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ITERATIVE ALGORITHM FOR ZEROS OF BOUNDED MULTI-VALUED ACCRETIVE OPERATORS

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Abstract. Let E be a uniformly smooth real Banach space and $A : E \to 2^E$ a multi-valued mapping. An efficient iteration algorithm for approximating zeros of A, in the case that A is m-accretive and bounded, is studied and the sequence of the algorithm is proved to converge strongly to a point in $A^{-1}(0)$. We achieve this by using the celebrated result of Simeon Reich.

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1. Introduction

For several years, the study of fixed point theory for *multi-valued nonlinear mappings* has attracted, and continues to attract, the interest of several well known mathematicians, (see e.g., Brouwer [2], Downing and Kirk [13], Geanakoplos [16], Kakutani [20], Nadler [28], Nash [29, 30] and the references therein).
Interest in such studies stems, perhaps, mainly from the usefulness of such fixed point theory in real-world applications, such as in *Game Theory, Market Economy* and other areas of mathematics,(for details, see e.g.,[11] and the references contained therein).
Consider the following problem:

$$\text{Find } u \in H \text{ such that } 0 \in Au \tag{1.1}$$

where H is a real Hilbert space and A is a *maximal monotone operator* on H. It is well known that fixed point theory for nonlinear maps is closely related to the theory of existence and approximation of solution of problem (1.1) for certain nonlinear operator, A (see e.g., [4, 8, 30] and the references therein). Several methods of approximating solution of (1.1) assuming existence have been proposed and studied by many authors (see e.g. [18, 22, 27, 33, 35, 38, 39] and the references therein).
Recently, Chidume and Djitte [9] proved the following result for approximating solution of (1.1) assuming existence.

Theorem 1.1. *Let E be a 2-uniformly smooth real Banach space and let $A : E \to E$ be a bounded m-accretive map. For arbitrary $x_1 \in E$, define the sequence $\{x_n\}$ iteratively by*

$$x_{n+1} := x_n - \lambda_n A x_n - \lambda_n \theta_n (x_n - x_1),\ n \geq 1, \tag{1.2}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(1) $\lim \theta_n = 0$; *and $\{\theta_n\}$ is decreasing;*

(2) $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty, \quad \lambda_n = o(\theta_n)$;

(3) $\lim_{n\to\infty} \frac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n \theta_n} = 0, \sum_{n=1}^{\infty} \lambda_n^2 < \infty.$

Suppose that the equation $Ax = 0$ has a solution. Then, there exists a constant $\gamma_0 > 0$ such that if $\lambda_n \leq \gamma_0 \theta_n \ \forall\, n \geq 1$, $\{x_n\}$ converges strongly to a solution of the equation $Ax = 0$.

Here, we continue the study of the problem $0 \in Au$ for the much more general case where A is multi-valued *m-accretive* and bounded and in a more general uniformly smooth real Banach space.

Definition 1.1. Let E be a real normed linear space. A map $T : D(T) \subset E \to E$ is called *pseudo-contractive* (see, *e.g.*, [4]) if the inequality

$$\|x - y\| \leq \|x - y + t((x - Tx)) - (y - Ty))\| \tag{1.3}$$

holds for each $x, y \in D(T)$ and for all $t > 0$. As a result of Kato [17], it follows from inequality (1.3) that T is *pseudo-contractive* if and only if for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2, \tag{1.4}$$

where $J : E \to 2^{E^*}$ is the normalized duality mapping.

Definition 1.2. (see, *e.g.*, [31]) Let E be a normed space. A multi-valued mapping $T : D(T) \to 2^E$ is called *pseudo-contractive* if for all $x, y \in D(T)$, we have

$$\langle u - v, j(x - y)\rangle \leq \|x - y\|^2 \quad \forall\, u \in Tx,\ v \in Ty. \tag{1.5}$$

The class of pseudo-contractive mappings is deeply connected with the class of accretive operators, where an operator A with domain $D(A)$ in E is called *accretive* if the inequality $\|x-y\| \leq \|x-y+s(u-v)\|$ holds for each $x, y \in D(A)$, $u \in Ax$, $v \in Ay$ and for all $s > 0$ (see e.g.,[4]). In Hilbert spaces, accretive operators are called *monotone*. We remark that A is accretive if and only if $T := I - A$ is pseudo-contractive and thus, the set of zeros of A, $N(A) := \{x \in D(A) : x \in A^{-1}(0)\}$, coincides with the fixed point set of T (see [4, 8] for more details). Accretive operators were introduced and studied independently by Browder and Kato (see [3, 4, 17]).

