Fixed Point Theory, 15(2014), No. 1, 233-252
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

BOUNDEDNESS AND GLOBAL ATTRACTIVITY FOR A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we are concerned with the boundedness and global attractivity of some vector functional differential equations. Using fixed point technique instead of the usual Lyapunov direct method, some quite general criteria for boundedness and attractivity are established. We then apply these results to some special delay equations, for which some existing results are included.
Key Words and Phrases: Nonlinear functional differential equation, contraction mapping principle, boundedness, global attractivity.
2010 Mathematics Subject Classification: 34D20, 34K20, 37C25, 47H10.

1. Introduction

In ordinary and functional differential equations, the stability for equilibrium solution, the boundedness of solution, and the existence of periodic solutions have been the most concerned issues from both mathematics and applied mathematics point of views. The Lyapunov direct method is most often used in dealing with these problems. However, there are some big deals that seem difficult to overcome by the Lyapunov function method. Examples can be found in [8] where the functions in the equations are unbounded with time, and [11] where the delay is unbounded, and [9] where the derivative of the delay is not small. In the past few years, some efforts have been made to cope with these difficulties by means of fixed point theory, see for instance, the works in [2, 3, 4, 5, 6, 7] where the focus are putted on some specific equations with significant applications. Other investigations on the study of stability by fixed point theory can be found in [10, 12, 13] and the references therein, to just a few.

Very recently, Burton [1] deals with the stability of the following scalar delay equations

$$x'(t) = -\int_{t-L}^{t} p(s-t)g(x(s))\,\mathrm{d}s, \tag{1.1}$$

$$x'(t) = -\int_{0}^{t} e^{-a(t-s)}\sin(t-s)g(x(s))\,\mathrm{d}s, \tag{1.2}$$

$$x'(t) = -\int_{-\infty}^{t} p(s-t)g(x(s))\,\mathrm{d}s \tag{1.3}$$

and

$$x'(t) = -a(t)g(x(q(t))). \tag{1.4}$$

In Equations (1.1)-(1.4), if we add the term $g(x)$, then these equations can be written as follows, respectively

$$x'(t) = -g(x(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\int_{-L}^{0} p(s)\int_{t+s}^{t} g(x(u))\,\mathrm{d}u\mathrm{d}s, \tag{1.5}$$

$$x'(t) = -g(x(t))\int_{0}^{\infty} e^{-av}\sin v\,\mathrm{d}v + \frac{\mathrm{d}}{\mathrm{d}t}\int_{0}^{t}\int_{t-s}^{\infty} e^{-av}\sin v g(x(s))\,\mathrm{d}v\mathrm{d}s, \tag{1.6}$$

$$x'(t) = -g(x(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t}\int_{-\infty}^{s-t} p(u)g(x(s))\,\mathrm{d}u\mathrm{d}s \tag{1.7}$$

and

$$x'(t) = -a(h(t))h'(t)g(x(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\int_{h(t)}^{t} a(s)g(x(q(s)))\,\mathrm{d}s. \tag{1.8}$$

It is seen that Equations (1.5)-(1.8) contain a asymptotical stable linear term, so they can be written as a fixed point of some continuous mappings by means of the variation of constants. The stability is then obtained by contraction mapping principle. We refer this to [1]. This also shows that the contraction mapping principle is suitable in studying some scalar delay equations, especially for those with distributed bounded delay, distributed unbounded delay, distributed infinite delay, or even pointwise variable delay.

In this paper, motivated from the fact aforementioned, we consider the boundedness and global attractivity of some delay vector equations by the fixed point technique. These models are described by the following equations:

$$x'(t) = A(t)x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{-L}^{0} P(s)\int_{t+s}^{t} G(x(u))\,\mathrm{d}u\mathrm{d}s, \tag{1.9}$$

$$x'(t) = A(t)x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{0}^{t}\int_{t-s}^{\infty} P(v)\,\mathrm{d}vG(x(s))\,\mathrm{d}s, \tag{1.10}$$

$$x'(t) = A(t)x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t}\int_{-\infty}^{s-t} Q(u)\,\mathrm{d}uG(x(s))\,\mathrm{d}s \tag{1.11}$$

