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FIXED POINT THEOREMS FOR 1-SET CONTRACTIONS IN BANACH SPACES

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Abstract. This work presents some new fixed point theorems for 1-set contractions in Banach spaces. We first prove an existence and uniqueness result for a 1-set contraction mapping f when $I - f$ is ψ-expansive, extending [13, Proposition 3.4]. More generally, when $I - f$ is α-ψ-expansive, an existence result for 1-set contraction mappings is obtained. We then derive several fixed point results for the sum (in Banach spaces) and the product (in Banach algebras) of two operators, one of them is completely continuous and the other one is a 1-set contraction. In this context, the Furi-Pera boundary condition is investigated and comparison with recent results is given. Finally, [13, Proposition 3.4] is obtained as a consequence of a compactness result proved in [8]. The proofs essentially use the properties of the Kuratowski measure of noncompactness.
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1. Introduction

Nonexpansive mappings appear in many nonlinear integral equations for which the Banach fixed point theorem of contraction mappings fails (see [17] for extension and equivalence conditions on contractivity conditions). However, the fixed point theory for nonexpansive mappings is well developed in the literature (see, e.g., [10, 27, 28] and references therein) and it is well known that the geometric properties of the Banach space play a key role in such a theory (see [14]). As for the fixed point theory for k-set contraction mappings, it started in 1955 when G. Darbo [5] generalized the Schauder fixed point theorem using the measure of noncompactness introduced by the polish mathematician Kuratowskii around 1930. Later in 1967, Sadovskii [24] introduced the notion of condensing mappings which are more general than k-set contractions and he established a fixed point theorem on a bounded convex subset of a Banach space. Indeed, the class of nonexpansive mappings appear as a special case of 1-set contractions while compact operators are 0-set contractions. Moreover, notice that the case of 1-set contractions turns out to be very interesting and useful since this class of mappings encompasses completely continuous, condensing as well as nonexpansive mappings. In 1973, Petryshyn [23] established some new theorems and surveyed the fixed point theory for 1-set contractions satisfying the classical Leray-Schauder boundary condition. Recently, some interesting results have also been obtained, e.g., in [20] for convex-power 1-set contractions, in [25] for demi-closed 1-set contractions with sublinear growth and in [26] for demi-closed 1-set contractions satisfying some boundary conditions.

Our objective in this paper is to contribute in this theory by investigating the existence of some fixed points for some of 1-set contraction mappings defined in Banach spaces. In Section 2, we start with some notions and definitions used in this paper, including some notations and preliminary lemmas. In Section 3, we present our first result of existence and uniqueness of a fixed point. Let X be a Banach space, Q a nonempty convex closed subset of X, and $f : Q \longrightarrow X$ a mapping. Motivated by the recent work of J. García-Falset [13] for nonexpansive mappings, we shall prove Proposition 3.1 which generalizes Proposition 3.4 in [13]; here we consider a 1-set contraction f (always assumed continuous throughout this work) instead of a nonexpansive map and suppose that $I - f$ is ψ-expansive. When f is a 1-set contraction mapping while $I - f$ is rather what we call α-ψ-expansive, we obtain an existence result without uniqueness; here α denoted the measure of noncompactness of Kuratowski. Proposition 3.1 allows us to prove the existence of fixed points for the sum (in Banach spaces, Sec. 3.3) and the product (in Banach algebras, Sec. 3.4) of two operators when one of them is completely continuous and the other one is a 1-set contraction. We end this section with some existence results for accretive and dissipative operators and with an example of application to the study of an integral equation. Section 4 is devoted to similar results for the class of 1-set contractions $f : Q \longrightarrow X$ satisfying the well-known Furi-Pera boundary condition in case $I - f$ is ψ-expansive. In Section 5, we recapture Lemma 3.3 given in [13] by another method showing that it can be obtained as a consequence of a compactness result proved in [8] and then we conclude.

2. Preliminaries

Let X be a Banach space. We say that a mapping $f : X \longrightarrow X$ is bounded if it maps bounded sets into bounded sets.

Definition 2.1. Let $f : X \longrightarrow X$ be a mapping.

(a) f is called $\mathcal{D}$-Lipschitzian with $\mathcal{D}$-function ϕ if there exists a continuous nondecreasing function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\phi(0) = 0$ and

$$\|f(x) - f(y)\| \leq \phi(\|x - y\|),\ \forall\, (x, y) \in X^2.$$

(b) If further $\phi(r) < r$, $\forall\, r > 0$, f is called a nonlinear contraction.

(c) In particular, if $\phi(r) = kr$ for some constant $0 < k < 1$, then f is a contraction.

(d) f is said to be a nonexpansive map if $\phi(r) = r$, that is

$$\|f(x) - f(y)\| \leq \|x - y\|,\ \forall\, (x, y) \in X^2.$$

A more precise definition of a nonlinear contraction is given by:

Definition 2.2. A mapping $f : X \longrightarrow X$ is called a ϕ-contraction if there exists a function $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying at least one of the following two conditions:

(a) either ϕ is continuous and $\phi(t) < t,\ t > 0$ $(\phi(t) \leq t,\ t > 0)$,

(b) or there exists a nondecreasing function $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\psi(0) = 0$ and $0 < \psi(r) \leq r - \phi(r),\ r > 0$ $(0 \leq \psi(r) \leq r - \phi(r),\ r > 0)$ such that

$$\|f(x) - f(y)\| \leq \phi(\|x - y\|),\ \forall\, (x, y) \in X^2.$$

We point out that in the survey paper [17], the authors present a detailed analysis of several contraction conditions, including the one given in the above definition. Also, they prove equivalence of some contractivity properties in metric spaces and give some approximates fixed point point theorems with application to the domain invariance theorem for contractive fields.

Arguing as in [7, Lemma 3.1], we can prove that

Lemma 2.1. *Every ϕ-contraction mapping is bounded.*

Definition 2.3. (see, e.g., [13, Definition 2.5]) An operator $f : D(f) \subset X \longrightarrow X$ is said to be ψ-expansive if there exists a function $\psi : [0, \infty) \longrightarrow [0, \infty)$ such that for every $x,\ y \in D(f)$

$$\|fx - fy\| \geq \psi(\|x - y\|),\ \forall\, (x, y) \in X^2$$

with

(a) $\psi(0) = 0$ and $\psi(r) > 0,\ \forall\, r > 0$,

(b) ψ is either continuous or nondecreasing.

The following interesting existence and uniqueness result (with several consequences) was recently proved by J. García-Falset (see also [[1], Corollary 2.25]):

Proposition 2.2. [13, Proposition 3.4] *Let X be a Banach space, Q a nonempty closed bounded convex subset, and $f : Q \longrightarrow Q$ a mapping. If f is nonexpansive and $I - f$ is ψ-expansive, then f has a unique fixed point in Q.*

Throughout this paper, we will make use of the Kuratowski measure of noncompactness ($\alpha - MNC$ for short). For the main properties of the $\alpha - MNC$, we refer the reader to [3, 4, 6].

Now a natural extension of Definition 2.1 is given by the following

Definition 2.4. Let X, Y be two Banach spaces and $f : X \longrightarrow Y$ a continuous bounded mapping.

