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EXISTENCE OF POSITIVE MONOTONIC SOLUTIONS OF FUNCTIONAL HYBRID FRACTIONAL INTEGRAL EQUATIONS OF QUADRATIC TYPE

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Abstract. We present an existence result for positive monotonic solutions for a certain nonlinear functional hybrid integral equation of quadratic type via fixed point theoretic technique of Dhage [10, 11] in Banach algebras. Our result generalizes the existence result proved in Darwish and Ntouyas [7] and thereby several results as special cases with a different method.
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1. Introduction

Given a closed and bounded interval $J = [0, T]$ in $\mathbb{R}$, the set of real numbers and given a real number $q > 0$, consider the following functional hybrid fractional integral equation of quadratic type (in short HFIE),

$$\begin{aligned} x(t) &= k(t, x(t), x(\alpha(t))) \\ &\quad + \big[f(t, x(t), x(\beta(t)))\big] \left(\frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}} g(s, x(s), x(\eta(s)))\, ds \right) \end{aligned} \tag{1.1}$$

for all $t \in J$, where the functions $k, f, g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\alpha, \beta, \eta : J \to J$ and $v : J \times J \to \mathbb{R}$ are continuous and Γ is the Gamma function.

By a *solution* of HFIE (1.1) we mean a function $x \in C(J, \mathbb{R})$ that satisfies (1.1) on J, where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

The integral equations of quadratic type has been discussed in the literature for a long time. In this connection we refer the readers to the work of Chandrasekhar [5] who established his famous quadratic equation in the theory of radiative transfer. The study of fractional integral equation of quadratic type gained momentum because of their occurrence in some natural problems of kinetic theory of gases, neutron transport, traffic theory and some biological phenomenon (cf. [6, 7]). The study

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of quadratic integral equations using the measure of noncompactness is initiated by Dhage in [9] and since then several authors have studied quadratic integral equations via a measure theoretic technique established by Darbo [8, 12]. See for example [1], [2], [3], [4], [6] and the references given therein. Recently Dhage established some fixed point theorems in Banach algebras and they are employed for proving the existence theorems for quadratic differential and integral equations. In this connection, the readers are referred to the work of Dhage [10]. It is known that fixed point theoretic technique is more powerful than the measure theoretic technique established by Darbo [12] which is further generalized by Dhage [11]. The main difference between these two techniques is that the former needs the nonlinear function under integral sign to be Carathéodory whereas the later needs the same function to be continuous on the domain of it's definition.

It can be shown as in Darwish and Ntouyas [7] that the functional HFIE (1.1) is more general and new to the theory of fractional integral equations and includes several well-known fractional integral equations as special cases. In this article, we prove the existence as well as positivity and monotonic results for the functional HFIE (1.1) with a different method than [7].

2. Auxiliary Results

We place the problems in the function space $E = C(J, \mathbb{R})$ of continuous real-valued functions defined on J. Define a norm $\|\cdot\|$ and a multiplication in E by

$$||x|| = \sup_{t\in J} |x(t)| \tag{2.1}$$

and

$$(xy)(t) = x(t)\, y(t), \ \forall\, t \in J. \tag{2.2}$$

Clearly E is a Banach algebra with respect to above supremum norm and the multiplication in it. Let $L^1(S)$ we denote the Banach space of all Lebesgue integrable functions defined on J equipped with the norm $\|\cdot\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^T |x(s)|\, ds.$$

We need the following definitions in what follows.

Definition 2.1. A mapping $A : E \to E$ is said to be $\mathcal{D}$-Lipschitz if there exists an upper semi-continuous nondecreasing functions $\psi_A : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|Ax - Ay\| \le \psi_A(\|x - y\|) \tag{2.3}$$

for all $x, y \in E$. The function ψ_A is called a $\mathcal{D}$-function of A on E. In particular if $\psi_A(r) = \lambda r$, then A is called a Lipschitz mapping on E with a Lipschitz constant λ. If $\lambda < 1$ then A is called a contraction on A. Further if $\psi_A(r) < r$ for $r > 0$ then ψ_A is called a nonlinear $\mathcal{D}$-contraction on A with a $\mathcal{D}$-function ψ_A.

Definition 2.2. An operator $B : E \to E$ is called compact if $B(E)$ is a relatively compact subset of E. B is said to be totally bounded if $B(S)$ is a totally bounded subset of E for any bounded subset $S(E)$. B is said to be completely continuous if it is totally bounded and continuous on E.

Remark 2.3. It is known that every compact operator is totally bounded, but the converse may not be true. However, both the concepts coincide for an operator A on a bounded subset S of E. Thus complete continuity is a weaker concept than the compactness of continuous operator B on the Banach space E.