and

$$x'(t) = A(t)x - \frac{\mathrm{d}}{\mathrm{d}t} \int_{h(t)}^{t} B(s)G(x(q(s)))\,\mathrm{d}s, \tag{1.12}$$

where $A(t)$, $B(t)$, $P(t)$ and $Q(t)$ are continuous $n \times n$ matrix functions defined over $\mathbb{R} = (-\infty, \infty)$, $G(u)$ is a continuous n-dimensional vector function defined over $\mathbb{R}^n$ and $G(0) = 0$, $q(t)$ is a continuous and strictly increasing function defined over $\mathbb{R}^+ = [0, \infty)$ with $q(t) < t$ for all $t > 0$, and $h(t)$ is the inverse function of $q(t)$ such that $q(h(t)) = t$.

We proceed as follows. In Section 2, some sufficient conditions are given for the boundedness and global attractivity of the systems (1.9)-(1.12) by the contraction mapping principle. Section 3 is devoted to the application of the results presented in Section 2 to the boundedness and global attractivity of some special functional differential equations.

2. Main results

For any vector $a = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$ and matrix $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, define the norms of a and A by

$$|a| = \max_{1\le i\le n} |a_i|, \quad |A| = \max_{1\le i,j\le n} |a_{ij}|.$$

Given constant $L > 0$, we define $C^n[-L, 0]$ to be the Banach space consisting of all continuous vector functions $\phi = (\phi_1, \phi_2, \cdots, \phi_n) : [-L, 0] \to \mathbb{R}^n$ with the norm

$$\|\phi\| = \sup_{t\in[-L,\, 0]} |\phi(t)|,$$

where $|\phi(t)| = \max_{1\le i\le n} |\phi_i(t)|$.

The following assumptions are applied to all Equations (1.9)-(1.12).

(A_1). There is a constant $k > 0$ such that

$$|G(x) - G(y)| \le k|x - y| \quad \text{for all} \quad x, y \in \mathbb{R}^n.$$

(A_2). Let $\Phi(t)$ be the fundamental matrix solution of the following linear ordinary differential equation

$$x'(t) = A(t)x \tag{2.1}$$

with $\Phi(0) = E$, where E is the $n \times n$ identity matrix. Suppose that there are positive constants a and b such that

$$|\Phi(t)\Phi^{-1}(s)| \le a e^{-b\int_s^t |A(u)|\,\mathrm{d}u} \quad \text{for all} \quad t > s.$$

(A_3). The matrix function $A(t)$ satisfies $\int_0^\infty |A(t)|\,\mathrm{d}t = +\infty$.

Remark. *In fact, we note that the condition (A_2) is a hard proposition to test. So, in the next section, we will consider some special cases of matrix function $A(t)$ to simplifies condition (A_2).*

The following Theorem 2.1 is on the boundedness of all solutions and global attractivity of the zero solution for Equation (1.9).

Theorem 2.1. *Suppose that (A_1) and (A_2) hold. If*

$$k\left(1 + \frac{a}{b}\right) \int_{-L}^{0} |P(s)s|\,\mathrm{d}s < 1, \tag{2.2}$$

then every solution of Equation (1.9) is bounded. In addition, if (A_3) *also holds, then every solution tends to zero as time goes to infinity.*

Proof. For any $\phi \in C^n[-L,\ 0]$, by the fundamental theory of functional differential equations, we know that Equation (1.9) admits a unique solution $x(t,\phi)$ with the initial condition $x_0 = \phi$. By the variation of constants, we can get

$$x(t,\phi) = \Phi(t)\phi(0) + \Phi(t)\int_0^t \Phi^{-1}(v)\frac{\mathrm{d}}{\mathrm{d}v}\int_{-L}^0 P(s)\int_{v+s}^v G(x(u))\,\mathrm{d}u\mathrm{d}s\mathrm{d}v.$$

