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KY FAN'S INEQUALITIES FOR VECTOR-VALUED MULTIFUNCTIONS IN TOPOLOGICAL ORDERED SPACES

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Abstract. The aim of this paper is using the cone semicontinuity and cone quasiconvexity for multivalued mappings to present four variants of Ky Fan's type inequality for vector-valued multifunctions in topological ordered spaces.
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1. Introduction

In 1972, Ky Fan [5] proved the following minimax inequality for real-valued functions.

Theorem 1.1. *Let X be a Hausdorff topological vector space, and let K be a nonempty compact convex subset of X. Suppose that $f : K \times K \to \mathbb{R}$ satisfies the following:*

(1) $f(x,x) \leq 0, \ \forall x \in K$;

(2) $\forall y \in K$, $f(.,y)$ *is quasiconcave;*

(3) $\forall x \in K$, $f(x,.)$ *is lower semicontinuous.*

Then there exists $y^* \in K$ such that $f(x,y^*) \leq 0, \ \forall x \in K$.

The above Ky Fan minimax inequality is well known. It plays a very important role in many fields, such as variational inequalities, game theory, mathematical economics, optimization theory, and fixed point theory. Because of wide applications, this inequality has been generalized in a number ways (e.g., see Allen [1], Aubin and Ekeland [2], Chang [3], Ding and Tan [4], Tian [14], Yen [16], Yuan [17], and Zhou and Chen [19], Horvath [7], Georgiev and Tanaka [6]). In the framework of topological semilattices, Horvath and Llinares Ciscar (1996, [8]) first established an order theoretical version of the classical result of Knaster-Kuratowski-Mazurkiewicz, as well as fixed point theorems for multivalued mappings.

In 2001, by using Horvath and Llinares Ciscar's results, Luo [10] proved a similar result to Theorem 1.1 in topological semilattices. In 2006, Luo [11] studied Ky Fan's minimax inequalities for vector multivalued mappings in topological semilattices. However, his results does not imply scalar Ky Fan minimax inequality in the setting of topological semilattice spaces.

Recently, Song and Wang [13] gave an extension of Ky Fan minimax inequality but only for vector single-valued mappings in topological semilattices.

Motivated and inspired by research works mentioned above, in this paper, we will use the cone semicontinuity and cone quasiconvexity for multivalued mappings to show four kinds of vector valued Ky Fan's type inequality for multivalued mappings. Any of our Theorems 3.1-3.4 implies the scalar Fan minimax inequality in topological semilattices, while the main result in [11] does not imply it in its full generality, but only for continuous functions.

The rest of the paper is organized as follows. In Section 2, we introduce about topological semilattices and recall some concepts of cone semicontinuity and cone convexity. In Section 3, we prove the existence of solutions for multivalued Ky Fan inequalities (SKFI) as an application by means of Browder-Fan fixed point theorem in the setting of topological semilattices. We also give some examples to illustrate our results.

2. Preliminaries

Definition 2.1. ([8]) A partially ordered set $(X, \leqslant)$ is called a sup-semilattice if any two elements x, y of X have a least upper bound, denoted by $\sup\{x, y\}$. The partially ordered set $(X, \leqslant)$ is a topological semilattice if X is a sup-semilattice equipped with a topology such that the mapping

$$X \times X \to X$$
$$(x, y) \mapsto \sup\{x, y\}$$

is continuous.

We have given the definition of a sup-semilattice, we could obviously also consider inf-semilattices. When no confusion can arise we will simply use the word semilattice. It is also evident that each nonempty finite set A of X will have a least upper bound, denoted by $\sup A$.

In a partially ordered set $(X, \leq)$, two arbitrary elements x and x' do not have to be comparable but, in the case where $x \leq x'$, the set

$$[x, x'] = \{y \in X : x \leq y \leq x'\}$$

is called an order interval or simply, an interval. Now assume that $(X, \leq)$ is a semilattice and A is a nonempty finite subset; then the set

$$\Delta(A) = \bigcup_{a \in A} [a, \sup A]$$

is well defined and it has the following properties:

(1) $A \subseteq \Delta(A)$;

(2) if $A \subset A'$, then $\Delta(A) \subseteq \Delta(A')$.