(a) f is said to be α-Lipschitz if there exists a function $\phi = \phi_f : [0, +\infty) \longrightarrow [0, +\infty)$ such that

$$\alpha(f(\Omega)) \leq \phi_f(\alpha(\Omega)),$$

for every bounded subset $\Omega \subset X$.

(b) f is called k-set contraction (with respect to α) if there exists some $k \geq 0$ such that $\phi(r) = kr$, i.e.

$$\alpha(f(\Omega)) \leq k\alpha(\Omega),$$

for every bounded subset $A \subset X$.

(c) f is a strict k-set contraction whenever $k < 1$.

(d) f is said to be α-condensing (or densifying) whenever

$$\alpha(f(A)) < \alpha(A),$$

for every bounded subset $A \subset X$ with $\alpha(A) \neq 0$.

1-set contractions are also called α-nonexpansive mappings since they extend nonexpansive mappings. As an example of a k-set contraction, one can consider the sum of a contraction and a compact mapping. The following Darbo fixed point theorem [1955] concerns α-contraction mappings; it will be used in this work. This theorem has been extended to condensing mappings in 1967 by Sadovskii [24].

Lemma 2.3. [5] *(See also* [2, Theorem 4.16]*) Let Q be a bounded closed convex subset of Banach space X and $f : Q \longrightarrow Q$ a k-set contraction mapping with $k < 1$. Then f has a fixed point.*

Before closing this section, we introduce

Definition 2.5. f is called α-ψ-expansive if there exists a function $\psi : [0, \infty) \longrightarrow [0, \infty)$ with $\psi(0) = 0$, $\psi(r) > 0$, $\forall\, r > 0$ and such that

$$\alpha(f(\Omega)) \geq \psi(\alpha(\Omega)),$$

for every bounded subset $\Omega \subset X$.

Definition 2.6. Let $f : Q \longrightarrow X$ be a mapping. We say that f has the property $(\mathcal{K})$ if there exists a nonempty bounded closed convex subset $K \subset X$ such that $f(K \cap Q) \subset K$.

As it will be seen in this paper, the property $(\mathcal{K})$ is a substitute to the boundedness of Q.

3. Main results

3.1. **Existence and uniqueness result.** The following result is a generalization of Proposition 2.2, since every nonexpansive map defined on a convex closed bounded set is a 1-set contraction. Also we provide a longer but different proof than the one given in [13, Proposition 3.4].

Proposition 3.1. *Let X be a Banach space, $Q \ni 0$ a convex closed subset of X (not necessarily bounded), and $f : Q \longrightarrow Q$ a 1-set contraction satisfying the property $(\mathcal{K})$ and such that $I - f$ is ψ-expansive. Then f has a unique fixed point in Q.*

Proof. **Step 1.** Let K be a convex bounded closed subset such that $Q \cap K$ is self-mapped by f and let $f_n := (1 - 1/n)f$, for $n \in \{1, 2, \ldots\}$ be a sequence of approximating mappings. Without loss of generality, assume that $0 \in K$. Since f is a 1-set contraction, f_n is a $(1 - 1/n)$-set contraction. By Darbo's fixed point theorem (Lemma 2.3), for every $n \in \{1, 2, \ldots\}$, f_n has at least one solution $x_n \in Q \cap K$. Let $S = \{x_n \mid n = 1, 2, \ldots\}$ be such bounded sequence. To prove its relative compactness, we show that $\alpha(S) = 0$. First, for given some $n_0 \in \{1, 2, \ldots\}$, we have $\alpha(S) = \alpha(\{x_n \mid n \geq n_0\})$. Now arguing by contradiction, assume that $\alpha(S) > 0$ and let $0 < \varepsilon < \alpha(S)$. Then using the definition of the MNC, we can find an $\varepsilon_1 > 0$

and a finite covering $(\Omega_i)_{i\in I}$ $(I = \{1,\ldots,m\})$ such that $\{x_n \mid n \geq n_0\} = \bigcup_{i\in I}\Omega_i$ with $\mathrm{diam}(\Omega_i) \leq \varepsilon_1, \forall i \in I$ and thus $\alpha(S) \leq \varepsilon_1 < \alpha(S) + \varepsilon$. As a consequence $\alpha(S) = \max\{\alpha(\Omega_i),\, i \in I\} = \alpha(\Omega_l)$ for some $l \in [1, m]$. By the characteristic property of the upper bound, there exist $y_l, z_l \in \Omega_l$ such that

$$\alpha(S) - \varepsilon = \mathrm{diam}(\Omega_l) - \varepsilon \leq \|y_l - z_l\| \leq \mathrm{diam}(\Omega_l) \leq \alpha(S) + \varepsilon. \tag{3.1}$$

In addition, for every $x_p, x_q \in \{x_n \,|\, n \geq n_0\}$, we have

$$\|(x_p - f(x_p)) - (x_q - f(x_q))\| \leq \frac{1}{n_0}(\|f(x_p)\| + \|f(x_q)\|).$$

Let $\gamma > 0$ be such that $K \subset B(0, \gamma/2)$. Then

$$\|(x_p - f(x_p)) - (x_q - f(x_q))\| \leq \frac{\gamma}{n_0}.$$

Step 2. Since $y_l = x_p$ and $z_l = x_q$ for some $p, q \geq n_0$ and $I - f$ is ψ-expansive, we deduce that

$$\psi(\|y_l - z_l\|) \leq \frac{\gamma}{n_0}. \tag{3.2}$$

According to the properties of ψ, we now distinguish between two cases:

(i) ψ *is nondecreasing*: from Eqns. (3.1) and (3.2) and the fact that $\psi(r) > 0$, $\forall\, r > 0$, we obtain

$$0 < \psi(\alpha(S) - \varepsilon) \leq \frac{\gamma}{n_0}$$

and a contradiction is reached by choosing $n_0 > \dfrac{\gamma}{\psi(\alpha(S) - \varepsilon)}$.

(ii) ψ *is continuous*: for all positive δ, there exists $\eta > 0$ such that

$$\forall\, r > 0,\ (|r - \alpha(S)| \leq \eta \implies |\psi(r) - \psi(\alpha(S))| \leq \delta)\,.$$

Let $\delta > 0$ and $0 < \varepsilon < \min(\alpha(S), \eta)$. Using again (3.1) and (3.2), we get

$$|\psi(\alpha(S)) - \psi(\|y_l - z_l\|)| \leq \delta$$

and

$$0 < \psi(\alpha(S)) \leq \psi(\|y_l - z_l\|)| + \delta \leq \frac{\gamma}{n_0} + \delta.$$

Since $\delta > 0$ is arbitrary, we deduce that $0 < \psi(\alpha(S)) \leq \dfrac{\gamma}{n_0}$. Choosing $n_0 > \dfrac{\gamma}{\psi(\alpha(S))}$ again leads to a contradiction.