Integration by parts yields

$$\begin{aligned} x(t,\phi) = &\int_{-L}^0 P(s)\int_{t+s}^t G(x(u))\,\mathrm{d}u\mathrm{d}s - \Phi(t)\int_{-L}^0 P(s)\int_s^0 G(x(u))\,\mathrm{d}u\mathrm{d}s \\ &-\Phi(t)\int_0^t (\Phi^{-1}(v))'\int_{-L}^0 P(s)\int_{v+s}^v G(x(u))\,\mathrm{d}u\mathrm{d}s\mathrm{d}v + \Phi(t)\phi(0). \end{aligned}$$

Since $(\Phi^{-1}(v))' = -\Phi^{-1}(v)A(v)$, it follows that

$$\begin{aligned} x(t,\phi) = &\int_{-L}^0 P(s)\int_{t+s}^t G(x(u))\,\mathrm{d}u\mathrm{d}s - \Phi(t)\int_{-L}^0 P(s)\int_s^0 G(x(u))\,\mathrm{d}u\mathrm{d}s \\ &+\Phi(t)\int_0^t \Phi^{-1}(v)A(v)\int_{-L}^0 P(s)\int_{v+s}^v G(x(u))\,\mathrm{d}u\mathrm{d}s\mathrm{d}v + \Phi(t)\phi(0). \end{aligned} \tag{2.3}$$

We first prove the boundedness of solutions for Equation (1.9). To this purpose let

$$M = \{z(t) : [-L,\infty) \to \mathbb{R}^n | z_0 = \phi \in C^n[-L,0], z(t) \text{ is bounded and continuous}\}.$$

M is a Banach space with the norm $\|z\| = \sup_{t\in[-L,\infty)} |z(t)|$ for any $z \in M$. With reference to (2.3), we define a continuous mapping $T : z \to \phi$ as following

$$\begin{aligned} T(z)(t) = &\int_{-L}^0 P(s)\int_{t+s}^t G(z(u))\,\mathrm{d}u\mathrm{d}s - \Phi(t)\int_{-L}^0 P(s)\int_s^0 G(\phi(u))\,\mathrm{d}u\mathrm{d}s \\ &+\int_0^t \Phi(t)\Phi^{-1}(v)A(v)\int_{-L}^0 P(s)\int_{v+s}^v G(z(u))\,\mathrm{d}u\mathrm{d}s\mathrm{d}v + \Phi(t)\phi(0) \end{aligned} \tag{2.4}$$

for all $t > 0$. Now, we prove that T maps M into itself. In fact, $T(z)(t)$ is defined for any $z \in M$ and $t > -L$ and $(T(z))_0 = \phi$. This follows from the fact (2.4) that

$$\begin{aligned} |T(z)(t)| \le\ &|\Phi(t)\phi(0)|\int_{-L}^0 |P(s)| + \int_{t+s}^t |G(z(u))|\,\mathrm{d}u\mathrm{d}s \\ &+|\Phi(t)|\int_{-L}^0 |P(s)|\int_s^0 |G(\phi(u))|\,\mathrm{d}u\mathrm{d}s \\ &+\int_0^t |\Phi(t)\Phi^{-1}(v)||A(v)|\int_{-L}^0 |P(s)|\int_{v+s}^v |G(z(u))|\,\mathrm{d}u\mathrm{d}s\mathrm{d}v. \end{aligned} \tag{2.5}$$

By assumptions (A_1) and (A_2), we have

$$|G(z(t))| \le k|z(t)| + |G(0)|, \quad |\Phi(t)| \le ae^{-b\int_0^t |A(s)|\,\mathrm{d}s}. \tag{2.6}$$

