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EXISTENCE OF MILD SOLUTIONS FOR FRACTIONAL EVOLUTION EQUATIONS

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Abstract. In this article, we establish sufficient conditions for the existence of mild solutions for fractional evolution differential equations by using a new fixed point theorem. The results obtained here improve and generalize many known results. An example is also given to illustrate our results.
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1. Introduction

Our aim in this paper is to study the nonlocal initial value problem

$$\begin{cases} D^q x(t) = Ax(t) + f(t, x(t)), & t \in [0,1], \\ x(0) = g(x), \end{cases} \tag{1.1}$$

where D^q is the Caputo fractional derivative of order $0 < q < 1$, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operator (i.e. C_0-semigroup) $T(t)$ in Banach space X, $f : [0,1] \times X \to X$ and $g : C([0,1]; X) \to X$ are appropriate functions to be specified later.

Fractional differential equations have appeared in many branches of physics, economics and technical sciences [1, 2]. There has been a considerable development in fractional differential equations in the last decades. Recently, Many authors are interested in the existence of mild solutions for fractional evolution equations. In [3], El-Borai discussed the following equation in Banach X,

$$\begin{cases} D^\alpha u(t) = Au(t) + B(t)u(t), \\ u(0) = u_0, \end{cases}$$

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where A generates an analytic semigroup and the solution was given in terms of some probability densities. In [4], Zhou and Jiao concerned the existence and uniqueness of mild solutions for fractional evolution equations by some fixed point theorems. Cao et al. [5] studied the α-mild solutions for a class of fractional evolution equations and optimal controls in fractional powder space. For more information on this subjects, the readers may refer to [6]-[10] and the references therein.

Very recently, Zhu [11] used the measure of noncompactness to discuss problem (1.1) when $q = 1$. Motivated by this paper we continue to study the existence of mild solutions for problem (1.1) with a fixed point theorem related to the measure of noncompactness which is firstly used to deal with fractional evolution equations. We obtain the existence results without the compactness on $T(t)$ which are different from many existing papers such as [4, 6, 7]. The rest of the paper will be organized as follows. In section 2 we will recall some basic definitions and lemmas from the measure of noncompactness, fractional derivation and integration. Section 3 is devoted to the existence results for problem (1.1). We shall present in Section 4 an example which illustrates our main theorems.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary results which are used in the rest of the paper.

Throughout this paper, we denote by $\mathbf{R}^+$ and $\mathbf{N}$ the set of positive real numbers and the set of positive integers. Let $(X, \|\cdot\|)$ be a real Banach space. We denote by $C([0,1];X)$ the space of X-valued continuous functions on $[0,1]$ with the $\|x\|_\infty = \sup\{\|x(t)\| : t \in [0,1]\}$. Let $L^p([0,1];X)$ be the space of X-valued Bochner function on [0,1] with the norm $\|x\|_{L^p} = (\int_0^1 \|x(s)\|^p ds)^{\frac{1}{p}}$, $1 \leq p < \infty$.

Definition 2.1 ([2]). The Riemann-Liouville fractional integral of order $q \in \mathbf{R}^+$ of a function $f : \mathbf{R}^+ \to X$ is defined by

$$I_0^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \quad t > 0,$$

provided the right-hand side is pointwise defined on $\mathbf{R}^+$, where Γ is the gamma function.

Definition 2.2 ([2]). The Caputo fractional derivative of order $0 < q < 1$ of a function $f : C^1(\mathbf{R}^+; X)$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} f'(s) ds, \quad t > 0.$$

Let α define the Hausdorff measure of noncompactness on both X and $C([0,1];X)$. To prove our results we need the following lemmas.

Lemma 2.3 ([12]). If $W \subseteq C([0,1];X)$ is bounded, then $\alpha(W(t)) \leq \alpha(W)$ for every $t \in [0,1]$, where $W(t) = \{x(t); x \in W\}$. Furthermore if W is equicontinuous on $[0,1]$, then $\alpha(W(t))$ is continuous on $[0,1]$ and $\alpha(W) = \sup\{\alpha(W(t)); t \in [0,1]\}$.