Conclusion. We have proved that $\alpha(S) = 0$ and so the set $\overline{S}$ is compact. As a consequence and taking if need be a subsequence, the sequence $(x_n)_n$ converges to some limit x. By continuity of f, x is the desired fixed point. Since $I - f$ is ψ-expansive, the uniqueness follows immediately. $\square$

Remark 3.1. In Proposition 3.1, we may assume that $I - f$ is ψ-expansive only on the set $\{(x, y) \in Q^2 \mid \alpha(S) - \varepsilon_0 \leq \|x - y\| \leq \alpha(S) + \varepsilon_0\}$ for some $\varepsilon_0 > 0$. In this case, we loose the uniqueness of the fixed point.

3.2. **Existence result.** We begin with

Lemma 3.2. *Let $f : X \longrightarrow X$ be an ψ-expansive map, where $\psi : [0,\infty) \to [0,\infty)$ is either nondecreasing, unbounded or continuous. Then f is α-ψ-expansive.*

Proof. Let $\varepsilon > 0$ be fixed and $\Omega \subset X$ a nonempty bounded subset. Then there exist bounded subsets $(Y_i)_{1\leq i\leq n}$ and $\varepsilon_0 > 0$ such that

$$f(\Omega) = \bigcup_{i=1}^{n} Y_i \text{ with } \operatorname{diam}(Y_i) \leq \varepsilon_0,\ \forall\, i \in \{1,\ldots,n\}$$

and

$$\alpha(f(\Omega)) \leq \varepsilon_0 < \alpha(f(\Omega)) + \varepsilon;$$

hence

$$\operatorname{diam}(Y_i) \leq \alpha(f(\Omega)) + \varepsilon,\ \forall\, i \in \{1,\ldots,n\}.$$

Moreover

$$\begin{array}{rcl} \Omega \subset f^{-1}(f(\Omega)) & \subset & f^{-1}\left(\bigcup_{i=1}^n Y_i\right) \\ & = & \bigcup_{i=1}^n f^{-1}(Y_i) \\ & := & \bigcup_{i=1}^n \Omega_i. \end{array} \tag{3.3}$$

For every $i \in \{1,\ldots,n\}$, let $x_1^i, x_2^i \in \Omega_i$; then there exist $y_1^i, y_2^i \in Y_i$ such that $f(x_1^i) = y_1^i$ and $f(x_2^i) = y_2^i$. Since f is ψ-expansive

$$\psi(\|x_1^i - x_2^i\|) \leq \|y_1^i - y_2^i\| \leq \operatorname{diam} Y_i \leq \alpha(f(\Omega) + \varepsilon).$$

Now we distinguish between two cases:

(a) *ψ is invertible and nondecreasing* (and so is ψ^{-1}). Then, for all $i \in \{1,\ldots,n\}$

$$\|x_1^i - x_2^i\| \leq \operatorname{diam}(\Omega_i) \leq \psi^{-1}(\alpha(f(\Omega) + \varepsilon)).$$

(3.3) guarantees

$$\alpha(\Omega) \leq \max_{1\leq i\leq n} \alpha(\Omega_i) \leq \psi^{-1}(\alpha(f(\Omega) + \varepsilon)).$$

$\varepsilon > 0$ being arbitrary, we finally deduce that

$$\psi(\alpha(\Omega)) \leq \alpha(f(\Omega))$$

for all bounded subsets $\Omega \subset X$, as claimed.

(b) *ψ is continuous.* We argue as in the proof of Proposition 3.1. For $i \in \{1,\ldots,n\}$, put $\Theta_i = \Omega_i \cap \Omega$; then $\{\Theta_i\}_{i=1}^n$ is a covering of Ω with $\Omega = \bigcup_{i=1}^n \Theta_i$. Hence $\alpha(\Omega) = \max\{\alpha(\Theta_i),\ i \in \{1,\ldots,n\}\} = \alpha(\Theta_l)$ for some $l \in [1,n]$. By the characteristic property of the upper bound, there exist $y_l, z_l \in \Theta_l$ such that

$$\operatorname{diam}(\Theta_l) - \varepsilon \leq \|y_l - z_l\| \leq \operatorname{diam}(\Theta_l).$$

Hence

$$\alpha(\Omega) - \varepsilon = \alpha(\Theta_l) - \varepsilon \leq \operatorname{diam}(\Theta_l) - \varepsilon \leq \|y_l - z_l\| \leq \operatorname{diam}(\Theta_l) \leq \alpha(\Omega) + \varepsilon.$$

This implies that

$$|\|y_l - z_l\| - \alpha(\Omega)| \leq \varepsilon. \tag{3.4}$$

Let $y_1^l = f(y_l)$ and $y_2^l = f(z_l)$. Since f is ψ-expansive, we obtain as in case (a)

$$\psi(\|y_l - z_l\|) \leq \|y_1^l - y_2^l\| \leq \operatorname{diam} Y_l \leq \alpha(f(\Omega)) + \varepsilon. \tag{3.5}$$

ψ being continuous, for all positive η, there exists $\delta > 0$ such that

$$\forall\, r > 0,\ (|r - \alpha(\Omega)| \leq \delta \implies |\psi(r) - \psi(\alpha(S))| \leq \eta)\,.$$

Given $\eta > 0$, choosing $0 < \varepsilon \leq \delta$ and using (3.4), (3.5), we finally get

$$|\psi(\alpha(\Omega)) - \psi(\|y_l - z_l\|)| \leq \eta$$

and

$$0 < \psi(\alpha(\Omega)) \leq \psi(\|y_l - z_l\|)| + \eta \leq \alpha(f(\Omega)) + \eta + \varepsilon.$$

Since $\eta, \varepsilon > 0$ are arbitrary, we conclude that

$$\psi(\alpha(\Omega)) \leq \alpha(f(\Omega).$$

The proof of the lemma is completed. □

Lemma 3.2 makes connection between Proposition 3.1 and the following one:

Proposition 3.3. *Let X be a Banach space, $Q \ni 0$ a convex closed subset of X, and $f : Q \longrightarrow Q$ a 1-set contraction satisfying the property $(\mathcal{K})$. If $I - f$ is α-ψ-expansive, then f has a fixed point in Q.*

Proof. The proof is almost analogous to that of Proposition 3.1. Recall that $\alpha(S) = \max\{\alpha(\Omega_i),\ i \in I\}$ and let $I_0 \subset I$ be such that $\alpha(S) = \alpha(\Omega_i)$, $\forall\, i \in I_0$. For every $0 < \varepsilon < \alpha(S)$ and all $i \in I_0$, there exist $y_i, z_i \in \Omega_l$ satisfying (3.1), that is for every $0 < \varepsilon < \alpha(S)$ and all $i \in I_0$

$$(\Omega_i \times \Omega_i) \bigcap \{(x, y) \in Q^2 |\, \alpha(S) - \varepsilon \leq \|x - y\| \leq \alpha(S) + \varepsilon\} \neq \emptyset.$$

Then the first step of the proof of Proposition 3.1 leads to the estimate:

$$\|(I - f)(y_i) - (I - f)(z_i)\| \leq \frac{\gamma}{n_0},\ \forall\, i \in I_0,$$

where $K \subset B(0, \gamma/2)$ for some $\gamma > 0$. This with the fact that f is α-ψ-expansive guarantee

$$\psi\left(\alpha(\Omega_i)\right) \leq \alpha\left((I - f)(\Omega_i)\right) \leq \frac{\gamma}{n_0},\ \forall\, i \in I_0.$$