It is easy to see that every nonexpansive map is pseudocontractive. In general, pseudocontractive maps are not continuous. It suffices, for example, to consider the map $T:[0,1]\to\mathbb{R}$ defined by

$$Tx=\begin{cases} x-\frac{1}{2} & \text{if } \quad x\in[0,\frac{1}{2});\\ \\ x-1 & \text{if } \quad x\in[\frac{1}{2},1].\end{cases}$$

Let K be a nonempty subset of a normed space E. The set K is called *proximinal* (see, *e.g.*, [32, 34, 36]) if for each $x\in E$, there exists $u\in K$ such that

$$d(x,u)=\inf\{\|x-y\|:y\in K\}=d(x,K),$$

where $d(x,y)=\|x-y\|$ for all $x,y\in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximinal. Let $CB(K)$ and $P(K)$ denote the families of nonempty, closed and bounded subsets and nonempty, proximinal and bounded subsets of K, respectively. The *Hausdorff metric* on $CB(K)$ is defined by:

$$D(A,B)=\max\Big\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(b,A)\Big\}$$

for all $A,B\in CB(K)$. Let $T:D(T)\subseteq E\to CB(E)$ be a *multi-valued mapping* on E. A point $x\in D(T)$ is called a *fixed point of* T if $x\in Tx$. The fixed point set of T is denoted by $F(T):=\{x\in D(T):x\in Tx\}$.
A multi-valued mapping $T:D(T)\subseteq E\to CB(E)$ is called L- *Lipschitzian* if there exists $L>0$ such that

$$D(Tx,Ty)\leq L\|x-y\|\;\;\forall x,y\in D(T). \tag{1.6}$$

When $L\in(0,1)$ in (1.6), we say that T is a *contraction*, and T is called *nonexpansive* if $L=1$.
Several results have been proved for the problem of *approximating* fixed points of *multi-valued nonexpansive* mappings and their generalizations, when the operator is defined using the Hausdorff metric and when it is defined without the Hausdorff metric, using either the Mann-type sequence,[24] or the Ishikawa-type sequence [19], (see, *e.g.*, [1, 15, 21, 32, 34, 36], and the references therein).

Remark 1.3. We note that for approximating fixed point of a *multi-valued Lipschitz pseudo-contractive map* in a real Hilbert space, an example of Chidume and Mutangadura [5] shows that, even in the single-valued case, the Mann iteration method does not always converge strongly.
Chidume and Zegeye [6] later introduced an iteration algorithm which converges in this setting. Motivated by this algorithm, Ofoedu and Zegeye [31] introduced an iteration scheme for approximating a fixed point of a *multi-valued* Lipschitz pseudo-contractive mapping. They proved the following theorem.

Theorem 1.4. (Ofoedu and Zegeye [31]) *Let E be a reflexive real Banach space having uniformly Gâteaux differentiable norm, D be a nonempty open convex subset of E, such that every closed convex bounded nonempty subset of $\overline{D}$ has the fixed point property for nonexpansive self-mappings. Let $T:\overline{D}\to K(\overline{D})$ be a pseudo-contractive Lipschitzian mapping with constant $L>0$ and let $u\in\overline{D}$ be fixed. Let $\{x_n\}$ be* *generated from arbitrary* $x_0 \in \overline{D}$, $w_0 \in Tx_0$ *by*

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n w_n - \lambda_n\theta_n(x_n - u),\ w_n \in Tx_n. \tag{1.7}$$

Suppose that $\|w_n - w_{n-1}\| = d(w_{n-1}, Tx_n)$, $n \geq 1$. *If* $F(T) \neq \emptyset$, *then* $\{x_n\}$ *converges strongly to some fixed point of* T.

Remark 1.5. Nadler [28] remarked that requiring a *multivalued mapping* to be *Lipschitz* is placing a *strong continuity condition* on the mapping.
Recently, Chidume *et al*, [10] weakened the *Lipschitz* continuity assumption on T in theorem OZ and proved a strong convergence theorem for multi-valued *continuous* and *bounded* pseudocontractive mapping T. Precisely, they proved the following result.