Lemma 2.4 ([13]). If $\{u_n\}_{n=1}^{\infty} \subset L^1([0,1];X)$ is uniformly integrable, then $\alpha(\{u_n\}_{n=1}^{\infty})$ is measurable and

$$\alpha\left(\left\{\int_0^t u_n(s)ds\right\}_{n=1}^{\infty}\right) \leq 2\int_0^t \alpha(\{u_n(s)\}_{n=1}^{\infty})ds.$$

Lemma 2.5 ([14]). If W is bounded, then for each $\epsilon > 0$, there is a sequence $\{u_n\}_{n=1}^{\infty} \subseteq W$ such that

$$\alpha(W) \leq 2\alpha(\{u_n\}_{n=1}^{\infty}) + \epsilon.$$

Lemma 2.6 ([15]). Suppose that $x \geq 1$, then

$$(\frac{x}{e})^x\sqrt{2\pi x}(1+\frac{1}{12x}) < \Gamma(x+1) < (\frac{x}{e})^x\sqrt{2\pi x}(1+\frac{1}{12x-0.5}).$$

Lemma 2.7 ([16] Fixed Point Theorem). Let G be a closed and convex subset of a real Banach space X, let $A : G \to G$ be a continuous operator and $A(G)$ be bounded. For each bounded subset $B \subset G$, set

$$A^1(B) = A(B),\ A^n(B) = A(\overline{co}(A^{n-1}(B))), \quad n = 2, 3, \ldots,$$

if there exist a constant $0 \leq k < 1$ and a positive integer n_0 such that for each bounded subset $B \subset G$,

$$\alpha(A^{n_0}(B)) \leq k\alpha(B),$$

then A has a fixed point in G.

3. Main results

In this section we will establish the existence results by using the Hausdorff measure of noncompactness. Based on reference [6], we give the definition of the mild solutions of problem (1.1) as follows.

Definition 3.1. By the mild solution of problem (1.1), we mean that the function $x \in C([0,1];X)$ which satisfies

$$x(t) = \mathfrak{S}(t)g(x) + \int_0^t (t-s)^{q-1}\mathfrak{T}(t-s)f(s,x(s))ds, \quad t \in [0,1],$$

where

$$\mathfrak{S}(t) = \int_0^{\infty} \xi_q(\theta)T(t^q\theta)d\theta, \quad \mathfrak{T}(t) = q\int_0^{\infty} \theta\xi_q(\theta)T(t^q\theta)d\theta, \tag{3.1}$$

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\Psi_q(\theta^{-\frac{1}{q}}),$$

$$\Psi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q),\ \theta \in \mathbf{R}^+.$$

Remark 3.2 ([6]). $\xi_q(\theta)$ is the probability density function defined on $\mathbf{R}^+$ and

$$\int_0^{\infty} \theta\xi_q(\theta)d\theta = \int_0^{\infty} \frac{1}{\theta^q}\Psi_q(\theta)d\theta = \frac{1}{\Gamma(1+q)}.$$

To state and prove our main results for the existence of mild solutions of problem (1.1), we need the following hypotheses:

(H1) The C_0-semigroup $\{T(t)\}_{t\geq 0}$ generated by A is equicontinuous and $M =$

$\sup\{\|T(t)\|; t \in [0,\infty)\} < +\infty$.
(H2) The function $g : C([0,1];X) \to X$ is completely continuous, moreover there exist positive constants c and d such that $\|g(x)\| \le c\|x\|_\infty + d$, for every $x \in C([0,1];X)$.
(H3) The function $f : [0,1] \times X \to X$ satisfies the Carathéodory type conditions, i.e. $f(t,\cdot) : X \to X$ is continuous for a.e. $t \in [0,1]$ and $f(\cdot,x) : [0,1] \to X$ is strongly measurable for each $x \in C([0,1],X)$.
(H4) There exist a function $m \in L^{\frac{1}{p}}([0,1];\mathbf{R}^+)$, $0 < p < q$ and a nondecreasing continuous function $\Omega : \mathbf{R}^+ \to \mathbf{R}^+$ such that $\|f(t,x)\| \le m(t)\Omega(\|x\|)$ for all $x \in X$ and a.e. $t \in [0,1]$.
(H5) There exists $L \in L^1([0,1];\mathbf{R}^+)$ such that for each bounded $D \subset X$,

