Fixed Point Theory, 15(2014), No. 1, 285-296
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A NEW METHOD IN THE STUDY OF IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

JINRONG WANG*, JIANHUA DENG* AND YONG ZHOU**

* Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P.R. China
E-mail: wjr9668@126.com, jhdengmath@126.com

** Department of Mathematics, Xiangtan University, Xiangtan, Hunan 411105, P.R. China
E-mail: yzhou@xtu.edu.cn

Abstract. In this paper, a new method named by Picard and weakly Picard operators methods is applied to revisit impulsive fractional differential equations in Banach spaces. By using this powerful tool, some interesting existence and uniqueness theorems and data dependence results are established in a suitable piecewise continuous functions space with piecewise Bielecki norm.
Key Words and Phrases: Impulsive fractional differential equations, solvability, data dependence, Picard and weakly Picard operators.

2010 Mathematics Subject Classification: 26A33, 34A37, 34G20, 47H10.

1. Introduction

Recently, many researchers show great interest in the subject of impulsive fractional differential equations (see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]). However, Fečkan et al. [15] point an error in former solutions for some impulsive fractional differential equations by construct a counterexample and establish a general framework to seek a nature solution for such problems.

It is remarkable that Picard and weakly Picard operators methods is a powerful tool to study the nonlinear differential equations. It can be widely used to discuss existence and uniqueness and the data dependence on data of the solutions for nonlinear differential equations. For more details, one can see Rus et al. [16, 17, 18, 19, 20, 21, 22], Şerban et al. [23], Mureşan [24, 25], Olaru [26] and Wang et al. [27].

In this paper, we will use Picard and weakly Picard operators methods to discuss existence and uniqueness and the data dependence on data of the solutions for the following impulsive fractional Cauchy problems

$$\begin{cases} {}^cD_t^q u(t) = f(t,u(t)),\ t \in J' := J \setminus \{t_1,\cdots,t_m\},\ J := [0,T], \\ u(t_k^+) = u(t_k^-) + I_k(u(t_k^-)),\ k = 1,2,\cdots,m, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

in a Banach space X, where ${}^cD_t^q$ is the Caputo fractional derivative of order $q \in (0,1)$ with the lower limit zero, $u_0 \in X$, $f : J \times X \to X$ is a Carathéodory function, I_k :

$X \to X$ and t_k satisfy $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $u(t_k^+) = \lim_{\epsilon\to 0^+} u(t_k + \epsilon)$ and $u(t_k^-) = \lim_{\epsilon\to 0^-} u(t_k + \epsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$.

We emphasize that fractional derivative was firstly introduced at the end of the nineteenth century by Liouville and Riemann, but the concept of non-integer derivative and integral, as a generalization of the traditional integer order differential and integral calculus was mentioned by Leibniz and L'Hospital over 300 years.

Let us recall the following known fractional calculus definitions. For more details see Baleanu et al. [28], Diethelm [29], Kilbas et al. [30], Lakshmikantham et al. [31], Miller and Ross [32], Michalski [33], Podlubny [34] and Tarasov [35].

Definition 1.1. *A function $f(t)$ is said to be in the space C_γ, $\gamma \in R$ if there exists a real number $\kappa > \gamma$, such that $f(t) = t^\kappa g(t)$, where $g \in C[0,\infty)$ and it is said to be in the space C_γ^m iff $f^{(m)} \in C_\gamma$, $m \in N$.*

Definition 1.2. *The fractional integral of order γ with the lower limit zero for a function $f \in C_\gamma$, $\gamma \geq -1$ is defined as*

$$I_t^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds,\ t > 0,\ \gamma > 0,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 1.3. *The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f \in C_{-1}^\gamma$, $\gamma \in N$, can be written as*

$${}^L D_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds,\ t > 0,\ n-1 < \gamma < n.$$

Definition 1.4. *The Caputo derivative of order γ for a function $f \in C_{-1}^\gamma$, $\gamma \in N$, can be written as*

$${}^C D_t^\gamma f(t) = {}^L D_t^\gamma \left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right],\ t > 0,\ n-1 < \gamma < n.$$

Remark 1.5. *(i) If $f(t) \in C^n[0,\infty)$, then*

$${}^C D_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds = I^{n-\gamma} f^{(n)}(t),\ t > 0,\ n-1 < \gamma < n.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in Banach space X, then integrals which appear in Definitions 1.2 and 1.3 are taken in Bochner's sense.

Motivated by Fečkan et al. [15], we give a definition of a solution for the problem (1.1).

Definition 1.6. *A function $u \in PC(J,X)$ is said to be a solution of the problem (1.1) if $u(t) = u_k(t)$ for $t \in (t_k, t_{k+1})$ and $u_k \in C([0,t_{k+1}],X)$, $k = 0,1,2,\cdots,m$ satisfies ${}^c D_t^q u_k(t) = f(t, u_k(t))$ a.e. on $(0,t_{k+1})$ with the restriction of $u_k(t)$ on $[0,t_k)$ is just $u_{k-1}(t)$, and the conditions $u(t_k^+) = u(t_k^-) + y_k$, $k = 1,2,\cdots,m$, and $u(0) = u_0$.*

Similar to the Lemma 2.7 in Fečkan et al. [15], a function $u \in PC(J,X)$ is a solution of the problem (1.1) if and only if u is a solution of the integral equation

$$u(t) = u_0 + \sum_{0<t_i<t} I_i(u(t_i)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))ds, \ t \in (t_k, t_{k+1}]. \quad (1.2)$$

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts.

