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COUPLING EXTRAGRADIENT METHODS WITH CQ METHODS FOR EQUILIBRIUM POINTS, PSEUDOMONOTONE VARIATIONAL INEQUALITIES AND FIXED POINTS

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Abstract. In this paper, we suggest a hybrid method for finding a common element of the set of solution of an equilibrium problem, the set of solution of a pseudomonotone variational inequality problem and the set of common fixed points of an infinite family of nonexpansive mappings. The constructed iterative method combines two well-known methods: extragradient method and CQ method. We derive a necessary and sufficient condition for the strong convergence of the sequences generated by the proposed method.

Key Words and Phrases: Equilibrium problem, pseudomonotone variational inequality, fixed point, pseudomonotone mapping, nonexpansive mapping, extragradient method, CQ method.

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1. Introduction

In the recent past, equilibrium problems, variational inequality problems and fixed points problems have been attracted so much attention. How to construct algorithms for finding the common element of the set of solution of an equilibrium problem, the set of solution of a variational inequality problem and the set of common fixed points of nonexpansive mappings is an interesting topic. Some related works have been studied extensively in the literature. See, for instance, [1]-[26] and the references therein. It is our main purpose in this paper that we construct a hybrid method for finding a common element of the set of solution of an equilibrium problem, the set of solution of a pseudomonotone, Lipschitz-continuous variational inequality problem and the set of common fixed points of an infinite family of nonexpansive mappings. We will show the strong convergence of the proposed algorithm to the common element of the set of solution of an equilibrium problem, the set of solution of a pseudomonotone

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variational inequality problem and the set of common fixed points of an infinite family of nonexpansive mappings.

We next briefly review some historic approaches in the literature which relate to the variational inequality problems and fixed points problems. Let us start with Korpelevich's extragradient method which was introduced by Korpelevich [13] in 1976 and which generates a sequence $\{x_n\}$ via the recursion:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \quad \forall n \geq 0, \end{cases} \tag{1.1}$$

where P_C is the metric projection from R^n onto C, $A : C \to H$ is a monotone operator and λ is a constant. Korpelevich [13] proved that the sequence $\{x_n\}$ converges strongly to a solution of $VI(C, A)$. Note that the setting of the space is a finite dimension Euclid space.

Korpelevich's extragradient method has extensively been studied in the literature for solving a more general problem that consists of finding a common point that lies in the solution set of a variational inequality and the set of fixed points of a nonexpansive mapping. This type of problem aries in various theoretical and modeling contexts, see e.g., [27]-[33] and references therein. Especially, Nadezhkina and Takahashi [34] introduced the following iterative method:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x), \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

where P_C is the metric projection from H onto C, $A : C \to H$ is a monotone k-Lipschitz-continuous mapping, $S : C \to C$ is a nonexpansive mapping, $\{\lambda_n\}$ and $\{\alpha_n\}$ are two real number sequences. They proved the strong convergence of the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ to a common element of the set of solution of a variational inequality problem and the set of fixed points of a nonexpansive mapping.

Very recently, Ceng, Teboulle and Yao [35] further suggested a new iterative method as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) S_n P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ \text{find } x_{n+1} \in C_n \text{ such that} \\ \langle x_n - x_{n+1} + e_n - \sigma_n A x_{n+1}, x_{n+1} - x \rangle \geq -\epsilon_n, \quad \forall x \in C_n, \end{cases} \tag{1.3}$$

where $A : C \to H$ is a pseudomonotone, k-Lipschitz-continuous and (w, s)-sequentially-continuous mapping, $\{S_i\}_{i=1}^N : C \to C$ are N nonexpansive mappings. Under some mild conditions, they proved the weak convergence of the sequences $\{x_n\}$,

$\{y_n\}$ and $\{z_n\}$ to a common element of the set of solution of a pseudomonotone variational inequality problem and the set of fixed points of a finite family of nonexpansive mappings if and only if $\liminf_n \langle Ax_n, x - x_n\rangle \geq 0$, $\forall x \in C$.

On the algorithms (1.2) and (1.3), we have the following remarks.

Remark 1.1. (1) We note that Nadezhkina and Takahashi's method (1.2) combines Korpelevich's extragradient method and a CQ method. They obtained the strong convergence of their method. It is observed that Nadezhkina and Takahashi [34] employed the monotonicity and Lipschitz-continuity of A to define a maximal monotone operator T as follows:

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C, \end{cases}$$

where $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$ is the normal cone to C at $v \in C$ (see, [36]). However, if the mapping A is a pseudomonotone Lipschitz-continuous, then T is not necessarily a maximal monotone operator. This fact implies that the approach used in [34] cannot be applied.