$$\alpha(f(t,D)) \le L(t)\alpha(D), \quad \text{for a.e. } \ t \in [0,1].$$

Remark 3.3. (i) If A generates an analytic semigroup or a differentiable semigroup $\{T(t)\}_{t\ge 0}$, then $\{T(t)\}_{t\ge 0}$ is an equicontinuous (see [18]).
(ii) If $\|f(t,x) - f(t,y)\| \le L(t)\|x-y\|$, $L(t) \in L^1([0,1];\mathbf{R}^+)$, $x,y \in X$, then we can get $\alpha(f(t,D)) \le L(t)\alpha(D)$ for each bounded $D \in X$ and a.e. $t \in [0,1]$ (see [11]).

For each positive constant r, let $B_r = \{x \in C([0,1],X); \|x\|_\infty \le r\}$, then B_r is clearly a bounded closed and convex subset in $C([0,1],X)$.

Lemma 3.4. Assume that hypotheses (H1)-(H4) hold, then
(i) For any fixed $t \ge 0$, $\mathfrak{S}(t)$ and $\mathfrak{T}(t)$ defined in (3.1) are linear and bounded operators, i.e. for any $x \in X$,

$$\|\mathfrak{S}(t)x\| \le M\|x\|, \quad \|\mathfrak{T}(t)x\| \le \frac{M}{\Gamma(q)}\|x\|.$$

(ii) $\mathfrak{S}(t)$ and $\mathfrak{T}(t)$ are strongly continuous.
(iii) The set $\{t \to \int_0^t (t-s)^{q-1}\mathfrak{T}(t-s)f(s,x(s))ds; x \in B_r\}$ is equicontinuous on $[0,1]$.

Proof. (i) and (ii) were given in [6], we only check (iii) as follows.

For $x \in B_r$, $0 \le t_1 < t_2 \le 1$, we have

$$\begin{aligned}
&\| \int_0^{t_2} (t_2-s)^{q-1}\mathfrak{T}(t_2-s)f(s,x(s))ds - \int_0^{t_1} (t_1-s)^{q-1}\mathfrak{T}(t_1-s)f(s,x(s))ds\| \\
&= \left\| q\int_0^{t_2}\int_0^\infty \theta(t_2-s)^{q-1}\xi_q(\theta)T((t_2-s)^q\theta)f(s,x(s))d\theta ds \right. \\
&\quad \left. -q\int_0^{t_1}\int_0^\infty \theta(t_1-s)^{q-1}\xi_q(\theta)T((t_1-s)^q\theta)f(s,x(s))d\theta ds \right\| \\
&\le \left\| q\int_{t_1}^{t_2}\int_0^\infty \theta(t_2-s)^{q-1}\xi_q(\theta)T((t_2-s)^q\theta)f(s,x(s))d\theta ds \right\| \\
&\quad + \left\| q\int_0^{t_1}\int_0^\infty \theta(t_2-s)^{q-1}\xi_q(\theta)T((t_2-s)^q\theta)f(s,x(s))d\theta ds \right. \\
&\quad \left. -q\int_0^{t_1}\int_0^\infty \theta(t_1-s)^{q-1}\xi_q(\theta)T((t_2-s)^q\theta)f(s,x(s))d\theta ds \right\| \\
&\quad + \left\| q\int_0^{t_1}\int_0^\infty \theta(t_1-s)^{q-1}\xi_q(\theta)T((t_2-s)^q\theta)f(s,x(s))d\theta ds \right.
\end{aligned}$$

