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WEAKLY PICARD OPERATORS METHOD FOR MODIFIED FRACTIONAL ITERATIVE FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Boundary value problems for modified fractional iterative functional differential equations is offered. Using weakly Picard operators method, some new existence and uniqueness theorems and data dependence results are presented. Further, examples are given to illustrate our results.
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1. Introduction

Picard and weakly Picard operators methods have been a powerful tool to study the nonlinear differential equations. For more details on this novel methods to discuss existence and uniqueness and the data dependence on data of the solutions for some differential equations and integral equations, one can see Rus et al. [1, 2, 3, 4, 5, 6, 7], Şerban et al. [8], Muresan [9, 10] and Olaru [11]. It is remarkable that Wang et al. [12] apply this interesting methods to study nonlocal Cauchy problems and impulsive Cauchy problems for nonlinear differential equations.

On the other hand, a strong motivation for studying fractional differential equations comes from the fact they have been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics and economics. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. For more details on basic theory of fractional differential equations, one can see the monographs of Diethelm [13], Kilbas et al. [14], Miller and Ross [15], Podlubny [16] and Tarasov [17], and the references [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

However, functional differential equations with fractional derivative have not been studied extensively. In particular, weakly Picard operators methods have not been used to study such problems. Motivated by [7, 12, 31, 32], we offer to study boundary value problems for the following modified fractional iterative functional differential equations

$$\begin{cases} {}^cD^q_{a,t}x(t) = f(t, x(t), x(x^v(t))),\ t \in [a, b],\ v \in R \setminus \{0\},\ q \in (1, 2), \\ x(t) = \varphi(t),\ t \in [a_1, a], \\ x(t) = \psi(t),\ t \in [b, b_1], \end{cases} \tag{1.1}$$

where ${}^cD^q_{a,t}$ is the Caputo fractional derivative of order q with the lower limit a (see Definition 2.3) and

(C_1) $a, b, a_1, b_1 \in R$, $a_1 \le a < b \le b_1$, a function $\Upsilon(z) = z^v$ satisfies $\Upsilon \in C([a_1, b_1], [a_1, b_1])$;

(C_2) $f \in C([a, b] \times [a_1, b_1]^2, R)$;

(C_3) $\varphi \in C([a_1, a], [a_1, b_1])$ and $\psi \in C([b, b_1], [a_1, b_1])$;

(C_4) there exists $L_f > 0$ such that $|f(t, u_1, w_1) - f(t, u_2, w_2)| \le L_f(|u_1 - u_2| + |w_1 - w_2|)$ for all $t \in [a, b]$, $u_i, w_i \in [a_1, b_1]$, $i = 1, 2$.

A function $x \in C([a_1, b_1], [a_1, b_1])$ is said to be a solution of the problem (1.1) if x satisfies the equation ${}^cD^q_{a,t}x(t) = f(t, x(t), x(x^v(t)))$ on $[a, b]$, and the conditions $x(t) = \varphi(t), t \in [a_1, a]$, $x(t) = \psi(t), t \in [b, b_1]$.

It is easy to verify that the problem (1.1) is equivalent with the following fixed point equation

$$x(t) = \begin{cases} \varphi(t),\ \text{for } t \in [a_1, a], \\ w(\varphi, \psi)(t) + \frac{1}{\Gamma(q)} \int_a^b G(t, s) f(s, x(s), x(x^v(s)))ds, \text{for } t \in [a, b], \\ \psi(t),\ \text{for } t \in [b, b_1], \end{cases} \tag{1.2}$$

and $x \in C([a_1, b_1], [a_1, b_1])$, where $w(\varphi, \psi)(t) := \varphi(a) + \frac{\psi(b) - \varphi(a)}{b - a}(t - a)$, G is the Green function defined by

$$G(t, s) := \begin{cases} (t - s)^{q-1} - \frac{t-a}{b-a}(b - s)^{q-1},\ \text{for } a \le s \le t \le b, \\ -\frac{t-a}{b-a}(b - s)^{q-1},\ \text{for } a \le t \le s \le b, \end{cases}$$

and

$$\begin{aligned} \int_a^b G(t, s) f(s, x(s), x(x^v(s)))ds &= -\frac{t - a}{b - a} \int_a^b (b - s)^{q-1} f(s, x(s), x(x^v(s)))ds \\ &\quad + \int_a^t (t - s)^{q-1} f(s, x(s), x(x^v(s)))ds. \end{aligned}$$

Remark 1.1. *Note $G(t, s) \le 0$ for $a \le t \le s \le b$, while $G(t, t) = -\frac{t-a}{b-a}(b - t)^{q-1} < 0$ and $G(t, a) = (t - a)\left((t - a)^{q-2} - (b - a)^{q-2}\right) > 0$ for $a < t < b$. Next $\frac{\partial}{\partial s}G(t, s) = (q - 1)\frac{(b-s)^{q-2}(t-a) - (t-s)^{q-2}(b-a)}{b-a} < 0$ for $a < s < t < b$. So for any $t \in (a, b)$ there is a unique $s(t) \in (a, t)$ such that $G(t, s(t)) = 0$, $G(t, s) < 0$ for $s(t) < s < b$ and $G(t, s) > 0$ for $s(t) > s \ge a$. Note $G(t, b) = 0$ and*

$$s(t) = b + \frac{(b - a)^{\frac{1}{q-1}}(b - t)}{(t - a)^{\frac{1}{q-1}} - (b - a)^{\frac{1}{q-1}}}.$$

Furthermore, $s(t) = a$ for $q = 2$, and this is a great difference for the Green function when $q \in (1,2)$. Since we cannot expect monotonicity of the integral operator B_f defined in (3.1) below.