Theorem 1.6. *Let* E *be a* q*-uniformly smooth real Banach space and* D *be a nonempty, open and convex subset of* E. *Assume that* $T : \overline{D} \to CB(\overline{D})$ *is a multi-valued continuous, bounded and pseudo-contractive mapping with* $F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated iteratively from arbitrary* $x_1 \in \overline{D}$ *by*

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n u_n - \lambda_n\theta_n(x_n - x_1),\ u_n \in Tx_n, \tag{1.8}$$

where $\{\lambda_n\}$ *and* $\{\theta_n\}$ *are real sequences in* $(0,1)$ *satisfying the following conditions:*

(i) $\lim\theta_n = 0$;

(ii) $\lambda_n(1+\theta_n) < 1, \sum_{n=1}^{\infty}\lambda_n\theta_n = \infty,\ \ \lambda_n^{q-1} = o(\theta_n)$;

(iii) $\limsup_{n\to\infty} \dfrac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n\theta_n} \leq 0,\ \ \sum_{n=1}^{\infty}\lambda_n^q < \infty.$

Then, there exists a real constant $\gamma_0 > 0$ *such that if* $\lambda_n^{q-1} < \gamma_0\theta_n$, *for all* $n \geq 1$, *the sequence* $\{x_n\}$ *converges strongly to a fixed point of* T.

Remark 1.7. It is known that if $A : D(A) \subseteq E \to 2^E$ is a multivalued *continuous accretive* map, then it is always single-valued in the interior of its domain. In fact, this result holds if continuity is replaced by lower semi-continuity (see, *e.g.,* Chidume and Morales [7], or Chidume [8], chapter 23).

Definition 1.8. A multi-valued map A defined on a normed linear space E is called m-accretive if it is accretive and $R(I+rA) = E$ for some $r > 0$ and it is said to satisfy the *range condition* $R(I+rA) = E$ for all $r > 0$.
Example. Let $A : \mathbb{R} \to 2^{\mathbb{R}}$ defined by

$$Ax = \begin{cases} sgn(x), & x \neq 0 \\ [-1,1], & x = 0, \end{cases} \tag{1.9}$$

where A is the subdifferential of the absolute value function, $\partial|.|$, then A is m-accretive.
It is known that if $R(I+rA) = E$ for some $r > 0$, then $R(I+rA) = E$ for all $r > 0$, (see e.g.,[9]). Hence, m-accretive condition implies range condition.

Motivated by remark 1.7 and the on-going research in this direction, it is our purpose in this paper to extend the result of Chidume *et al*, [9, 10] and that of Ofoedu and Zegeye [31] to the case where the operator A is m-accretive, multi-valued and bounded in uniformly smooth real Banach space without any continuity assumption on the operator A.

2. Preliminaries

Let E be a real normed space with dual E^* and let $S := \{x \in E : \|x\| = 1\}$. The space E is said to have *Gâteaux differentiable norm* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S$; E is said to have *uniformly Gâteaux differentiable norm* if for each $y \in S$, the limit is attained uniformly for $x \in S$.
The space E is said to have *Frêchet differentiable norm* if for each $x \in S(E) := \{u \in E : ||u|| = 1\}$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $y \in S(E)$.
Let E be a real normed linear space of dimension ≥ 2. The *modulus of smoothness* of E , ρ_E, is defined by:

$$\rho_E(\tau) := \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \quad \tau > 0.$$

A normed linear space E is called *uniformly smooth* if $\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$. It is well known (see, *e.g.* [8], [23]) that ρ_E is nondecreasing. If there exist a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth.* Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p \ (or\ l_p)\ or\ W_p^m \text{ is } \begin{cases} 2 - \text{uniformly smooth} & \text{if } \ 2 \leq p < \infty; \\ p - \text{uniformly smooth} & \text{if } \ 1 < p < 2. \end{cases}$$

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\right\}$$

where $\langle ., .\rangle$ denotes the generalized duality pairing. J_2 is called the *normalized duality mapping* and is denoted by J. It is well known that if E is smooth, J_q is single-valued. Every uniformly smooth real normed space has uniformly Gâteaux differentiable norm (see, *e.g.*, [8]).
In the sequel we shall need the following results.
Lemma 2.1. (Reich, [14]) *Let E be a uniformly smooth Banach space, and let $A \subset E \times E$ be m-accretive. If $0 \in R(A)$, then for each $x \in E$ the strong $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}(0)$, where J_t stands for the resolvent operator of A with parameter t.*