The following hybrid fixed point theorem in a Banach algebra E is well-known in the literature on fixed point theory. See Dhage [10] and the references therein.

Theorem 2.4 (Hybrid Fixed Point Theorem)**.** *Let S be a nonempty closed convex and bounded subset of a Banach algebra E and let $A, C : E \to E$ and $B : S \to E$ be three operators satisfying:*

(a) *A and C are $\mathcal{D}$-Lipschitz with $\mathcal{D}$-functions ψ_A and ψ_C respectively,*
(b) *B is completely continuous,*
(c) *$Ax\,By + Cx = x \;\forall\, y \in S \Longrightarrow\; x \in S$ and*
(d) *$M\psi_A(r) + \psi_C(r) < r$ for $r > 0$, where $M = \|B(S)\| = \sup\{\|x\| : x \in S\}$.*

Then the operator equation

$$Ax\,Bx + Cx = x \tag{2.4}$$

has a solution.

Remark 2.5. If we take $\psi_A(r) = \dfrac{L_1 r}{K + r}$ and $\psi_C(r) = L_2\, r$, then hypothesis (d) of the above hybrid fixed point theorem takes the form $\dfrac{L_1 M}{K + r} + L_2 < 1$ for each real number $r > 0$. Similarly, if $\psi_A(r) = L_1\, r$, and $\psi_C(r) = \dfrac{L_2 r}{K + r}$, then hypothesis (d) of the above hybrid fixed point theorem takes the form $M\,L_1 + \dfrac{L_2 M}{K + r} < 1$ for each real number $r > 0$.

We close this section with the following special case of Theorem 2.4 which is frequently used in the existence theory for quadratic differential and integral equations.

Corollary 2.6. *Let S be a nonempty closed convex and bounded subset of a Banach algebra E and let $A, C : E \to E$ and $B : S \to E$ be three operators satisfying*

(a) *A and C are Lipschitz with Lipschitz constants α and β respectively,*
(b) *B is compact and continuous,*
(c) *$Ax\,By + Cx \in S$ for all $x \in E$ and $y \in S$, and*
(d) *$\alpha M + \beta < 1$, where $M = \|B(S)\|$.*

Then the operator equation (2.4) *has a solution.*

3. Main Results

Before going to the main results we give some useful definitions. First we recall a basic definition of fractional calculus [13, 14, 15].

Definition 3.1. Let $h \in L^1(I)$, $I = [a, b]$, and let $q > 0$ be a real number. Then the Riemann-Liouville fractional integral of order q of the function $h(t)$ is defined as

$$I^q h(t) = \frac{1}{\Gamma(q)} \int_a^t \frac{h(s)}{(t - s)^{1-q}}\, ds, \quad a < t < b. \tag{3.1}$$

An equation containing the fractional integral of unknown function is called a **fractional integral equation**. If the unknown function in a fractional integral equation occurs nonlinearly, then it is called **nonlinear fractional integral equation**. A fractional integral equation involving two or more nonlinearities which satisfy different characteristics from different branches of mathematics is called **hybrid fractional integral equation**.

Definition 3.2. A function $g(t,x,y)$ is called Carathéodory if:

(i) The map $t \mapsto g(t,x,y)$ is measurable for each $x, y \in \mathbb{R}$ and
(ii) the map $(x,y) \mapsto g(t,x,y)$ is continuous for each $t \in J$.

A Caratheódory function g is called L^2-Carathéodory if:

(iii) There exists a function $h \in L^2(J,\mathbb{R})$ such that
$$|g(t,x,y)| \leq h(t) \text{ a.e. } t \in J$$
for all $x, y \in \mathbb{R}$.

We need the following hypotheses in what follows.

(H_0) The functions $\alpha, \beta, \eta : J \to J$ are continuous.

(H_1) The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist constants $L_1 > 0$, $K_1 > 0$ such that
$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leq \frac{L_1 \max\{|x_1 - y_1|, |x_2 - y_2|\}}{K_1 + \max\{|x_1 - y_1|, |x_2 - y_2|\}}$$
for all $t \in J$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(H_2) The function $k : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist constants $L_2 > 0$, $K_2 > 0$ such that
$$|k(t,x_1,x_2) - k(t,y_1,y_2)| \leq \frac{L_2 \max\{|x_1 - y_1|, |x_2 - y_2|\}}{K_2 + \max\{|x_1 - y_1|, |x_2 - y_2|\}}$$
for all $t \in J$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(H_3) The function v is continuous on $J \times J$. Moreover, $V = \sup_{t,s \in J} |v(t,s)|$.

(H_4) The function g is L^2-Carathéodory on $J \times \mathbb{R} \times \mathbb{R}$.

