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CORRECTION OF "FRACTIONAL EQUATIONS AND A THEOREM OF BROUWER-SCHAUDER TYPE"

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Abstract. In a recent paper we offered a theorem which was intended to be a direct parallel of Brouwer's fixed point theorem applied to certain mappings of sets in a Banach space of bounded continuous functions mapping $[0, \infty) \to \Re$. The mappings were generated by integral equations having roots in fractional differential equations of Caputo type. Brouwer's theorem in the simplest form shows that the continuous mapping of the closed $n - ball$ in E^n has a fixed point. We started with a set in the Banach space which was not a ball and we had an error in the proof. In this correction our mapping set is in the Banach space of bounded continuous functions with the supremum norm, $(BC, \| \cdot |)$, and is defined by $M = \{\phi \in BC | a \leq \phi(t) \leq b\}$ for constants $a < b$. We show that if the continuous mapping of M into M is generated by our integral equations, then it has a fixed point in M.

Key Words and Phrases: Fixed points, fractional differential equations, Schauder's theorem, Brouwer's theorem.
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1. Introduction: Description of the error

In the simplest form Brouwer's fixed point theorem [4, p. 11] states that a continuous mapping of the closed ball in E^n has a fixed point. In [3] we sought to show the same for certain mappings in a Banach space of continuous functions $\phi : [0, \infty) \to \Re$.

Starting with $(BC, \| \cdot \|)$, the Banach space of bounded continuous functions $\phi : [0, \infty) \to \Re$ with the supremum norm, we selected a closed convex nonempty set M. The mapping was generated by the natural mapping defined by an integral equation

$$x(t) = F(t) + \int_0^t L(t - s)v(s, x(s))ds \tag{1}$$

having its roots in a fractional differential equation of Caputo type. Under conditions imposed in that work the integral mapped functions from M into equicontinuous sets.

We changed the norm to a weighted norm and noted that the mapping of the set M in the new space was compact. It was at this point that we made an error. We stated that M was closed in the new space. That is true for the parts of the paper which deal with the Banach space on a finite interval, say $[0, T]$, instead of $[0, \infty)$.

If we had asked that M be a closed ball in $(BC, \|\cdot\|)$, the result would have been true and completely parallel to Brouwer's theorem in E^n. But allowing any closed convex nonempty set for M was simply too general.

This correction asks that there be constants $a < b$ with

$$M = \{\phi \in BC : a \leq \phi(t) \leq b\}. \tag{*}$$

As it is shown in Smart [4, pp. 11-12] for Brouwer's theorem, other sets are allowed using the theory of retracts. Also, there are less severe changes such as simply saying that M is closed in both spaces. But the main point of the paper was to show that for a wide class of problems occurring in applied mathematics, Schauder's theorem is as simple as Brouwer's. Before we proceed, we remark that the explicit error occurs in the proof of Theorem 4.1 part (iii) where we state in the middle of the proof that our set M is closed in the weighted space.

2. The corrected theorems

Theorem 3.1 is an abbreviated version of Theorem 4.1 so we will state here only that in Theorem 3.1 we replace the conditions on M with (*) when the functions are considered on the entire interval $[0, \infty)$.

In Theorem 4.1 (iii) we are dealing with a set M defined in (*) and with the mapping obtained from (1) so that for $\phi \in M$ then

$$(Q\phi)(t) = F(t) + \int_0^t L(t-s)v(s, \phi(s))ds \tag{2}$$

where $F : [0, \infty) \to \Re$ is uniformly continuous, L is completely monotone with $L(t) \geq 0$, $\int_0^\infty L(s)ds = 1$, $v : [0, \infty) \times \Re \to \Re$ is continuous and bounded for $\phi \in M$.

Theorem. *Let M be defined in (*) and Q in (2) with the stated conditions. If $Q : M \to M$ is continuous in the supremum norm, then QM is equicontinuous and Q has a fixed point in M.*

Proof. We define a Banach space $(W, |\cdot|_g)$ as follows. Let $g : [0, \infty) \to [1, \infty)$ be an arbitrary continuous function with $g(0) = 1$, g strictly increasing, and $g(t) \to \infty$ as $t \to \infty$. This space contains all continuous functions $\phi : [0, \infty) \to \Re$ for which

$$|\phi|_g := \sup_{0 \leq t < \infty} \frac{|\phi(t)|}{g(t)} < \infty.$$

Notice that M resides in this space and $Q : M \to M$. We will use Schauder's second fixed point theorem [4, p. 25] to show that Q has a fixed point in M.

It is shown in [1] that QM is equicontinuous and it is shown in [2] that Q is continuous in this weighted norm. If we can show that M is closed in the weighted space, then a proof in [2] will show that QM is contained in a compact subset of the weighted space.

To see that M is closed in the weighted space, suppose that $\{\phi_n\}$ is a sequence residing in M and that sequence converges to ψ so that

$$\sup_{0 \leq t < \infty} \frac{|\phi_n(t) - \psi(t)|}{g(t)} \to 0$$

as $n \to \infty$. Notice that on compact subsets the sequence converges uniformly in the supremum norm to ψ and so ψ is continuous. We must show that ψ resides entirely in M. If it does not, then there is a t_1 with $\psi(t_1)$ outside the closed interval $[a, b]$ and so there is a $D > 0$ with

$$\frac{|\phi_n(t_1) - \psi(t_1)|}{g(t_1)} \geq D,$$

a contradiction to that quantity tending to zero as $n \to \infty$.

This will complete the proof.

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