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CHARACTERIZATIONS OF COMMON FIXED POINTS OF ONE-PARAMETER NONEXPANSIVE SEMIGROUPS

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Abstract. We prove characterizations of common fixed points of one-parameter nonexpansive semigroups. These results are sharper than some previous results.
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1. Introduction

Throughout this paper we denote by N the set of all positive integers. For a real number t, we denote by $[t]$ the maximum integer not exceeding t.

Let E be a real Banach space. We denote by E^* the dual of E. Let T be a nonexpansive mapping on a subset C of E, i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. $F(T)$ is denoted by the set of all fixed points of T. In 1965, Kirk [1] proved that T has a fixed point when C is weakly compact and convex, and has normal structure. See also [4,6,11,12] and others. C is said to have the fixed point property for nonexpansive mappings (FPP, for short) if for every bounded closed convex subset D of C, every nonexpansive mapping on D has a fixed point. That is, a weakly compact convex subset C with normal structure has FPP.

A family of mappings $\{T(t) : t \geq 0\}$ is called a one-parameter strongly continuous semigroup of nonexpansive mappings (nonexpansive semigroup, for short) on C if the following are satisfied:

(A1) for each $t \geq 0, T(t)$ is a nonexpansive mapping on C;

(A2) $T(s + t) = T(s) \circ T(t)$ for all $s, t \geq 0$;

(A3) for each $x \in C$, the mapping $t \mapsto T(t)x$ from $[0, \infty)$ into C is continuous.

Bruck's famous fixed point theorem in [8] yields that $\{T(t) : t \geq 0\}$ has a common fixed point when C is weakly compact and convex, and has FPP. See also Browder [6]. A remarkable application of fixed-point theorems is to prove the existence of fixed-points in best approximation (see [3-5]), which has special significance for the spaces that are not strictly convex (see [5]). As generalization of fixed-points, common fixed-points of two maps f and g satisfying some contractive or nonexpansive type condition have been studied by many authors and applied to various problems, especially to those associated with best approximation; see [2-5,13] and others.

Let C be a closed convex subset of a Banach space E and T a nonexpansive mapping of C into itself. Halpern [3] introduced the following iterative scheme for approximating a fixed point of T :

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n \tag{1.1}$$

for all $n \in N$, where $x_1 = x \in C$ and $\{\alpha_n\}$ is a sequence of [0, 1]. In [11], the authors introduce the following iterative sequence:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_n x_n \tag{1.2}$$

for all $n \in N$, where $x_1 = x \in C$ and $\{\alpha_n\}$ is a sequence of [0, 1], C is a nonempty closed convex subset of a Banach space, and $\{T_n : n \in N\}$ is a sequence of nonexpansive mappings with some conditions.

In this paper, motivated by these results related to Halpern type iterative schemes, we introduce the following iterative sequence:

$$x_{n+1} = \alpha_n P x_n + (1 - \alpha_n) Q x_n \tag{1.3}$$

for all $n \in N$, where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence of [0, 1], $Px = \int_0^\tau \eta(s)T(s)xds$, $Qx = \sum_{j=1}^\infty \theta(j)T(\tau_j)x$, $\{T(t) : t \geq 0\}$ is a sequence of nonexpansive mappings, and C is a nonempty closed convex subset of a Banach space E. Then we prove that $\{x_n\}$ defined by (1.3) converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$. Further we apply our result to the problem of finding a common fixed point of a countable or uncountable family of nonexpansive mappings.

2. Main results

In this section, using the Bochner integral, we present characterizations of common fixed points of nonexpansive semigroups.

Theorem 2.1. *Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on a subset C of a Banach space E. Let η be a continuous function from $[0, \tau]$ into $[0, \infty)$ such that $\int_0^\tau \eta(s)ds = \alpha$, where τ is some positive real number, and let $\{\tau_j\}$ be a sequence in $[0, \infty)$ such that the closure of the set $\{\tau_j : j \in N\}$ is $[0, \tau]$. Let θ be a function from N into $[0, \infty)$ satisfying $\sum_{j=1}^\infty \theta(j) = \beta$. Let $Px = \int_0^\tau \eta(s)T(s)xds, Qx = \sum_{j=1}^\infty \theta(j)T(\tau_j)x$. Define a nonexpansive mapping S from C into E by*

$$Sx = Px + Qx = \int_0^\tau \eta(s)T(s)xds + \sum_{j=1}^\infty \theta(j)T(\tau_j)x \tag{2.1}$$

for $x \in C$. If $\alpha + \beta = 1$, then

$$F(S) = \bigcap_{t \geq 0} F(T(t)) \tag{2.2}$$

holds.