(2) Ceng, Teboulle and Yao's method (1.3) combines Korpelevich's extragradient and approximate proximal method. It is interesting that they have overcome the difficulty mentioned above, i.e., they assumed the involved operator A is pseudomonotone (not monotone). However, Ceng, Teboulle and Yao's method has only weak convergence.

It is an interesting problem: could we construct a new algorithm involving pseudomonotone operators such that the strong convergence is guaranteed?

Motivated and inspired by the works of Nadezhkina and Takahashi [34] and Ceng, Teboulle and Yao [35], in this paper we suggest a hybrid method which combines two well-known methods: extragradient method and CQ method. We derive a necessary and sufficient condition for the strong convergence of the proposed sequences for finding a common element of the set of solution of an equilibrium problem, the set of solution of a pseudomonotone, Lipschitz-continuous variational inequality problem and the set of common fixed points of an infinite family of nonexpansive mappings.

2. Preliminaries

In this section, we will recall some basic notations and collect some conclusions that will be used in the next section.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Recall that a mapping $A : C \to H$ is called *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

A mapping $A : C \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

A mapping $A : C \to H$ is called *pseudomonotone* if

$$\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, y - x\rangle \geq 0, \quad \forall x, y \in C.$$

It is clear that if a mapping A is monotone, then it is pseudomonotone.

Recall also that a mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Denote by $Fix(S)$ the set of fixed points of S, that is, $Fix(S) = \{x \in C : Sx = x\}$.

Let $B : C \to H$ be a nonlinear mapping and $F : C \times C \to R$ be a bifunction. The equilibrium problem is to find $z \in C$ such that

$$F(z, y) + \langle Bz, y - z\rangle \geq 0, \quad \forall y \in C. \tag{2.1}$$

The solution set of (2.1) is denoted by $EP(F, B)$. If $B = 0$, then (2.1) reduces to the following equilibrium problem of finding $z \in C$ such that

$$F(z, y) \geq 0, \quad \forall y \in C. \tag{2.2}$$

The solution set of (2.2) is denoted by $EP(F)$. If $F = 0$, then (2.1) reduces to the variational inequality problem of finding $z \in C$ such that

$$\langle Bz, y - z\rangle \geq 0, \quad \forall y \in C. \tag{2.3}$$

The solution set of variational inequality (2.3) is denoted by $VI(C, B)$.

Throughout this paper, we assume that a bifunction $F : C \times C \to R$ satisfies the following conditions:

(H1) $F(x, x) = 0$ for all $x \in C$;

(H2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(H3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(H4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

It is well known that, for any $u \in H$, there exists a unique $u_0 \in C$ such that

$$\|u - u_0\| = \inf\{\|u - x\| : x \in C\}.$$

We denote u_0 by $P_C(u)$, where P_C is called the *metric projection* of H onto C. The metric projection P_C of H onto C has the following basic properties:

(1) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$ for all $x, y \in H$.

(2) $\langle x - P_C(x), y - P_C(x)\rangle \leq 0$ for all $x \in H$, $y \in C$.

(3) The property (2) is equivalent to

$$\|x - P_C(x)\|^2 + \|y - P_C(x)\|^2 \leq \|x - y\|, \quad \forall x \in H,\ y \in C.$$

(4) In the context of the variational inequality problem, the characterization of the projection implies that $u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda Au), \quad \forall \lambda > 0$.

Recall that H satisfies the Opial condition [37]; i.e., for any sequence $\{x_n\}$ with x_n converges weakly to x, the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$ holds for every $y \in H$ with $y \neq x$.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_i\}_{i=1}^{\infty}$ be infinite family of nonexpansive mappings of C into itself and let $\{\xi_i\}_{i=1}^{\infty}$ be real number sequences such that $0 \leq \xi_i \leq 1$ for every $i \in \mathbb{N}$. For any $n \in \mathbb{N}$, define a mapping W_n of C into itself as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \xi_n S_n U_{n,n+1} + (1-\xi_n)I, \\ U_{n,n-1} = \xi_{n-1} S_{n-1} U_{n,n} + (1-\xi_{n-1})I, \\ \quad \vdots \\ U_{n,k} = \xi_k S_k U_{n,k+1} + (1-\xi_k)I, \\ U_{n,k-1} = \xi_{k-1} S_{k-1} U_{n,k} + (1-\xi_{k-1})I, \\ \quad \vdots \\ U_{n,2} = \xi_2 S_2 U_{n,3} + (1-\xi_2)I, \\ W_n = U_{n,1} = \xi_1 S_1 U_{n,2} + (1-\xi_1)I. \end{cases} \tag{2.4}$$

Such W_n is called the W-*mapping* generated by $\{S_i\}_{i=1}^{\infty}$ and $\{\xi_i\}_{i=1}^{\infty}$.