On the other hand, we derive

$$\int_a^b G(t,s)ds = \frac{t-a}{q}\left((t-a)^{q-1} - (b-a)^{q-1}\right).$$

Hence $\int_a^b G(t,s)ds \leq 0$ for $t \in [a,b]$ and $\int_a^b G(t,s)ds < 0$ for $t \in (a,b]$, which holds also for $q = 2$. So $G(t,\cdot)$ is nonpositive in average on $[a,b]$.

On the other hand, the first equation of the problem (1.1) is equivalent with

$$x(t) := \begin{cases} x(t), \text{ for } t \in [a_1, a], \\ w(x|_{[a_1,a]}, x|_{[b,b_1]})(t) \\ +\frac{1}{\Gamma(q)}\int_a^b G(t,s)f(s,x(s),x(x^v(s)))ds, \text{for } t \in [a,b], \\ x(t), \text{ for } t \in [b, b_1], \end{cases} \tag{1.3}$$

and $x \in C([a_1,b_1],[a_1,b_1])$.

We will apply a new method to study the equations (1.2) and (1.3). More precisely, we will use the weakly Picard operator technique to obtain some new existence, uniqueness and data dependence results for the solution of the problem (1.1).

2. Notation, definitions and auxiliary facts

To end this section, we recall some basic definitions of the fractional calculus theory which are used further in this paper. For more details, see Kilbas et al. [14].

Definition 2.1. *The fractional order integral of the function $h \in L^1([a,b],R)$ of order $q \in R^+$ is defined by*

$$I_{a,t}^q h(t) = \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)ds$$

where Γ is the Gamma function.

Definition 2.2. *For a function h given on the interval $[a,b]$, the qth Riemann-Liouville fractional order derivative of h, is defined by*

$$^L(D_{a,t}^q h)(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-q-1}h(s)ds,$$

here $n = [q]+1$ and $[q]$ denotes the integer part of q.

Definition 2.3. *The Caputo derivative of order q for a function $f : [a,b] \to R$ can be written as*

$$^cD_{a,t}^q h(t) = {}^LD_{a,t}^q\left(h(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}h^{(k)}(a)\right), \ t > 0, \ n-1 < q < n.$$

We need some notions and results from the weakly Picard operator theory (for more details see Rus [5, 6]).

Let (X,d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A = \{x \in X \mid A(x) = x\}$−the fixed point set of A;
$I(A) = \{Y \in P(X) \mid A(Y) \subseteq Y, Y \neq \emptyset\}$;
$A^{n+1} = A^n \circ A$, $A^1 = A$, $A^0 = I$, $n \in N$
$P(X) = \{Y \subseteq X \mid Y \neq \emptyset\}$;
$O_A(x) = \{x, A(x), A^2(x), \cdots, A^n(x), \cdots\}$−the A−orbit of $x \in X$;
$H : P(X) \times P(X) \to R_+ \cup \{+\infty\}$;
$H(Y,Z) = \max\left\{\sup_{y\in Y} \inf_{z\in Z} d(y,z), \sup_{z\in Z} \inf_{y\in Y} d(y,z)\right\}$ −the Pompeiu-Hausdorff functional on $P(X) \times P(X)$.

Definition 2.4. *Let (X, d) be a metric space. An operator $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that $F_A = \{x^*\}$ and the sequence $(A^n(x_0))_{n\in N}$ converges to x^* for all $x_0 \in X$.*

Theorem 2.5. *(Contraction principle) Let (X, d) be a complete metric space and $A : X \to X$ a γ−contraction. Then*
(i) $F_A = \{x^*\}$;
(ii) $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$;
(iii) $d(x^*, A^n(x_0)) \leq \frac{\gamma^n}{1-\gamma} d(x_0, A(x_0))$, for all $n \in N$.

Remark 2.6. *Accordingly to the Definition 2.4, the contraction principle insures that, if $A : X \to X$ is a γ−contraction on the complete metric space X, then it is a Picard operator.*

Theorem 2.7. *Let (X, d) be a complete metric space and $A, B : X \to X$ two operators. We suppose the following:*
(i) A is a contraction with contraction constant γ and $F_A = \{x_A^*\}$.
(ii) B has fixed points and $x_B^* \in F_B$.
(iii) There exists $\eta > 0$ such that $d(A(x), B(x)) \leq \eta$, for all $x \in X$.
Then $d(x_A^*, x_B^*) \leq \frac{\eta}{1-\gamma}$.

Definition 2.8. *Let (X, d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator if the sequence $(A^n(x_0))_{n\in N}$ converges for all $x_0 \in X$ and its limit (which may depend on x_0) is a fixed point of A.*

Theorem 2.9. *Let (X, d) be a metric space. Then $A : X \to X$ is a weakly Picard operator if and only if there exists a partition $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ of X such that*
(i) $X_\lambda \in I(A)$, for all $\lambda \in \Lambda$;
(ii) $A\,|_{X_\lambda} : X_\lambda \to X_\lambda$ is a Picard operator, for all $\lambda \in \Lambda$.