We say that a subset $E \subseteq X$ is Δ-convex if for any nonempty finite subset $A \subseteq E$ we have $\Delta(A) \subseteq E$.

Example 2.2. We consider $\mathbb{R}^2$ with usual order defined by

$$(a, b), (c, d) \in \mathbb{R}^2, (a, b) \leq (c, d) \Leftrightarrow a \leq c;\ b \leq d.$$

Clearly, $(\mathbb{R}^2, \leq)$ is a topological semilattice.

(1) The set

$$X = \{(x, 1) : 0 \leq x \leq 1\} \cup \{(1, y) : 0 \leq y \leq 1\}$$

is Δ-convex but not convex in the usual sense.

(2) The set

$$X = \{(x, y) : 0 \leq x \leq 1;\ y = 1 - x\}$$

is convex in the usual sense but not Δ-convex.

Lemma 2.3. ([18], Lemma 1.1) *Let Y be a topological vector space and C a closed, convex and pointed cone of Y with* $\operatorname{int} C \neq \emptyset$*, where* $\operatorname{int} C$ *denotes the interior of C. Then we have* $\operatorname{int} C + C \subset \operatorname{int} C$.

We now recall some concepts of generalized convexity of multivalued mappings. Let X be a nonempty convex subset of a vector space E, C be a convex cone of a vector space Y, and $F : X \to 2^Y$ be a set-valued mapping with nonempty values.

The mapping F is called C-quasiconvex if for all $x_i \in X$, $i = 1, 2$ and $x \in conv\{x_1, x_2\}$,

either

$$F(x) \subset F(x_1) - C,$$

or

$$F(x) \subset F(x_2) - C.$$

The mapping F is called C-quasiconcave if for all $x_i \in X$, $i = 1, 2$, and $x \in conv\{x_1, x_2\}$,

either

$$F(x_1) \subset F(x) + C,$$

or

$$F(x_2) \subset F(x) + C.$$

Similarly, in the setting of topological semilattices, we introduce the following definition.

Definition 2.4. Let X be a topological semilattice or a Δ-convex subset of a topological semilattice, Y be a topological vector space, $C \subset Y$ be a convex cone. Let $F : X \to 2^Y$ be a multivalued mapping with nonempty values.

(1) F is called C_Δ-quasiconvex mapping if, for any pair $x_1, x_2 \in X$ and for any $x \in \Delta(\{x_1, x_2\})$, we have either

$$F(x) \subset F(x_1) - C$$

or

$$F(x) \subset F(x_2) - C.$$

(2) F is called C_Δ-quasiconcave mapping if, for any pair $x_1, x_2 \in X$ and for any $x \in \Delta(\{x_1, x_2\})$, we have either

$$F(x_1) \subset F(x) + C,$$

or

$$F(x_2) \subset F(x) + C.$$

We use $\in$ instead of $\subset$ when F is single-valued.

Remark 2.5. If $Y = \mathbb{R} = (-\infty, +\infty)$ and $C = [0, +\infty)$, and F is a real function, then the C_Δ-quasiconvexity of φ is equivalent to the Δ-quasiconvexity of φ (see [10]).

Example 2.6. Let $X = [0,1] \times [0,1]$. We set $x^1 \le x^2$ denoting that $x^2 \in x^1 + \mathbb{R}^2_+, \forall x^1, x^2 \in X$, where $\mathbb{R}^2_+ = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \ge 0, y_2 \ge 0\}$. It is obvious that $(X, \le)$ is a topological semilattice, in which

$$x^1 \vee x^2 = (\max(x^1_1, x^2_1), \max(x^1_2, x^2_2)),\ \forall x^i = (x^i_1, x^i_2) \in X,\ i = 1, 2.$$

(1) Let $F : X \to 2^{\mathbb{R}}$ and $C = \mathbb{R}_+$ such that

$$F(x) = [(1 - x_1)(1 - x_2), +\infty),\ \forall x = (x_1, x_2) \in X.$$

It is clear that F is C_Δ-quasiconcave mapping but not C-quasiconcave. Indeed, for $x^1 = (0, 1)$, $x^2 = (1, 0)$, $x = \frac{1}{2}x^1 + \frac{1}{2}x^2 = (\frac{1}{2}, \frac{1}{2})$, we see that

$$F(x^1) = F(x^2) = [0, +\infty),\ F(x) = \left[\frac{1}{4}, +\infty\right)$$

while

$$F(x^1) = F(x^2) = [0, +\infty) \not\subset F(x) + C = \left[\frac{1}{4}, +\infty\right).$$

(2) Let $F : X \to 2^{\mathbb{R}}$ and $C = \mathbb{R}_+$ such that

$$F(x) = \{x_1^2 + x_2^2\},\ \ \forall x = (x_1, x_2) \in X.$$

It is easy to see that F is C-quasiconvex but not C_Δ-quasiconvex.