For sufficiently large n_0 and $i \in I_0$, we deduce that $\psi\left(\alpha(\Omega_i)\right) = 0$. Since $\psi(0) = 0$, we conclude that $\alpha(\Omega_i) = 0$. Finally,

$$\{x_n \mid n \geq n_0\} \subset \left(\bigcup_{i \in I_0} \Omega_i\right) \bigcup \left(\bigcup_{i \in I_1} \Omega_i\right),$$

where $\operatorname{diam} \Omega_i \leq \alpha(S) - \varepsilon, \forall\, i \in I_1$. Therefore $\alpha(S) = \alpha(\bigcup\limits_{i \in I_0} \Omega_i) = 0$ which implies that $\{x_n | n \geq n_0\}$ is relatively compact (whenever ψ is either continuous or nondecreasing). As a consequence and taking if need be a subsequence, $(x_n)_{n \geq n_0}$ converges to some limit $x \in \overline{Q} = Q$. □

3.3. **Sum of operators.** Many problems in physical sciences are modeled by equations of the form:

$$Ax + Bx = x, \quad x \in Q \tag{3.6}$$

where Q is a closed convex subset of a Banach space X and A, B are two nonlinear operators. In some context, a useful tool to solve Problem (3.6) is the following celebrated fixed point theorem proved by Krasnozels'kĭi in 1958 (see [19]). Initially, Krasnoselskii was interested in the inversion of a perturbed differential operator and noticed that this may be obtained by the sum of a contraction and a compact mapping.

Theorem 3.4. *Let Q be a nonempty closed convex subset of a Banach space X and A, B be two mappings from M to X such that*
(a) A is compact and continuous,
(b) B is a contraction,
(c) $AQ + BQ \subset Q$.
Then $A + B$ has at least one fixed point in Q.

The proof of Theorem 3.4 combines the Banach contraction mapping principle and the Schauder fixed point theorem [28] and uses the fact that if $g\colon D \longrightarrow X$ a contraction, where D is a subset of a Banach space X, then the mapping $I - g\colon D \longrightarrow (I-g)(D)$ is a homeomorphism. For recent fixed point theorems for the sum of two operators, we refer to [1, 13, 22]. In the results of this section, one of the operators will be 1-set contraction. However, before sating a new existence result for the sum of two operators, we first derive two direct consequences from Proposition 3.3:

Corollary 3.5. *Let X be a Banach space and $Q \subset X$ be a nonempty closed bounded convex subset. Let A, $B : Q \longrightarrow X$ be two operators such that*
(a) A is completely continuous.
(b) B is a 1-set contraction and $I - (A + B)$ is α-ψ-expansive.
(c) $x, y \in Q \implies Bx + Ay \in Q$.
Then $A + B$ has a fixed point $x \in Q$.

Proof. By condition (c), $A + B$ maps Q into itself. Since A is completely continuous and B is a 1-set contraction, $A + B$ is a 1-set contraction too. Moreover $I - (A + B)$ is α-ψ-expansive. By Proposition 3.3, $A + B$ has a fixed point in Q. □

Definition 3.1. An operator $A : X \to X$ on a Banach space X is said to be accretive if the inequality $\|x-y+\lambda(Ax-Ay)\| \geq \|x-y\|$ holds for all $\lambda \geq 0$ and all $x, y \in D(A)$ ($I + \lambda A$ is injective and $(I + \lambda A)^{-1}$ is nonexpansive for all $\lambda \geq 0$). An operator B is said to be dissipative if $-B$ is accretive.

Following García-Falset [12, 13], for an accretive operator $A : D(A) \longrightarrow X$ and $\lambda > 0$, we denote

$$J_\lambda^A = (I + \lambda A)^{-1} : \mathcal{R}(I + \lambda A) \longrightarrow D(A) \text{ and } A_\lambda = \frac{I - J_\lambda^A}{\lambda}.$$

The operators J_λ^A, A_λ are the resolvent and the Yoshida approximant of A, respectively.

Corollary 3.6. *Let X be a Banach space and $Q \subset X$ a nonempty closed bounded convex subset. Let A, $B : Q \longrightarrow X$ be two operators such that*
(a) A is a 1-set contraction.
(b) B is dissipative and $I - (J_1^{-B} \circ A)$ is α-ψ-expansive.
(c) $x, y \in Q \implies Bx + Ay \in Q$.
Then $A + B$ has a fixed point $x \in Q$.

Proof. Since B is dissipative, then J^{-B} is nonexpansive and so $J_1^{-B} \circ A$ is a 1-set contraction. Moreover $I - (J_1^{-B} \circ A)$ is α-ψ-expansive. By Proposition 3.3, $J_1^{-B} \circ A$ has a fixed point and so $A + B$ has a fixed point. □

Now, our first existence result in this section is:

Theorem 3.7. *Let X be a Banach space and $Q \subset X$ a nonempty closed bounded convex subset. Let A, $B : Q \longrightarrow X$ be two operators such that*
(a) A is completely continuous.
(b) B is a 1-set contraction and $I - B$ is ψ-expansive.
(c) $x \in Q$ whenever $x = Bx + Ay \in Q$ for some $y \in Q$.
Then $A + B$ has a fixed point $x \in Q$.

Proof. For fixed $x \in Q$, let $A_x : Q \longrightarrow Q$ be the mapping defined by $A_x(y) = Ax + By$. Since B is a 1-set contraction, A_x is a 1-set contraction too. Moreover $I - B$ ψ-expansive implies $I - A_x$ ψ-expansive. According to Proposition 3.1, A_x admits a unique fixed point z in Q i.e. $z = Ax + Bz$. Consequently $Ax = (I - B)z$ and thus the map $(I - B) : Q \longrightarrow (I - B)(Q)$ is bijective; indeed, by definition it is surjective and since $(I - B)$ is ψ-expansive, then it is one-to-one. Let $J^B = (I - B)^{-1} : (I - B)(Q) \longrightarrow Q$. Then $J^B Ax = z \in Q$. By Assumption (c), $T = J^B \circ A$ maps Q into itself. According to the proof of Theorem 3.7 in [13], we know that J^B is continuous and since A is completely continuous, we deduce that T is completely continuous too. By the Schauder fixed point theorem, there exists $x \in Q$ such that $Tx = x$. Hence $x = Bx + Ax$, as claimed. □

As a consequence, we recover

Corollary 3.8 ([13, Theorem 3.7])**.** *Let X be a Banach space and $Q \subset$ a nonempty closed bounded convex subset. Let A, $B : X \longrightarrow X$ be two operators such that*
(a) A is completely continuous.
(b) B is nonexpansive and $I - B$ is ψ-expansive.
(c) $x, y \in Q \implies Bx + Ay \in Q$.
Then there exists $x \in Q$ such that $x = Ax + Bx$.