Definition 2.10. *If A is a weakly Picard operator, then we consider the operator A^∞ defined by $A^\infty : X \to X$, $A^\infty(x) = \lim_{n\to\infty} A^n(x)$.*

It is clear that $A^\infty(X) = F_A$ and $\omega_A(x) = \{A^\infty(x)\}$ where $\omega_A(x)$ is the ω−limit point set of mapping A for point x.

Definition 2.11. *Let A be a weakly Picard operator and $c > 0$. The operator A is c−weakly Picard operator if $d(x, A^\infty(x)) \leq cd(x, A(x))$, $\forall\ x \in X$.*

Remark 2.12. *Let (X, d) be a complete metric space and $A : X \to X$ a continuous operator. We suppose that there exists $\gamma \in [0, 1)$ such that*

$$d(A^2(x), A(x)) \leq \gamma d(x, Ax),\ \forall\ x \in X.$$

Then A is $c-$weakly Picard operator with $c = \frac{1}{1-\gamma}$.

Theorem 2.13. *Let (X, d) be a complete metric space and $A_i : X \to X$, $i = 1, 2$. We suppose that*
(i) A_i is c_i-weakly Picard operator, $i = 1, 2$;
(ii) There exists $\alpha > 0$ such that $d(A_1(x), A_2(x)) \leq \alpha$, $\forall\, x \in X$.
Then $H(F_{A_1}, F_{A_2}) \leq \alpha \max(c_1, c_2)$.

3. Existence

In what follows we consider the fixed point equation (1.2). Consider the operator

$$B_f : C([a_1, b_1], [a_1, b_1]) \to C([a_1, b_1], [a_1, b_1])$$

where

$$B_f(x)(t) := \begin{cases} \varphi(t), \text{ for } t \in [a_1, a], \\ w(\varphi, \psi)(t) + \int_a^b G(t, s) f(s, x(s), x(x^v(s)))ds, \text{ for } t \in [a, b], \\ \psi(t), \text{ for } t \in [b, b_1]. \end{cases} \tag{3.1}$$

It is clear that x is a solution of the problem (1.1) if and only if x is a fixed point of the operator B_f. So, the problem is to study the fixed point equation $x = B_f(x)$.

Let $L > 0$ and introduce the following notation:

$$C_L([a_1, b_1], [a_1, b_1]) = \{x \in C([a_1, b_1], [a_1, b_1]) : |x(t_1) - x(t_2)| \leq L|t_1 - t_2|\},$$

for all $t_1, t_2 \in [a_1, b_1]$. Remark that $C_L([a_1, b_1], [a_1, b_1]) \subseteq C([a_1, b_1], R)$ is also a complete metric space with respect to the metric,

$$d(x_1, x_2) := \max_{a_1 \leq t \leq b_1} |x_1(t) - x_2(t)|.$$

Theorem 3.1. *We suppose that*
(i) the conditions (C_1)–(C_4) are satisfied but in addition $v \geq 1$;
(ii) $\varphi \in C_L([a_1, a], [a_1, b_1])$, $\psi \in C_L([b, b_1], [a_1, b_1])$;
(iii) there are m_f, $M_f \in R$ such that

$$m_f \leq f(t, u, w) \leq M_f, \ \forall\, t \in [a, b], u, w \in [a_1, b_1],$$

and moreover,

$$a_1 \leq \min(\varphi(a), \psi(b)) - \max\left(0, \frac{M_f(b-a)^q}{\Gamma(q+1)}\right) + \min\left(0, \frac{m_f(b-a)^q}{\Gamma(q+1)}\right),$$

$$\max(\varphi(a), \psi(b)) - \min\left(0, \frac{m_f(b-a)^q}{\Gamma(q+1)}\right) + \max\left(0, \frac{M_f(b-a)^q}{\Gamma(q+1)}\right) \leq b_1;$$

(iv) $\dfrac{|\psi(b) - \varphi(a)|}{b-a} + \dfrac{(b-a)^{q-1}(1+q)\max\{|m_f|, |M_f|\}}{\Gamma(q+1)} < L;$

(v) $\dfrac{2(b-a)^q L_f(Lv\max\{|a_1|, |b_1|\}^{v-1} + 2)}{\Gamma(q+1)} < 1.$

Then the problem (1.1) has in $C_L([a_1, b_1], [a_1, b_1])$ a unique solution.

Moreover, the operator B_f,

$$B_f : C_L([a_1, b_1], [a_1, b_1]) \to C_L([a, b], [a_1, b_1])$$

is a $c-$Picard operator with

$$c := \frac{\Gamma(q+1)}{\Gamma(q+1) - 2(b-a)^q L_f(Lv\max\{|a_1|, |b_1|\}^{v-1} + 2)}.$$

Proof. First of all we remark that the condition (iii) and (iv) imply that $C_L([a_1, b_1], [a_1, b_1])$ is an invariant subset for B_f. Indeed, for $t \in [a_1, a] \cup [b, b_1]$, we have $B_f(x)(t) \in [a_1, b_1]$. Furthermore, we obtain $a_1 \le B_f(x)(t) \le b_1, \ \forall\, t \in [a, b]$, if and only if

$$a_1 \le \min_{t\in[a,b]} B_f(x)(t) \tag{3.2}$$

and

$$\max_{t\in[a,b]} B_f(x)(t) \le b_1 \tag{3.3}$$

hold.