Proof. We note that S is well defined because of (A3). For $x, y \in C$, we have

$$\begin{aligned}\|Sx - Sy\| &= \left\| \int_0^\tau \eta(s)\Big(T(s)x - T(s)y\Big)ds + \sum_{j=1}^{\infty}\theta(j)\Big(T(\tau_j)x - T(\tau_j)y\Big)\right\| \\ &\le \int_0^\tau \eta(s)\|T(s)x - T(s)y\|ds + \sum_{j=1}^{\infty}\theta(j)\|T(\tau_j)x - T(\tau_j)y\| \\ &\le \int_0^\tau \eta(s)\|x-y\|ds + \sum_{j=1}^{\infty}\theta(j)\|x-y\| \\ &= (\alpha+\beta)\|x-y\| = \|x-y\|,\end{aligned}$$

and hence S is nonexpansive. We first show that

$$F(S) \supset \bigcap_{t\ge 0} F(T(t)). \tag{2.3}$$

Let $x \in \bigcap_{t\ge 0} F(T(t))$, then

$$\begin{aligned}Sx &= \int_0^\tau \eta(s)T(s)xds + \sum_{j=1}^{\infty}\theta(j)T(\tau_j)x = \int_0^\tau \eta(s)xds + \sum_{j=1}^{\infty}\theta(j)x \\ &= \alpha x + \beta x = x\end{aligned}$$

then $x \in F(s)$, so (2.3) holds.

In the following, we prove the converse inclusion. Let $z \in C$ be a fixed point of S. Since $\{T(t)z : t \in [0,\tau]\}$ is compact, there exists $\mu \in [0,\tau]$ such that

$$\|T(\mu)z - z\| = \max_{t\in[0,\tau]} \|T(t)z - z\|.$$

We assume $\delta := \|T(\mu)z - z\| > 0$. From the Hahn-Banach theorem, there exists $f \in E^*$ with

$$\|f\| = 1 \text{ and } f(T(\mu)z - z) = \|T(\mu)z - z\| = \delta.$$

Since $t \mapsto T(t)z$ is continuous, there exist $\mu_1, \mu_2 \in [0,\tau]$ with

$$\mu_1 < \mu_2, \mu \in [\mu_1, \mu_2] \text{ and } \|T(\mu)z - T(t)z\| \le \delta/2$$

for $t \in [\mu_1, \mu_2]$. Then we have

$$\begin{aligned}
\delta &= \|T(\mu)z - z\| = f(T(\mu)z - z) = f(T(\mu)z - Sz) \\
&= \int_0^\tau \eta(s) f(T(\mu)z - T(s)z) ds + \sum_{j=1}^\infty \theta(j) f(T(\mu)z - T(\tau_j)z) \\
&\le \int_0^\tau \eta(s) \|T(\mu)z - T(s)z\| ds + \sum_{j=1}^\infty \theta(j) \|T(\mu)z - T(\tau_j)z\| \\
&= \int_0^{\mu_1} \eta(s) \|T(\mu)z - T(s)z\| ds + \int_{\mu_1}^{\mu_2} \eta(s) \|T(\mu)z - T(s)z\| ds \\
&\quad + \int_{\mu_2}^{\tau} \eta(s) \|T(\mu)z - T(s)z\| ds + \sum \{\theta(j) \|T(\mu)z - T(\tau_j)z\| : \tau_j \in [\mu_1, \mu_2]\} \\
&\quad + \sum \{\theta(j) \|T(\mu)z - T(\tau_j)z\| : \tau_j \notin [\mu_1, \mu_2]\} \\
&= \int_0^{\mu_1} \eta(s) \|T(s) \circ T(\mu - s)z - T(s)z\| ds + \int_{\mu_1}^{\mu_2} \eta(s) \|T(\mu)z - T(s)z\| ds \\
&\quad + \int_{\mu_2}^{\tau} \eta(s) \|T(\mu)z - T(\mu) \circ T(s - \mu)z\| ds \\
&\quad + \sum \{\theta(j) \|T(\mu)z - T(\tau_j)z\| : \tau_j \in [\mu_1, \mu_2]\} \\
&\quad + \sum \{\theta(j) \|T(\mu)z - T(\tau_j)z\| : \tau_j \notin [\mu_1, \mu_2]\} \\
&\le \int_0^{\mu_1} \eta(s) \|T(\mu - s)z - z\| ds + \int_{\mu_1}^{\mu_2} \eta(s) \frac{\delta}{2} ds + \int_{\mu_2}^{\tau} \eta(s) \|T(s - \mu)z - z\| ds \\
&\quad + \sum \{\theta(j) \frac{\delta}{2} : \tau_j \in [\mu_1, \mu_2]\} + \sum \{\theta(j) \|T(|\mu \tau_j|)z - z\| : \tau_j \notin [\mu_1, \mu_2]\} \\
&\le \int_0^{\mu_1} \eta(s) \delta ds + \int_{\mu_1}^{\mu_2} \eta(s) \frac{\delta}{2} ds + \int_{\mu_2}^{\tau} \eta(s) \delta ds + \sum \{\theta(j) \frac{\delta}{2} : \tau_j \in [\mu_1, \mu_2]\} \\
&\quad + \sum \{\theta(j) \delta : \tau_j \notin [\mu_1, \mu_2]\} \\
&\le \alpha \delta - \frac{\delta}{2} \int_{\mu_1}^{\mu_2} \eta(s) ds + \beta \delta - \frac{\delta}{2} \sum \{\theta(j) : \tau_j \in [\mu_1, \mu_2]\} \\
&= (\alpha + \beta) \delta - \frac{\delta}{2} \left(\int_{\mu_1}^{\mu_2} \eta(s) ds + \sum \{\theta(j) : \tau_j \in [\mu_1, \mu_2]\} \right) \\
&< \delta.
\end{aligned}$$