We have the following crucial Lemmas 3.1 and 3.2 concerning W_n which can be found in [38]. Now we only need the following similar version in Hilbert spaces.

Lemma 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S_1, S_2, \cdots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} Fix(S_n)$ is nonempty, and $\xi_1, \xi_2, \cdots$ be real numbers such that $0 < \xi_i \le b < 1$ for any $i \in \mathbb{N}$. Then, for every $x \in C$ and $k \in \mathbb{N}$, $\lim_{n\to\infty} U_{n,k}x$ exists.*

Lemma 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S_1, S_2, \cdots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} Fix(S_n)$ is nonempty, and $\xi_1, \xi_2, \cdots$ be real numbers such that $0 < \xi_i \le b < 1$ for any $i \in N$. Then $Fix(W) = \bigcap_{n=1}^{\infty} Fix(S_n)$.*

Lemma 2.3. ([39]) *Using Lemmas 2.1 and 2.2, one can define a mapping W of C into itself as: $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$ for every $x \in C$. If $\{x_n\}$ is a bounded sequence in C, then we have $\lim_{n\to\infty} \|Wx_n - W_n x_n\| = 0$.*

We also need the following well-known lemmas for proving our main results.

Lemma 2.4. ([6]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to R$ be a bifunction which satisfies conditions* (H1)-(H4). *Let $\mu > 0$ and $x \in C$. Then there exists $z \in C$ such that $F(z,y) + \frac{1}{\mu}\langle y-z, z-x\rangle \ge 0, \quad \forall y \in C$. Further, if $T_\mu(x) = \big\{z \in C : F(z,y) + \frac{1}{\mu}\langle y-z, z-x\rangle \ge 0$ for all $y \in C\big\}$, then the following hold:*

(a) T_μ is single-valued and T_μ is firmly nonexpansive, i.e., for any $x, y \in C$,
$\|T_\mu x - T_\mu y\|^2 \le \langle T_\mu x - T_\mu y, x-y\rangle$;

(b) $EP(F)$ is closed and convex and $EP(F) = Fix(T_\mu)$.

Lemma 2.5. ([40]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping with $Fix(S) \ne \emptyset$. Then S is demiclosed on C, i.e., if $y_n \to z \in C$ weakly and $y_n - Sy_n \to y$ strongly, then $(I-S)z = y$.*

Lemma 2.6. ([41]) *Let C be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C(u)$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition $\|x_n - u\| \le \|u - q\|, \quad \forall n$. Then $x_n \to q$.*

We adopt the following notation:

1. $x_n \rightharpoonup x$ stands for the weak convergence of (x_n) to x.

2. $x_n \to x$ stands for the strong convergence of (x_n) to x.

3. For a given sequence $\{x_n\} \subset H$, $\omega_w(x_n)$ denotes the weak ω-limit set of $\{x_n\}$; that is, $\omega_w(x_n) := \{x \in H : \{x_{n_j}\}$ converges weakly to x for some subsequence $\{n_j\}$ of $\{n\}\}$.

3. Main results

In this section we will state and prove our main results.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and $F : C \times C \to R$ be a bifunction satisfying* (H1)-(H4). *Let $A : C \to H$ be a pseudomonotone, k-Lipschitz-continuous and (w, s)-sequentially-continuous mapping. Let $B : C \to H$ be an α-inverse-strongly monotone mapping and $\{S_n\}_{n=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings such that $\Omega := EP(F, B) \cap VI(C, A) \cap \bigcap_{n=1}^{\infty} Fix(S_n) \neq \emptyset$. For $x_0 \in C$, let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{\mu}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(I - \lambda_n A)u_n, \\ z_n = \alpha_n x_n + (1 - \alpha_n)W_n P_C(u_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad n \geq 0, \end{cases} \tag{3.1}$$

where $\mu \in (0, 2\alpha)$ is a constant and W_n is W-mapping defined by (2.4). *Assume the following conditions are satisfied:*

(*a*) *$\{\lambda_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{k}\right)$;*

(*b*) *$\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$.*

Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ generated by (3.1) *converge strongly to the same point $P_\Omega(x_0)$ if and only if $\liminf_n \langle Ax_n, x - x_n\rangle \geq 0$, $\forall x \in C$.*

Next, we will divide our detail proofs into several Lemmas. In the sequel, we assume that all conditions of Theorem 3.1 are satisfied.