Since

$$\begin{aligned}\min_{t\in[a,b]} B_f(x)(t) \quad &\ge \quad \min(\varphi(a), \psi(b)) \\ &\quad - \max\left(0, \frac{M_f(b-a)^q}{\Gamma(q+1)}\right) + \min\left(0, \frac{m_f(b-a)^q}{\Gamma(q+1)}\right),\end{aligned}$$

respectively

$$\begin{aligned}\max_{t\in[a,b]} B_f(x)(t) \quad &\le \quad \max(\varphi(a), \psi(b)) \\ &\quad - \min\left(0, \frac{m_f(b-a)^q}{\Gamma(q+1)}\right) + \max\left(0, \frac{M_f(b-a)^q}{\Gamma(q+1)}\right),\end{aligned}$$

the requirements (3.2) and (3.3) are equivalent with the conditions appearing in (iii).

Now, consider $a_1 \le t_1 < t_2 \le a$. Then,

$$\begin{aligned}|B_f(x)(t_2) - B_f(x)(t_1)| \quad &= \quad |\varphi(t_2) - \varphi(t_1)| \\ &\le \quad L|t_1 - t_2|\end{aligned}$$

as $\varphi \in C_L([a_1, a], [a_1, b_1])$, due to (ii).

Similarly, for $b \le t_1 < t_2 \le b_1$,

$$\begin{aligned}|B_f(x)(t_2) - B_f(x)(t_1)| \quad &= \quad |\psi(t_2) - \psi(t_1)| \\ &\le \quad L|t_1 - t_2|\end{aligned}$$

that follows from (ii), too.

On the other hand, for $a \le t_1 < t_2 \le b$,

$$\begin{aligned}&|B_f(x)(t_2) - B_f(x)(t_1)| \le |w(\varphi,\psi)(t_2) - w(\varphi,\psi)(t_1)| \\ &+\frac{1}{\Gamma(q)}\int_a^b |G(t_2, s) - G(t_1, s)||f(s, x(s), x(x^v(s))))|ds \\ \le\ & \frac{|\psi(b) - \varphi(a)|}{b-a}|t_2 - t_1| + \frac{|t_2 - t_1|}{(b-a)\Gamma(q)}\int_a^b (b-s)^{q-1}|f(s, x(s), x(x^v(s)))|ds \\ &+\frac{1}{\Gamma(q)}\int_a^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}]|f(s, x(s), x(x^v(s)))|ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}|f(s,x(s),x(x^v(s)))|ds\\
&\le \frac{|\psi(b)-\varphi(a)|}{b-a}|t_2-t_1|+\frac{(b-a)^{q-1}\max\{|m_f|,|M_f|\}|t_2-t_1|}{\Gamma(q+1)}\\
&+\frac{\max\{|m_f|,|M_f|\}}{\Gamma(q+1)}\left((t_2-a)^q-(t_1-a)^q-(t_2-t_1)^q\right)+\frac{\max\{|m_f|,|M_f|\}}{\Gamma(q+1)}(t_2-t_1)^q\\
&\le\left(\frac{|\psi(b)-\varphi(a)|}{b-a}+\frac{(b-a)^{q-1}(1+q)\max\{|m_f|,|M_f|\}}{\Gamma(q+1)}\right)|t_1-t_2|.
\end{aligned}
$$

where we use the inequality

$$r^q-s^q\le qr^{q-1}(r-s)$$

for all $r\ge s\ge 0$. Therefore, due to (iv), the function $B_f(x)$ is L–Lipschitz in t. Thus, according to the above, we have $C_L([a_1,a],[a_1,b_1])\in I(B_f)$.