Let (X,d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$P(X) := \{Y \subseteq X \mid Y \neq \emptyset\}$; $F_A := \{x \in X \mid A(x) = x\}$−the fixed point set of A;

$I(A) := \{Y \in P(X) \mid A(Y) \subseteq Y\}$;

$O_A(x) := \{x, A(x), A^2(x), \cdots, A^n(x), \cdots\}$ −the A−orbit of $x \in X$;

$H : P(X) \times P(X) \to R_+ \cup \{+\infty\}$;

$H_d(Y,Z) := \max\{\sup_{a\in Y} \inf_{b\in Z} d(a,b), \sup_{b\in Z} \inf_{a\in Y} d(a,b)\}$ - the Pompeiu-Hausdorff functional on $P(X)$.

Let us recall the following known definitions. For more details, see Rus [17, 18].

Definition 2.1. *Let (X,d) be a metric space. An operator $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that $F_A = \{x^*\}$ and the sequence $(A^n(x_0))_{n\in N}$ converges to x^* for all $x_0 \in X$.*

Definition 2.2. *Let (X,d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator if the sequence $(A^n(x_0))_{n\in N}$ converges for all $x_0 \in X$ and its limit (which may depend on x_0) is a fixed point of A.*

Definition 2.3. *If A is a weakly Picard operator, then we consider the operator*

$$A^\infty : X \to X, \ A^\infty(x) = \lim_{n\to\infty} A^n(x).$$

It is clear that $A^\infty(X) = F_A$ and $\omega_A(x) = \{A^\infty(x)\}$ where $\omega_A(x)$ is the ω−limit point set of mapping A for point x.

The following results appeared in Rus [16, 18, 19] are useful in this paper.

Theorem 2.4. *Let (Y,d) be a complete metric space and $A, B : Y \to Y$ two operators. We suppose the following:*

(i) A is a contraction with contraction constant α and $F_A = \{x_A^*\}$.

(ii) B has fixed points and $x_B^* \in F_B$.

(iii) There exists $\gamma > 0$ such that $d(A(x), B(x)) \leq \gamma$, for all $x \in Y$.

Then $d(x_A^*, x_B^*) \leq \frac{\gamma}{1-\alpha}$.

Theorem 2.5. *Let (X,d) be a metric space. Then $A : X \to X$ is a weakly Picard operator if and only if there exists a partition of X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, where Λ is the indices' set of partition, such that*

(i) $X_\lambda \in I(A)$,

(ii) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ is a Picard operator, for all $\lambda \in \Lambda$.

Theorem 2.6. *Let (X,d) be a complete metric space and $A, B : X \to X$ two orbitally continuous operators. We suppose the following:*

(i) There exists $\alpha \in [0,1)$ such that

$$d(A^2(x), A(x)) \le \alpha d(x, A(x)), \quad d(B^2(x), B(x)) \le \alpha d(x, B(x))$$

for all $x \in X$

(ii) There exists $\gamma > 0$ such that $d(A(x), B(x)) \le \gamma$ for all $x \in X$.

Then $H_d(F_A, F_B) \le \frac{\gamma}{1-\alpha}$ where H_d denotes the Pompeiu-Hausdorff functional.

3. Solvability

In this section, we use $\|\phi\|_{L^p(J)}$ to denote the $L^p(J, R_+)$ norm of ϕ whenever $\phi \in L^p(J, R_+)$ for some p with $1 < p < \infty$. Let $C(J, X)$ be the space of all X-valued continuous functions from J into X. Let $\|\cdot\|_B$ and $\|\cdot\|_C$ be the Bielecki and Chebyshev norms on $C(J, X)$ defined by

$$\|x\|_B := \sup_{t\in J}\left\{\sup_{t\in J}\{\|x(t)\|\}e^{-\tau t}\right\} (\tau > 0) \text{ and } \|x\|_C := \sup_{t\in J}\{\|x(t)\|\}$$

and denote by d_B and d_C their corresponding metrics.

We consider the set

$$C_L^{q-q^*}(J, X) := \left\{x \in C(J, X) : \|x(t_1) - x(t_2)\| \le L|t_1 - t_2|^{q-q^*},\ t_1, t_2 \in J\right\}$$

where $L > 0$, $q^* \in (0, q)$.

If $d \in \{d_C, d_B\}$, then $(C(J, X), d)$ and $(C_L^{q-q^*}(J, X), d)$ are complete metric spaces.