Lemma 3.2. (*a*) *C_n and Q_n are closed and convex, $\forall n \geq 0$;*

(*b*) *$\Omega \subset C_n \cap Q_n$, $\forall n \geq 0$;*

(*c*) *$\{x_n\}$ is well-defined.*

Proof. It is obvious that C_n is closed and Q_n is closed and convex for every $n \geq 0$. From (3.1), we can rewrite C_n as

$$C_n = \left\{ z \in C : \left\langle z - \frac{x_n + z_n}{2}, z_n - x_n \right\rangle \geq 0 \right\}.$$

It is clear that C_n is a half space. Hence, C_n is convex. Therefore, C_n and Q_n are closed and convex, $\forall n \geq 0$. Next we show that $\Omega \subset C_n \cap Q_n, \forall n \geq 0$.

From Lemma 2.4, we have $u_n = T_\mu(I - \mu B)x_n$, $\forall n \geq 0$. Set $t_n = P_C(u_n - \lambda A y_n)$ for all $n \geq 1$. Pick up $u \in \Omega$. From property (3) of P_C, we have

$$\begin{aligned} \|t_n - u\|^2 &\leq \|u_n - \lambda_n A y_n - u\|^2 - \|u_n - \lambda_n A y_n - t_n\|^2 \\ &= \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n \langle Ay_n, u - t_n\rangle \end{aligned}$$

$$= \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n \langle Ay_n, u - y_n\rangle + 2\lambda_n \langle Ay_n, y_n - t_n\rangle. \tag{3.2}$$

Since $u \in VI(C, A)$ and $y_n \in C$, we get $\langle Au, y_n - u\rangle \geq 0$.
This together with the pseudomonotonicity of A imply that

$$\langle Ay_n, y_n - u\rangle \geq 0. \tag{3.3}$$

Combine (3.2) with (3.3) to deduce

$$\begin{aligned} \|t_n - u\|^2 &\leq \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n \langle Ay_n, y_n - t_n\rangle \\ &= \|u_n - u\|^2 - \|u_n - y_n\| - 2\langle u_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 \\ &\quad + 2\lambda_n \langle Ay_n, y_n - t_n\rangle \\ &= \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &\quad + 2\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle. \end{aligned} \tag{3.4}$$

Note that $y_n = P_C(u_n - \lambda_n Au_n)$ and $t_n \in C$. Then, by using the property (2) of P_C, we have $\langle u_n - \lambda_n Au_n - y_n, t_n - y_n\rangle \leq 0$. Hence

$$\begin{aligned} \langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle &= \langle u_n - \lambda_n Au_n - y_n, t_n - y_n\rangle + \langle \lambda_n Au_n - \lambda_n Ay_n, t_n - y_n\rangle \\ &\leq \langle \lambda_n Au_n - \lambda_n Ay_n, t_n - y_n\rangle \leq \lambda_n k\|u_n - y_n\|\|t_n - y_n\|. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we get

$$\begin{aligned} \|t_n - u\|^2 &\leq \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k\|u_n - y_n\|\|t_n - y_n\| \\ &\leq \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n^2 k^2 \|u_n - y_n\|^2 \\ &+ \|y_n - t_n\|^2 = \|u_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \leq \|u_n - u\|^2. \end{aligned} \tag{3.6}$$

Notice that $I - \mu B$ is nonexpansive and for all $x, y \in C$

$$\|(I - \mu B)x - (I - \mu B)y\|^2 \leq \|x - y\|^2 + \mu(\mu - 2\alpha)\|Bx - By\|^2.$$

Then we have

$$\begin{aligned} \|u_n - u\|^2 &= \|T_\mu(x_n - \mu Bx_n) - T_\mu(u - \mu Bu)\|^2 \\ &\leq \|(I - \mu B)x_n - (I - \mu B)u\|^2 \\ &\leq \|x_n - u\|^2 + \mu(\mu - 2\alpha)\|Bx_n - Bu\|^2 \\ &\leq \|x_n - u\|^2. \end{aligned} \tag{3.7}$$