Remark 3.3. The conditions given on functions f and g in hypothesis (H_1) and (H_2) are more general than Lipschitz condition. In particular, if $L_1 < K_1$ and $L_2 < K_2$, then we obtain the Lipschitz conditions of the functions f and g respectively.

Theorem 3.4. *Assume that the hypothesis $(H_0)-(H_4)$ hold. Furthermore, if $q > 1/2$ and*
$$\left(\frac{V \, \|h\|_{L^2} \, T^{q-1/2}}{\Gamma(q)(2q-1)^{1/2}} \right) \frac{L_1}{K_1 + r} + \frac{L_2}{K_2 + r} < 1, \quad r > 0, \tag{3.2}$$
then the functional HFIE (1.1) has a solution defined on J.

Proof. Denote by $E = C(J,\mathbb{R})$. Define a subset S of E
$$S = \{x \in E \mid \|x\| \leq \rho\} \tag{3.3}$$

where

$$\rho = L_2 + K_0 + \frac{V[L_1 + F_0]\|h\|_{L^2}\, T^{q-1/2}}{\Gamma(q)(2q-1)^{1/2}},$$

and $K_0 = \sup_{t\in J} |k(t,0,0)|$, $F_0 = \sup_{t\in J} |f(t,0,0)|$.

Define three operators $A : E \to E$, $B : S \to E$ and $C : E \to E$ by

$$Ax(t) = f(t, x(t), x(\alpha(t))),\ t \in J, \tag{3.4}$$

$$Bx(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\ g(s, x(s), x(\eta(s)))\, ds,\ t \in J \tag{3.5}$$

and

$$Cx(t) = k(t, x(t), x(\alpha(t))),\ t \in J. \tag{3.6}$$

Then the HFIE (1.1) is transformed into an operator equation as

$$Cx(t) + Ax(t)\, Bx(t) = x(t),\ t \in J. \tag{3.7}$$

We shall show that the operators A, B and C satisfy all the conditions of Theorem 2.4. This will be achieved in the series of following steps.

Step I. A, B and C define the mappings $A, C : E \to E$ and $B : S \to E$.

Since the function f is continuous, the map $t \mapsto f(t, x_1, x_2)$ is continuous for each $x_1, x_2 \in \mathbb{R}$. As a result $Ax : J \to \mathbb{R}$ is a continuous function and that $Ax \in E$. Similarly, $Cx \in E$. Hence, A and C define the mappings $A, C : E \to E$. We will show, in step V, that the function Bx is continuous on J for each $x \in S$. As g is L^2-Carathéodory, the function $s \mapsto g(s, x(s), x(\eta(s)))$ is integrable and therefore, the integral

$$\int_0^t \frac{v(t,s)}{(t-s)^{1-q}} g(s, x(s), x(\eta(s)))\, ds$$

exists for each $x \in E$. Again the map

$$t \mapsto \int_0^t \frac{v(t,s)}{(t-s)^{1-q}} g(s, x(s), x(\eta(s)))\, ds$$

is continuous, so the function (Bx) is continuous on J, whence $Bx \in C(J, \mathbb{R})$ for each $x \in C(J, \mathbb{R})$. Hence B defines a mapping $B : S \to E$.

Step II. A and C are $\mathcal{D}$-Lipschitz on E.

Let $x, y \in E$ be arbitrary. Then, by (H_1), for all $x, y \in E$, we have

$$\begin{aligned} |Ax(t) - Ay(t)| &= |f(t, x(t), x(\beta(t))) - f(t, y(t), y(\beta(t)))| \\ &\le \frac{L_1 \max\{|x(t) - y(t)|, |x(\beta(t)) - y(\beta(t))|\}}{K_1 + \max\{|x(t) - y(t)|, |x(\beta(t)) - y(\beta(t))|\}} \\ &\le \frac{L_1 \|x - y\|}{K_1 + \|x - y\|} = \psi_A(\|x - y\|), \end{aligned}$$

where ψ_A is a $\mathcal{D}$-function defined by $\psi_A(r) = \dfrac{L_1 r}{K_1 + r} < r,\ r > 0$. This shows that A is a $\mathcal{D}$-Lipschitz operator on E into itself. Similarly C is also a $\mathcal{D}$-Lipschitz operator on E into itself with $\psi_C(r) = \dfrac{L_2 r}{K_2 + r} < r,\ r > 0$.

Step III. $B(S)$ is a uniformly bounded set in E.

Let $x \in S$ be any point. Then, we have, for all $t \in J$, that

$$\begin{aligned}
|Bx(t)| &\leq \left| \frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\, ds \right| \\
&\leq \frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, h(s)\, ds \leq \frac{1}{\Gamma(q)} \int_0^T \frac{v(t,s)}{(t-s)^{1-q}} h(s)\, ds.
\end{aligned}$$

Thus

$$\|Bx\| \leq \frac{1}{\Gamma(q)} \int_0^T \frac{v(t,s)}{(t-s)^{1-q}} h(s)\, ds.$$

Step IV. $B(S)$ is an equicontinuous set in E.