Set $PC(J, X) := \big\{x : J \to X \mid x$ is continuous at $t \in J \setminus D$, and x is continuous from left and has right hand limits at $t \in D\big\}$. Let $\|\cdot\|_{PB}$ and $\|\cdot\|_{PC}$ be the piecewise Bielecki and piecewise Chebyshev norms on $PC(J, X)$ defined by

$$\begin{aligned}
\|x\|_{PB} &:= \sup_{t\in J}\left\{\sup_{t\in J}\{\|x(t^+)\|\}e^{-\tau t},\ \sup_{t\in J}\{\|x(t^-)\|\}e^{-\tau t}\right\} (\tau > 0),\\
\|x\|_{PC} &:= \max\left\{\sup_{t\in J}\{\|x(t^+)\|\},\ \sup_{t\in J}\{\|x(t^-)\|\}\right\},
\end{aligned}$$

and denote by d_{PB} and d_{PC} their corresponding metrics. Set

$$PC_L^{q-q^*}(J, X) := \left\{x \in PC(J, X) : \|x(t_1) - x(t_2)\| \le L|t_1 - t_2|^{q-q^*},\ t_1, t_2 \in J\right\}$$

where $L > 0$, $q^* \in (0, q)$.

If $d \in \{d_{PC}, d_{PB}\}$, one can apply the above results on each interval $(t_k, t_{k+1}](k = 0, 1, 2, \cdots, m)$ to obtain that $(PC(J, X), d)$ and $(PC_L^{q-q^*}(J, X), d)$ are complete metric spaces.

We list the following assumptions.

(C1) $f : J \times X \to X$ is a Carathéodory function.

(C2) There exist a constant $q_1 \in (0, q)$ and a function $\overline{m}(\cdot) \in L^{\frac{1}{q_1}}(J, R_+)$ such that $\|f(t, u)\| \le \overline{m}(t)$ for all $u \in PC(J, X)$ and all $t \in J$. Moreover, $M := \|\overline{m}\|_{L^{\frac{1}{q_1}}(J)}$.

(C3) There exist constants $L > 0$ and $\beta := \frac{q-1}{1-q_1}$ such that

$$L \ge \frac{2M}{\Gamma(q)(1+\beta)^{1-q_1}}.$$

(C4) There exists a function $l(\cdot) \in C(J, R_+)$ such that

$$\|f(t,u_1) - f(t,u_2)\| \leq l(t)\|u_1 - u_2\|$$

for all $u_1, u_2 \in PC(J,X)$ and all $t \in J$.

(C5) I_i: $X \to X$ and there exists a constant $L_I > 0$ such that

$$\|I_i(u) - I_i(v)\| \leq L_I\|u - v\|,$$

for all $u, v \in X$ and $i = 1, 2, \cdots, m$.

(C6) There exist constants q_1 and τ such that

$$\frac{L_0}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} + mL_I < 1$$

where $L_I > 0$ and $L_0 := \max_{t\in J}\{l(t)\}$.

The main result of this section is contained in the below given theorem.

Theorem 3.1. *Let the assumptions (C1)–(C6) hold. Then the problem (1.1) has a unique solution u^* in $PC_L^{q-q_1}(J,X)$, and this solution can be obtained by the successive approximation method, starting from any element of $PC_L^{q-q_1}(J,X)$.*

Proof. Consider the operator

$$A : (PC_L^{q-q_1}(J,X), \|\cdot\|_{PB}) \to (PC_L^{q-q_1}(J,X), \|\cdot\|_{PB})$$

defined by

$$A(u)(t) = u_0 + \sum_{0<t_i<t} I_i(u(t_i)) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,u(s))ds, \ for\ t \in (t_k, t_{k+1}].$$

It is easy to see the operator A is well defined due to (C1)–(C2).

Step 1. We check that $Au \in PC(J,X)$ for every $u \in PC_L^{q-q_1}(J,X)$.

If $t \in (0, t_1]$ then for any $\delta > 0$ and $0 < t < t+\delta < t_1$, every $u \in PC_L^{q-q_1}((0,t_1],X)$, by (C2), Hölder inequality,

$$\begin{aligned}
\|A(u)(t+\delta) - A(u)(t)\| &\leq \frac{1}{\Gamma(q)}\int_0^t ((t-s)^{q-1} - (t+\delta-s)^{q-1})\overline{m}(s)ds \\
&\quad \frac{1}{\Gamma(q)}\int_t^{t+\delta} (t+\delta-s)^{q-1}\overline{m}(s)ds \\
&\leq \frac{1}{\Gamma(q)}\left(\int_0^t [(t-s)^{q-1} - (t+\delta-s)^{q-1}]^{\frac{1}{1-q_1}} ds\right)^{1-q_1}\left(\int_0^t (\overline{m}(s))^{\frac{1}{q_1}} ds\right)^{q_1} \\
&\quad + \frac{1}{\Gamma(q)}\left(\int_t^{t+\delta} [(t+\delta-s)^{q-1}]^{\frac{1}{1-q_1}} ds\right)^{1-q_1}\left(\int_t^{t+\delta} (\overline{m}(s))^{\frac{1}{q_1}} ds\right)^{q_1} \\
&\leq \frac{\|\overline{m}\|_{L^{\frac{1}{q_1}}([0,t_1])}}{\Gamma(q)}\left(\int_0^t \delta^\beta ds\right)^{1-q_1} + \frac{\|\overline{m}\|_{L^{\frac{1}{q_1}}([0,t_1])}}{\Gamma(q)}\left(\int_t^{t+\delta} (t+\delta-s)^\beta ds\right)^{1-q_1} \\
&\leq \frac{2\|\overline{m}\|_{L^{\frac{1}{q_1}}([0,t_1])}}{\Gamma(q)(1+\beta)^{1-q_1}}\delta^{(1+\beta)(1-q_1)}.
\end{aligned}$$

It is easy to see that the right-hand side of the above inequality tends to zero as $\delta \to 0$. Therefore $Au \in PC((0, t_1], X)$.