$$-q\int_0^{t_1}\int_0^\infty \theta(t_1-s)^{q-1}\xi_q(\theta)T((t_1-s)^q\theta)f(s,x(s))d\theta ds\Big\|$$
$$=\Big\|q\int_{t_1}^{t_2}\int_0^\infty \theta(t_2-s)^{q-1}\xi_q(\theta)T((t_2-s)^q\theta)f(s,x(s))d\theta ds\Big\|$$
$$+\Big\|q\int_0^{t_1}\int_0^\infty \theta[(t_2-s)^{q-1}-(t_1-s)^{q-1}]\xi_q(\theta)T((t_2-s)^q\theta)f(s,x(s))d\theta ds\Big\|$$
$$+\Big\|q\int_0^{t_1}\int_0^\infty \theta(t_1-s)^{q-1}\xi_q(\theta)[T((t_2-s)^q\theta)-T((t_1-s)^q\theta)]f(s,x(s))d\theta ds\Big\|$$
$$=q(I_1+I_2+I_3),$$

where

$$\begin{aligned}
I_1 &= \Big\|\int_{t_1}^{t_2}\int_0^\infty \theta(t_2-s)^{q-1}\xi_q(\theta)T((t_2-s)^q\theta)f(s,x(s))d\theta ds\Big\|,\\
I_2 &= \Big\|\int_0^{t_1}\int_0^\infty \theta[(t_2-s)^{q-1}-(t_1-s)^{q-1}]\xi_q(\theta)T((t_2-s)^q\theta)f(s,x(s))d\theta ds\Big\|,\\
I_3 &= \Big\|\int_0^{t_1}\int_0^\infty \theta(t_1-s)^{q-1}\xi_q(\theta)[T((t_2-s)^q\theta)-T((t_1-s)^q\theta)]f(s,x(s))d\theta ds\Big\|.
\end{aligned}$$

From hypothesis (H4), we have

$$\begin{aligned}
I_1 &\le \frac{M\Omega(r)}{\Gamma(1+q)}\int_{t_1}^{t_2}|(t_2-s)^{q-1}m(s)|ds\\
&\le \frac{M\Omega(r)}{\Gamma(1+q)(1+\eta)^{1-p}}(t_2-t_1)^{(1+\eta)(1-p)}\|m\|_{L^{\frac{1}{p}}},\\
I_2 &\le \frac{M\Omega(r)}{\Gamma(1+q)}\left(\int_0^{t_1}((t_1-s)^{q-1}-(t_2-s)^{q-1})^{\frac{1}{1-p}}ds\right)^{1-p}\|m\|_{L^{\frac{1}{p}}}\\
&\le \frac{M\Omega(r)\|m\|_{L^{\frac{1}{p}}}}{\Gamma(1+q)}\left(\int_0^{t_1}((t_1-s)^{\eta}-(t_2-s)^{\eta})ds\right)^{1-p}\\
&= \frac{M\Omega(r)\|m\|_{L^{\frac{1}{p}}}}{\Gamma(1+q)(1+\eta)^{1-p}}(t_1^{1+\eta}-t_2^{1+\eta}+(t_2-t_1)^{1+\eta})^{1-p}\\
&\le \frac{M\Omega(r)\|m\|_{L^{\frac{1}{p}}}}{\Gamma(1+q)(1+\eta)^{1-p}}(t_2-t_1)^{(1+\eta)(1-p)},
\end{aligned}$$

where $\eta=\frac{q-1}{1-p}\in(-1,0)$. Hence $\lim_{t_2\to t_1} I_1=0$ and $\lim_{t_2\to t_1} I_2=0$.

On the other hand, from (H1) and the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
\lim_{t_2\to t_1} I_3 &\le \lim_{t_2\to t_1}\int_0^{t_1}\int_0^\infty \theta(t_1-s)^{q-1}\xi_q(\theta)\|T((t_2-s)^q\theta)f(s,x(s))\\
&\quad -T((t_1-s)^q\theta f(s,x(s))\|d\theta ds\\
&\le \int_0^{t_1}\int_0^\infty \theta(t_1-s)^{q-1}\xi_q(\theta)\lim_{t_2\to t_1}\|T((t_2-s)^q\theta)f(s,x(s))
\end{aligned}$$

$$-T((t_1-s)^q\theta)f(s,x(s))\|d\theta ds$$
$$= \quad 0.$$

Hence, $\| \int_0^{t_2}(t_2-s)^{q-1}\mathfrak{T}(t_2-s)f(s,x(s))ds - \int_0^{t_1}(t_1-s)^{q-1}\mathfrak{T}(t_1-s)f(s,x(s))ds\| \to 0$ independently of $x \in B_r$ as $t_2 \to t_1$. This completes the proof.