Our second existence result is

Theorem 3.9. *Let X be a Banach space and $Q \subset X$ a nonempty closed bounded convex subset. Let A, $B : Q \longrightarrow X$ be two operators such that*
(a) B is continuous, J_1^{-B} exists, and $I - B$ is α-ψ-expansive.
(b) A is completely continuous.
(c) $x, y \in Q \implies Bx + Ay \in Q$.
Then $A + B$ has a fixed point $x \in Q$.

Proof. Since J_1^{-B} exists and A is completely continuous, $J_1^{-A} \circ B$ is completely continuous. In fact

$$(I - B) \circ J_1^{-B} \circ A(Q) = A(Q)$$

and thus

$$\alpha\left((I - B) \circ J_1^{-B} \circ A(Q)\right) = \alpha(A(Q)).$$

Now, since $I - B$ is α-ψ-expansive and A is completely continuous, we have $\alpha(A(Q)) = 0$ and then

$$\psi\left(\alpha\left(J_1^{-B} \circ A(Q)\right)\right) \leq 0$$

which implies that $\psi\left(\alpha\left(J_1^{-B} \circ A(Q)\right)\right) = 0$. Since $\psi(0) = 0$, we deduce that $\alpha\left(J_1^{-B} \circ A(Q)\right) = 0$ and then the mapping $(J_1^{-B} \circ A)(Q)$ is completely continuous. By the Schauder fixed point theorem, we conclude that $A + B$ has a fixed point in Q. □

Example 3.1. Let $X = C([0,T], \mathbb{R})$ be the Banach space of continuous functions endowed with the sup-norm and consider the following integral equation:

$$u(t) = f(t, u(t)) + \int_0^t g(s, u(s))ds, \ t \in [0,T], \tag{3.7}$$

where $f : [0,T] \times X \longrightarrow X$ and $g : [0,T] \times X \longrightarrow X$ are continuous functions. As usual, we denote by $f(t, u(t)) := f(t,u)(t)$ and $g(t, u(t)) := g(t,u)(t)$. Let the operators A, $B : X \longrightarrow X$ be defined by

$$A(u)(t) = \int_0^t g(s, u(s))ds \ \text{ and } \ B(u)(t) = f(t, u(t)).$$

Assume that the nonlinear functions f and g satisfy the following conditions:

(H_1) $f : [0,T] \times X \longrightarrow X$ is a 1-set contraction map and $I - f(t, \cdot)$ is ψ-expansive, for $t \in [0,T]$.

(H_2) $\|u\| \leq \|u - f(t,u\|$, for all $(t,u) \in [0,T] \times X$.

(H_3) $A(Q) \subset Q$.

To prove existence of solution to Equation (3.7), we will apply Theorem 3.7. We just check the hypotheses. Since f is continuous, it is evident that A is completely continuous. Assumption (H_1) implies that

$$\|(u - B(u)) - (v - B(v))\| \geq \psi(\|u(t) - v(t)\|), \ \text{ for all } t \in [0,T].$$

Hence the operator $I - B$ is ψ-expansive. Moreover, from Assumption (H_1), the operator B is a 1-set contraction. Finally suppose that $u = B(u) + A(y)$ for some $y \in Q$. The fact that $u \in Q$ follows from (H_2) and (H_3); indeed,

$$\|u\| \leq \|u(t) - B(u)(t)\| = \|A(y)(t)\|.$$

Since $A(Q) \subset Q$, we conclude that $u \in Q$.

3.4. **Product of operators.** In this subsection, we suppose that X has the structure of a Banach algebra and for all subsets $X_1, X_2 \subset X$, we denote

$$X_1X_2 = \{ab \mid \ a \in X_1,\, b \in X_2\}.$$

Our main existence result is

Theorem 3.10. *Let $Q \ni 0$ be a closed convex bounded subset of a Banach algebra X and let $A : X \longrightarrow X,\ B : Q \longrightarrow X$ be two operators such that*
(a) B is completely continuous.
(b) A is an α-Lipschitz mapping with function ϕ_A, $I(\cdot) - A(\cdot)By$ is ψ-expansive, for every $y \in Q$, and ψ is invertible and unbounded.
(c) For every $y \in Q$, $(z = AzBy \Longrightarrow z \in Q)$.
Then the abstract equation $x = AxBx$ has a solution $x \in Q$ provided

$$(\mathcal{H}_0) \quad \|B(Q)\|\phi_A(r) \leq r,\ \forall\, r > 0.$$

In Theorem 3.10, notice that ϕ_A need not be continuous.

Proof. For every $x \in Q$, let the mapping A_x be defined on Q by $A_xy = AyBx$.
Claim 1. A_x is a 1-set contraction. Let $\Omega \subset Q$ be a bounded subset, $\varepsilon > 0$ be arbitrary, and $(\Omega_i)_{i=1}^{i=N}$ a finite covering of $A(\Omega)$ with $\operatorname{diam}(\Omega_i) \leq \varepsilon$ for all $i = 1, \ldots, N$. Then $\widetilde{(\Omega_i)}_{i=1}^{i=N}$ is a finite covering of $A_x(\Omega)$, where $\widetilde{(\Omega_i)} = \Omega_i.Bx = \{z \in X | z = yBx,\ y \in \Omega_i\}$. Then clearly

$$\operatorname{diam}\widetilde{(\Omega_i)} \leq M \operatorname{diam}(\Omega_i) \leq M\varepsilon, \quad \text{for all } \ i = 1, \ldots, N,$$

where $M = \|BQ\|$. Hence $\alpha(A_x(\Omega)) \leq M\varepsilon$ and passing to the infinimum over ε, we get

$$\alpha(A_x(\Omega)) \leq M\alpha(A(\Omega)).$$

Notice that this estimate can also deduced from the observation that

$$A_x(\Omega) \subset co(\{0\} \cup MA(\Omega)).$$

Now A α-Lipschitz both with $(\mathcal{H}_0)$ guarantee

$$\alpha(A_x(\Omega)) \leq M\phi_A(\alpha(A(\Omega)) \leq \alpha(A(\Omega)),$$

as claimed. Moreover $I - A_x$ is ψ-expansive; indeed

$$\begin{aligned} \|(I - A_x)(y_1) - (I - A_x)(y_2)\| &= \|(y_1 - Ay_1Bx) - (y_2 - Ay_2Bx)\| \\ &\geq \psi\left(\|y_1 - y_2\|\right). \end{aligned}$$

By Proposition 3.1, we conclude that A_x has only one fixed point $y \in Q$. Claim 1 allows us to define, for $x \in Q$, the mapping $N : Q \longrightarrow X$ by $Nx = y$ where y is the unique solution of the equation $y = AyBx$ (Proposition 3.1).
Claim 2. N is bounded. Let $y = AyBx$, for some $x \in D$, where D is a bounded subset of the Banach space X and assume without loss of generality that $0 \in Q$. Then

$$\|(y - AyBx) - (0 - A0Bx)\| = \|A0Bx\|.$$

From Assumption (b), we deduce that

$$\psi(\|y\|) \leq \|(y - AyBx) - (0 - A0Bx)\| \leq M\|A0\|.$$

Since ψ is invertible $\|y\| \leq \psi^{-1}(M\|A0\|)$ and Assumption (c) conclude that $N(Q) \subset Q$.