Lemma 2.2. (Cholamjiak and Suantai, [12]) *Let E be a real Banach space with Frêchet differentiable norm. For $x \in E$, let $\beta^*(t)$ be defined for $0 < t < \infty$ by*

$$\beta^*(t) := \sup\Big\{\frac{(||x+ty||^2 - ||x||^2)}{t} - 2Re\langle y, J(x)\rangle : ||y|| = 1\Big\}.$$

Then, $\lim_{t\to 0^+} \beta^*(t) = 0,$ and,

$$||x+h||^2 \le ||x||^2 + 2\langle h, j(x)\rangle + ||h||\beta^*(||h||)\forall h \in E\ \{0\}. \tag{2.1}$$

Remark 2.3. In a real Hilbert space, we see that $\beta^*(t) = t$ for $t > 0$.
In $L_p, 2 \le p < \infty$, β^* in (2.1) is estimated by $\beta^*(t) \le (p-1)t$ for $t > 0$.
For the rest of this paper, we shall assume that $\beta(t) \le b_0 t,\ t > 0,$ for some $b_0 > 1$.
Lemma 2.4. *Let E be a real normed linear space. Then, the following inequality holds:*

$$||x+y||^2 \le ||x||^2 + 2\langle y, j(x+y)\rangle\ \ \forall\, j(x+y) \in J(x+y),\ \forall\, x,y \in E. \tag{2.2}$$

Lemma 2.5. (Xu, [38]) *Let $\{\rho_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$\rho_{n+1} \le (1-\alpha_n)\rho_n + \alpha_n\sigma_n + \gamma_n,\ \ n \ge 0,$$

where,
(i) $\{\alpha_n\} \subset (0,1),\ \sum \alpha_n = \infty$*; (ii)* $\limsup_{n\to\infty} \sigma_n \le 0$*;*
(ii) $\gamma_n \ge 0,\ \sum \gamma_n < \infty$*. Then,* $\rho_n \to 0$ *as* $n \to \infty$*.*

3. Main results

We now prove our main result.
Theorem 3.1. *Let E be a uniformly smooth real Banach space and let $A : E \to 2^E$ be a multi-valued bounded m-accretive map. Assume $A^{-1}(0) \ne \emptyset$. For arbitrary $x_1 \in E$, define the sequence $\{x_n\}$ iteratively by*

$$x_{n+1} := x_n - \lambda_n u_n - \lambda_n\theta_n(x_n - x_1),\ \ u_n \in Ax_n\ n \ge 1, \tag{3.1}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(1) $\lim_{\infty} \theta_n = 0$*; $\{\theta_n\}$ is decreasing;*
(2) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty;\ \ \lambda_n = o(\theta_n)$*;*
(3) $\lim_{n\to\infty} \frac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n\theta_n} = 0;\ \sum_{n=1}^{\infty} \lambda_n^2 < \infty.$

Then, there exists a constant $\gamma_0 > 0$ such that if $\lambda_n < \gamma_0\theta_n$, the sequence $\{x_n\}$ converges strongly to x^*, where $x^* \in A^{-1}(0)$.
Proof. Let $x^* \in E$ such that $x^* \in A^{-1}(0)$. Then, there exists $r > 0$ sufficiently large such that $x_1 \in B(x^*, r/2)$. Set $B := \overline{B(x^*, r)}$. Since A is bounded, it follows that

$A(B)$ is bounded. Define

$$\begin{aligned} M_1 &:= \sup\{||u + \theta(x - x_1)|| : x \in B,\ u \in Ax,\ 0 < \theta < 1\} + 1 \\ M &:= b_0 M_1^2 \text{ and } \gamma_0 = \frac{r^2}{4M}. \end{aligned}$$

Step1. We prove that $\{x_n\}$ is bounded. Indeed, it suffices to prove by induction that x_n is in B for all $n \geq 1$. By construction, $x_1 \in B$. Suppose that $x_n \in B$ for some $n \geq 1$. We prove that $x_{n+1} \in B$.
Using Lemma 2.2 and the recursion formula (3.6), we have:

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||x_n - x^* - \lambda_n(u_n + \theta_n(x_n - x_1))||^2 \\ &\leq ||x_n - x^*||^2 - 2\lambda_n\langle u_n, j(x_n - x^*)\rangle - 2\lambda_n\theta_n\langle x_n - x_1, j(x_n - x^*)\rangle \\ &+ ||\lambda_n[u_n + \theta_n(x_n - x_1)]||\beta^*\Big(||\lambda_n[u_n + \theta_n(x_n - x_1)]||\Big) \\ &\leq ||x_n - x^*||^2 - 2\lambda_n\langle u_n, j(x_n - x^*)\rangle - 2\lambda_n\theta_n\langle x_n - x_1, j(x_n - x^*)\rangle \\ &+ \lambda_n||u_n + \theta_n(x_n - x_1)||\beta^*\Big(\lambda_n||u_n + \theta_n(x_n - x_1)||\Big). \end{aligned} \tag{3.2}$$

Since A is accretive and $x^* \in A^{-1}(0)$, then $\langle u_n, j(x_n - x^*)\rangle \geq 0$. Hence, we obtain that

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq ||x_n - x^*||^2 - 2\lambda_n\theta_n||x_n - x^*||^2 + 2\lambda_n\theta_n\langle x_1 - x^*, j(x_n - x^*)\rangle \\ &+ b_0 M_1^2 \lambda_n^2 \\ &\leq (1 - 2\lambda_n\theta_n)||x_n - x^*||^2 + \lambda_n\theta_n(||x_1 - x^*||^2 + ||x_n - x^*||^2) + \lambda_n^2 M \\ &\leq (1 - \lambda_n\theta_n)r^2 + \lambda_n\theta_n\frac{r^2}{4} + \lambda_n\theta_n\frac{r^2}{4} \\ &= \Big(1 - \frac{\lambda_n\theta_n}{2}\Big)r^2 \leq r^2. \end{aligned}$$

This implies that $x_{n+1} \in B$, so by induction, $x_n \in B\ \forall\ n \geq 1$. Therefore, $\{x_n\}$ is bounded.
Step 2. We prove that $\{x_n\}$ converges strongly to $x^* \in A^{-1}(0)$. Since A is m-accretive, using Lemma 2, there exists a sequence $\{y_n\}$ in E satisfying the following properties:
(i) $\theta_n(y_n - x_1) + w_n = 0$, for some $w_n \in Ay_n,\ \forall\, n \geq 1$,
(ii) $y_n \to x^*$ with $x^* \in A^{-1}(0)$.
Indeed, applying Lemma 2, with $t = \frac{1}{\theta_n}$, the sequence $\{y_n\}$ defined by

$$y_n := \Big(I + \frac{1}{\theta_n}A\Big)^{-1}(x_1)$$

has the properties (i) and (ii).

Claim. $\|x_{n+1} - y_n\| \to 0$ as $n \to 0$. Using Lemma 2, we have

$$\begin{aligned}\|x_{n+1} - y_n\|^2 &= \|x_n - y_n - \lambda_n(u_n + \theta_n(x_n - x_1))\|^2 \\ &\leq \|x_n - y_n\|^2 - 2\lambda_n\langle u_n + \theta_n(x_n - x_1), j(x_n - y_n)\rangle \\ &+ ||\lambda_n[u_n + \theta_n(x_n - x_1)]||\beta^*\Big(||\lambda_n[u_n + \theta_n(x_n - x_1)]||\Big) \\ &= \|x_n - y_n\|^2 - 2\lambda_n\langle u_n - w_n + w_n + \theta_n(x_n - x_1), j(x_n - y_n)\rangle \\ &+ ||\lambda_n[u_n + \theta_n(x_n - x_1)]||\beta^*\Big(||\lambda_n[u_n + \theta_n(x_n - x_1)]||\Big)\end{aligned}$$

Since A is accretive, using conclusion (i), we have

$$\langle u_n - w_n + w_n + \theta_n(x_n - x_1), j(x_n - y_n)\rangle \geq \theta_n||x_n - y_n||^2 \geq \frac{1}{2}\theta_n||x_n - y_n||^2.$$

Furthermore, since $\{x_n\}$ is bounded and A is bounded, there exists a positive constant K such that:

$$\|x_{n+1} - y_n\|^2 \leq (1 - \lambda_n\theta_n)||x_n - y_n||^2 + K\lambda_n^2 \tag{3.3}$$

Using again the fact that A is accretive, we obtain:

$$\|y_{n-1} - y_n\| \leq \Big\|y_{n-1} - y_n + \frac{1}{\theta_n}\Big(w_{n-1} - w_n\Big)\Big\|.$$