From the condition (v) it follows that B_f is an L_{B_f}–contraction with

$$L_{B_f}:=\frac{2(b-a)^qL_f(Lv\max\{|a_1|,|b_1|\}^{v-1}+2)}{\Gamma(q+1)}$$

Indeed, for all $t\in[a_1,a]\cup[b,b_1]$, we have

$$|B_f(x_1)(t)-B_f(x_2)(t)|=0.$$

Moreover, for $t\in[a,b]$ we get

$$
\begin{aligned}
&|B_f(x_1)(t)-B_f(x_2)(t)|\\
&\le\frac{1}{\Gamma(q)}\int_a^b|G(t,s)f(s,x_1(s))-G(t,s)f(s,x_2(s))|ds\\
&\le\frac{t-a}{(b-a)\Gamma(q)}\int_a^b(b-s)^{q-1}\left|f(s,x_1(s),x_1(x_1^v(s)))-f(s,x_2(s),x_2(x_2^v(s)))\right|ds\\
&+\frac{1}{\Gamma(q)}\int_a^t(t-s)^{q-1}\left|f(s,x_1(s),x_1(x_1^v(s)))-f(s,x_2(s),x_2(x_2^v(s)))\right|ds\\
&\le\frac{L_f}{\Gamma(q)}\int_a^b(b-s)^{q-1}\left[|x_1(s)-x_2(s)|+|x_1(x_1^v(s))-x_2(x_2^v(s))|\right]ds\\
&+\frac{L_f}{\Gamma(q)}\int_a^t(t-s)^{q-1}\left[|x_1(s)-x_2(s)|+|x_1(x_1^v(s))-x_2(x_2^v(s))|\right]ds\\
&\le\frac{L_f}{\Gamma(q)}\int_a^b(b-s)^{q-1}\Big[|x_1(s)-x_2(s)|+|x_1(x_1^v(s))-x_1(x_2^v(s))|\\
&\qquad+|x_1(x_2^v(s))-x_2(x_2^v(s))|\Big]ds\\
&+\frac{L_f}{\Gamma(q)}\int_a^t(t-s)^{q-1}\Big[|x_1(s)-x_2(s)|+|x_1(x_1^v(s))-x_1(x_2^v(s))|\\
&\qquad+|x_1(x_2^v(s))-x_2(x_2^v(s))|\Big]ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{L_f}{\Gamma(q)}\int_a^b (b-s)^{q-1}\Big[2\|x_1-x_2\|_C + L\,|x_1^v(s)-x_2^v(s)|\Big]ds \\
&\quad +\frac{L_f}{\Gamma(q)}\int_a^t (t-s)^{q-1}\Big[2\|x_1-x_2\|_C + L\,|x_1^v(s)-x_2^v(s)|\Big]ds \\
&\leq \frac{L_f}{\Gamma(q)}\int_a^b (b-s)^{q-1}\Big[2\|x_1-x_2\|_C + Lv\max\{|a_1|,|b_1|\}^{v-1}\,|x_1(s)-x_2(s)|\Big]ds \\
&\quad +\frac{L_f}{\Gamma(q)}\int_a^t (t-s)^{q-1}\Big[2\|x_1-x_2\|_C + Lv\max\{|a_1|,|b_1|\}^{v-1}\,|x_1(s)-x_2(s)|\Big]ds \\
&\leq \frac{L_f}{\Gamma(q)}\int_a^b (b-s)^{q-1}\Big[2\|x_1-x_2\|_C + Lv\max\{|a_1|,|b_1|\}^{v-1}\|x_1-x_2\|_C\Big]ds \\
&\quad +\frac{L_f}{\Gamma(q)}\int_a^t (t-s)^{q-1}\Big[2\|x_1-x_2\|_C + Lv\max\{|a_1|,|b_1|\}^{v-1}\|x_1-x_2\|_C\Big]ds \\
&\leq \frac{2(b-a)^q L_f(Lv\max\{|a_1|,|b_1|\}^{v-1}+2)}{\Gamma(q+1)}\|x_1-x_2\|_C.
\end{aligned}
$$

So, B_f is a $c-$Picard operator, with

$$c = \frac{1}{1-\frac{2(b-a)^q L_f(Lv\max\{|a_1|,|b_1|\}^{v-1}+2)}{\Gamma(q+1)}}.$$

This completes the proof. □

In what follows, consider the following operator

$$E_f : C_L([a_1,b_1],[a_1,b_1]) \to C_L([a_1,b_1],[a_1,b_1])$$

where

$$(E_f x)(t) := \begin{cases} x(t), \text{ for } t\in[a_1,a], \\ w(x|_{[a_1,a]}, x|_{[b,b_1]})(t) \\ +\frac{1}{\Gamma(q)}\int_a^b G(t,s)f(s,x(s),x(x^v(s))))ds, \text{ for } t\in[a,b], \\ x(t), \text{ for } t\in[b,b_1]. \end{cases}$$

Theorem 3.2. *Under the conditions of Theorem 3.1, the operator $E_f : C_L([a_1,b_1],[a_1,b_1]) \to C_L([a_1,b_1],[a_1,b_1])$ is a weakly Picard operator.*

Proof. The operator E_f is a continuous operator but it is not a contraction operator.

Let us take the following notation:

$$X_{\varphi,\psi} := \{x\in C_L([a_1,b_1],[a_1,b_1]) : x|_{[a_1,a]} = \varphi,\ x|_{[b,b_1]} = \psi\}.$$

Then we can write

$$C_L([a_1,b_1],[a_1,b_1]) = \bigcup_{\varphi\in C_L([a_1,a],[a_1,b_1]);\psi\in C_L([a_1,a],[a_1,b_1])} X_{\varphi,\psi}.$$

We have that $X_{\varphi,\psi}\in I(E_f)$ and $E_f|_{X_{\varphi,\psi}}$ is a Picard operator, because it is the operator which appears in the proof of Theorem 3.1. By applying Theorem 2.9, we obtain that E_f is weakly Picard operator. This completes the proof. □

Finally, in general, we have immediately from the proof of Theorem 3.1 and Schauder fixed point theorem the following existence result.

Theorem 3.3. *Suppose that the conditions (C_1)–(C_4) are satisfied together with assumptions (ii)–(iv) of Theorem 3.1. Then the problem (1.1) has a solution in* $C_L([a_1, b_1], [a_1, b_1])$.