Now, we recall the semicontinuous properties of multivalued mappings (see Ref. [2]). Let $F : X \to 2^Y$ be a multivalued mapping between topological spaces X and Y. The domain of F is defined to be the set $domF = \{x \in D : F(x) \neq \emptyset\}$.

The mapping F is upper semicontinuous (shortly, usc) at $x_0 \in domF$ if, for any open set V of Y with $F(x_0) \subset V$, there exists a neighborhood U of x_0 such that $F(x) \subset V$ for all $x \in U$.

The mapping F is lower semicontinuous (shortly, lsc) at $x_0 \in domF$ if, for any open set V of Y with $F(x_0) \cap V \neq \emptyset$, there exists a neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for all $x \in U$.

The mapping F is continuous at $x_0 \in domF$ if it is both usc and lsc at x_0. The mapping F is continuous (resp. usc, lsc) if $domF = X$ and if F is continuous (resp. usc, lsc) at each point $x \in X$.

If Y is a partially ordered topological vector space, then the above definitions of semicontinuous can be weakened. More precisely, we can introduce the following definitions taken from Ref. [9, 12].

Definition 2.7. Let X be a topological space, Y be a topological vector space with a cone C. Let $F : X \to 2^Y$. We say that

(1) F is C-upper semicontinuous (shortly, C-usc) at $x_0 \in domF$ if for any open set V of Y with $F(x_0) \subset V$ there exists a neighborhood U of x_0 such that

$$F(x) \subset V + C \quad \text{for each } x \in domF \cap U.$$

(2) F is C-lower semicontinuous (shortly, C-lsc) at $x_0 \in domF$ if for any open set V of Y with $F(x_0) \cap V \neq \emptyset$ there exists a neighborhood U of x_0 such that

$$F(x) \cap [V - C] \neq \emptyset \quad \text{for each } x \in domF \cap U.$$

(3) F is C-usc (resp. C-lsc) if $domF = X$ and if F is C-usc (resp. C-lsc) at each point of $domF$.

Remark 2.8. If $Y = \mathbb{R}$ and $C = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ (resp. $C = -\mathbb{R}_+$), F is single-valued and C-usc at x_0, then F is lower semicontinuous (resp. upper semicontinuous) at x_0 in the usual sense.

Remark 2.9. The upper (resp. lower) semicontinuity of F implies the C-upper (resp. C-lower) semicontinuity of F. Example 3.1 in Section 3 will show that the converse statement is no longer true.

Definition 2.10. Let X, Y be two topological spaces; $F : X \to 2^Y$ is said to have open lower sections if $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open for any $y \in Y$.

The following lemma is a special case of [8, Corollary 1, pp. 298].

Lemma 2.11. *(Browder-Fan fixed point theorem) Let K be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals M, $F : K \to 2^K$ with nonempty Δ-convex values, and $F^{-1}(y) \subset K$ be open, for any $y \in K$. Then F has a fixed point.*

3. Ky Fan's inequalities for vector-valued multifunctions

Let X be a topological semilattice, $K \subset X$ a nonempty Δ-convex subset, Y a topological vector space, $A : K \to 2^K, f : K \times K \to 2^Y$, C a closed, pointed and convex cone in Y with $\operatorname{int} C \neq \emptyset$.