From conclusion (i) and observing that

$$y_{n-1} - y_n + \frac{1}{\theta_n}\Big(w_{n-1} - w_n\Big) = \frac{\theta_n - \theta_{n-1}}{\theta_n}(y_{n-1} - x_1),$$

it follows that

$$\|y_{n-1} - y_n\| \leq \frac{\theta_{n-1} - \theta_n}{\theta_n}\|y_{n-1} - x_1\|. \tag{3.4}$$

By Lemma 2.4, we have

$$\begin{aligned}\|x_n - y_n\|^2 &= \|(x_n - y_{n-1}) + (y_{n-1} - y_n)\|^2 \\ &\leq \|x_n - y_{n-1}\|^2 + 2\langle y_{n-1} - y_n, j(x_n - y_n)\rangle.\end{aligned}$$

Using Schwartz's inequality, we obtain:

$$\|x_n - y_n\|^2 \leq \|x_n - y_{n-1}\|^2 + 2\|y_{n-1} - y_n\|\|x_n - y_n\|. \tag{3.5}$$

Using (3.3), (3.4), (3.5) and the fact that $\{x_n\}$ and $\{y_n\}$ are bounded, we have:

$$\begin{aligned}\|x_{n+1} - y_n\|^2 &\leq (1 - \lambda_n\theta_n)\|x_n - y_{n-1}\|^2 + K_1\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big) + K\lambda_n^2. \\ &= (1 - \lambda_n\theta_n)\|x_n - y_{n-1}\|^2 + (\lambda_n\theta_n)\sigma_n + \gamma_n\end{aligned}$$

for some positive constant $K_1 > 0$, where

$$\sigma_n := \frac{K_1\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)}{\lambda_n\theta_n} = K_1\Big(\frac{\frac{\theta_{n-1}}{\theta_n} - 1}{\lambda_n\theta_n}\Big), \ \ \gamma_n := K\lambda_n^2.$$

Thus, using Lemma 2.5, the conditions $\lim_{n\to\infty}\dfrac{\left(\frac{\theta_{n-1}}{\theta_n}-1\right)}{\lambda_n\theta_n}=0$ and $\sum_{n=1}^{\infty}\lambda_n^2<\infty$, it follows that $x_{n+1}-y_n\to 0$. Hence from conclusion (ii), we have that $x_n\to x^*$ with $x^*\in A^{-1}(0)$. This completes the proof.

Corollary 3.2. *Let $E=L_p, 2\leq p<\infty$ and let $A:E\to 2^E$ be a bounded m-accretive map. Assume $A^{-1}(0)\neq\emptyset$. For arbitrary $x_1\in E$, define the sequence $\{x_n\}$ iteratively by*

$$x_{n+1}:=x_n-\lambda_n Ax_n-\lambda_n\theta_n(x_n-x_1)\ n\geq 1, \tag{3.6}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(1) $\lim\theta_n=0$*; $\{\theta_n\}$ is decreasing;*

(2) $\sum_{n=1}^{\infty}\lambda_n\theta_n=\infty;\quad \lambda_n=o(\theta_n)$*;*

(3) $\lim_{n\to\infty}\dfrac{\left(\frac{\theta_{n-1}}{\theta_n}-1\right)}{\lambda_n\theta_n}=0;\ \sum_{n=1}^{\infty}\lambda_n^2<\infty.$

Then, there exists a real constant $\gamma_0>0$ such that if $\lambda_n<\gamma_0\theta_n\ \ \forall\, n\geq 1$, the sequence $\{x_n\}$ converges strongly to $x^*\in A^{-1}(0)$.

Corollary 3.3. *Let E be a uniformly smooth real Banach space and let $A:E\to E$, be a bounded m-accretive map. Assume $N(A)\neq\emptyset$. For arbitrary $x_1\in E$, define the sequence $\{x_n\}$ iteratively by*

$$x_{n+1}:=x_n-\lambda_n Ax_n-\lambda_n\theta_n(x_n-x_1)\ n\geq 1, \tag{3.7}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(1) $\lim\theta_n=0$*; $\{\theta_n\}$ is decreasing;*

(2) $\sum_{n=1}^{\infty}\lambda_n\theta_n=\infty;\quad \lambda_n=o(\theta_n)$*;*

(3) $\lim_{n\to\infty}\dfrac{\left(\frac{\theta_{n-1}}{\theta_n}-1\right)}{\lambda_n\theta_n}=0;\ \sum_{n=1}^{\infty}\lambda_n^2<\infty.$

Then, there exists a real constant $\gamma_0>0$ such that if $\lambda_n<\gamma_0\theta_n\ \ \forall\, n\geq 1$, the sequence $\{x_n\}$ converges strongly to x^*, a solution of the equation $Ax=0$.