Let $x \in S$ be arbitrary. Given $t_1, t_2 \in J$ with $t_2 > t_1$, we obtain,

$$\begin{aligned}
\Big|Bx(t_2) - Bx(t_1)\Big| &= \left| \frac{1}{\Gamma(q)} \int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\, ds \right. \\
&\quad \left. - \frac{1}{\Gamma(q)} \int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\, ds \right| \\
&\leq \left| \frac{1}{\Gamma(q)} \int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\, ds \right. \\
&\quad \left. - \frac{1}{\Gamma(q)} \int_0^{t_2} \frac{v(t_1,s)}{(t_2-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\, ds \right| \\
&\quad + \left| \frac{1}{\Gamma(q)} \int_0^{t_2} \frac{v(t_1,s)}{(t_2-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\, ds \right. \\
&\quad \left. - \frac{1}{\Gamma(q)} \int_0^{t_1} \frac{v(t_1,s)}{(t_2-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\, ds \right| \\
&\quad + \left| \frac{1}{\Gamma(q)} \int_0^{t_1} \frac{v(t_1,s)}{(t_2-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\, ds \right. \\
&\quad \left. - \frac{1}{\Gamma(q)} \int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{1-q}}\, g(s,x(s),x(\eta(s)))\, ds \right| \\
&\leq \frac{1}{\Gamma(q)} \int_0^{t_2} \frac{|v(t_2,s) - v(t_1,s)|}{(t_2-s)^{1-q}}\, |g(s,x(s),x(\eta(s)))|\, ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} \frac{|v(t_1,s)|}{(t_2-s)^{1-q}}\, |g(s,x(s),x(\eta(s)))|\, ds \\
&\quad + \frac{1}{\Gamma(q)} \int_0^{t_1} |v(t_1,s)|\, |[(t_2-s)^{q-1} - (t_1-s)^{q-1}]|\, |g(s,x(s),x(\eta(s)))|\, ds \\
&\leq \frac{1}{\Gamma(q)} \int_0^T \frac{|v(t_2,s) - v(t_1,s)|}{(t_2-s)^{1-q}}\, h(s)\, ds + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} \frac{|v(t,s)|}{(t_2-s)^{1-q}}\, h(s)\, ds \\
&\quad + \frac{1}{\Gamma(q)} \int_0^T |v(t_1,s)|\, \Big|\big[(t_2-s)^{q-1} - (t_1-s)^{q-1}\big]\Big|\, h(s)\, ds \longrightarrow 0 \quad \text{as} \quad t_1 \to t_2,
\end{aligned}$$

uniformly for all $t_1, t_2 \in J$ and $x \in S$. As a result $B(S)$ is an equicontinuous set in E. Since $B(S)$ is uniformly bounded and equicontinuous set, by an application of

Arzelá-Ascoli theorem, $B(S)$ is a compact set in E. Hence B is a compact operator on S into E.

Step V. B is continuous on S.

Let $\{x_n\}$ be a sequence in S converging to a point x in S. Then, by the dominated convergence theorem, we have, for all $t \in J$ that

$$\begin{aligned}
\lim_{n\to\infty} Bx_n &= \lim_{n\to\infty} \frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, g(s, x_n(s), x_n(\eta(s)))\, ds \\
&\leq \lim_{n\to\infty} \frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}} \lim_{n\to\infty} g(s, x_n(s), x_n(\eta(s)))\, ds \\
&= \lim_{n\to\infty} \frac{1}{\Gamma(q)} \int_0^t \frac{v(t,s)}{(t-s)^{1-q}}\, g(s, x(s), x(\eta(s)))\, ds = Bx(t),
\end{aligned}$$

This shows that $Bx_n(t) \to Bx(t)$ pointwise on J. Since, $\{Bx_n\} \subset B(S)$, $\{Bx_n\}$ is an equicontinuous sequence in E. Hence $Bx_n(t) \to Bx$ uniformly, whence B is a continuous operator on S.

Step VI. $x = Ax\,By + Cx,\ \forall\, y \in S \Rightarrow x \in S$.