Therefore, from (3.6) and (3.7), together with the convexity of the norm, we get

$$\begin{aligned} &\|z_n - u\|^2 \\ &= \|\alpha_n(x_n - u) + (1 - \alpha_n)(W_n t_n - u)\|^2 \\ &\leq \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\|W_n t_n - u\|^2 \\ &\leq \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\|t_n - u\|^2 \\ &\leq \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)[\|u_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2] \\ &\leq \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)[\|x_n - u\|^2 + \mu(\mu - 2\alpha)\|Bx_n - Bu\|^2 \\ &\quad + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2] \\ &\leq \|x_n - u\|^2, \end{aligned} \tag{3.8}$$

which implies that

$$u \in C_n.$$

So,

$$\Omega \subset C_n, \quad \forall n \geq 0.$$

Next, let us show by mathematical induction that $\{x_n\}$ is well-defined and $\Omega \subset C_n \cap Q_n$ for every $n \geq 0$. For $n = 0$ we have $Q_0 = C$. Hence we obtain

$$\Omega \subset C_0 \cap Q_0.$$

Suppose that x_k is given and $\Omega \subset C_k \cap Q_k$ for some $k \in N$. Since Ω is nonempty, $C_k \cap Q_k$ is a nonempty closed convex subset of C. So, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k}(x_0)$. It is also obvious that there holds $\langle x_{k+1} - u, x_0 - x_{k+1} \geq 0$ for every $u \in C_k \cap Q_k$. Since $\Omega \subset C_k \cap Q_k$, we have

$$\langle x_{k+1} - u, x_0 - x_{k+1} \rangle \geq 0, \quad \forall u \in \Omega$$

and hence

$$\Omega \subset Q_{k+1}.$$

Therefore, we obtain

$$\Omega \subset C_{k+1} \cap Q_{k+1}.$$

Lemma 3.3. *The sequences $\{x_n\}$, $\{z_n\}$, $\{u_n\}$ and $\{t_n\}$ are all bounded and $\lim_{n\to\infty} \|x_n - x_0\|$ exists.*

Proof. From $x_{n+1} = P_{C_n \cap Q_n}(x_0)$, we have

$$\langle x_0 - x_{n+1}, x_{n+1} - y \rangle \geq 0, \quad \forall y \in C_n \cap Q_n.$$

Since $\Omega \subset C_n \cap Q_n$, we also have

$$\langle x_0 - x_{n+1}, x_{n+1} - u \rangle \geq 0, \quad \forall u \in \Omega.$$

So, for $u \in \Omega$, we have

$$\begin{aligned}
0 &\leq \langle x_0 - x_{n+1}, x_{n+1} - u \rangle \\
&= \langle x_0 - x_{n+1}, x_{n+1} - x_0 + x_0 - u \rangle \\
&= -\|x_0 - x_{n+1}\|^2 + \langle x_0 - x_{n+1}, x_0 - u \rangle \\
&\leq -\|x_0 - x_{n+1}\|^2 + \|x_0 - x_{n+1}\| \|x_0 - u\|.
\end{aligned}$$

Hence

$$\|x_0 - x_{n+1}\| \leq \|x_0 - u\|, \quad \forall u \in \Omega, \tag{3.9}$$

which implies that $\{x_n\}$ is bounded. From (3.6)-(3.8), we can deduce that $\{z_n\}, \{u_n\}$ and $\{t_n\}$ are also bounded.

Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n}(x_0)$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|.$$

This together with the boundedness of the sequence $\{x_n\}$ imply that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

Lemma 3.4. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|x_n - z_n\| = \lim_{n\to\infty} \|x_n - t_n\| = 0$ *and* $\lim_{n\to\infty} \|x_n - W_n x_n\| = \lim_{n\to\infty} \|x_n - W x_n\| = 0.$

Proof. It is well-known that in Hilbert spaces H, the following identity holds:

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \quad \forall x, y \in H.$$

Therefore

$$\begin{aligned}\|x_{n+1}-x_n\|^2 &= \|(x_{n+1}-x_0)-(x_n-x_0)\|^2\\ &= \|x_{n+1}-x_0\|^2 - \|x_n-x_0\|^2 - 2\langle x_{n+1}-x_n, x_n-x_0\rangle.\end{aligned}$$