This together with (2.5) gives

$$\begin{aligned}
|T(z)(t)| \leq\ & a\phi(0)e^{-b\int_0^t |A(s)|\,\mathrm{d}s} + \int_{-L}^{0} |P(s)| \int_{t+s}^{t} (k|z(u)| + |G(0)|)\,\mathrm{d}u\mathrm{d}s \\
& + ae^{-b\int_0^t |A(s)|\,\mathrm{d}s} \int_{-L}^{0} |P(s)| \int_{s}^{0} (k|\phi(u)| + |G(0)|)\,\mathrm{d}u\mathrm{d}s \\
& + \int_0^t ae^{-b\int_v^t |A(s)|\,\mathrm{d}s} |A(v)| \int_{-L}^{0} P(s) \int_{v+s}^{v} (k|z(u)| + |G(0)|)\,\mathrm{d}u\mathrm{d}s\mathrm{d}v. \\
\leq\ & a\phi(0)e^{-b\int_0^t |A(s)|\,\mathrm{d}s} + (kz^u + |G(0)|) \left(\int_{-L}^{0} |P(s)s|\mathrm{d}s \right. \\
& + ae^{-b\int_0^t |A(s)|\,\mathrm{d}s} \int_{-L}^{0} |P(s)s|\,\mathrm{d}s \\
& \left. + \int_0^t ae^{-b\int_v^t |A(s)|\,\mathrm{d}s} |A(v)|\,\mathrm{d}v \int_{-L}^{0} |P(s)s|\,\mathrm{d}s \right)
\end{aligned}$$

with $z^u = \sup_{t\geq -L} |z(t)|$, where we used the facts $e^{-b\int_0^t |A(s)|\,\mathrm{d}s} \leq 1$ for all $t \geq 0$, and

$$\int_0^t ae^{-b\int_v^t |A(s)|\,\mathrm{d}s} |A(v)|\,\mathrm{d}v = \frac{a}{b}\, e^{-b\int_v^t |A(s)|\,\mathrm{d}s}\Big|_0^t = \frac{a}{b}\left(1 - e^{-b\int_0^t |A(s)|\,\mathrm{d}s}\right) \leq \frac{a}{b}. \quad (2.7)$$

This shows that $T(z)(t)$ is bounded on $[-L, \infty)$ with the upper bound

$$B = a|\phi(0)| + \left(1 + a + \frac{a}{b}\right)(kz^u + |G(0)|) \int_{-L}^{0} |P(s)s|\,\mathrm{d}s.$$

Therefore, T maps M into itself.

Secondly, we prove that T is a contraction mapping on M. For any $z_1, z_2 \in M$ and $z_{10} = z_{20} = \phi$, let $T(z_i)(t)$ $(i = 1, 2)$ be defined from Equation (2.4) with z replaced by z_1 and z_2, respectively. From assumptions (A_1), (A_2), and (2.7), we have

$$\begin{aligned}
& |T(z_1)(t) - T(z_2)(t)| \\
\leq\ & \int_{-L}^{0} |P(s)| \int_{t+s}^{s} |G(z_1(u)) - G(z_2(u))|\,\mathrm{d}u\mathrm{d}s \\
& + \int_0^t |\Phi(t)\Phi^{-1}(v)||A(v)| \int_{-L}^{0} P(s) \int_{v+s}^{v} |G(z_1(u)) - G(z_2(u))|\,\mathrm{d}u\mathrm{d}s\mathrm{d}v \\
\leq\ & k\|z_1 - z_2\| \left(\int_{-L}^{0} |P(s)s|\,\mathrm{d}s + \int_0^t |\Phi(t)\Phi^{-1}(v)||A(v)|\,\mathrm{d}v \int_{-L}^{0} |P(s)s|\,\mathrm{d}s \right) \\
\leq\ & k\left(1 + \frac{a}{b}\right)\|z_1 - z_2\| \int_{-L}^{0} |P(s)s|\,\mathrm{d}s
\end{aligned}$$

for all $t > 0$. This together with (2.2) shows that T is a contraction mapping on M. By virtue of the contraction mapping principle, T has a unique fixed point which is the solution of Equation (1.9). This shows that all solutions of Equation (1.9) are bounded.