If $t \in (t_1, t_2]$ then for any $\delta > 0$ and $t_1 < t < t+\delta < t_2$, every $u \in PC_L^{q-q_1}((t_1, t_2], X)$, repeating the same process, one can obtain

$$\|A(u)(t+\delta) - A(u)(t)\| \le \frac{2\|\overline{m}\|_{L^{\frac{1}{q_1}}([t_1,t_2])}}{\Gamma(q)(1+\beta)^{1-q_1}}\delta^{(1+\beta)(1-q_1)},$$

which implies that $Au \in PC((t_1, t_2], X)$.

If $t \in (t_k, t_{k+1}]$ then for any $\delta > 0$ and $t_k < t < t+\delta < t_{k+1}$, every $u \in PC_L^{q-q_1}((t_k, t_{k+1}], X)$, repeating the same process, one can obtain

$$\|A(u)(t+\delta) - A(u)(t)\| \le \frac{2\|\overline{m}\|_{L^{\frac{1}{q_1}}([t_k,t_{k+1}])}}{\Gamma(q)(1+\beta)^{1-q_1}}\delta^{(1+\beta)(1-q_1)},$$

which implies that $Au \in PC((t_k, t_{k+1}], X)$.

From the above discussion, we must have $Au \in PC(J, X)$ for every $u \in PC_L^{q-q_1}(J, X)$.

Step 2. We show that $Au \in PC_L^{q-q_1}(J, X)$.

Without lose of generality, for any $\tau_1 < \tau_2$, $\tau_1, \tau_2 \in (t_k, t_{k+1}]$, applying (C2) and Hölder inequality, we have

$$\begin{aligned}
\|A(u)(\tau_2) - A(u)(\tau_1)\| &\le \frac{1}{\Gamma(q)}\int_0^{\tau_1}[(\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1}]\overline{m}(s)ds \\
&\quad + \frac{1}{\Gamma(q)}\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{q-1}\overline{m}(s)ds \\
&\le \frac{1}{\Gamma(q)}\left(\int_0^{\tau_1}[(\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1}]^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\left(\int_0^{\tau_1}(\overline{m}(s))^{\frac{1}{q_1}}ds\right)^{q_1} \\
&\quad + \frac{1}{\Gamma(q)}\left(\int_{\tau_1}^{\tau_2}[(\tau_2 - s)^{q-1}]^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\left(\int_{\tau_1}^{\tau_2}(\overline{m}(s))^{\frac{1}{q_1}}ds\right)^{q_1} \\
&\le \frac{M}{\Gamma(q)}\left(\int_0^{\tau_1}((\tau_1 - s)^{\beta} - (\tau_2 - s)^{\beta})ds\right)^{1-q_1} + \frac{M}{\Gamma(q)}\left(\int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\beta}ds\right)^{1-q_1} \\
&\le \frac{M}{\Gamma(q)(1+\beta)^{1-q_1}}\left(\tau_1^{1+\beta} - \tau_2^{1+\beta} + (\tau_2 - \tau_1)^{1+\beta}\right)^{1-q_1} \\
&\quad + \frac{M}{\Gamma(q)(1+\beta)^{1-q_1}}(\tau_2 - \tau_1)^{(1+\beta)(1-q_1)} \\
&\le \frac{2M}{\Gamma(q)(1+\beta)^{1-q_1}}|\tau_1 - \tau_2|^{(1+\beta)(1-q_1)} = \frac{2M}{\Gamma(q)(1+\beta)^{1-q_1}}|\tau_1 - \tau_2|^{q-q_1}.
\end{aligned}$$

Similarly, for any $\tau_1 < \tau_2$, $\tau_1, \tau_2 \in (t_k, t_{k+1}]$, we also have the above inequality. This implies that Au is belong to $PC_L^{q-q_1}(J, X)$ due to (C3).

Step 3. We show that A is continuous.

For that, let $\{u_n\}$ be a sequence of $B_R := \{u \in PC(J, X) : \|u\|_{PC} \le R\}$ such that $u_n \to u$ in B_R. Then, $f(s, u_n(s)) \to f(s, u(s))$ as $n \to \infty$ due to (C1). On the one other hand using (C2), we get for each $s \in [0, t]$, $\|f(s, u_n(s)) - f(s, u(s))\| \le 2\overline{m}(s)$.

On the other hand, using the fact that the functions $2(t-s)^{q-1}\overline{m}(s)$ is integrable on $[0,t]$, by means of the Lebesgue Dominated Convergence Theorem yields

$$\int_0^t (t-s)^{q-1}\|f(s,u_n(s)) - f(s,u(s))\|ds \to 0.$$

For all $t \in J$, by (C5) we have

$$\begin{aligned}\|A(u_n)(t) - A(u)(t)\| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,u_n(s)) - f(s,u(s))\|ds \\ &\quad + mL_I\|u_n - u\|_{PC} \longrightarrow 0,\end{aligned}$$

as $n \to \infty$. Thus, A is continuous.