Let $x \in E$ be a fixed element such that $x = Ax\,By + Cx$ for all $y \in S$. Then,

$$\begin{aligned}
|x(t)| &\leq |Ax(t)|\,|By(t)| + |Cx(t)| \leq \big|k(t, x(t), x(\alpha(t)))\big| \\
&\quad + \big|f(t, x(t), x(\beta(t)))\big| \frac{1}{\Gamma(q)} \int_0^t \frac{|v(t,s)|}{(t-s)^{1-q}} |g(s, x(s), x(\eta(s)))|\, ds \\
&\leq \big|k(t, x(t), x(\alpha(t))) - k(t,0,0)\big| + |k(t,0,0)| \\
&\quad + \big[f(t, x(t), x(\alpha(t))) - f(t,0,0)\big| + |f(t,0,0)|\big] \frac{1}{\Gamma(q)} \int_0^t \frac{|v(t,s)|}{(t-s)^{1-q}} h(s)\, ds \\
&\leq \frac{L_2 \max\{|x(t)|, |x(\alpha(t))|\}}{K_2 + \max\{|x(t)|, |x(\alpha(t))|\}} + K_0 \\
&\quad + \frac{1}{\Gamma(q)} \left[\frac{L_1 \max\{|x(t)|, |x(\alpha(t))|\}}{K_1 + \max\{|x(t)|, |x(\alpha(t))|\}} + F_0 \right] \int_0^t \frac{|v(t,s)|}{(t-s)^{1-q}} h(s)\, ds \\
&\leq \frac{L_2\|x\|}{K_2 + \|x\|} + K_0 + \frac{1}{\Gamma(q)} \left[\frac{L_1\|x\|}{K_1 + \|x\|} + F_0 \right] \int_0^t \frac{|v(t,s)|}{(t-s)^{1-q}} h(s)\, ds \\
&\leq L_2 + K_0 + \frac{1}{\Gamma(q)} [L_1 + F_0] \int_0^t \frac{|v(t,s)|}{(t-s)^{1-q}} h(s)\, ds \\
&\leq L_2 + K_0 + \frac{V}{\Gamma(q)} [L_1 + F_0] \int_0^t \frac{h(s)}{(t-s)^{1-q}}\, ds \\
&\leq L_2 + K_0 + \frac{V}{\Gamma(q)} [L_1 + F_0] \left(\int_0^t h^2(s)\, ds \right)^{1/2} \left(\int_0^t \frac{1}{(t-s)^{2-2q}} ds \right)^{1/2} \\
&\leq L_2 + K_0 + \frac{V}{\Gamma(q)} [L_1 + F_0] \left(\int_0^T h^2(s)\, ds \right)^{1/2} \left(\int_0^t (t-s)^{2q-2}\, ds \right)^{1/2} \\
&= L_2 + K_0 + \frac{V}{\Gamma(q)} [L_1 + F_0] \|h\|_{L^2} \left(\frac{t^{2q-1}}{2q-1} \right)^{1/2}
\end{aligned}$$

$$\leq L_2 + K_0 + \frac{V}{\Gamma(q)}[L_1 + F_0]\frac{\|h\|_{L^2} T^{q-1/2}}{(2q-1)^{1/2}} = \rho,$$

which implies that $x \in S$ and so, hypothesis (c) of Theorem (2.4) is satisfied.

Step VII. $M\psi_A(r) + \psi_C(r) < r$ for $r > 0$.

Here, by hypothesis (H_4), we obtain

$$\begin{aligned}
M &= \|B(S)\| = \sup_{x\in S}\|Bx\| = \sup_{x\in S}\left\{\sup_{t\in J}|Bx(t)|\right\} \\
&= \sup_{x\in S}\sup_{t\in J}\left\{\frac{1}{\Gamma(q)}\int_0^t \frac{|v(t,s)|}{(t-s)^{1-q}}|g(s,x(s),x(\eta(s)))|\,ds : t\in J\right\} \\
&\leq \frac{1}{\Gamma(q)}\int_0^T \frac{|v(t,s)|}{(t-s)^{1-q}}h(s)\,ds \leq \frac{V}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)\,ds \leq \frac{V\,\|h\|_{L^2} T^{q-1/2}}{\Gamma(q)(2q-1)^{1/2}}.
\end{aligned}$$

Therefore,

$$M\psi_A(r) + \psi_C(r) \leq \left(\frac{V\,\|h\|_{L^2} T^{q-1/2}}{\Gamma(q)(2q-1)^{1/2}}\right)\frac{L_1 r}{K_1 + r} + \frac{L_2 r}{K_2 + r} < r$$

for $r > 0$, and so, hypothesis (d) of Theorem (2.4) is satisfied.

Thus all the conditions of Theorem 2.4 are satisfied. Hence we apply it to the operator equation (3.7) and conclude that the HFIE (1.1) has a solution defined on J. This completes the proof. □

Next we prove the positivity and monotonic character of the solutions of the functional HFIE (1.1) defined on J. To prove the positivity we need the following hypotheses.