Since $x_{n+1} \in Q_n$, we have

$$\langle x_n - x_{n+1}, x_0 - x_n\rangle \geq 0.$$

It follows that

$$\|x_{n+1}-x_n\|^2 \leq \|x_{n+1}-x_0\|^2 - \|x_n-x_0\|^2.$$

Since $\lim_{n\to\infty}\|x_n-x_0\|$ exists, we get $\|x_{n+1}-x_0\|^2 - \|x_n-x_0\|^2 \to 0$. Therefore,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|,$$

and hence

$$\begin{aligned}\|x_n - z_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\|\\ &\leq 2\|x_{n+1}-x_n\|\\ &\to 0.\end{aligned}$$

For each $u \in \Omega$, from (3.8), we have

$$\begin{aligned}&(1-\alpha_n)\mu(2\alpha-\mu)\|Bx_n - Bu\|^2 + (1-\alpha_n)(1-\lambda_n^2k^2)\|u_n - y_n\|^2\\ &\leq \|x_n-u\|^2 - \|z_n-u\|^2\\ &\leq (\|x_n-u\| + \|z_n-u\|)\|x_n-z_n\|.\end{aligned}$$

Since $\|x_n - z_n\| \to 0$, $\liminf_{n\to\infty}(1-\alpha_n)\mu(2\alpha-\mu) > 0$, $\liminf_{n\to\infty}(1-\alpha_n)(1-\lambda_n^2k^2) > 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|Bx_n - Bu\| \to 0$ and $\|u_n - y_n\| \to 0$.

From Lemma 2.4, we obtain

$$\begin{aligned}&\|u_n-u\|^2\\ &= \|T_\mu(x_n-\mu Bx_n) - T_\mu(u-\mu Bu)\|^2\\ &\leq \langle (x_n-\mu Bx_n)-(u-\mu Bu), u_n-u\rangle\\ &= \frac{1}{2}\big(\|(x_n-\mu Bx_n)-(u-\mu Bu)\|^2 + \|u_n-u\|^2\\ &\quad - \|(x_n-u_n)-\mu(Bx_n-Bu)\|^2\big)\\ &\leq \frac{1}{2}\big(\|x_n-u\|^2 + \|u_n-u\|^2 - \|(x_n-u_n)-\mu(Bx_n-Bu)\|^2\big)\\ &= \frac{1}{2}\big(\|x_n-u\|^2 + \|u_n-u\|^2 - \|x_n-u_n\|^2 + 2\mu\langle x_n-u_n, Bx_n-Bu\rangle\\ &\quad - \mu^2\|Bx_n-Bu\|^2\big).\end{aligned}$$

It follows that

$$\begin{aligned}
&\|u_n - u\|^2 \\
&\le \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2\mu\langle x_n - u_n, Bx_n - Bu\rangle - \mu^2\|Bx_n - Bu\|^2 \\
&\le \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2\mu\|x_n - u_n\|\|Bx_n - Bu\| \\
&\le \|x_n - u\|^2 - \|x_n - u_n\|^2 + M\|Bx_n - Bu\|,
\end{aligned}$$

where $M > 0$ is some constant. Therefore

$$\begin{aligned}
&\|z_n - u\|^2 \\
&\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|u_n - u\|^2 \\
&\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)[\|x_n - u\|^2 - \|x_n - u_n\|^2 + M\|Bx_n - Bu\|] \\
&\le \|x_n - u\|^2 - (1-\alpha_n)\|x_n - u_n\|^2 + M\|Bx_n - Bu\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(1-\alpha_n)\|x_n - u_n\|^2 \\
&\le \|x_n - u\|^2 - \|z_n - u\|^2 + M\|Bx_n - Bu\| \\
&\le (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\| + M\|Bx_n - Bu\|,
\end{aligned}$$

which implies that

$$\|x_n - u_n\| \to 0.$$

We note that

$$\begin{aligned}
&\|t_n - u\|^2 \\
&\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k\|u_n - y_n\|\|t_n - y_n\| \\
&\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + \|u_n - y_n\|^2 + \lambda_n^2 k^2\|y_n - t_n\|^2 \\
&= \|u_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - t_n\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\|z_n - u\|^2 &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|t_n - u\|^2 \\
&\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)(\|u_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - t_n\|^2) \\
&\le \|x_n - u\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|y_n - t_n\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|t_n - y_n\|^2 &\le \frac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - u\|^2 - \|z_n - u\|^2) \\
&\le \frac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\| \\
&\to 0.
\end{aligned}$$