This is a contradiction. Therefore $\delta = 0$. This implies that z is a common fixed point of $\{T(t) : t \ge 0\}$. This completes the proof.

As a direct consequence of Theorem 2.1, we obtain the following:

Corollary 2.1. *Let $\{T(t) : t \ge 0\}$ be a sequence of nonexpansive mappings, and C a nonempty closed convex subset of a Banach space E, then the following iterative sequence:*

$$x_{n+1} = \alpha_n P x_n + (1 - \alpha_n) Q x_n \tag{2.4}$$

converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$*, where* $x_1 \in C$ *and* $\{\alpha_n\}$ *is a sequence of [0, 1],* $Px = \int_0^\tau \eta(s)T(s)xds, Qx = \sum_{j=1}^\infty \theta(j)T(\tau_j)x$.

Corollary 2.2. *Let* $\{T(t) : t \geq 0\}$ *be a nonexpansive semigroup on a subset* C *of a Banach space* E *and Let* γ *be an irrational number. Fix* $\tau > 0$ *and define a nonexpansive mapping* S *from* C *into* E *by*

$$Sx = \frac{1}{2\tau}\int_0^\tau T(s)xds + \sum_{j=1}^\infty \frac{1}{2^{j+1}}T(j\gamma - [j\gamma]\tau)x \tag{2.5}$$

for $x \in C$*. Then (2.2) holds.*

If we let $\alpha = 1, \beta = 0$ in Theorem 2.1, we obtain the following:

Corollary 2.3. *Let* $\{T(t) : t \geq 0\}$ *be a nonexpansive semigroup on a subset* C *of a Banach space* E*. Fix* $\tau > 0$ *and define a nonexpansive mapping* S *from* C *into* E *by*

$$Sx = \frac{1}{\tau}\int_0^\tau T(s)xds \tag{2.6}$$

for $x \in C$*. Then (2.2) holds.*

If we let $\alpha = 0, \beta = 1$ in Theorem 2.1, we obtain the following:

Corollary 2.4. *Let* $\{T(t) : t \geq 0\}$ *be a nonexpansive semigroup on a subset* C *of a Banach space* E *and let* γ *be an irrational number. Fix* $\tau > 0$ *and define a nonexpansive mapping* S *from* C *into* E *by*

$$Sx = \sum_{j=1}^\infty \frac{1}{2^j}T(j\gamma - [j\gamma]\tau)x \tag{2.7}$$

for $x \in C$*. Then (2.2) holds.*

Remark 2.1. In this paper, we introduce the following iterative sequence:

$$x_{n+1} = \alpha_n Px_n + (1 - \alpha_n)Qx_n$$

which is sufficiently general to cover and improve the two following iterative sequences:

$$x_{n+1} = \lambda Px_n + (1 - \lambda)x_n, \tag{2.8}$$

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_n x_n \tag{2.9}$$

by letting $Q = I$ and $P = I$ respectively, where I denotes the identity map. Note that (2.8) was introduced in Theorem 5 in [12] and (2.9) in Theorem 3.4 in [11]. In fact, by Theorem 2.1, if we write $T_n = \sum_{j=1}^n \theta(j)T(\tau_j)$, we can easily obtain the iterative sequence (2.9). Therefore, our results in this paper generalize the results in [11,12].

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