We consider the following multivalued Ky Fan inequalities (SKFI):

(**SKFI1**) Find $x \in K$ such that

$$x \in A(x), \quad f(x, y) \not\subset \operatorname{int} C, \quad \forall y \in A(x).$$

(**SKFI2**) Find $x \in K$ such that

$$x \in A(x), \quad f(x,y) \cap \operatorname{int} C = \emptyset, \quad \forall y \in A(x).$$

(**SKFI3**) Find $x \in K$ such that

$$x \in A(x), \quad f(x,y) \cap (-C) \neq \emptyset, \quad \forall y \in A(x).$$

(**SKFI4**) Find $x \in K$ such that

$$x \in A(x), \quad f(x,y) \subset -C, \quad \forall y \in A(x).$$

The existence of solutions for the problems (SKFI1), (SKFI2), (SKFI4) were studied by Luo in [11]. However, he used either upper semicontinuous or lower semicontinuous multifunctions. So, in the scalar case, the single-valued function f is continuous with respect to the first variable, and therefore, his results are weaker than the original form.

In this paper, we use the cone semicontinuity and cone convexity of multivalued mappings to give some genuine generalizations of scalar Ky Fan minimax inequality in the setting of topological semilattices.

Theorem 3.1. *Let K be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals M, Y a topological vector space, $A : K \to 2^K$ with nonempty Δ-convex values, $f : K \times K \to 2^Y$, C a closed, pointed and convex cone in Y with $\operatorname{int} C \neq \emptyset$. Assume that*

(1) *A has open lower sections and $B := \{x \in K : x \in A(x)\}$ is closed;*

(2) *$f(x,x) \not\subset \operatorname{int} C$, $\forall x \in K$;*

(3) *$\forall x \in K$, $f(x,.)$ is $-C_\Delta$-quasiconvex;*

(4) *$\forall y \in K$, $f(.,y)$ is C-upper semicontinuous.*

Then there exists $x^* \in K$ such that $x^* \in A(x^*)$ and $f(x^*,y) \not\subset \operatorname{int} C$, $\forall y \in A(x^*)$.

Proof. Define $P : K \to 2^K$ by

$$P(x) = \{y \in K : f(x,y) \subset \operatorname{int} C\}, \quad \forall x \in K.$$

Suppose that there exists $x' \in K$ such that $P(x')$ is not Δ-convex; then there exist $y_1, y_2 \in P(x')$ such that $\Delta(\{y_1,y_2\}) \not\subset P(x')$, i.e., there exists $z \in \Delta(\{y_1,y_2\})$ and $z \notin P(x')$; hence $f(x',z) \not\subset \operatorname{int} C$. By (3), we have either

$$f(x',z) \subset f(x',y_1) + C$$

or

$$f(x',z) \subset f(x',y_2) + C.$$

By Lemma 2.1, we have either

$$f(x',z) \subset f(x',y_1) + C \subset \operatorname{int} C + C \subset \operatorname{int} C$$

or

$$f(x',z) \subset f(x',y_2) + C \subset \operatorname{int} C + C \subset \operatorname{int} C$$

which is a contradiction. Therefore, for any $x \in X, P(x)$ is Δ-convex.

Next, we prove that $P^{-1}(y)$ is open for each $y \in K$. We have

$$P^{-1}(y) = \{x \in K : f(x,y) \subset \operatorname{int} C\}$$

For each $y \in K$ and each $x \in P^{-1}(y)$, we have $f(x,y) \subset \operatorname{int} C$. By (4), there exists a neighborhood $U(x)$ of x such that $f(x',y) \subset \operatorname{int} C + C \subset \operatorname{int} C$ whenever $x' \in U(x)$, which implies that $U(x) \subset P^{-1}(y)$, i.e., $P^{-1}(y)$ is open.

By Lemma 2.2, B is a nonempty set. Define $S : K \to 2^K$ by

$$S(x) = \begin{cases} A(x) \cap P(x), & \text{if } x \in B, \\ A(x), & \text{if } x \in K \setminus B. \end{cases}$$

Then for any $x \in K$, $S(x)$ is Δ-convex. And for any $y \in K$,

$$S^{-1}(y) = (A^{-1}(y) \cap P^{-1}(y)) \cup ((K \setminus B) \cap A^{-1}(y))$$

is open.