Step 4. A is a Picard operator.

In fact, for all $x, z \in PC_L^{q-q_1}(J,X)$, using (C4), (C6) and Hölder inequality we have

$$\begin{aligned}\|A(x)(t) - A(z)(t)\| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} l(s)\|x(s) - z(s)\|ds + \sum_{0<t_i<t} L_I\|x(t_i) - z(t_i)\| \\ &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\sup_{s\in[0,t]}\{l(s)\}\left[\|x(s) - z(s)\|e^{-\tau s}\right]e^{\tau s}ds \\ &\quad + \sum_{0<t_i<t} L_I\left[\|x(t_i) - z(t_i)\|e^{-\tau t_i}\right]e^{\tau t_i} \\ &\leq \frac{L_0}{\Gamma(q)}\|x - z\|_{PB}\int_0^t (t-s)^{q-1}e^{\tau s}ds + L_I\sum_{0<t_i<t} e^{\tau t_i}\|x - z\|_{PB} \\ &\leq \left[\frac{L_0}{\Gamma(q)}\left(\int_0^t (t-s)^{\beta}ds\right)^{1-q_1}\left(\int_0^t e^{\frac{\tau s}{q_1}}ds\right)^{q_1} + L_I\sum_{0<t_i<t} e^{\tau t_i}\right]\|x - z\|_{PB} \\ &\leq \frac{L_0}{\Gamma(q)}\frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}}\left(\frac{q_1}{\tau}\right)^{q_1}e^{\tau t}\|x - z\|_{PB} + L_I\sum_{0<t_i<t} e^{\tau t_i}\|x - z\|_{PB}.\end{aligned}$$

It follows that

$$\|A(x)(t) - A(z)(t)\|e^{-\tau t} \leq \left[\frac{L_0}{\Gamma(q)}\frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}}\left(\frac{q_1}{\tau}\right)^{q_1} + L_I\sum_{0<t_i<t} e^{-(t-t_i)\tau}\right]\|x - z\|_{PB}$$

for all $t \in J$. So we have

$$\|A(x) - A(z)\|_{PB} \leq \left[\frac{L_0}{\Gamma(q)}\frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}}\left(\frac{q_1}{\tau}\right)^{q_1} + mL_I\right]\|x - z\|_{PB}$$

for all $x,\ z \in PC_L^{q-q_1}(J,X)$.

The operator A is of Lipschitz type with constant

$$L_A = \frac{L_0}{\Gamma(q)}\frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}}\left(\frac{q_1}{\tau}\right)^{q_1} + mL_I \tag{3.1}$$

and $0 < L_A < 1$ due to (C6). By applying the Contraction Principle to this operator we obtain that A is a Picard operator. This completes the proof. $\square$

4. Application to data dependence

Now we turn to consider another impulsive fractional Cauchy problem

$$\begin{cases} {}^cD_t^q u(t) = h(t, u(t)),\ t \in J', \\ u(t_k^+) = u(t_k^-) + \overline{I}_k(u(t_k^-)),\ k = 1, 2, \cdots, m, \\ u(0) = v_0 \in X, \end{cases} \tag{4.1}$$

where $h : J \times X \to X$ is another Carathéodory function and $\overline{I}_k : X \to X$.

A function $u \in PC(J, X)$ is a solution of the problem (4.1) if and only if $u \in PC(J, X)$ is a solution of the integral equation

$$u(t) = v_0 + \sum_{0<t_i<t} \overline{I}_i(u(t_i)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s, u(s))ds, t \in (t_k, t_{k+1}]. \tag{4.2}$$

We introduce the following assumptions.

(D1) All conditions in Theorem 3.1 are satisfied and $x^* \in PC_L^{q-q_1}(J, X)$ is the unique solution of the integral equation (1.2).

(D2) With the same function $\overline{m}(\cdot) \in L^{\frac{1}{q_1}}(J, R_+)$ as in Theorem 3.1, $\|h(t, u)\| \leq \overline{m}(t)$ for all $u \in X$ and all $t \in J$.

(D3) With the same function $l(\cdot) \in C(J, R_+)$ as in Theorem 3.1, $\|h(t, u_1) - h(t, u_2)\| \leq l(t)\|u_1 - u_2\|$ for all $u_1, u_2 \in X$ and all $t \in J$.

(D4) With the same constant L_I as in Theorem 3.1, $\|\overline{I}_i(u) - \overline{I}_i(v)\| \leq L_I\|u - v\|$ for all $u, v \in X$.

(D5) $L \geq \frac{2M}{\Gamma(q)(1+\beta)^{1-q_1}}$.

(D6) There exists a constant $L_\eta \in L^{\frac{1}{q_1}}(J, R_+)$ such that $\|f(t, u) - h(t, u)\| \leq L_\eta(t)$ for all $u \in X$ and $t \in J$.

(D7) There exists a constant $L_\mu > 0$ such that $\|I_i(u) - \overline{I}_i(u)\| \leq L_\mu$ for all $u \in X$.