Lemma 3.5. Suppose that $0<a<1$, $b>0$ are two fixed constants, let

$$S_n = \left(a^n + C_n^1\frac{a^{n-1}b}{\Gamma(q+1)} + C_n^2\frac{a^{n-2}b^2}{\Gamma(2q+1)} + \cdots + C_n^{n-1}\frac{ab^{n-1}}{\Gamma((n-1)q+1)} + C_n^n\frac{b^n}{\Gamma(nq+1)}\right).$$

Then, $\lim_{n\to\infty} S_n = 0$.

Proof. Since $0<a<1$, there exists a constant $\overline{b}>0$ with $a+\overline{b}<1$.

From $0<q<1$, we know that there exists $n_1 \in \mathbf{N}$ such that, if $n>n_1$ then $nq>1$. By Lemma 2.6 if $n>n_1$, then

$$\Gamma(nq+1) > \left(\frac{nq}{e}\right)^{nq}\sqrt{2\pi nq} > \left(\frac{nq}{e}\right)^{nq}.$$

Therefore, for $n>n_1$, we have

$$\frac{1}{\Gamma(nq+1)} < \frac{1}{((\frac{nq}{e})^q)^n}.$$

On the other hand, there exists $n_2 \in \mathbf{N}$ such that $\frac{b}{(\frac{nq}{e})^q} < \overline{b}$ for each $n>n_2$.

Set $n_3=\max\{n_1,n_2\}$, for $n>n_3$, we divide S_n into two parts

$$S_n = S_n' + S_n'',$$

where

$$S_n' = a^n + C_n^1\frac{a^{n-1}b}{\Gamma(q+1)} + C_n^2\frac{a^{n-2}b^2}{\Gamma(2q+1)} + \cdots + C_n^{n_3}\frac{a^{n-n_3}b^{n_3}}{\Gamma(n_3q+1)},$$

$$S_n'' = C_n^{n_3+1}\frac{a^{n-n_3-1}b^{n_3+1}}{\Gamma((n_3+1)q+1)} + C_n^{n_3+2}\frac{a^{n-n_3-2}b^{n_3+2}}{\Gamma((n_3+2)q+1)} + \cdots + C_n^n\frac{b^n}{\Gamma(nq+1)}.$$

For $n>n_3$, we have

$$\begin{aligned} S_n'' &= C_n^{n_3+1}\frac{a^{n-n_3-1}b^{n_3+1}}{\Gamma((n_3+1)q+1)} + C_n^{n_3+2}\frac{a^{n-n_3-2}b^{n_3+2}}{\Gamma((n_3+2)q+1)} + \cdots + C_n^n\frac{b^n}{\Gamma(nq+1)} \\ &\le C_n^{n_3+1}\frac{a^{n-n_3-1}b^{n_3+1}}{((\frac{(n_3+1)q}{e})^q)^{n_3+1}} + C_n^{n_3+2}\frac{a^{n-n_3-2}b^{n_3+2}}{((\frac{(n_3+2)q}{e})^q)^{n_3+2}} + \cdots + C_n^n\frac{b^n}{((\frac{nq}{e})^q)^n} \\ &\le C_n^{n_3+1}a^{n-n_3-1}\overline{b}^{n_3+1} + C_n^{n_3+2}a^{n-n_3-2}\overline{b}^{n_3+2} + \cdots + C_n^n\overline{b}^n \\ &\le (a+\overline{b})^n. \end{aligned}$$

In view of $a+\overline{b}<1$, we have $\lim_{n\to+\infty} S_n''=0$. Since $\lim_{n\to+\infty} S_n'=0$ is obvious, we obtain $\lim_{n\to+\infty} S_n = 0$. The proof is completed.