Suppose that $\forall x \in K,\ S(x)$ is nonempty. By Lemma 2.2, we deduce that S has a fixed point, i.e., there exists $x_0 \in K$ such that $x_0 \in S(x_0)$. If $x_0 \in B$, then $x_0 \in S(x_0) = A(x_0) \cap P(x_0)$. Hence $x_0 \in P(x_0),\ f(x_0,x_0) \cap -\operatorname{int} C \neq \emptyset$ which contradicts our assumption (2). If $x_0 \in K \setminus B$, then $x_0 \in S(x_0) = A(x_0) \subset A(x_0)$, and hence $x_0 \in B$ which contradicts $x_0 \in K \setminus B$. Therefore, there exists $x^* \in K$ such that $S(x^*) = \emptyset$. Since $A(x)$ is nonempty for all $x \in K$, hence $x^* \in B$, $S(x^*) = A(x^*) \cap P(x^*) = \emptyset$, i.e., $x^* \in A(x^*)$ and for any $y \in A(x^*),\ y \notin P(x^*)$, we have

$$x^* \in A(x^*), \quad f(x^*,y) \cap \operatorname{int} C = \emptyset, \quad \forall y \in A(x^*).$$

Therefore, the assertion of Theorem 3.1 is true.

In Theorem 3.1, when f is single-valued, we have the following corollary.

Corollary 3.2. *Let K be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals M, Y a topological vector space, $A : K \to 2^K$ with nonempty Δ-convex values, $f : K \times K \to Y$, C a closed, pointed and convex cone in Y with $\operatorname{int} C \neq \emptyset$. Assume that*

(1) *A has open lower sections and $B := \{x \in K : x \in A(x)\}$ is closed;*

(2) *$f(x,x) \notin \operatorname{int} C,\ \forall x \in K$;*

(3) *$\forall x \in K,\ f(x,.)$ is $-C_\Delta$-quasiconvex;*

(4) *$\forall y \in K,\ f(.,y)$ is C-upper semicontinuous.*

Then there exists $x^* \in K$ such that $x^* \in A(x^*)$ and $f(x^*,y) \notin \operatorname{int} C,\ \forall y \in A(x^*)$.

Now we give an example to explain that Corollary 3.2 is applicable.

Example 3.3. Let X be given in Example 2.6 and $Y = \mathbb{R}$ with $C = \mathbb{R}_+$. For each $x \in X$, let $A(x) = [(0,1),(1,1)] \cup [(1,0),(1,1)]$, where $[(0,1),(1,1)]$ denotes the line segment joining points $(0,1)$ and $(1,1)$. Then we have:

(1) for each $x \in X$, $A(x)$ is nonempty and Δ-convex;

(2) for $y = (y_1, y_2) \in X$,

$$A^{-1}(y) = \begin{cases} X & \text{if } \ y \in [(0,1),(1,1)] \cup [(1,0),(1,1)] \\ \emptyset & \text{if } \ y \in X \setminus \{[(0,1),(1,1)] \cup [(1,0),(1,1)]\} \end{cases}$$

Therefore, for each $y \in X$, $A^{-1}(y)$ is open in X.

(3) The set $B = \{x \in X : x \in A(x)\} = [(0,1),(1,1)] \cup [(1,0),(1,1)]$ is closed.
For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$, we define $f : X \times X \to Y$ by

$$f(x,y) = \begin{cases} -(1 + x_1 - y_1)(1 + x_2 - y_2), & \text{if } (x,y) \neq (0,0) \\ -2, & \text{if } (x,y) = (0,0) \end{cases}$$

Then all the assumptions of Corollary 3.2 are satisfied. So Corollary 3.2 is applicable. The set of solutions for the (SKFI1) is the overall B.

Remark 3.4. For every fixed x, following the same argument as Example 2.1 in Ref. [13], we see that $f(x, .)$ is not a usual quasiconcave function. Indeed, for $x = 0$, we have

$$f(0,y) = \begin{cases} -(1 - y_1)(1 - y_2), & \text{if } (y_1, y_2) \neq (0,0) \\ -2, & \text{if } (y_1, y_2) = (0,0) \end{cases}$$

Clearly, for $y^1 = (1,0)$, $y^2 = (0,1)$, $y = \frac{1}{2}y^1 + \frac{1}{2}y^2 = (\frac{1}{2}, \frac{1}{2})$, we see that $f(0, y^1) = 0$, $f(0, y^2) = 0$, while $f(0, y) = -\frac{1}{4}$.