Theorem 4.1. *Let the assumptions (D1)–(D7) hold. If y^* is the solution of the integral equation (4.2), then*

$$\|x^* - y^*\|_{PB} \leq \frac{\|u_0 - v_0\| + \frac{T^{(1+\beta)(1-q_1)}\|L_\eta\|_{L^{\frac{1}{q_1}}(J)}}{\Gamma(q)(1+\beta)^{1-q_1}} + mL_\mu}{1 - L_A} \tag{4.3}$$

and L_A is given by (3.1) with a $\tau = \tau_* > 0$ such that $0 < L_A < 1$.

Proof. Consider the following two continuous operators

$$A, B : (PC_L^{q-q_1}(J, X), \|\cdot\|_{PB}) \to (PC_L^{q-q_1}(J, X), \|\cdot\|_{PB})$$

defined by

$$\begin{aligned} A(u)(t) &:= u_0 + \sum_{0<t_i<t} I_i(u(t_i)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))ds, \\ B(u)(t) &:= v_0 + \sum_{0<t_i<t} \overline{I}_i(u(t_i)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s, u(s))ds, \end{aligned}$$

for $t \in (t_k, t_{k+1}]$. Clearly,

$$\|A(u)(t) - B(u)(t)\| \leq \|u_0 - v_0\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s, u(s)) - h(s, u(s))\| ds$$
$$+ \sum_{i=1}^{m} \|I_i(u(t_i)) - \overline{I}_i(u(t_i))\| \leq \|u_0 - v_0\| + \frac{T^{(1+\beta)(1-q_1)} \|L_\eta\|_{L^{\frac{1}{q_1}}(J)}}{\Gamma(q)(1+\beta)^{1-q_1}} + mL_\mu$$

for $t \in J$. It follows that

$$\|Au - Bu\|_{PB} \leq \|u_0 - v_0\| + \frac{T^{(1+\beta)(1-q_1)} \|L_\eta\|_{L^{\frac{1}{q_1}}(J)}}{\Gamma(q)(1+\beta)^{1-q_1}} + mL_\mu.$$

So we can apply Theorem 2.4 to obtain the inequality (4.3) which completes the proof. □

Next, we consider another integral equation

$$u(t) = u(0) + \sum_{0<t_i<t} I_i(u(t_i)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))\, ds, t \in (t_k, t_{k+1}], \quad (4.4)$$

where f, I_i are as in the problem (1.1).

Theorem 4.2. *Suppose that for the integral equation (4.4) the same conditions as in Theorem 3.1 are satisfied. Then this equation has solutions in $PC_L^{q-q_1}(J, X)$. If $\widetilde{\mathcal{S}} \subset PC_L^{q-q_1}(J, X)$ is its solutions set, then card $\widetilde{\mathcal{S}}$= card X.*

Proof. Consider the operator $A_* : (PC_L^{q-q_1}(J, X), \|\cdot\|_{PB}) \to (PC_L^{q-q_1}(J, X), \|\cdot\|_{PB})$ defined by

$$A_*(u)(t) := u(0) + \sum_{0<t_i<t} I_i(u(t_i)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))\, ds, t \in (t_k, t_{k+1}]. \quad (4.5)$$

This is a continuous operator, but not a Lipschitz one. We can write $PC_L^{q-q_1}(J, X) = \bigcup_{\alpha \in X} X_\alpha$, $X_\alpha = \left\{u \in PC_L^{q-q_1}(J, X) : u(0) = \alpha\right\}$. We have that X_α is an invariant set of A_* and we apply Theorem 3.1 to $A_*|_{X_\alpha}$. By using Theorem 2.5 we obtain that A_* is a weakly Picard operator. Consider the operator $A_*^\infty : PC_L^{q-q_1}(J, X) \to PC_L^{q-q_1}(J, X)$, $A_*^\infty(u) = \lim_{n\to\infty} A_*^n(u)$. From $A_*^{n+1}(u) = A_*(A_*^n(u))$ and the continuity of A_*, $A_*^\infty(u) \in F_{A_*}$. Then $A_*^\infty(PC_L^{q-q_1}(J, X)) = F_{A_*} = \widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{S}} \neq \emptyset$. So, *card* $\widetilde{\mathcal{S}}$= *card* X. □

In order to study data dependence for the solutions set of the integral equation (4.4) we consider both (4.4) and the following integral equation

$$u(t) = u(0) + \sum_{0<t_i<t} \overline{I}_i(u(t_i)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s, u(s))\, ds,\ t \in (t_k, t_{k+1}],$$

where $h, \overline{I}_i$ are as in the problem (4.1). Let $\widetilde{\mathcal{S}}_1$ be the solutions set of this equation.

We make the following assumptions.

(E1) There exists a constant $L^* > 0$ such that $\|f(t, u_1) - f(t, u_2)\| \leq L^* \|u_1 - u_2\|$ and $\|h(t, u_1) - h(t, u_2)\| \leq L^* \|u_1 - u_2\|$ for all $u_1, u_2 \in X$ and all $t \in J$.

(E2) There exists a constant $\overline{L}^* > 0$ such that $\|I_i(u) - I_i(v)\| \le \overline{L}^*\|u - v\|$ and $\|\overline{I}_i(u) - \overline{I}_i(v)\| \le \overline{L}^*\|u - v\|$ for all $u, v \in X$.