Finally, we prove that the zero solution $x = 0$ is globally attractive. Define

$$\overline{M} = \{z(t) : [-L, \infty) \to \mathbb{R}^n | z(t) \in M \text{ and } \lim_{t\to\infty} z(t) = 0\}.$$

$\overline{M}$ is a Banach space with the norm $\|z\| = \sup_{t\in[-L,\infty)} |z(t)|$ for any $z \in \overline{M}$. According to (2.4), we limit the previous defined continuous mapping T on $\overline{M}$. We show that T maps $\overline{M}$ into itself also. In fact, by assumptions (A_2) and (A_3), it can be easily shown that

$$\lim_{t\to\infty} |\Phi(t)| = 0. \tag{2.8}$$

Moreover, we have $\lim_{t\to\infty} \Phi(t)\phi(0) = 0$ and

$$\lim_{t\to\infty} \Phi(t) \int_{-L}^{0} P(s) \int_{s}^{0} G(\phi(u))\, \mathrm{d}u\mathrm{d}s = 0.$$

Since $G(0) = 0$ and $\lim_{t\to\infty} z(t) = 0$, we have $\lim_{t\to\infty} G(z(t))$=0. Therefore,

$$\lim_{t\to\infty} \int_{-L}^{0} P(s) \int_{t+s}^{t} G(z(u))\, \mathrm{d}u\mathrm{d}s = 0.$$

Given $\varepsilon > 0$, there is a constant $t_0 > 0$ such that $|G(z(t))| < \varepsilon/2B_0$ for all $t \geq t_0$, where

$$B_0 = \frac{a}{b} \int_{-L}^{0} |P(s)|\, \mathrm{d}s.$$

By (2.8), there is a constant $t_1 \geq t_0 + L$ such that for all $t \geq t_1$,

$$\left| \Phi(t) \int_{0}^{t_0+L} \Phi^{-1}(v) A(v) \int_{-L}^{0} P(s) \int_{v+s}^{v} G(z(u))\, \mathrm{d}u\mathrm{d}s\mathrm{d}v \right| < \frac{\varepsilon}{2},$$

which results in

$$\begin{aligned}
& \left| \int_{0}^{t} \Phi(t)\Phi^{-1}(v) A(v) \int_{-L}^{0} P(s) \int_{v+s}^{v} G(z(u))\, \mathrm{d}u\mathrm{d}s\mathrm{d}v \right| \\
\leq\ & \left| \int_{0}^{t_0+L} \Phi(t)\Phi^{-1}(v) A(v) \int_{-L}^{0} P(s) \int_{v+s}^{v} G(z(u))\, \mathrm{d}u\mathrm{d}s\mathrm{d}v \right| \\
& + \left| \int_{t_0+L}^{t} \Phi(t)\Phi^{-1}(v) A(v) \int_{-L}^{0} P(s) \int_{v+s}^{v} G(z(u))\, \mathrm{d}u\mathrm{d}s\mathrm{d}v \right| \\
<\ & \frac{\varepsilon}{2} + \frac{\varepsilon}{2B_0} \int_{t_0+L}^{t} a e^{-b\int_v^t |A(s)|\,\mathrm{d}s} |A(v)|\, \mathrm{d}v \int_{-L}^{0} |P(s)s|\, \mathrm{d}s \\
<\ & \frac{\varepsilon}{2} + \frac{a\varepsilon}{2B_0 b} \left(1 - e^{-\int_{t_0+L}^{t} |A(s)|\,\mathrm{d}s}\right) \int_{-L}^{0} |P(s)s|\, \mathrm{d}s \\
<\ & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

for all $t \geq t_1$. This shows that

$$\lim_{t\to\infty} \int_{0}^{t} \Phi(t)\Phi^{-1}(v) A(v) \int_{-L}^{0} P(s) \int_{v+s}^{v} G(z(u))\, \mathrm{d}u\mathrm{d}s\mathrm{d}v = 0.$$

Therefore, $\lim_{t\to\infty} T(z)(t) = 0$. That is, T maps $\overline{M}$ into itself. Moreover, from the above discussion, T is contraction mapping on $\overline{M}$. So, there is a unique $\overline{z}(t) \in \overline{M}$ such that $\overline{z}(t) = T(\overline{z})(t)$ and $\overline{z}_0 = \phi$. By the existence and uniqueness of the solution, we have $x(t,\phi) = \overline{z}(t)$ for all $t \geq 0$. Thus, $\lim_{t\to\infty} x(t,\phi) = 0$. This shows that the zero solution $x = 0$ is globally attractive. The proof is complete. □

The following Theorem 2.2 is on the boundedness of all solutions, and global attractivity of the zero solution of Equation (1.10).