When $Y = (-\infty, +\infty)$, $C = [0, +\infty)$ and $A(x) = K$, $\forall x \in K$, from Corollary 3.2, we get scalar Ky Fan inequality for real-valued functions in topological semilattices (see, for instance, [10, 15]).

Corollary 3.5. *Let K be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals M and let $f : K \times K \to \mathbb{R}$ be such that*

(1) $f(x,x) \leq 0, \ \forall x \in K$;

(2) $\forall x \in K$, $f(x, .)$ *is Δ-quasiconcave;*

(3) $\forall y \in K$, $f(., y)$ *is lower semicontinuous.*

Then there exists $x^* \in K$ such that $f(x^*, y) \leq 0, \ \forall y \in K$.

Theorem 3.6. *Let K be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals M, Y a topological vector space, $A : K \to 2^K$ with nonempty Δ-convex values, $f : K \times K \to 2^Y$, C a closed, pointed and convex cone in Y with $\operatorname{int} C \neq \emptyset$. Assume that*

(1) *A has open lower sections and $B := \{x \in K : x \in A(x)\}$ is closed;*

(2) $f(x,x) \cap \operatorname{int} C = \emptyset, \ \forall x \in K$;

(3) $\forall x \in K$, $f(x, .)$ *is C_Δ-quasiconvex;*

(4) $\forall y \in K$, $f(., y)$ *is $-C$-lower semicontinuous.*

Then there exists $x^* \in K$ such that $x^* \in A(x^*)$ and $f(x^*, y) \cap \operatorname{int} C = \emptyset,\ \forall y \in A(x^*)$.

Proof. Define $P : K \to K$ by

$$P(x) = \{y \in K : f(x, y) \cap \operatorname{int} C \neq \emptyset\},\ \forall x \in K.$$

Suppose that there exists $x' \in K$ such that $P(x')$ is not Δ-convex; then there exist $y_1, y_2 \in P(x')$ such that $\Delta(\{y_1, y_2\}) \not\subset P(x')$, i.e., there exists $z \in \Delta(\{y_1, y_2\})$ and $z \notin P(x')$; hence $f(x', z) \cap \operatorname{int} C = \emptyset$. By (3), we have either

$$f(x', y_1) \subset f(x', z) - C$$

or

$$f(x', y_2) \subset f(x', z) - C.$$

Since $f(x', y_i) \cap \operatorname{int} C \neq \emptyset$, take $u_i \in f(x', y_i) \cap \operatorname{int} C,\ i = 1, 2$. Then there exist $v_i \in f(x', z)$ and $w_i \in C$ such that either $u_1 = v_1 - w_1$ or $u_2 = v_2 - w_2$. By Lemma 2.1, we have either $v_1 = u_1 + w_1 \in \operatorname{int} C$ or $v_2 = u_2 + w_2 \in \operatorname{int} C$ which contradicts $f(x', z) \cap \operatorname{int} C = \emptyset$. Therefore, for any $x \in X,\ P(x)$ is Δ-convex.

Next, we prove that $P^{-1}(y)$ is open for each $y \in K$. We have

$$P^{-1}(y) = \{x \in K : f(x, y) \cap \operatorname{int} C \neq \emptyset\}.$$

Take arbitrarily $x \in P^{-1}(y)$, we have $f(x, y) \cap \operatorname{int} C \neq \emptyset$. By assumption (4), there exists an open neighborhood $U(x)$ such that

$$f(x', y) \cap (\operatorname{int} C + C) = f(x', y) \cap \operatorname{int} C \neq \emptyset,$$

for all $x' \in U(x)$.

Let $\{x_\alpha\}$ be any net in D converging to $\bar{x}$, hence there exists β such that $x_\alpha \in U$, $\forall \alpha \geq \beta$ and then $f(x_\alpha, y) \cap \operatorname{int} C \neq \emptyset$, which contradicts $x_\alpha \in D$. Therefore, $\bar{x} \in D$ and D is closed. Consequently, we infer that $P^{-1}(y)$ is open for each $y \in K$. The rest of the proof is similar to that of Theorem 3.1. Hence the proof is complete.