(H_5) f, g, k define the functions $f, g, k : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$.

(H_6) v defines a function $v : J \times J \to \mathbb{R}_+$.

Theorem 3.5. *Assume that all the conditions of Theorem 3.4 hold. Furthermore if hypotheses (H_5) and (H_6) hold, then the functional HFIE* (1.1) *has a positive solution defined on J.*

Proof. By Theorem 3.4, the functional HFIE (1.1) has a solution x defined on J. We show that x is positive on J. To achieve this, it is enough to prove that

$$|x(t)| = x(t), \text{ for all } t \in J. \tag{3.8}$$

Now by (H_4) and (H_5),

$$\begin{aligned}
|x(t)| - x(t) &= |k(t,x(t),x(\alpha(t)))| - k(t,x(t),x(\alpha(t))) \\
&\quad + \left|\left[f(t,x(t),x(\beta(t)))\right]\left(\frac{1}{\Gamma(q)}\int_0^t \frac{v(t,s)}{(t-s)^{1-q}}g(s,x(s),x(\eta(s)))\,ds\right)\right| \\
&\quad - \left[f(t,x(t),x(\beta(t)))\right]\left(\frac{1}{\Gamma(q)}\int_0^t \frac{v(t,s)}{(t-s)^{1-q}}g(s,x(s),x(\eta(s)))\,ds\right) = 0
\end{aligned}$$

for all $t \in J$. Hence $|x(t)| = x(t)$ for all $t \in J$. Consequently x is a positive solution of the functional HFIE (1.1) defined on J. This completes the proof. □

Finally we prove the monotonic character of the solutions of the functional HFIE (1.1) on J. We consider the following hypotheses in the sequel.

(H_7) The maps $t \mapsto f(t,x,y), t \mapsto g(t,x,y), t \mapsto k(t,x,y)$ are monotonic increasing for each $x, y \in \mathbb{R}$.

(H_8) The map $t \mapsto v(t,s)$ is nondecreasing for each $s \in J$ with $t > s$.

Theorem 3.6. *Assume that all the conditions of Theorem 3.5 hold. Further if the hypotheses (H_7) and (H_8) hold, then the functional HFIE (1.1) has a monotonic increasing positive solution defined on J.*

Proof. By Theorem 3.5, the functional HFIE (1.1) has a positive solution defined on J. We show that x is monotonic increasing on J. To finish, it is enough to prove that

$$|x(t) - x(s)| = x(t) - x(s) \tag{3.9}$$

for all $t, s \in J$ with $t > s$.

In what follows, fix an arbitrary $x \in X$ and $t_1,\ t_2 \in J$ with $t_2 > t_1$. Then, taking into account our hypotheses, we have

$$\begin{aligned}
|x(t_2) - x(t_1)| - [x(t_2) - x(t_1)] = \Bigg| & k(t_2, x(t_2), x(\alpha(t_2))) \\
+[f(t_2, x(t_2), x(\beta(t_2)))] & \left(\frac{1}{\Gamma(q)} \int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{1-q}} g(s, x(s), x(\eta(s)))ds \right) \\
& -k(t_1, x(t_1), x(\alpha(t_1))) \\
-[f(t_1, x(t_1), x(\beta(t_1)))] & \left(\frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{1-q}} g(s, x(s), x(\eta(s)))ds \right) \Bigg| \\
& - \Bigg[k(t_2, x(t_2), x(\alpha(t_2))) \\
+[f(t_2, x(t_2), x(\beta(t_2)))] & \left(\frac{1}{\Gamma(q)} \int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{1-q}} g(s, x(s), x(\eta(s)))ds \right) \\
& -k(t_1, x(t_1), x(\alpha(t_1))) \\
-[f(t_1, x(t_1), x(\beta(t_1)))] & \left(\frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{1-q}} g(s, x(s), x(\eta(s)))ds \right) \Bigg] \\
& \leq \big\{ |k(t_2, x(t_2), x(\alpha(t_2))) - k(t_1, x(t_1), x(\alpha(t_1)))| \\
& \quad -[k(t_2, x(t_2), x(\alpha(t_2))) - k(t_1, x(t_1), x(\beta(t_1)))] \big\} \\
+ \Bigg| [f(t_2, x(t_2), x(\beta(t_2)))] & \left(\frac{1}{\Gamma(q)} \int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{1-q}} g(s, x(s), x(\eta(s)))ds \right) \\
- [f(t_1, x(t_1), x(\beta(t_1)))] & \left(\frac{1}{\Gamma(q)} \int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{1-q}} g(s, x(s), x(\eta(s)))ds \right) \Bigg| \\
+ \Bigg[f(t_1, x(t_1), x(\beta(t_1)))] & \left(\frac{1}{\Gamma(q)} \int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{1-q}} g(s, x(s), x(\eta(s)))ds \right) \\
- [f(t_1, x(t_1), x(\beta(t_1)))] & \left(\frac{1}{\Gamma(q)} \int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{1-q}} g(s, x(s), x(\eta(s)))ds \right) \Bigg] \\
& \leq \big\{ |k(t_2, x(t_2), x(\alpha(t_2))) - k(t_1, x(t_1), x(\alpha(t_1)))|
\end{aligned}$$