(E3) There exists a function $\overline{m}(\cdot) \in L^{\frac{1}{q_1}}(J, R_+)$ such that $\|f(t, u)\| \le \overline{m}(t)$ and $\|h(t, u)\| \le \overline{m}(t)$ for all $u \in X$ and all $t \in J$. Moreover, $M := \|\overline{m}\|_{L^{\frac{1}{q_1}}(J)}$.

(E4) There exists a constant $L > 0$ such that $L \ge \frac{2M}{\Gamma(q)(1+\beta)^{1-q_1}}$.

(E5) There exists a constant $L_\eta^* \in L^{\frac{1}{q_1}}(J, R_+)$ such that $\|f(t, u) - h(t, u)\| \le L_\eta^*(t)$ for all $u \in X$ and $t \in J$.

(E6) There exists a constant $L_\mu^* > 0$ such that $\|I_i(u) - \overline{I}_i(u)\| \le L_\mu^*$ for all $u \in X$.

(E7) $\frac{L^*}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} + m\overline{L}^* < 1$.

Theorem 4.3. *Suppose the assumptions (E1)–(E7) are satisfied. Then*

$$H_{\|\cdot\|_{PB}}(\widetilde{\mathcal{S}}, \widetilde{\mathcal{S}}_1) \;\le\; \frac{mL_\mu^* + \frac{T^{(1+\beta)(1-q_1)}\|L_\eta^*\|_{L^{\frac{1}{q_1}}(J)}}{\Gamma(q)(1+\beta)^{1-q_1}}}{1 - m\overline{L}^* - \frac{L^*}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1}}.$$

where by $H_{\|\cdot\|_{PB}}$ we denote the Pompeiu-Hausdorff functional with respect to $\|\cdot\|_{PB}$ on $PC_L^{q-q_1}(J, X)$.

Proof. Consider the operator $B_* : (PC_L^{q-q_1}(J, X), \|\cdot\|_{PB}) \to (PC_L^{q-q_1}(J, X), \|\cdot\|_{PB})$ defined by

$$B_*(u)(t) := u(0) + \sum_{0<t_i<t} \overline{I}_i(u(t_i)) + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} h\left(s, u(s)\right) ds, t \in (t_k, t_{k+1}].$$

Moreover, we have

$$\begin{aligned} &\|A_*^2(u)(t) - A_*(u)(t)\| \\ &\le \frac{L^*}{\Gamma(q)} \int_0^t (t - s)^{q-1}\|A_*(u)(s) - u(s)\|ds + \sum_{0<t_i<t} \|I_i(A_*(u)(t_i)) - I_i(u(t_i)\| \\ &\le \left(\frac{L^*}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} e^{\tau t} + \overline{L}^* \sum_{0<t_i<t} e^{\tau t_i}\right) \|A_*(u) - u\|_{PB} \end{aligned}$$

for all $x \in PC_L^{q-q_1}(J, X)$. Similarly,

$$\|B_*^2(u)(t) - B_*(u)(t)\| \le \left(\frac{L^*}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} e^{\tau t} + \overline{L}^* \sum_{0<t_i<t} e^{\tau t_i}\right) \|B_*(u) - u\|_{PB}$$

for all $x \in PC_L^{q-q_1}(J, X)$.

Thus,

$$\|A_*^2(u) - A_*(u)\|_{PB} \le \left(\frac{L^*}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} + m\overline{L}^*\right) \|A_*(u) - u\|_{PB},$$

$$\|B_*^2(u) - B_*(u)\|_{PB} \le \left(\frac{L^*}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} + m\overline{L}^*\right) \|B_*(u) - u\|_{PB}.$$

Because of (E5)–(E6), $\|A_*(u) - B_*(u)\|_{PB} \leq mL_\mu^* + \frac{T^{(1+\beta)(1-q_1)}\|L_\eta^*\|_{L^{\frac{1}{q_1}}(J)}}{\Gamma(q)(1+\beta)^{1-q_1}}$ for all $x \in PC_L^{q-q_1}(J, X)$. By (E7) and applying Theorem 2.6 we obtain

$$H_{\|\cdot\|_{PB}}(F_{A_*}, F_{B_*}) \leq \frac{mL_\mu^* + \frac{T^{(1+\beta)(1-q_1)}\|L_\eta^*\|_{L^{\frac{1}{q_1}}(J)}}{\Gamma(q)(1+\beta)^{1-q_1}}}{1 - m\overline{L}^* - \frac{L^*}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1}}.$$

This completes the proof. □

5. Conclusions

This paper revisits some impulsive fractional differential equations in Banach spaces by applying a powerful tool named by Picard and weakly Picard operators methods. After introducing a suitable piecewise continuous functions space with piecewise Bielecki norm, some new existence and uniqueness theorems and data dependence results are obtained.

Acknowledgement. The first and second author's work was supported by National Natural Science Foundation of China (11201091) and Key Projects of Science and Technology Research in the Chinese Ministry of Education (211169); The third author's work was supported by National Natural Science Foundation of China (11271309), Specialized Research Fund for the Doctoral Program of Higher Education (20114301110001) and Key Projects of Hunan Provincial Natural Science Foundation of China (12JJ2001).

References

[1] R.P. Agarwal, M. Benchohra, S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta. Appl. Math., **109**(2010), 973-1033.