Theorem 3.7. *Let K be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals M, Y a topological vector space, $A : K \to 2^K$ with nonempty Δ-convex values, $f : K \times K \to 2^Y$, C a closed, pointed and convex cone in Y with $\operatorname{int} C \neq \emptyset$. Assume that*

1. *A has open lower sections and $B := \{x \in K : x \in A(x)\}$ is closed;*
2. *$f(x, x) \cap (-C) \neq \emptyset,\ \forall x \in K$;*
3. *$\forall x \in K,\ f(x, .)$ is $-C_\Delta$-quasiconvex;*
4. *$\forall y \in K,\ f(., y)$ is C-upper semicontinuous.*

Then there exists $x^* \in K$ such that $x^* \in A(x^*)$ and $f(x^*, y) \cap (-C) \neq \emptyset,\ \forall y \in A(x^*)$.

Proof. Define $P : K \to 2^K$ by

$$P(x) = \{y \in K : f(x, y) \cap (-C) = \emptyset\},\quad \forall x \in K.$$

Suppose that there exists $x' \in K$ such that $P(x')$ is not Δ-convex; then there exist $y_1, y_2 \in P(x')$ such that $\Delta(\{y_1, y_2\}) \not\subset P(x')$, i.e., there exists $z \in \Delta(\{y_1, y_2\})$ and

$z \notin P(x')$; hence $f(x',z) \cap (-C) \neq \emptyset$. Take $u \in f(x',z) \cap (-C)$. By (3), we have either

$$f(x',z) \subset f(x',y_1) + C$$

or

$$f(x',z) \subset f(x',y_2) + C.$$

Since $u \in f(x',z)$, then there exist $v_i \in f(x',y_i)$ and $w_i \in C$ such that either $u = v_1 + w_1$ or $u = v_2 + w_2$. Therefore either $v_1 = u - w_1 \in -C$ or $v_2 = u - w_2 \in -C$, which is a contradiction. Therefore, for any $x \in X, P(x)$ is Δ-convex.

Next, we prove that $P^{-1}(y)$ is open for each $y \in K$. We have

$$P^{-1}(y) = \{x \in K : f(x,y) \cap (-C) = \emptyset\}.$$

Take arbitrarily $x \in P^{-1}(y)$, we have

$$f(x,y) \cap (-C) = \emptyset \ \text{ or } \ f(x,y) \subset Y \setminus (-C).$$

By assumption (4), there exists an open neighborhood $U(x)$ such that

$$f(x',y) \subset Y \setminus (-C) + C \subset Y \setminus (-C),$$

for all $x' \in U(x)$, it means $f(x',y) \cap (-C) = \emptyset$ for all $x' \in U(x)$. We infer that $P^{-1}(y)$ is open for each $y \in K$.

The rest of the proof is similar to that of Theorem 3.1. Hence our proof is finished.

Theorem 3.8. *Let K be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals M, Y a topological vector space, $A : K \to 2^K$ with nonempty Δ-convex values, $f : K \times K \to 2^Y$, C a closed, pointed and convex cone in Y with $\operatorname{int} C \neq \emptyset$. Assume that*

(1) *A has open lower sections and $B := \{x \in K : x \in A(x)\}$ is closed;*

(2) *$f(x,x) \subset -C, \ \forall x \in K$;*

(3) *$\forall x \in K$, $f(x,.)$ is $-C_\Delta$-quasiconcave;*

(4) *$\forall y \in K$, $f(.,y)$ is $-C$-lower semicontinuous.*

Then there exists $x^* \in K$ such that $x^* \in A(x^*)$ and $f(x^*,y) \subset -C, \ \forall y \in A(x^*)$.

Proof. Define $P : K \to 2^K$ by

$$P(x) = \{y \in K : f(x,y) \not\subset -C\}, \quad \forall x \in K.$$

Suppose that there exists $x' \in K$ such that $P(x')$ is not Δ-convex; then there exist $y_1, y_2 \in P(x')$ such that $\Delta(\{y_1,y_2\}) \not\subset P(x')$, i.e., there exists $z \in \Delta(\{y_1,y_2\})$ and $z \notin P(x')$; hence $f(x',z) \subset -C$. By (3), we have either

$$f(x',y_1) \subset f(x',z) - C$$

or

$$f(x',y_2) \subset f(x',z) - C.$$

Consequently, we have either

$$f(x',y_1) \subset f(x',z) - C \subset -C - C \subset -C$$

or

$$f(x', y_2) \subset f(x', z) - C \subset -C - C \subset -C,$$

which is a contradiction. Therefore, for any $x \in X, P(x)$ is Δ-convex.