Since A is k-Lipschitz-continuous, we have $\|Ay_n - At_n\| \to 0$. From

$$\|x_n - t_n\| \le \|x_n - u_n\| + \|u_n - y_n\| + \|y_n - t_n\|,$$

we also have

$$\|x_n - t_n\| \to 0.$$

Since $z_n = \alpha_n x_n + (1-\alpha_n)W_n t_n$, we have

$$(1-\alpha_n)(W_n t_n - t_n) = \alpha_n(t_n - x_n) + (z_n - t_n).$$

Then

$$\begin{aligned}(1-c)\|W_n t_n - t_n\| &\le (1-\alpha_n)\|W_n t_n - t_n\| \\ &\le \alpha_n\|t_n - x_n\| + \|z_n - t_n\| \\ &\le (1+\alpha_n)\|t_n - x_n\| + \|z_n - x_n\|\end{aligned}$$

and hence $\|t_n - W_n t_n\| \to 0$. Observe also that

$$\begin{aligned}\|x_n - W_n x_n\| &\le \|x_n - t_n\| + \|t_n - W_n t_n\| + \|W_n t_n - W_n x_n\| \\ &\le \|x_n - t_n\| + \|t_n - W_n t_n\| + \|t_n - x_n\| \\ &\le 2\|x_n - t_n\| + \|t_n - W_n t_n\|.\end{aligned}$$

So, we have $\|x_n - W_n x_n\| \to 0$. On the other hand, since $\{x_n\}$ is bounded, from Lemma 2.3, we have $\lim_{n\to\infty}\|W_n x_n - W x_n\| = 0$. Therefore we have

$$\lim_{n\to\infty}\|x_n - W x_n\| = 0.$$

Finally, according to Lemmas 3.2-3.4, we prove the remainder of Theorem 3.1.

Proof. First, we claim that the necessity of Theorem 3.1 holds. Indeed, suppose that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ converge strongly to the same element $\tilde{u} \in \Omega$. From the (w,s)-sequential continuity of A, we have $Ax_n \to A\tilde{u}$. Observe that, for every $x \in C$,

$$\begin{aligned}&|\langle Ax_n, x - x_n\rangle - \langle A\tilde{u}, x - \tilde{u}\rangle| \\ &\le |\langle Ax_n, x - x_n\rangle - \langle A\tilde{u}, x - x_n\rangle| + |\langle A\tilde{u}, x - x_n\rangle - \langle A\tilde{u}, x - \tilde{u}\rangle| \\ &= |\langle Ax_n - A\tilde{u}, x - x_n\rangle| + |\langle A\tilde{u}, \tilde{u} - x_n\rangle| \\ &\le \|Ax_n - A\tilde{u}\|\|x - x_n\| + |\langle A\tilde{u}, \tilde{u} - x_n\rangle|.\end{aligned}$$

This implies that

$$\liminf_{n\to\infty}\langle Ax_n, x - x_n\rangle = \lim_{n\to\infty}\langle Ax_n, x - x_n\rangle = \langle A\tilde{u}, x - \tilde{u}\rangle, \quad \forall x \in C.$$

Consequently, the necessity holds.

Next, we claim the the sufficiency of Theorem 3.1 holds. Indeed, by Lemmas 3.2-3.4, we have proved that

$$\lim_{n\to\infty}\|x_n - W x_n\| = 0.$$

Furthermore, since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_j}\}$ which converges weakly to some $\tilde{u} \in C$, hence, we have $\lim_{j\to\infty}\|x_{n_j} - W x_{n_j}\| = 0$. Note that, from Lemma 2.5, it follows that $I - W$ is demiclosed at zero. Thus $\tilde{u} \in Fix(W) = \bigcap_{n=1}^{\infty} Fix(S_n)$. Observe that, for every $x \in C$,

$$\begin{aligned}&|\langle Ax_{n_j}, x - x_{n_j}\rangle - \langle A\tilde{u}, x - \tilde{u}\rangle| \\ &\le |\langle Ax_{n_j}, x - x_{n_j}\rangle - \langle A\tilde{u}, x - x_{n_j}\rangle| + |\langle A\tilde{u}, x - x_{n_j}\rangle - \langle A\tilde{u}, x - \tilde{u}\rangle| \\ &= |\langle Ax_{n_j} - A\tilde{u}, x - x_{n_j}\rangle| + |\langle A\tilde{u}, \tilde{u} - x_{n_j}\rangle| \\ &\le \|Ax_{n_j} - A\tilde{u}\|\|x - x_{n_j}\| + |\langle A\tilde{u}, \tilde{u} - x_{n_j}\rangle|.\end{aligned}$$