We do not know about uniqueness. But this is not so surprising, since B_f is not Lipschitzian in general. So we cannot apply metric fixed point theorems, only topological one. This can be simply illustrated on the problems

$$x'(t) = Ax(x^2(t)), \quad x(0) = 0, \tag{3.4}$$

and

$$x'(t) = Ax(\sqrt[4]{x(t)}), \quad x(0) = 0, \tag{3.5}$$

for $A > 0$. Rewriting (3.4) as $x(t) = B_1(x)(t) = A\int_0^t x(x^2(s))ds$, and applying the above procedure to B_1, it follows that $B_1 : C_{Ab_1}([0, b_1], [0, b_1]) \to C_{Ab_1}([0, b_1], [0, b_1])$, $Ab_1 \leq 1$ $0 < b_1 \leq 1$ is $Ab_1(1 + 2Ab_1^2)$–Lipschitzian, so its only solution is $x(t) = 0$ in that space when $Ab_1(1 + 2Ab_1^2) < 1$. On the other hand, rewriting (3.5) as $x(t) = B_2(x)(t) = A\int_0^t x(\sqrt[4]{x(s)})ds$, it follows that $B_2 : C_{Ab_1}([0, b_1], [0, b_1]) \to C_{Ab_1}([0, b_1], [0, b_1])$, $b_1 \geq 1$, $Ab_1 \leq 1$ satisfies

$$\|B_2(x_1) - B_2(x_2)\|_C \leq Ab_1 \left(\|x_1 - x_2\|_C + Ab_1 \sqrt[4]{\|x_1 - x_2\|_C}\right),$$

so it is not Lipschitzian. Hence (3.5) should have a nonzero solution, and it does have $x(t) = \frac{4}{A}t^2$.

4. Data dependence

In this section, we consider the problem (1.1) and suppose the conditions of Theorem 3.1 are satisfied. Denote by $x(\cdot; \varphi, \psi, f)$ the solution of this problem.

Theorem 4.1. *Let $\varphi_i, \psi_i, f_i, i = 1, 2$, be as in Theorem 3.1. Furthermore, we suppose that*

(i) there exists $\eta_1 > 0$, such that

$$|\varphi_1(t) - \varphi_2(t)| \leq \eta_1, \ t \in [a_1, a],$$

and

$$|\psi_1(t) - \psi_2(t)| \leq \eta_1, \ t \in [b, b_1];$$

(ii) there exists $\eta_2 > 0$ such that

$$|f_1(t, u, w) - f_2(t, u, w)| \leq \eta_2, \ \forall \ t \in [a, b], \ u, w \in [a_1, b_1].$$

Then

$$|x(t; \varphi_1, \psi_1, f_1) - x(t; \varphi_2, \psi_2, f_2)| \quad \leq \quad \frac{3\eta_1 + \frac{2(b-a)^q}{\Gamma(q+1)}\eta_2}{1 - \frac{2(b-a)^q L_f(Lv \max\{|a_1|,|b_1|\}^{v-1}+2)}{\Gamma(q+1)}},$$

where $L_f = \min\{L_{f_1}, L_{f_2}\}$.

Proof. Consider the operators $B_{\varphi_i,\psi_i,f_i}, i = 1, 2$. From Theorem 3.1 these operators are contractions. Additionally,

$$\begin{aligned}
& \|B_{\varphi_1,\psi_1,f_1}(x) - B_{\varphi_2,\psi_2,f_2}(x)\|_C \\
\leq\ & |w(\varphi_1,\psi_1)(t) - w(\varphi_2,\psi_2)(t)| \\
& + \frac{1}{\Gamma(q)} \int_a^b G(t,s)|f_1(s,x(s),x(x^v(s)))) - f_2(s,x(s),x(x^v(s))))|ds \\
\leq\ & 3\eta_1 + \frac{2(b-a)^q}{\Gamma(q+1)}\eta_2.
\end{aligned}$$

Now, the proof follows from Theorem 2.7, with

$$A := B_{\varphi_1,\psi_1,f_1},\ B := B_{\varphi_2,\psi_2,f_2},\ \eta := 3\eta_1 + \frac{2(b-a)^q}{\Gamma(q+1)}\eta_2,$$

and

$$\gamma := L_A = \frac{2(b-a)^q L_f(Lv\max\{|a_1|,|b_1|\}^{v-1} + 2)}{\Gamma(q+1)},$$

where we can suppose that $L_f = L_{f_1} = \min\{L_{f_1}, L_{f_2}\}$.

Remark 4.2. *Let $\varphi_i, \psi_i, f_i, i \in N$, and φ, ψ, f be as in Theorem 3.1. We suppose that*

$$\varphi_i \stackrel{unif.}{\longrightarrow} \varphi,\ \psi_i \stackrel{unif.}{\longrightarrow} \psi,\ f_i \stackrel{unif.}{\longrightarrow} f.$$

Then

$$x(\cdot;\varphi_i,\psi_i,f_i) \stackrel{unif.}{\longrightarrow} x(\cdot;\varphi,\psi,f),\ \text{as } i \to \infty.$$

Theorem 4.3. *Let f_1 and f_2 be as in Theorem 3.1. Let $F_{E_{f_i}}$ be the solution set of the first equation of the problem (1.1) corresponding to f_i, $i = 1, 2$. Suppose that there exists $\eta > 0$ such that*

$$|f_1(t,u_1,w_1) - f_2(t,u_2,w_2)| \leq \eta,\ \forall\ t \in [a,b],\ u_i, w_i \in [a_1,b_1],\ i = 1,2.$$