Theorem 2.2. *Assume* (A_1) *and* (A_2). *If*

$$\sup_{t\geq 0} \int_0^t \int_{t-s}^{\infty} |P(v)|\, \mathrm{d}v \mathrm{d}s = p^* < \infty, \quad \left(1 + \frac{a}{b}\right) kp^* < 1, \tag{2.9}$$

then every solution of Equation (1.10) is bounded. In addition, if (A_3) *also holds, then every solution tends to zero as time goes to infinity.*

Proof. By the fundamental theory of functional differential equations, we know that for any $x_0 \in \mathbb{R}^n$, Equation (1.10) admits a unique solution $x(t, x_0)$ with initial condition $x(0) = x_0$. By variation of constants, we can get

$$x(t, x_0) = \Phi(t)x_0 + \Phi(t) \int_0^t \Phi^{-1}(u) \frac{\mathrm{d}}{\mathrm{d}u} \int_0^u \int_{u-s}^{\infty} P(v)G(x(s))\, \mathrm{d}v \mathrm{d}s \mathrm{d}u.$$

By integration by part and noticing the fact $(\Phi^{-1}(u))' = -\Phi^{-1}(u)A(u)$, we have

$$\begin{aligned} x(t, x_0) = \Phi(t)x_0 + &\int_0^t \int_{t-s}^{\infty} P(v)G(x(s))\, \mathrm{d}v \mathrm{d}s \\ +\Phi(t) \int_0^t \Phi^{-1}(u)A(u) &\int_0^u \int_{u-s}^{\infty} P(v)G(x(s))\, \mathrm{d}v \mathrm{d}s \mathrm{d}u. \end{aligned} \tag{2.10}$$

Now we prove the boundedness of solutions for Equation (1.10). To this purpose let

$$M = \{z(t) : [0, \infty) \to \mathbb{R}^n | z_0 = x_0 \in \mathbb{R}^n, z(t) \text{ is bonded and continuous}\}.$$

M is a Banach space with the norm $\|z\| = \sup_{t\in\mathbb{R}^+} |z(t)|$ for any $z \in M$. For any $z \in M$, define a continuous mapping $T : M \to M$ using the Equation (2.10) with x replaced by z. Now, we prove that T maps M into itself. In fact, by definition, $T(z)(t)$ is defined for any $z \in M$ and $t > 0$ and $T(z)(0) = x_0$. By assumption (A_2), (2.6) and (2.10), we can obtain that

$$\begin{aligned} |T(z)(t)| \leq \quad & a|x_0|e^{-b\int_0^t |A(s)|\, \mathrm{d}s} + (kz^u + |G(0)|) \left(\int_0^t \int_{t-s}^{\infty} |P(v)|\, \mathrm{d}v \mathrm{d}s \right. \\ & \left. + \int_0^t ae^{-b\int_u^t |A(s)|\, \mathrm{d}s} |A(u)| \int_0^u \int_{u-s}^{\infty} |P(v)|\, \mathrm{d}v \mathrm{d}s \mathrm{d}u \right), \end{aligned}$$

where $z^u = \sup_{t\in\mathbb{R}^+} |z(t)|$. From (2.7), it follows that $T(z)(t)$ is bounded on $\mathbb{R}^+$ with upper bound

$$B = a|x_0| + (1 + \frac{a}{b})(kz^u + |G(0)|)p^*.$$

Therefore, T maps M into itself.

Secondly, we show that T is a contraction mapping on M. Actually, for any $z_1, z_2 \in M$ and $z_{10} = z_{20} = x_0$, let $T(z_i)(t)$ $(i = 1, 2)$ be defined by Equation (2.10) with x replaced by z_1 and z_2, respectively. By assumptions (A_1), (A_2), (2.7), (2.9) and (2.10), we have

$$\begin{aligned} |T(z_1)(t) - T(z_2)(t)| \leq \; & k\|z_1 - z_2\| \left(\int_0^t \int_{t-s}^{\infty} |P(v)| \, \mathrm{d}v\mathrm{d}s \right. \\ & \left. + \int_0^t a e^{-b \int_u^t |A(s)| \, \mathrm{d}s} |A(u)| \int_0^u \int_{u-s}^{\infty} |P(v)| \, \mathrm{d}v\mathrm{d}s\mathrm{d}u \right) \\ \leq \; & k\left(1 + \frac{a}{b}\right) p^* \|z_1 - z_2\|. \end{aligned}$$

By (2.9), T is contraction on M. By virtue of the contraction mapping principle, there is a unique fixed point of T which is the bounded solution of Equation (1.9). That means that all solutions of Equation (1.9) are bounded.