Remark 3.4. Let E be a real Banach space and $A:E\to E$. It is known (see e.g., [25], [26]) that if A is single-valued, continuous and accretive, then A satisfies range condition. Consequently, A is m-accretive.

Remark 3.5. The main result of this paper, Theorem 3.1 extends Theorem 1.1 from single valued m-accretive map to the much more general class of multi-valued accretive map and from 2-uniformly smooth real Banach spaces to uniformly smooth real Banach spaces.

Recall that an operator A defined on a Banach space E is accretive if $I - A$ is pseudocontractive, where I is the identity map on E. Therefore, Theorem 3.1 improves on Theorem 1.6 in the sense that continuity assumption in Theorem 1.3 is dispensed with and from q-uniformly smooth real Banach space to uniformly smooth real Banach space.

Prototype. Real sequences that satisfy the hypotheses of our theorems are

$$\lambda_n = \frac{1}{(n+1)^a},\ n \geq 1, \quad \theta_n = \frac{1}{(n+1)^b},\ n \geq 1,$$

with $0 < b < a, 1/2 < a < 1$ and $a + b < 1$.

4. Numerical example

Let $E = \mathbb{R}$, the set of real numbers in Corollary 3 and $A : E \to E$ be defined by $Ax = \tanh(x)$. Then, A is continuous, monotone and bounded. Using the prototypes of our iteration parameters defined above with $a = \frac{3}{5},\ b = \frac{1}{4}$. Then the sequence $\{x_n\}$ generated by

$$x_{n+1} \quad = \quad x_n - \frac{1}{(n+1)^{\frac{3}{5}}}\tanh(x_n) - \frac{1}{(n+1)^{\frac{3}{5}}}\frac{1}{(n+1)^{\frac{1}{4}}}(x_n - x_1),\ n \geq 1 \tag{4.1}$$

converges strongly to $x^* = 0$, where $x^* \in A^{-1}(0)$.

Using Matlab 7.6, to analyze the convergence of the sequence (4.1), we obtain the figures; fig.1, fig.2 and fig.3 respectively with different initial points $x_1 = 5000cm, x_1 = 1000cm$ and $x_1 = .25cm$. From the figures, we observe that the sequence converges to 0 with each of the initial points but the closer the initial point is to 0, the better approximation we obtain.

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References

[1] M. Abbas, S.H. Khan, A.R. Khan, R.P. Agarwal, *Common fixed points of two multi-valued nonexpansive mappings by one-step itrative scheme*, Appl. Math. Letters, **24**(2011), 97-102.

[2] L.E.J. Brouwer, *Uber Abbildung von Mannigfaltigkeiten*, Mathematische Annalen, **71**(1912), no. 4, 598.

[3] F.E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc., **73**(1967), 875-882.

[4] F.E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*, Proc. of Symposia in pure Math., **16**(1976), part 2.

[5] C.E. Chidume, S.A. Mutangadura, *An example on the Mann iteration method for Lipschitz pseudocontrations,* Proc. Amer. Math. Soc., **129**(2001), no. 8, 2359-2363.

[6] C.E. Chidume, H. Zegeye, *Approximate fixed point sequence and and convergence theorems for lipschitz pseudo-contractive maps*, Proc. Amer. Math. Soc., **132**(2004), no. 3, 831-840.

[7] C.E. Chidume, C.H. Morales, *Accretive operators which are always single-valued in normed spaces*, Nonlinear Anal., **67**(12)(2007), 3328-3334.

[8] C. Chidume, *Geometric Properties of Banach spaces and Nonlinear Iterations*, Springer Verlag, Series: Lecture Notes in Mathematics, **1965**(2009).

[9] C.E. Chidume, N. Djitte, *Strong convergence theorems for zeros of bounded maximal monotone nonlinear operators*, Abstract and Applied Analysis, **2012**, Article ID 681348, 19 pages, doi:10.1155/2012/681348.

[10] C.E. Chidume, C.O. Chidume, N. Djitte, M. Minjibir, *Iterative algorithms for zeros of multivalued nonlinear mappings in Banach spaces*, (to appear).