Claim 3. N is continuous. Two cases need to be discussed:

(a) *If ψ (in the definition of ψ-expansive) is continuous*, then consider a sequence $(x_n)_n$ such that $x_n \longrightarrow x$ in Q, as $n \to +\infty$. We have

$$\begin{aligned} Nx_n - Nx &= ANx_nBx_n - ANxBx \\ &= ANx_nBx_n - ANxBx_n + ANxBx_n - ANxBx. \end{aligned}$$

Hence

$$(I-A)Nx_nBx_n - (I-A)NxBx_n = ANx(Bx_n - Bx).$$

Since $I(\cdot) - A(\cdot)By$ is ψ-expansive, we have

$$\begin{aligned} \psi(\|Nx_n - Nx\|) &\leq \|(I-A)Nx_nBx_n - (I-A)NxBx_n\| \\ &\leq \|ANx\|\|Bx_n - Bx\|. \end{aligned}$$

According to the fact that N, A are bounded and B is continuous, we obtain that $\psi(\|Nx_n - Nx\|) \longrightarrow 0$, as $n \to +\infty$. In addition ψ is continuous and $\psi(0) = 0$; thus $\|Nx_n - Nx\| \longrightarrow 0$, i.e. N is continuous, as claimed.

(b) *If ψ (in the definition of ψ-expansive) is a nondecreasing map*, then consider a sequence $(x_n)_n$ in Q such that $x_n \longrightarrow x$, as $n \to +\infty$. Reasoning by contradiction, if Nx_n does not converge to Nx, then there exists $\varepsilon > 0$ and a subsequence $(x_{n_k})_{n_k}$ of $(x_n)_n$ such that $\|Nx_{n_k} - Nx\| > \varepsilon$, for all $k \in \mathbb{N}$. Since $(x_n)_n$ converges to x, AN is bounded, and B is continuous, then $\|ANx\|\|Bx_n - Bx\| < \delta = \psi(\varepsilon)/2$. Consequently

$$2\delta = \psi(\varepsilon) \leq \psi(\|Nx_{n_k} - Nx\|) \leq \|ANx\|\|Bx_{n_k} - Bx\| < \delta,$$

which is a contradiction; therefore N is a continuous mapping.

Claim 4. N is compact. From Lemma 2.1 and the fact that N and A are bounded, there exists some positive constant k_1 such that $\|ANx\| \leq k_1$, $\forall\, x \in Q$. Let $\varepsilon > 0$ be given. Since Q is bounded and B is completely continuous, $B(Q)$ is relatively compact, hence totally bounded. Then there exists a finite set $\mathcal{E} = \{x_1, ..., x_n\} \subset Q$ such that

$$B(Q) \subset \bigcup_{i=1}^{n} \mathcal{B}_\delta(w_i),$$

where $\mathcal{B}_\delta(w_i) = B(w_i, \delta)$, $w_i := B(x_i)$, and $\delta := k_2\varepsilon$ for some constant k_2 to be selected later on. Then, for every $x \in Q$, there exists $x_i \in \mathcal{E}$ such that

$$0 \leq \|Bx - Bx_i\| \leq k_2\varepsilon.$$

We have

$$\begin{aligned} \psi(\|Nx_i - Nx\|) &\leq \|(I-A)Nx_iBx_i - (I-A)NxBx_i\| \\ &\leq \|ANx\|\|Bx_i - Bx\| \\ &\leq k_1k_2\varepsilon. \end{aligned}$$

The map ψ being invertible, we have $\|Nx_i - Nx\| \leq \psi^{-1}(k_1k_2\varepsilon)$.

Choosing $0 < k_2 \leq \dfrac{\psi(\varepsilon)}{k_1\varepsilon}$ yields

$$\|Nx_i - Nx\| \leq \varepsilon.$$

We have proved that $N(Q) \subset \bigcup_{i=1}^{n} \mathcal{B}_\varepsilon(Nx_i)$, showing that N is totally bounded.

Conclusion. With the Schauder fixed point theorem, we conclude that N has a fixed point, ending the proof of the theorem. $\square$

Corollary 3.11. *Let Q be a closed convex bounded subset of a Banach algebra X with $0 \in Q$ and let $A : X \longrightarrow X,\ B : Q \longrightarrow X$ be two operators such that*
(a) B is completely continuous.
(b) A is α-Lipschitz with function ϕ_A, $I(\cdot) - A(\cdot)By$ is ψ-expansive for every $y \in S$, and ψ is invertible.
(c) ($z = AzBy \Longrightarrow z \in Q$), for all $y \in Q$.
Then the abstract equation $x = AxBx$ has a solution $x \in Q$ on condition that

$$\|B(Q)\|\phi_A(r) \leq r,\ \forall\, r > 0.$$

4. Furi-Pera fixed point theorems

Let X be a Banach space. First we recall a definition (1987), Mönch's fixed point theorem (1980), and the NRP property.

Definition 4.1. [11] A map $f : \overline{Q} \longrightarrow X$ is said to satisfy the Furi-Pera condition if the following boundary condition holds:

$$(\mathcal{FP}) \begin{cases} \text{if } \{(x_j, \mu_j)\}_{j\geq 1} \text{ is a sequence in } \partial Q \times [0,1] \\ \text{converging to } (x, \mu) \text{ with } x = \mu f(x) \text{ and } 0 \leq \mu < 1, \\ \text{then } \mu_j f(x_j) \in Q \text{ for j sufficiently large.} \end{cases}$$

Lemma 4.1. *(See, e.g.,* [3, 21]*) Suppose that $Q \subset X$ is a closed convex subset with some $x_0 \in Q$. Suppose there is a continuous map $f : Q \longrightarrow Q$ such that*

$$(D \subset Q, D \text{ countable, } D \subset \overline{co}(\{x_0\} \cup f(D))) \Longrightarrow \overline{D} \text{ compact.}$$

Then f has a fixed point in Q.

Definition 4.2. (a) A subset Q of a Banach space X is a nonexpansive retract of X if there exists a nonexpansive mapping $r : X \longrightarrow A$ such that $rx = x$ for all $x \in Q$. The map r is called a nonexpansive retraction. Using the Minkowskii functional (see [27], Lemma 4.2.5, p. 27), r may be chosen such that $r(x) \in \partial Q$ whenever $x \notin Q$.

(b) We say that a Banach space X has the nonexpansive retract property "NRP" (for short) if there exists a convex closed subset $Q \subset X$ which is a nonexpansive retraction of X.

Our main existence theorem in this section is:

Theorem 4.2. *Let X be a Banach space satisfying the NRP, $Q \ni 0$ a convex closed bounded nonexpansive retract of X, and $f : Q \longrightarrow X$ a 1-set contraction satisfying the Furi-Pera condition $(\mathcal{FP})$ and such that $I - f$ is ψ-expansive. Then f has a unique fixed point in Q.*

Of course we can replace the boundedness of Q by the property $\mathcal{K}$. Several consequences of this result will also be provided.