Theorem 3.6. If hypotheses (H1)-(H5) are satisfied, then there is at least one mild solution for problem (1.1) provided that there exists a constant r such that

$$M(cr+d) + \frac{M\Omega(r)}{(1+\eta)^{1-p}\Gamma(q)}\|m\|_{L^{\frac{1}{p}}} \le r, \tag{3.2}$$

where $\eta = \frac{q-1}{1-p}$ is defined in the proof of Lemma 3.4.

Proof. Define operator $F : C([0;1], X) \to C([0,1]; X)$ by

$$(Fx)(t) = \mathfrak{S}(t)g(x) + \int_0^t (t-s)^{q-1}\mathfrak{T}(t-s)f(s,x(s))ds, \quad t \in [0,1].$$

We can easily show that F is continuous by the usual techniques (see [4]). For any $x \in B_r$, we have

$$\begin{aligned}
\|(Fx)(t)\| &\le \|\mathfrak{S}(t)g(x)\| + \left\|\int_0^t (t-s)^{q-1}\mathfrak{T}(t-s)f(s,x(s))ds\right\| \\
&= \left\|\int_0^\infty \xi_q(\theta)T(t^q\theta)g(x)d\theta\right\| \\
&\quad + \left\|q\int_0^t (t-s)^{q-1}\int_0^\infty \theta\xi_q(\theta)T((t-s)^q\theta)d\theta f(s,x(s))ds\right\| \\
&\le M(cr+d) + \frac{M\Omega(r)}{\Gamma(q)}\int_0^t (t-s)^{q-1}m(s)ds \\
&\le M(cr+d) + \frac{M\Omega(r)}{\Gamma(q)}\left(\int_0^t (t-s)^{\frac{q-1}{1-p}}ds\right)^{1-p}\|m\|_{L^{\frac{1}{p}}} \\
&\le M(cr+d) + \frac{M\Omega(r)}{(1+\eta)^{1-p}\Gamma(q)}\|m\|_{L^{\frac{1}{p}}}.
\end{aligned}$$

Then from (3.2) we get $\|Fx\|_\infty \le r$ which means that $F : B_r \to B_r$ is a bounded operator.

Let $B_0 = \overline{co}FB_r$. By Lemma 2.5 and the condition $g(x)$ is compact, we get for any $B \subset B_0$ and $\epsilon > 0$, there is a sequence $\{x_n\}_{n=1}^\infty \subset B$ such that

$$\begin{aligned}
\alpha(F^1B(t)) &= \alpha(FB(t)) \\
&\le 2\alpha\left(\int_0^t (t-s)^{q-1}\mathfrak{T}(t-s)f(s,\{x_n\}_{n=1}^\infty)ds\right) + \epsilon \\
&\le 4\int_0^t (t-s)^{q-1}\alpha(\mathfrak{T}(t-s)f(s,\{x_n\}_{n=1}^\infty))ds + \epsilon \\
&\le \frac{4M}{\Gamma(q)}\int_0^t (t-s)^{q-1}L(s)\alpha(\{x_n\}_{n=1}^\infty)ds + \epsilon \\
&\le \frac{4M}{\Gamma(q)}\alpha(B)\int_0^t (t-s)^{q-1}L(s)ds + \epsilon.
\end{aligned}$$

From the fact that there is a continuous function $\phi : [0,1] \to \mathbf{R}^+$ such that for any $\gamma > 0$,

$$\int_0^t (t-s)^{q-1}|L(s) - \phi(s)|ds < \gamma.$$

We choose $\gamma < \frac{\Gamma(q)}{4M}$ and let $\overline{M} = \max\{|\phi(t)| : t \in [0,1]\}$, then

$$\begin{aligned}
\alpha(F^1B(t)) &\le \frac{4M}{\Gamma(q)}\alpha(B)\left[\int_0^t (t-s)^{q-1}|L(s)-\phi(s)|ds + \int_0^t (t-s)^{q-1}|\phi(s)|ds\right] + \epsilon \\
&\le \frac{4M}{\Gamma(q)}\alpha(B)\left(\gamma + \frac{\overline{M}t^q}{q}\right) + \epsilon.
\end{aligned}$$

From $\epsilon > 0$ is arbitrary, it follows that

$$\alpha(F^1B(t)) \leq (a + \frac{b}{\Gamma(q+1)}t^q)\alpha(B),$$

where $a = \frac{4M\gamma}{\Gamma(q)}$, $b = 4M\overline{M}$.