From the (w,s)-sequential continuity of A, it follows that $\lim_{j\to\infty}\|Ax_{n_j}-A\tilde{u}\|=0$. Hence, we have

$$\langle A\tilde{u}, x-\tilde{u}\rangle = \lim_{j\to\infty}\langle Ax_{n_j}, x-x_{n_j}\rangle \geq \liminf_{n\to\infty}\langle Ax_n, x-x_n\rangle \geq 0, \quad \forall x\in C.$$

This implies that $\tilde{u}\in VI(C,A)$.

Now we show $\tilde{u}\in EP(F,B)$. Since $u_n=T_\mu(x_n-\mu Bx_n)$, for any $y\in C$ we have

$$F(u_n,y)+\frac{1}{\mu}\langle y-u_n, u_n-(x_n-\mu Bx_n)\rangle \geq 0.$$

From the monotonicity of F, we have

$$\frac{1}{\mu}\langle y-u_n, u_n-(x_n-\mu Bx_n)\rangle \geq F(y,u_n), \quad \forall y\in C.$$

Hence

$$\left\langle y-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{\mu}+Bx_{n_i}\right\rangle \geq F(y,u_{n_i}), \quad \forall y\in C. \tag{3.10}$$

Put $v_t=ty+(1-t)\tilde{u}$ for all $t\in(0,1]$ and $y\in C$. Then, we have $v_t\in C$. So, from (3.10) we have

$$\begin{aligned}\langle v_t-u_{n_i}, Bv_t\rangle &\geq \langle v_t-u_{n_i}, Bv_t\rangle - \left\langle v_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{\mu}+Bx_{n_i}\right\rangle\\ &\quad + F(v_t,u_{n_i})\\ &= \langle v_t-u_{n_i}, Bv_t-Bu_{n_i}\rangle + \langle v_t-u_{n_i}, Bu_{n_i}-Bx_{n_i}\rangle\\ &\quad - \left\langle v_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{\mu}\right\rangle + F(v_t,u_{n_i}).\end{aligned} \tag{3.11}$$

Note that $\|Bu_{n_i}-Bx_{n_i}\|\leq\frac{1}{\beta}\|u_{n_i}-x_{n_i}\|\to 0$. Further, from monotonicity of B, we have $\langle v_t-u_{n_i}, Bv_t-Bu_{n_i}\rangle\geq 0$. Letting $i\to\infty$ in (3.11), we have

$$\langle v_t-\tilde{u}, Bv_t\rangle \geq F(v_t,\tilde{u}). \tag{3.12}$$

From (H1), (H4) and (3.12), we also have

$$\begin{aligned}0 = F(v_t,v_t) &\leq tF(v_t,y)+(1-t)F(v_t,\tilde{u})\\ &\leq tF(v_t,y)+(1-t)\langle v_t-\tilde{u}, Bv_t\rangle\\ &= tF(v_t,y)+(1-t)t\langle y-\tilde{u}, Bv_t\rangle\end{aligned}$$

and hence

$$0\leq F(v_t,y)+(1-t)\langle Bv_t, y-\tilde{u}\rangle. \tag{3.13}$$

Letting $t\to 0$ in (3.13), we have, for each $y\in C$,

$$0\leq F(\tilde{u},y)+\langle y-\tilde{u}, B\tilde{u}\rangle.$$

This implies that $\tilde{u}\in EP(F,B)$. Consequently, $\tilde{u}\in\Omega$. That is, $\omega_w(x_n)\subset\Omega$.

In (3.9), if we take $u=P_\Omega(x_0)$, we get

$$\|x_0-x_{n+1}\|\leq\|x_0-P_\Omega(x_0)\|. \tag{3.14}$$

Notice that $\omega_w(x_n) \subset \Omega$. Then, (3.14) and Lemma 2.6 ensure the strong convergence of $\{x_{n+1}\}$ to $P_\Omega(x_0)$. Consequently, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ also converge strongly to $P_\Omega(x_0)$. This completes the proof.

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