Proof. The proof is analogous to that of [9, Theorems 3.1 and 3.2] and [2, Theorem 5.10]. For the sake of completeness, we give the details in case of 1-set contractions.
Step 1. Approximate fixed points for fr. Let $r : X \longrightarrow Q$ be a nonexpansive retraction and, for each $n \in \{1, 2, \ldots\}$, consider the nonlinear equation

$$x = (1 - 1/n)fr(x), \tag{4.1}$$

Since r is nonexpansive and f is a 1-set contraction map, then the map $(1 - 1/n)fr : X \longrightarrow X$ is a $(1 - 1/n)$-set contraction. By Mönch's fixed point theorem (Lemma 4.1), for every $n \in \{1, 2 \ldots\}$, Equation (4.1) has at least one solution x_n. Then, for every fixed integer $n \in \{1, 2 \ldots\}$, consider the nonempty set $S_n = \{x \in X \mid x = (1 - 1/n)fr(x)\}$. By continuity of f and r, the set S_n is closed and even compact; indeed

$$\alpha(S_n) \leq (1 - 1/n)\alpha(r(S_n)) \leq (1 - 1/n)\alpha(S_n).$$

This implies that $\alpha(S_n) = 0$ and thus $\overline{S}_n = S_n$ is compact.
Step 2. Approximate fixed points of f. We shall prove that the following equation

$$x = (1 - 1/n)f(x) \tag{4.2}$$

has a solution in Q, for each $n \in \{1, 2, \ldots\}$. For this, it is enough to prove that the sequence $(x_n)_n$ obtained in Step 1 lies in Q where, for each $n \in \{1, 2 \ldots\}$, x_n is a solution of the equation (4.1). Arguing by contradiction, assume that $S_n \cap Q = \emptyset$. Since Q is closed and S_n is compact, there exists $0 < \delta < \mathrm{dist}(S_n, Q)$. Following the proof of [2, Theorem 5.10], let $N \in \{1, 2, \ldots\}$ be an integer such that $N > 1/\delta$. Subsequently, for all integer $i \geq N$, consider the open set $U_i = \{x \in E : d(x, Q) < \frac{1}{i}\}$. It is clear that $\mathrm{dist}(S_n, Q) > \delta$ and $1/i < \delta$ imply that $S_n \cap \overline{U}_i = \emptyset$. Thus the mapping $(1 - 1/n)fr : \overline{U}_i \longrightarrow X$ is a k-set contraction with $k = 1 - 1/n$. Since $S_n \cap \overline{U}_i = \emptyset$, a classical nonlinear alternative [2, Theorem 5.7] guarantees that there exists $(y_i, \mu_i) \in \partial U_i \times (0, 1)$ such that $y_i = \mu_i(1 - 1/n)fr(y_i)$. Notice that $y_i \in \partial U_i$ implies that $y_i \notin Q$. As a consequence

$$\mu_i(1 - 1/n)fr(y_i) \notin Q, \ \forall\, i \geq N. \tag{4.3}$$

Now let

$$D_n = \{x \in E \mid \exists\, \mu \in [0, 1], \ x = \mu(1 - 1/n)fr(x)\}.$$

Then D_n is nonempty because it contains $0, x_n$ and y_i, for every $i \geq N$. Moreover, D_n is compact. Indeed

$$D_n \subseteq \overline{co}\,((1 - 1/n)fr(D_n) \cup \{0\})$$

implies

$$\alpha(D_n) \leq \alpha\,(\overline{co}((1 - 1/n)fr(D_n) \cup \{0\}))\,,$$

where α is the Kuratowski MNC. However, since f is a 1-set contraction and r is nonexpansive, we have the estimates

$$\begin{aligned} \alpha(D_n) &\leq \alpha(\overline{co}\,((1 - 1/n)fr(D_n)) \\ &\leq (1 - 1/n)\alpha(r(D_n)) \\ &\leq (1 - 1/n)\alpha(D_n). \end{aligned}$$

Then $\alpha(D_n) = 0$ showing that D_n is compact since it is closed. Now, for given $i \geq N$ and $0 \leq \mu_i \leq 1$, we have that $d(y_i, Q) = \frac{1}{i}$ for $y_i \in \partial U_i \cap D_n$. Up to a subsequence, the sequence μ_i converges to $\mu^* \in [0,1]$ and, by the compactness of D_n, $y_i \longrightarrow y^* \in \partial Q$, as $i \to +\infty$. By continuity, $y_i = \mu_i(1-1/n)fr(y_i)$ tends to $\mu^*(1-1/n)fr(y^*)$. Hence $y^* = \mu^*(1-1/n)fr(y^*)$. In addition $S_n \cap Q = \emptyset$, implies that $\mu^* \neq 1$. Finally, let the sequence $(x_i)_i$ be such that $x_i = r(y_i) \in \partial Q$ (this follows from $y_i \notin Q$ and the definition of the retraction r) and $((x_i)_i)$ converges to $x = y^*$. In addition $d(y_i, Q) = \frac{1}{i}$; hence $y^* \in \partial Q$ and so $y^* = r(y^*)$ and $\mu'_i = (1-1/n)\mu_j$ with $\mu' = (1-1/n)\mu^*$. Since f satisfies the Furi-Pera condition, we infer that $\mu_i(1-1/n)fr(y_i) \in Q$ for i sufficiently large. This contradicts (4.3) and the fact that $y_i \notin Q$, for $i \geq N$. Thus, for each $n \in \{1,2,\ldots\}$, $S_n \cap Q \neq \emptyset$. Hence $r(x_n) = x_n$ and $x_n = (1-1/n)fr(x_n) = (1-1/n)f(x_n)$. To conclude, we have proved that Equation (4.2) has a solution for each $n \in \{1,2,\ldots\}$.

Step 3. Passage to the limit. Let $(x_n)_{n\in\mathbb{N}} \subset Q \cap S_n$ be the sequence obtained in Step 2 and $S_K = \{(x_n)_n\} \cap K$. Owing to Steps 1, 2, the set S_K is a nonempty bounded set. As in Proposition 3.1, we know that the set $\overline{S_K}$ is compact, hence sequentially compact. Therefore we can extract a sequence converging to x and, by continuity of f, we conclude that x is a fixed point of f. □

Remark 4.1. Even if the Banach space X does not satisfy the NRP property, one can replace the nonexpansive retraction by another retraction denoted P such that $P(A) \subseteq A$ or $P(A) \subseteq \overline{co}(A \cup \{p\})$, $p \in X$ for each subset (bounded) of X (see [16]); thus we obtain $\alpha\,(P(A)) \leq \alpha(A)$.

Corollary 4.3. *Let X be a Banach space satisfying the NRP on a convex closed bounded subset $Q \ni 0$ and let $f : Q \longrightarrow X$ be a nonexpansive satisfying the Furi-Pera condition $(\mathcal{FP})$ such that $I - f$ is ψ-expansive. Then f has a unique fixed point in Q.*

Also, we have the following existence results the proofs of which are analogous to that of Theorem 3.7. We omit the details.

Theorem 4.4. *Let X be a Banach space satisfying the NRP on a convex closed bounded subset $Q \ni 0$ and let A, $B : X \longrightarrow X$ be two operators such that*
(a) A is completely continuous.
(b) B is 1-set contraction and $I - (A + B)$ is ψ-expansive.
(c) $f = A + B$ satisfies the Furi-Pera condition.
Then $A + B$ has a fixed point in Q.