From Lemma 2.5, we know for any $\epsilon > 0$, there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset \overline{co}(F^1B)$ such that

$$\begin{aligned}
\alpha(F^2B(t)) &= \alpha(F\overline{co}(F^1B(t))) \\
&\leq 2\alpha\left(\int_0^t (t-s)^{q-1}\mathfrak{T}(t-s)f(s,\{y_n\}_{n=1}^{\infty})ds\right)+\epsilon \\
&\leq 4\int_0^t (t-s)^{q-1}\alpha(\mathfrak{T}(t-s)f(s,\{y_n\}_{n=1}^{\infty}))ds+\epsilon \\
&\leq \frac{4M}{\Gamma(q)}\int_0^t (t-s)^{q-1}L(s)\alpha(F^1B(s))ds+\epsilon \\
&\leq \frac{4M}{\Gamma(q)}\alpha(B)\int_0^t [(t-s)^{q-1}|L(s)-\phi(s)|+|\phi(s)|](a+\frac{b}{\Gamma(q+1)}s^q)ds+\epsilon \\
&\leq \frac{4M}{\Gamma(q)}\alpha(B)\Big[(a+\frac{bt^q}{\Gamma(q+1)})\int_0^t (t-s)^{q-1}|L(s)-\phi(s)|ds \\
&+\overline{M}\int_0^t (t-s)^{q-1}(a+\frac{b}{\Gamma(q+1)}s^q)ds\Big]+\epsilon \\
&\leq \left(a^2+2a\frac{bt^q}{\Gamma(q+1)}+\frac{b^2t^{2q}}{\Gamma(2q+1)}\right)\alpha(B)+\epsilon.
\end{aligned}$$

From $\epsilon > 0$ is arbitrary, it follows that

$$\alpha(F^2B(t)) \leq \left(a^2+2a\frac{bt^q}{\Gamma(q+1)}+\frac{b^2t^{2q}}{\Gamma(2q+1)}\right)\alpha(B).$$

By the method of mathematical induction, for any positive integer n and $t \in [0,1]$, we obtain

$$\begin{aligned}
\alpha(F^nB(t)) \quad \leq \quad & \Big(a^n + C_n^1a^{n-1}\frac{bt^q}{\Gamma(q+1)} + C_n^2a^{n-2}\frac{b^2t^{2q}}{\Gamma(2q+1)} + \cdots \\
& + C_n^{n-1}a\frac{b^{n-1}t^{(n-1)q}}{\Gamma((n-1)q+1)} + C_n^n\frac{b^nt^{nq}}{\Gamma(nq+1)}\Big)\alpha(B).
\end{aligned}$$

Therefore, by Lemma 3.4 and Lemma 2.3, we get

$$\begin{aligned}
\alpha(F^nB) \quad \leq \quad & \Big(a^n + C_n^1a^{n-1}\frac{b}{\Gamma(q+1)} + C_n^2a^{n-2}\frac{b^2}{\Gamma(2q+1)} + \cdots \\
& + C_n^{n-1}a\frac{b^{n-1}}{\Gamma((n-1)q+1)} + C_n^n\frac{b^n}{\Gamma(nq+1)}\Big)\alpha(B).
\end{aligned}$$

Then from Lemma 3.4, there exists a positive integer n_0 such that

$$\Big(a^{n_0} + C_{n_0}^1\frac{a^{n_0-1}b}{\Gamma(q+1)} + C_{n_0}^2\frac{a^{n_0-2}b^2}{\Gamma(2q+1)} + \cdots$$

$$+C_{n_0}^{n_0-1}\frac{ab^{n_0-1}}{\Gamma((n_0-1)q+1)}+C_{n_0}^{n_0}\frac{b^{n_0}}{\Gamma(n_0q+1)}\Bigg)=k<1.$$

Then $\alpha(F^{n_0}B)\le k\alpha(B)$. From Lemma 2.7 we conclude that F has at least one fixed point in B_0, i.e. the nonlocal value problem (1.1) has at least one mild solution in B_0. The proof is completed.

