map $H(\cdot,\cdot,1)\colon \mathbb{R}^4 \to \mathbb{R}^3$ vanishes only at $(0,0)$ and the restriction $h = -H(\cdot,\cdot,1)_{|_{S^3}} : S^3 \to S^2$ is the Hopf fibration (see, e.g. [18]). Hence,

$$deg\big(H(\cdot,\cdot,1),\overline{O_\varepsilon}\big) = deg\big(H(\cdot,\cdot,1),\overline{B^4}\big) = deg(-h,\overline{B^4}) = \Sigma[-h] \neq 0,$$

where Σ is defined in Example 1.
Therefore, $(0,0)$ is a bifurcation point for solutions of (9). Moreover, from the fact that relation (14) holds true for all $\varepsilon > 0$ it follows that $(0,0)$ is the unique bifurcation point for solutions of (9). Now, the application of Theorem 1 ends the proof. ■

Acknowledgment. The work on the paper was carried out during Dr. Nguyen Van Loi's and Professor V. Obukhovskii's visit to the Department of Applied Mathematics, National Sun Yat-Sen University, and the Center of Fundamental Science, Kaohsiung Medical University, Kaohsiung, Taiwan, in 2014. They would like to express their gratitude to the members of the Department of Applied Mathematics and the Center of Fundamental Science for their kind hospitality. The work of the second author is supported by NNS-RFBR grant 14-01-92004 and RFBR grant 14-01-00468.

References

[1] J.C. Alexander and P.M. Fitzpatrick, *Global bifurcation for solutions of equations involving several parameter multivalued condensing mappings*, in: Fixed Point Theory (Sherbrooke, Quebec 1980, E. Fadell, G. Fournier-Eds.) , Lecture Notes in Mathematics, 886, Springer, New York 1981, pp. 1-19.

[2] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces.* Noordhoff International Publishing, Leyden, 1976.

[3] Yu.G. Borisovich, B.D. Gelman, A.D. Myshkis, V.V. Obukhovskii, *Topological methods in the theory of fixed points of multivalued mappings.* (in Russian) Uspekhi Mat. Nauk, **35**(1980), 59-126. Translation: Russian Math. Surveys, **35**(1980), 65-143.

[4] Y.G. Borisovich, B.D. Gelman, A.D. Myshkis, V.V. Obukhovskii, *Introduction to the Theory of Multivalued Maps and Differential Inclusions*, Librokom, Moscow, 2011.

[5] K. Borsuk, *Theory of Retracts*, Monografie Mat. 44, PWN, Warszawa, 1967 (in Russian).

[6] Z. Denkowski, S. Migórski, N. S. Papageorgiou, *An Introduction to Nonlinear Analysis: Theory*, Kluwer Academic Publ., Boston, 2003.

[7] D. Gabor, *The coincidence index for fundamentally contractible multivalued maps with nonconvex values*, Ann. Polon. Math., **75**(2000), 143-166.

[8] D. Gabor and W. Kryszewski, *A coincidence theory involving Fredholm operators of nonnegative index*, Topol. Meth. Nonl. Anal., **15**(2000), 43-59.

[9] D. Gabor and W. Kryszewski, *Systems of inclusions involving Fredholm operators of nonnegative index and nonconvex-valued maps*, Set-Valued Anal., **13**(2005), 337-379.

[10] D. Gabor and W. Kryszewski, *A global bifurcation index for set-valued perturbations of Fredholm operators*, Nonlinear Anal., **73**(2010), 2714-2736.

[11] D. Gabor and W. Kryszewski, *Alexander invariant for perturbations of Fredholm operators*, Nonlinear Anal., **74**(2011), 6911-6932.

[12] R.E. Gaines, J.L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Lecture Notes in Mathematics, no. 568, Springer-Verlag, Berlin-New York, 1977.

[13] L. Górniewicz, A. Granas, W. Kryszewski, *On the homotopy method in the fixed point index theory of multi-valued mappings of compact absolute neighborhood retracts*, J. Math. Anal. Appl., **161**(1991), 457-473.

[14] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Springer, Dordrecht, 2006.

[15] L. Górniewicz, W. Kryszewski, *Topological degree theory for acyclic mappings related to the bifurcation problem*, Boll. Un. Mat. Ital. B, **6(7)**(1992), 579-595.
[16] L. Górniewicz, W. Kryszewski, *Bifurcation invariants for acyclic mappings*, Rep. Math. Phys., **31**(1992), 217-239.
[17] Ph. Hartman, *Ordinary Differential Equations*, Corrected reprint of the second edition (1982) [Birkha"user, Boston, MA], Classics in Applied Mathematics, 38. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2002.
[18] S.T. Hu, *Homotopy Theory*, Academic Press, New York, 1959.
[19] D. M. Hyman, *On decreasing sequences of compact absolute retracts*, Fund Math., **64**(1969), 91-97.
[20] M. Kamenskii, V. Obukhovskii, P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, W. de Gruyter Series in Nonlinear Analysis and Applications 7, Walter de Gruyter, Berlin-New York 2001.
[21] W. Kryszewski, *Homotopy properties of set-valued mappings*, Univ. N. Copernicus Publishing, Torun, 1997.
[22] W. Kryszewski, R. Skiba, *A cohomoligical index of Fuller type for set-valued dynamical systems*, Nonlinear Anal., **75**(2012), 684-716.
[23] A.D. Myshkis, *Generalizations of the theorem on a fixed point of a dynamical system inside of a closed trajectory*, (in Russian) Mat. Sb., **34**(1954), no. 3, 525-540.
[24] V. Obukhovskii, P. Zecca, N.V. Loi and S. Kornev, *Method of Guiding Functions in Problems of Nonlinear Analysis*, Lecture Notes in Math. 2076, Springer-Velag, Berlin-Heidelberg, 2013.
[25] E.H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

Received: 19 April 2014; Accepted: May 11, 2014.