Then

$$H_{\|\cdot\|_C}(F_{E_{f_1}}, F_{E_{f_2}}) \leq \frac{\eta 2(b-a)^q}{\Gamma(q+1) - 2(b-a)^q L_f(Lv\max\{|a_1|,|b_1|\}^{v-1} + 2)},$$

where $L_f = \max\{L_{f_1}, L_{f_2}\}$ and $H_{\|\cdot\|_C}$ denotes the Pompeiu-Hausdorff functional with respect to $\|\cdot\|_C$ on $C_L([a_1,b_1],[a_1,b_1])$.

Proof. We will look for those c_i, for which in condition of Theorem 3.1 the operators $E_{f_i}, i = 1, 2$, are c_i-weakly Picard operators.

Set

$$X_{\varphi,\psi} := \{x \in C_L([a_1,b_1],[a_1,b_1]) : x|_{[a_1,a]} = \varphi,\ x|_{[b,b_1]} = \psi\}.$$

It is clear that $E_{f_i}|_{X_{\varphi,\psi}} = B_{f_i}$. So, from Theorem 2.9 and Theorem 3.1 we have

$$\|E_{f_i}^2(x) - E_{f_i}(x)\|_C \leq \frac{2(b-a)^q L_{f_i}(Lv\max\{|a_1|,|b_1|\}^{v-1} + 2)}{\Gamma(q+1)}\|E_{f_i}(x) - x\|_C$$

for all $x \in C_L([a_1,b_1],[a_1,b_1])$ and $i = 1, 2$.

Now, choosing

$$\lambda_i = \frac{2(b-a)^q L_{f_i}(Lv\max\{|a_1|,|b_1|\}^{v-1}+2)}{\Gamma(q+1)},$$

we get that E_{f_i} are c_i-weakly Picard operators, with $c_i = \frac{1}{1-\lambda_i},\ i=1,2$.

Next, we obtain

$$\|E_{f_1}(x) - E_{f_2}(x)\|_C \leq \eta\frac{2(b-a)^q}{\Gamma(q+1)},$$

for all $x \in C_L([a_1,b_1],[a_1,b_1])$.

Applying Theorem 2.13 we have that

$$H_{\|\cdot\|_C}(F_{E_{f_1}}, F_{E_{f_2}}) \leq \frac{\eta 2(b-a)^q}{\Gamma(q+1) - 2(b-a)^q L_f(Lv\max\{|a_1|,|b_1|\}^{v-1}+2)}.$$

The proof is completed. □

5. Examples

Consider the following problem:

$$\begin{cases} {}^cD^{\frac{3}{2}}_{\frac{2}{5},t}x(t) = \mu x(x(t)),\ t\in[\frac{2}{5},\frac{3}{5}],\ \mu>0,\\ x(t) = \frac{1}{2},\ t\in[\frac{1}{5},\frac{2}{5}],\\ x(t)=\frac{1}{2},\ t\in[\frac{3}{5},\frac{4}{5}], \end{cases} \tag{5.1}$$

where $x \in C_L([\frac{1}{5},\frac{4}{5}],[\frac{1}{5},\frac{4}{5}])$.

Proposition 5.1. *Consider the problem (5.1). We suppose that*

$$\begin{array}{lll} \mu < \frac{3L\sqrt{5\pi}}{8} & \textit{for} & 0<L\leq\sqrt{6}-1,\\ \mu<\frac{15\sqrt{5\pi}}{8(L+2)} & \textit{for} & \sqrt{6}-1\geq L. \end{array} \tag{5.2}$$

Then the problem (5.1) has in $C_L([\frac{1}{5},\frac{4}{5}],[\frac{1}{5},\frac{4}{5}])$ a unique solution.

Proof. First of all notice that accordingly to Theorem 3.1 we have $v=1$, $q=\frac{3}{2}$, $a=\frac{2}{5}, b=\frac{3}{5}$, $\psi(\frac{3}{5})=\frac{1}{2}, \varphi(\frac{2}{5})=\frac{1}{2}$, $a_1=\frac{1}{5}$, $b_1=\frac{4}{5}$. Observe that the Lipschitz constant for the function $f(t,u_1,u_2)=\mu u_2$ is $L_f=\mu$ and $|f(t,u_1,u_2)-f(t,w_1,w_2)|\leq \mu|u_2-w_2|,\ u_i,w_i\in[\frac{1}{5},\frac{4}{5}]$. So we choose $m_f=\frac{\mu}{5}$ and $M_f=\frac{4\mu}{5}$. By a common check in the conditions of Theorem 3.1 we can make sure that

$$\mu \leq \frac{45\sqrt{5\pi}}{32}$$

$$\Longleftrightarrow \begin{cases} a_1 \leq \min(\varphi(a),\psi(b)) - \max\left(0,\frac{M_f(b-a)^q}{\Gamma(q+1)}\right) + \min\left(0,\frac{m_f(b-a)^q}{\Gamma(q+1)}\right),\\ \max(\varphi(a),\psi(b)) - \min\left(0,\frac{m_f(b-a)^q}{\Gamma(q+1)}\right)+\max\left(0,\frac{M_f(b-a)^q}{\Gamma(q+1)}\right)\leq b_1; \end{cases}$$

$$\frac{8}{3\sqrt{5\pi}}\mu < L \Longleftrightarrow \frac{|\psi(b)-\varphi(a)|}{b} + \frac{(b-a)^{q-1}(1+q)\max\{|m_f|,|M_f|\}}{\Gamma(q+1)} < L,$$

and

$$\frac{8}{15\sqrt{5\pi}}\mu < \frac{1}{L+2} \Longleftrightarrow \frac{2(b-a)^q L_f(Lv\max\{|a_1|,|b_1|\}^{v-1}+2)}{\Gamma(q+1)} < 1.$$