Corollary 3.7. If the hypotheses (H1)-(H5) are satisfied, then there is at least one mild solution for (1.1) provided that

$$\|m\|_{L^{\frac{1}{p}}}<\liminf_{T\to+\infty}\frac{[T-M(cT+d)](1+\eta)^{1-p}\Gamma(q)}{M\Omega(T)}. \tag{3.3}$$

Proof. (3.3) implies that there exists a constant $r>0$ such that

$$M(cr+d)+\frac{M\Omega(r)}{(1+\eta)^{1-p}\Gamma(q)}\|m\|_{L^{\frac{1}{p}}}\le r.$$

Then by Theorem 3.6 we know the corollary is true.

4. An example

Let $X=L^2(\mathbf{R}^n)$. Consider the following fractional parabolic nonlocal Cauchy problem.

$$\begin{cases} D^q u(t,z)=(\mathfrak{L}u)(t,z)+f(t,u(t,z)), & t\in[0,1], z\in\mathbf{R}^n,\\ u(0,z)=\sum_{i=1}^m\int_{\mathbf{R}^n}K(z,y)u(t_i,y)dy, & z\in\mathbf{R}^n,\end{cases} \tag{4.1}$$

where D^q is the Caputo fractional partial derivative of order $0<q<1$, f is a given function, m is a positive integer, $0<t_1<t_2<\cdots<t_m<1$, $K(z,y)\in L^2(\mathbf{R}^n\times\mathbf{R}^n;\mathbf{R}^+)$. Moreover,

$$(\mathfrak{L}u)(t,z)=\sum_{i,j=1}^n a_{ij}(z)\frac{\partial u}{\partial z_i\partial z_j}(t,z)+\sum_{i=1}^n b_i(z)\frac{\partial u}{\partial z_i}(t,z)+\overline{c}(z)u(t,z),$$

where given coefficients a_{ij}, b_i, $\overline{c}$, $i,j=1,2,\dots,n$ satisfy the usual uniformly ellipticity conditions.

We define an operator A by $A=L$ with the domain

$$D(A)=\{v(\cdot)\in X: H^2(\mathbf{R}^n)\}.$$

From [19], we know that A generates an analytic, noncompact semigroup $\{T(t)\}_{t\ge0}$ on $L^2(\mathbf{R}^n)$. In addition, there exists a constant $M>0$ such that $M=\sup\{\|T(t)\|; t\in[0,\infty)\}<+\infty$.

Then the system (4.1) can be reformulated as follows in X,

$$\begin{cases} D^q x(t)=Ax(t)+f(t,x(t)), & t\in[0,1],\\ x(0)=g(x),\end{cases}$$

where $x(t)=u(t,\cdot)$, that is $x(t)z=u(t,z)$, $z\in\mathbf{R}^n$. The function $g: C([0,1],X)\to X$ is given by

$$g(x)z=\sum_{i=0}^m K_g x(t_i)(z),$$

where $K_g v(z) = \int_{\mathbf{R}^n} K(z,y)v(y)dy$ for $v \in X, z \in \mathbf{R}^n$.

Let's take $q = \frac{1}{2}$, $f(t,x(t)) = t^{-\frac{1}{4}} \sin x(t)$.

Firstly, we have (H1) and (H3) are satisfied. Then from $\|f(t,x(t))\| \leq t^{-\frac{1}{4}}$, we get (H4) holds with $\Omega(\|x\|) = 1$. From $\|f(t,x(t)) - f(t,y(t))\| \leq t^{-\frac{1}{4}}\|x-y\|_\infty$ and Remark 3.3 we get that (H5) is satisfied. Furthermore, note that $K_g : X \to X$ is completely continuous and assume that $c = m(\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} K^2(z,y)dydz)^{\frac{1}{2}}$, we get (H2) is satisfied.

If $Mc < 1$, then there exists a constant r which satisfies (3.2). According to Theorem 3.6, problem (4.1) has at least one mild solution provided that $Mc < 1$.

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