Theorem 4.5. *Let X be a Banach space satisfying the NRP on a convex closed, bounded subset $Q \ni 0$ and let A, $B : X \longrightarrow X$ be two operators such that*
(a) A is completely continuous.
(b) B is 1-set contraction and $I - B$ is ψ-expansive.
(c) The operator $N : Q \longrightarrow X$ defined by $Nx = y$, where y is the unique solution of the equation $y = Ax + By$, satisfies the Furi-Pera condition.
Then $A + B$ has a fixed point in Q.

Corollary 4.6. *Let X be a Banach space satisfying the NRP on a convex closed, bounded subset $Q \ni 0$. If A, $B : X \longrightarrow X$ are two operators such that*
(a) A is completely continuous.
(b) B is nonexpansive and $I - B$ is ψ-expansive.

(c) The operator $N : Q \longrightarrow X$ defined by $Nx = y$, where y is the unique solution of the equation $y = Ax + By$, satisfies the Furi-Pera condition.
Then $A + B$ has a fixed point in Q.

Theorem 4.7. *Let X be a Banach space satisfying the NRP on a convex closed bounded subset $Q \ni 0$. Let A, $B : X \longrightarrow X$ be two operators such that*
(a) A is completely continuous.
(b) B is a 1-set contraction, $I(\cdot) - A(\cdot)By$ is ψ-expansive for each $y \in Q$, and ψ is invertible and unbounded.
(c) The operator $N : Q \longrightarrow X$ defined by $Nx = y$, where y is the unique solution of the equation $y = Ax\,By$, satisfies the Furi-Pera condition.
Then $A + B$ has a fixed point in Q.

5. Discussion

In this section, we show that Lemma 3.3 proved in [13] can be derived from a compactness result given in [8] and then we conclude. First, we introduce some notations. For some positive δ and a bounded subset $\Omega \subseteq Q$, denote by (see [15], [18], [27])

$$F_\delta(f, \Omega) = \{x \in \Omega : \|x - f(x)\| \leq \delta\}, \tag{5.1}$$

the set of the δ-fixed points of f in Ω,

$$S = \{(x_n)_{n\in\mathbb{N}} \subset Q \mid \ x_n = \left(1 - \frac{1}{n}\right) f(x_n), \ \forall\, n \in \{1, 2, \ldots\}, \tag{5.2}$$

the set of approximate fixed points, and let

$$S_K = S \cap K, \tag{5.3}$$

whenever K is a closed bounded convex subset. For some real parameters $\varepsilon > 0$ and $c > 0$ such that $0 < c < \alpha(\Omega) + \varepsilon$, define the sets:

$$N_\varepsilon(\Omega) = \{(x, y) \in \Omega^2 \mid \alpha(\Omega) - \varepsilon \leq \|x - y\| \leq \alpha(\Omega) + \varepsilon\}. \tag{5.4}$$

$$N^c_\varepsilon(\Omega) = \{(x, y) \in \Omega^2 \mid c \leq \|x - y\| \leq \alpha(\Omega) + \varepsilon\}, \tag{5.5}$$

where α is the measure of noncompactness of Kuratowski. The following technical lemma (see [8, Lemma 3.1]) has been used by the authors to prove the compactness of the set S_K and then the convergence of a sequence of approximate fixed points.

Lemma 5.1. *Let X be a Banach space, $Q \ni 0$ a convex, closed subset of E, and $f : Q \longrightarrow Q$ a nonexpansive mapping satisfying the property $(\mathcal{K})$. Assume that there exist $\delta_0, \varepsilon > 0$ such that for all $c \in (0, \alpha(S_K) + \varepsilon)$, we have*

$$[F_{\delta_0}(f, S_K) \times F_{\delta_0}(f, S_K)] \cap N^c_\varepsilon(f, S_K) = \emptyset. \tag{5.6}$$

Then $\alpha(S_K) = 0$.

In the next proposition, we give a sufficient condition for (5.6) be satisfied.

Proposition 5.2. *Let Ω be a nonempty bounded subset of X such that $\alpha(\Omega) > 0$ and let Q be a convex closed subset of X. Suppose that $f : Q \longrightarrow Q$ is a nonexpansive map such that $I - f$ is ψ-expansive. Then for every $\varepsilon > 0$, $0 < c < \alpha(\Omega) + \varepsilon$, and all $\delta, \delta' > 0$ with $0 < \delta + \delta' < \psi(c)$, we have*

$$[F_\delta(f, \Omega) \times F_{\delta'}(f, \Omega)] \cap N^c_\varepsilon(\Omega) = \emptyset.$$

Proof. Let $\varepsilon > 0$ and $c > 0$ be such that $0 < c < \alpha(\Omega) + \varepsilon$. Arguing by contradiction, suppose that there exist $\delta, \delta' > 0$ such that $0 < \delta + \delta' < \psi(c)$ and

$$[F_\delta(f, \Omega) \times F_{\delta'}(f, \Omega)] \cap N^c_\varepsilon(\Omega) \neq \emptyset.$$

For $(x, y) \in [F_{\delta'}(f, \Omega) \times F_\delta(f, \Omega)] \cap N^c_\varepsilon(\Omega)$, since $I - f$ is ψ-expansive, we have

$$\psi(\|x - y\|) \leq \|(x - f(x)) - (y - f(y))\| \leq \|x - f(x)\| + \|y - f(y)\| \leq \delta + \delta'. \quad (5.7)$$

(a) If ψ is nondecreasing, then since $(x, y) \in N^c_\varepsilon(\Omega)$, we have $c \leq \|x - y\|$ which implies that $\psi(c) \leq \psi(\|x-y\|) \leq \delta+\delta'$, leading to a contradiction to $0 < \delta+\delta' < \psi(c)$.

(b) If ψ is continuous, then let $(x, y) \in N^c_\varepsilon(\Omega)$ be such that $\|x-y\| = c+1/n$ for large enough $n \in \{1, 2, \ldots\}$. By continuity of ψ, for every $\eta > 0$, there is $n_0 \in \{1, 2, \ldots\}$ such that for all $n \geq n_0$

$$\psi(c) - \eta < \psi(\|x - y\|) < \psi(c) + \eta.$$

With (5.7), we get $\psi(c) - \eta \leq \delta + \delta'$ and a contradiction is reached by choosing $\eta > \psi(c) - (\delta + \delta')$. □

Conclusion. According to Proposition 5.2, we can see that the existence part of Proposition 2.2 (i.e. Lemma 3.3 and Proposition 3.4 in [13]) is a consequence of [8, Theorem 4.5] because every nonexpansive f mapping such that $I - f$ is ψ-expansive satisfies the geometric condition (5.6) of Lemma 5.1 which in turn implies the compactness of the set S_K and then yields the existence of a fixed point. Finally, since ψ-expansive mappings ψ are α-ψ-expansive, as shown in Lemma 3.2, we believe that Proposition 3.1 and related corollaries can make a contribution in the fixed point theory of the important class 1-set contractions.

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