$$
\begin{aligned}
&-[k(t_2, x(t_2), x(\alpha(t_2))) - k(t_1, x(t_1), x(\alpha(t_1)))]\Big\} \\
&+\Bigg\{|f(t_2, x(t_2), x(\beta(t_2))) - f(t_1, x(t_1), x(\beta(t_1)))|\times \\
&\times\left(\frac{1}{\Gamma(q)}\int_0^{t_2}\frac{v(t_2,s)}{(t_2-s)^{1-q}}g(s,x(s),x(\eta(s)))ds\right) \\
+|f(t_1, x(t_1), x(\beta(t_1)))|&\left|\left(\frac{1}{\Gamma(q)}\int_0^{t_2}\frac{v(t_2,s)}{(t_2-s)^{1-q}}g(s,x(s),x(\eta(s)))ds\right)\right. \\
&\left.-\left(\frac{1}{\Gamma(q)}\int_0^{t_1}\frac{v(t_1,s)}{(t_1-s)^{1-q}}g(s,x(s),x(\eta(s)))ds\right)\right|\Bigg\} \\
&+\Bigg\{[f(t_2, x(t_2), x(\beta(t_2))) - f(t_1, x(t_1), x(\beta(t_1)))]\times \\
&\times\left(\frac{1}{\Gamma(q)}\int_0^{t_2}\frac{v(t_2,s)}{(t_2-s)^{1-q}}g(s,x(s),x(\eta(s)))ds\right) \\
+[f(t_1, x(t_1), x(\beta(t_1)))]&\Bigg[\left(\frac{1}{\Gamma(q)}\int_0^{t_2}\frac{v(t_2,s)}{(t_2-s)^{1-q}}g(s,x(s),x(\eta(s)))ds\right) \\
&-\left(\frac{1}{\Gamma(q)}\int_0^{t_1}\frac{v(t_1,s)}{(t_1-s)^{1-q}}g(s,x(s),x(\eta(s)))ds\right)\Bigg]\Bigg\}.
\end{aligned}
$$

Now, we will prove that

$$
\int_0^{t_2}\frac{v(t_2,s)}{(t_2-s)^{1-q}}g(s,x(s),x(\eta(s)))ds - \int_0^{t_1}\frac{v(t_1,s)}{(t_1-s)^{1-q}}g(s,x(s),x(\eta(s)))ds \geq 0.
$$

In fact, we have

$$
\begin{aligned}
&\int_0^{t_2}\frac{v(t_2,s)}{(t_2-s)^{1-q}}g(s,x(s),x(\eta(s)))ds - \int_0^{t_1}\frac{v(t_1,s)}{(t_1-s)^{1-q}}g(s,x(s),x(\eta(s)))ds \\
= &\int_0^{t_2}\frac{v(t_2,s)}{(t_2-s)^{1-q}}g(s,x(s),x(\eta(s)))ds - \int_0^{t_2}\frac{v(t_1,s)}{(t_2-s)^{1-q}}g(s,x(s),x(\eta(s)))ds \\
&+\int_0^{t_2}\frac{v(t_1,s)}{(t_2-s)^{1-q}}g(s,x(s),x(\eta(s)))ds - \int_0^{t_1}\frac{v(t_1,s)}{(t_2-s)^{1-q}}g(s,x(s),x(\eta(s)))ds \\
&+\int_0^{t_1}\frac{v(t_1,s)}{(t_2-s)^{1-q}}g(s,x(s),x(\eta(s)))ds - \int_0^{t_1}\frac{v(t_1,s)}{(t_1-s)^{1-q}}g(s,x(s),x(\eta(s)))ds \\
= &\int_0^{t_2}\frac{v(t_2,s)-v(t_1,s)}{(t_2-s)^{1-q}}g(s,x(s),x(\eta(s)))ds + \int_{t_1}^{t_2}\frac{v(t_1,s)}{(t_2-s)^{1-q}}g(s,x(s),x(\eta(s)))ds \\
&+\int_0^{t_1}v(t_1,s)\,[(t_2-s)^{q-1}-(t_1-s)^{q-1}]g(s,x(s),x(\eta(s)))ds.
\end{aligned}
$$