Next, we prove that $P^{-1}(y)$ is open for each $y \in K$. We have

$$P^{-1}(y) = \{x \in K : f(x, y) \not\subset -C\}.$$

Take arbitrarily $x \in P^{-1}(y)$, we have $f(x, y) \not\subset -C$. It is equivalent to $f(x, y) \cap [Y \setminus (-C)] \neq \emptyset$. By assumption (4), there exists an open neighborhood $U(x)$ such that

$$f(x, y) \cap [Y \setminus (-C) + C] \neq \emptyset,$$

for all $x' \in U(x)$. Since $Y \setminus (-C) + C \subset Y \setminus (-C)$, it follows that

$$f(x, y) \cap [Y \setminus (-C)] \neq \emptyset,$$

for all $x' \in U(x)$. Therefore $f(x', y) \not\subset -C$ for all $x' \in U(x)$. We infer that $P^{-1}(y)$ is open for each $y \in K$.

The rest of the proof is similar to that of Theorem 3.1. Hence our proof is finished.

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References

[1] G. Allen, *Variational inequalities, complementarity problems and duality theorems*, J. Math. Anal. Appl., **58**(1977), 1-10.

[2] J.P. Aubin, I. Ekeland, *Applied Nonlinear Analysis*, John Wiley, New York, 1984.

[3] S.S. Chang, Y. Zhang, *Generalized KKM theorem and variational inequalities*, J. Math. Anal. Appl., **159**(1991), 208-223.

[4] X.P. Ding, K.K. Tan, *A minimax inequality with applications to existence of equilibrium point and fixed point theorems*, Colloq. Math., **63**(1992), 233-247.

[5] K. Fan, *A minimax inequality and applications*, in: Inequalities III. (O. Shisha, ed.) Proc. of the Third Symposium on Inequalities, Academic Press, New York, 1972.

[6] P.G. Georgiev, T. Tanaka, *Fan's inequality for set-valued maps*, Nonlinear Anal., **47**(2001), 607-618.

[7] C.D. Horvath, *Contractibility and generalized convexity*, J. Math. Anal. Appl., **156**(1991), 341-357.

[8] C.D. Horvath, J.V. Llinares Ciscar, *Maximal elements and fixed points for binary relations on topological ordered spaces*, J. Math. Econom., **25**(1996), 291-306.

[9] D.T. Luc, *Theory of Vector Optimization*, in Lecture Notes in Economics and Mathematical Systems, Vol. 319, Springer-Verlag, Berlin, 1989.

[10] Q. Luo, *KKM and Nash equilibria type theorems in topological ordered spaces*, J. Math. Anal. Appl., **264**(2001), 262-269.

[11] Q. Luo, *The applications of the Fan-Browder fixed point theorem in topological ordered spaces*, Appl. Math. Lett., **19**(2006), 1265-1271.

[12] P.H. Sach, *New nonlinear scalarization functions and applications*, Nonlinear Anal., **75**(2012), 2281-2292.

[13] Q.Q. Song, L.S. Wang, *The existence of solutions for the system of vector quasi-equilibrium problems in topological order spaces*, Comput. Math. Appl., **62**(2011), 1979-1983.

[14] G. Tian, *Generalized KKM theorems, minimax inequalities, and their applications*, J. Optim. Theory Appl., **83**(1994), 375-389.

[15] N.T. Vinh, *Matching theorems, fixed point theorems and minimax inequalities in topological ordered spaces*, Acta Math. Vietnam., **30**(2005), 211-224.

[16] C.L. Yen, *A minimax inequality and its applications to variational inequalities*, Pacific J. Math., **97**(1981), 477-481.

[17] X.Z. Yuan, *KKM principle, Ky Fan minimax inequalities and fixed point theorems*, Nonlinear World, **2**(1995), 131-169.

[18] H. Yang, J. Yu, *Essential component of the set of weakly Pareto-Nash equilibrium points*, Appl. Math. Lett., **15**(2002), 553-560.

[19] J. Zhou, G. Chen, *Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities*, J. Math. Anal. Appl., **132**(1988), 213-225.

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