[11] C.E. Chidume, C.O. Chidume, N. Djitte, M.S. Minjibir, *Convergence theorems for fixed points of multivalued strictly pseudo-contractive mappings in Hilbert Spaces*, Abstract and Applied Analysis, FPTA, (in press).

[12] P. Cholamjiak, S. Suantai, *Weak convergence theorems for a countable family of strict pseudo-contractions in Banach spaces*, Fixed Point Theory Appl., **2010**(2010), Article ID 632137.

[13] D. Downing, W.A. Kirk, *Fixed point theorems for set-valued mappings in metric and Banach spaces*, Math. Japon., **22**(1977), no. 1, 99-112.

[14] S. Reich, *Strong convergence theorems for resolvents of accretive mappings in Banach spaces,* J. Math. Anal. Appl., **75**(1980), 287-292.

[15] J. Garcia-Falset, E. Lorens-Fuster, T. Suzuki, *Fixed point theory for a classs of generalised nonexpansive mappings*, J. Math. Anal. Appl., **375**(2011), 185-195.

[16] J. Geanakoplos, *Nash and Walras equilibrium via Brouwer*, Economic Theory, **21**(2003), 585-603.

[17] T. Kato, *Nonlinear semi groups and evolution equations*, J. Math. Soc. Japan, **19**(1967), 508-520.

[18] O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim., **29**(1991), 403-419.

[19] S. Ishikawa, *Fixed points by a new iteration Method*, Proc. Amer. Math. Soc., **44**(1974), no. 1, 147-150.

[20] S. Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Mathematical Journal, **8**(1941), no. 3, 457-459.

[21] S.H. Khan, I. Yildirim, B.E. Rhoades, *A one-step iterative scheme for two multi-valued nonexpansive mappings in Banach spaces*, Comput. Math. Appl., **61**(2011), 3172-3178.

[22] N. Lehdili, A. Moudafi, *Combining the proximal algorithm and Tikhonov regularization*, Optimization, **37**(1996), 239-252.

[23] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces II: Function Spaces*, Ergebnisse Math. Grenzgebiete Bd., **97**, Springer-Verlag, Berlin, 1979.

[24] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4**(1953), 506-510.

[25] R.H. Martin, *Nonlinear operators and differential equations in Banach spaces*, Interscience, New York, 1976.

[26] R.H. Martin, *A global existence theorem for autonomous differential equations in Banach spaces*, Proc. Amer. Math. Soc., **26**(1970), 307-314.

[27] B. Martinet, *Rgularization d'inéquations variationelles par approximations successives*, Revue Francaise d'Informatique et de Recherche operationelle, **4**(1970), 154-159.

[28] S.B. Nadler Jr., *Multivaled contraction mappings*, Pacific J. Math., **30**(1969), 475-488.

[29] J.F. Nash, *Equilibrium points in n-person games*, Proceedings of the National Academy of Sciences of the United States of America, **36**(1950), no. 1, 48-49.

[30] J.F. Nash, *Non-coperative games*, Annals of Mathematics, Second series, **54**(1951), 286-295.

[31] E.U. Ofoedu, H. Zegeye, *Iterative algorithm for multi-valued pseudo-contractive mappings in Banach spaces*, J. Math. Anal. Appl., **372**(2010), 68-76.

[32] B. Panyanak, *Mann and Ishikawa iteration processes for multi-valued mappings in Banach Spaces*, Comput. Math. Appl., **54**(2007), 872-877.

[33] R.T. Rockafellar, *Monotone operator and the proximal point algorithm*, SIAM J. Control Optim., **14** (1976), 877-898.

[34] K.P.R. Sastry, G.V.R. Babu, *Convergence of Ishikawa iterates for a multi-valued mapping with a fixed point*, Czechoslovak Math. J., **55**(2005), 817-826.

[35] M.V. Solodov, B.F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilber space*, Math. Program., Ser. A, **87**(2000), 189-202.

[36] Y. Song, H. Wang, *Erratum to "Mann and Ishikawa iterative processes for multi-valued mappings in Banach Spaces*, Comput. Math. Appl., **54**(2007), 872-877, Comput. Math. Appl., **55**(2008), 2999-3002.

[37] H.K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Analysis, **16**(1991), 1127-1138.

[38] H.K. Xu, *Iterative algorithms for nonlinear operators,* J. London Math. Soc., **66**(2002), no. 2, 240-256.

[39] H.K. Xu, *A regularization Method for the proximal point algorithm*, J. Global Opt., **36**(2006), 115-125.

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