Since $v(t,s)$ is nondecreasing with respect to t, we have that $v(t_2,s) \geq v(t_1,s)$ and therefore

$$\int_0^{t_2} \frac{v(t_2,s)-v(t_1,s)}{(t_2-s)^{1-q}} g(s,x(s),x(\eta(s)))ds \geq 0. \tag{3.10}$$

On the other hand, since the term $(t_2-s)^{q-1}-(t_1-s)^{q-1}$ is negative for $0 \leq s < t_1$, we have

$$\begin{aligned}
& \int_0^{t_1} v(t_1,s)\ [(t_2-s)^{q-1}-(t_1-s)^{q-1}] g(s,x(s),x(\eta(s)))ds \\
& + \int_{t_1}^{t_2} \frac{v(t_1,s)}{(t_2-s)^{1-q}} g(s,x(s),x(\eta(s)))ds \\
\geq\ & \int_0^{t_1} v(t_1,t_1)\ [(t_2-s)^{q-1}-(t_1-s)^{q-1}] g(t_1,x(t_1),x(\eta(t_1))) \\
& + \int_{t_1}^{t_2} \frac{v(t_1,t_1)}{(t_2-s)^{1-q}} g(t_1,x(t_1),x(\eta(t_1)))ds \qquad (3.11) \\
=\ & v(t_1,t_1) g(t_1,x(t_1),x(\eta(t_1))) \left[\int_0^{t_2} \frac{ds}{(t_2-s)^{1-q}} - \int_0^{t_1} \frac{ds}{(t_1-s)^{1-q}}\right] \\
=\ & v(t_1,t_1) g(t_1,x(t_1),x(\eta(t_1))) \frac{t_2^q - t_1^q}{q} \geq 0.
\end{aligned}$$

Hence

$$\int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{1-q}} g(t,x(t),x(\eta(t)))ds - \int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{1-q}} g(t,x(t),x(\eta(t)))ds \geq 0.$$

This together with (3.10) yields

$$|x(t_2)-x(t_1)| - [x(t_2)-x(t_1)] \leq 0,$$

which proves that $|x(t_2)-x(t_1)| = [x(t_2)-x(t_1)]$, and so x is monotonic increasing on J. This completes the proof. □

4. Examples

In this section we illustrate the abstract theory developed in the previous section by giving some examples of nonlinear fractional integral equations.

Example 4.1. Given a closed and bounded interval $J=[0,1]$ of the real line $\mathbb{R}$, consider the fractional integral equation

$$x(t) = \frac{t}{3}|x(t)| + \frac{t^2}{1+t^2} \cdot \frac{|x(t)|}{1+|x(t)|} \left(\frac{1}{\Gamma(\frac{3}{2})} \int_0^t (t-s)^{\frac{1}{2}} \frac{\log(1+|x(s)|)}{1+|x(s)|}\, ds \right), t \in J \tag{4.1}$$

Let $q = \dfrac{3}{2}$ and define the functions $f,g,k : J \times \mathbb{R} \to \mathbb{R}$ by

$$f(t,x) = \left(\frac{t^2}{1+t^2}\right) \frac{|x|}{1+|x|}, g(t,x) = \frac{\log(1+|x|)}{1+|x|}, k(t,x) = \frac{t}{3}|x|.$$

It is easy to verify that the functions f, g and k satisfy all the hypotheses (H_1) through (H_8) of Theorem 3.6 with the values $v \equiv 1, L_2 = \dfrac{1}{3}, L_1 = \dfrac{1}{2}, K_1 = 1$ and $M = \dfrac{2}{\sqrt{\pi}}$, so that in view of Remark 2.5, we obtain, for every $r > 0$, that

$$\left(\frac{V\,\|h\|_{L^1}\,T^{q-1/2}}{\Gamma(q)(2q-1)^{1/2}}\right)\frac{L_1}{K_1+r}+L_2 \quad = \quad \frac{2}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\,\frac{1}{1+r}+\frac{1}{3}<1.$$

Hence the fractional integral equation (4.1) has a positive and monotone nondecreasing solution defined on J.

Example 4.2. Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, consider the nonlinear fractional integral equation

$$x(t) = \frac{t^2}{t^2+1}\cdot\frac{|x(t)|}{2+|x(t)|} + \left[\frac{t}{2}|x(t)|\right]\left(\frac{1}{\Gamma(\frac{3}{2})}\int_0^t \big(t-s\big)^{\frac{1}{2}}\ \frac{\log(1+|x(s)|)}{1+|x(s)|}\,ds\right) \tag{4.2}$$

for all $t \in J$. Now following the arguments similar to those given in Example 4.1 it is proved that the nonlinear fractional hybrid integral equation (4.2) has a positive and monotonic nondecreasing solution defined on J.

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