Note

$$\frac{8}{15\sqrt{5\pi}}\mu < \frac{1}{L+2} \Longrightarrow \mu \leq \frac{45\sqrt{5\pi}}{32}.$$

Hence we consider

$$\mu < \min\left\{\frac{3L\sqrt{5\pi}}{8}, \frac{15\sqrt{5\pi}}{8(L+2)}\right\} = \frac{3\sqrt{5\pi}}{8}\min\left\{L, \frac{5}{L+2}\right\}.$$

Note $L = \min\left\{L, \frac{5}{L+2}\right\}$ for $0 < L \leq \sqrt{6}-1$, and $\frac{5}{L+2} = \min\left\{L, \frac{5}{L+2}\right\}$ for $L \geq \sqrt{6}-1$. This gives (5.2) and therefore, by Theorem 3.1 we have the proof. □

Now take the following problems

$$\begin{cases} {}^cD^{\frac{3}{2}}_{\frac{2}{5},t}x(t) = \mu_1 x(x(t)),\ t \in [\frac{2}{5}, \frac{3}{5}],\ \mu_1 > 0, \\ x(t) = \varphi_1,\ [\frac{1}{5}, \frac{2}{5}], \\ x(t) = \psi_1,\ [\frac{3}{5}, \frac{4}{5}], \end{cases} \tag{5.3}$$

and

$$\begin{cases} {}^cD^{\frac{3}{2}}_{\frac{2}{5},t}x(t) = \mu_2 x(x(t)),\ t \in [\frac{2}{5}, \frac{3}{5}],\ \mu_2 > 0, \\ x(t) = \varphi_2,\ [\frac{1}{5}, \frac{2}{5}], \\ x(t) = \psi_2,\ [\frac{3}{5}, \frac{4}{5}]. \end{cases} \tag{5.4}$$

Suppose that we have satisfied the following assumptions

(H_1) $\varphi_i \in C_L([\frac{1}{5}, \frac{2}{5}], [\frac{1}{5}, \frac{4}{5}])$, $\psi_i \in C_L([\frac{3}{5}, \frac{4}{5}], [\frac{1}{5}, \frac{4}{5}])$ such that $\varphi_i(\frac{2}{5}) = \frac{1}{2}$, $\psi_i(\frac{3}{5}) = \frac{1}{2}$, $i = 1, 2$;

(H_2) we are in the conditions (5.2) of Proposition 5.1 for both of the problems (5.3) and (5.4).

Let x_1^*, be the unique solution of the problem (5.3) and x_2^* be the unique solution of the problem (5.4). We are looking for an estimation for $\|x_1^* - x_2^*\|_C$.

Then, build upon Theorem 4.1 and Theorem 4.3, by a common substitution one can make sure that we have

Proposition 5.2. *Consider the problems (5.3), (5.4) and suppose the requirements (H_1)–(H_2) hold. Additionally,*

(i) there exists $\eta_1 > 0$ such that

$$|\varphi_1(t) - \varphi_2(t)| \leq \eta_1,\ \forall\ t \in [\frac{1}{5}, \frac{2}{5}],$$

and

$$|\psi_1(t) - \psi_2(t)| \leq \eta_1,\ \forall\ t \in [\frac{3}{5}, \frac{4}{5}].$$

(ii) there exists $\eta_2 > 0$ such that

$$|\mu_1 - \mu_2| \leq \frac{5}{4}\eta_2.$$

Then

$$|x_1^*(\cdot; \psi_1, \psi_1, f_1) - x_2^*(\cdot; \psi_2, \psi_2, f_2)| \quad \leq \quad \frac{45\sqrt{5\pi}\eta_1 + 8\eta_2}{15\sqrt{5\pi} - 8(L+2)\min\{\mu_1, \mu_2\}}.$$

Further, let $F_{E_{f_1}}$ *be the solution set of the first equation of the problem (5.3) and* $F_{E_{f_2}}$ *be the solution set of the first equation of the problem (5.4). Then,*

$$H_{\|\cdot\|_C}(F_{E_{f_1}}, F_{E_{f_2}}) \leq \frac{15\sqrt{5\pi}\eta_2}{15\sqrt{5\pi} - 8(L+2)\max\{\mu_1, \mu_2\}},$$

where $H_{\|\cdot\|_C}$ *denotes the Pompeiu-Hausdorff functional with respect to* $C_L([\frac{1}{5}, \frac{4}{5}], [\frac{1}{5}, \frac{4}{5}])$.

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