[2] M. Benchohra, D. Seba, *Impulsive fractional differential equations in Banach spaces*, Electronic J. Qualitative Theory Diff. Eq., Spec. Ed. I, **8**(2009), No. 8, 1-14.

[3] K. Balachandran, S. Kiruthika, *Existence of solutions of abstract fractional impulsive semilinear evolution equations*, Electronic J. Qualitative Theory Diff. Eq., 2010, No.4, 1-12.

[4] B. Ahmad, S. Sivasundaram, *Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations*, Nonlinear Anal. Hybrid Systems, **3**(2009), 251-258.

[5] B. Ahmad, S. Sivasundaram, *Existence of solutions for impulsive integral boundary value problems of fractional order*, Nonlinear Anal. Hybrid Systems, **4**(2010), 134-141.

[6] B. Ahmad, G. Wang, *Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order*, Comput. Math. Appl., **59**(2010), 1341-1349.

[7] Y. Tian, Z. Bai, *Existence results for the three-point impulsive boundary value problem involving fractional differential equations*, Comput. Math. Appl., **59**(2010), 2601-2609.

[8] J. Cao, H. Chen, *Some results on impulsive boundary value problem for fractional differential inclusions*, E. J. Qualitative Theory Diff. Eq., 2010, No. 11, 1-24.

[9] G. Wang, B. Ahmad, L. Zhang, *Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order*, Nonlinear Anal., **74**(2011), 792-804.

[10] G. Wang, L. Zhang, G. Song, *Systems of first order impulsive functional differential equations with deviating arguments and nonlinear boundary conditions*, Nonlinear Anal., **74**(2011), 974-982.

[11] G. Wang, B. Ahmad, L. Zhang, *Some existence results for impulsive nonlinear fractional differential equations with mixed boundary conditions*, Comput. Math. Appl., **59**(2010), 1389-1397.
[12] X. Wang, *Impulsive boundary value problem for nonlinear differential equations of fractional order*, Comput. Math. Appl., **62**(2011), 2383-2391.
[13] J. Cao, H. Chen, *Impulsive fractional differential equations with nonlinear boundary conditions*, Math. Comput. Model., **55**(2012), 303-311.
[14] L. Yang, H. Chen, *Nonlocal boundary value problem for impulsive differential equations of fractional order*, Adv. Differ. Equ., 2011, Article ID 404917, 16 pp.
[15] M. Fečkan, Y. Zhou, J. Wang, *On the concept and existence of solution for impulsive fractional differential equations*, Commun. Nonlinear Sci. Numer. Simulat., **17**(2012), 3050-3060.
[16] I.A. Rus, *Metrical fixed point theorems*, Univ. of Cluj-Napoca, Romania, 1979.
[17] I.A. Rus, *Picard mappings: results and problems*, Preprint. Cluj-Napoca: Babeş-Bolyai Univ., Seminar on Fixed Point Theory, Preprint no. **3**(1987), 55-64.
[18] I.A. Rus, *Weakly Picard mappings*, Comment. Math. Univ. Carolinae, **34**(1993), 769-773.
[19] I.A. Rus, S. Mureşan, *Data dependence of the fixed points set of some weakly Picard operators*, In: Proc. Itinerant Seminar (Elena Popoviciu-Ed.), Srima Publishing House, Cluj-Napoca, 2000, 201-207.
[20] I.A. Rus, *Functional-differential equations of mixed type, via weakly Picard operators*, Seminar on Fixed Point Theory, Cluj-Napoca, 2002, 335-345.
[21] I.A. Rus, *Picard operators and applications*, Scienticae Mathematicae Japonicae, **58**(2003), 191-219.
[22] I.A. Rus, E. Egri, *Boundary value problems for iterative functional-differential equations*, Studia Univ. Babeş-Bolyai, Mathematica, **51**(2006), 109-126.
[23] M.A. Şerban, I.A. Rus, A. Petruşel, *A class of abstract Volterra equations, via weakly Picard operators technique*, Math. Ineq. Appl., **13**(2010), 255-269.
[24] V. Mureşan, *Existence, uniqueness and data dependence for the solutions of some integro-differential equations of mixed type in Banach space*, J. Anal. Appl., **23**(2004), 205-216.
[25] V. Mureşan, *Volterra integral equations with iterations of linear modification of the argument*, Novi Sad J. Math., **33**(2003), 1-10.
[26] I. M. Olaru, *An integral equation via weakly Picard operators*, Fixed Point Theory, **11**(2010), 97-106.
[27] J. Wang, Y. Zhou, M. Medveď, *Picard and weakly Picard operators technique for nonlinear differential equations in Banach spaces*, J. Math. Anal. Appl., **389**(2012), 261-274.
[28] D. Baleanu, J.A.T. Machado, A.C.-J. Luo, *Fractional Dynamics and Control*, Springer, 2012.
[29] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, 2010.
[30] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.
[31] V. Lakshmikantham, S. Leela, J.V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009.
[32] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.
[33] M.W. Michalski, *Derivatives of noninteger order and their applications*, Dissertationes Mathematicae, 328, Inst. Math., Polish Acad. Sci., 1993.
[34] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[35] V.E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, HEP, 2010.

Received: May 13, 2012; Accepted: August 16, 2012.