Finally, we prove the global attractivity of the zero solution $x = 0$. Define

$$\overline{M} = \{z(t) : [0, \infty) \to \mathbb{R}^n | z \in M \text{ and } \lim_{t\to\infty} z(t) = 0\}.$$

Again, $\overline{M}$ is a Banach space with the norm $\|z\| = \sup_{t\in\mathbb{R}^+} |z(t)|$ for any $z \in \overline{M}$. We limit the previous defined continuous mapping T on $\overline{M}$, then the mapping T is continued on $\overline{M}$ in a natural way. We show that T maps $\overline{M}$ into itself. In fact, by assumptions (A_2) and (A_3), it can be easily shown that

$$\lim_{t\to\infty} |\Phi(t)| = 0. \tag{2.11}$$

Moreover, $\lim_{t\to\infty} \Phi(t)x_0 = 0$.

Since $G(0) = 0$ and $\lim_{t\to\infty} z(t) = 0$, we have $\lim_{t\to\infty} G(z(t)) = 0$. This together with (2.9) shows that, given any $\varepsilon > 0$, there are positive constants t_0, t_1 and $t_1 > t_0$ such that $|G(z(t))| < \varepsilon$ for all $t \geq t_0$ and $\int_{t-t_0}^{\infty} |P(v)| \, \mathrm{d}v < \varepsilon/Bt_0$ for all $t \geq t_1$, where $B = \sup_{t\in\mathbb{R}^+} G(z(t))$. Hence, for all $t \geq t_1$

$$\begin{aligned} \left| \int_0^t \int_{t-s}^{\infty} P(v)G(z(s)) \, \mathrm{d}v\mathrm{d}s \right| = \; & \left| \left(\int_0^{t_0} \int_{t-s}^{\infty} + \int_{t_0}^{t} \int_{t-s}^{\infty} \right) P(v)G(z(s)) \, \mathrm{d}v\mathrm{d}s \right| \\ \leq \; & B \int_0^{t_0} \int_{t-t_0}^{\infty} |P(v)| \, \mathrm{d}v\mathrm{d}s + \varepsilon \int_{t_0}^{t} \int_{t-s}^{\infty} |P(v)| \, \mathrm{d}v\mathrm{d}s \\ \leq \; & (1 + p^*)\varepsilon. \end{aligned}$$

So

$$\lim_{t\to\infty} \int_0^t \int_{t-s}^{\infty} P(v)G(z(s)) \, \mathrm{d}v\mathrm{d}s = 0.$$

By (2.11), there is a constant $t_2 \geq t_1$ such that for all $t \geq t_2$,

$$\left| \Phi(t) \int_0^{t_0} \Phi^{-1}(u)A(u) \int_0^u \int_{u-s}^{\infty} P(v)G(z(s)) \, \mathrm{d}v\mathrm{d}s\mathrm{d}u \right| < \varepsilon.$$

This together with (2.6) and (2.9) shows that

$$\begin{aligned}
&\left|\int_0^t \Phi(t)\Phi^{-1}(u)A(u)\int_0^u\int_{u-s}^{\infty} P(v)G(z(s))\,\mathrm{d}v\mathrm{d}s\mathrm{d}u\right| \\
\leq\ & \left|\int_0^{t_0} \Phi(t)\Phi^{-1}(u)A(u)\int_0^u\int_{u-s}^{\infty} P(v)G(z(s))\,\mathrm{d}v\mathrm{d}s\mathrm{d}u\right| \\
& +\left|\int_{t_0}^{t} \Phi(t)\Phi^{-1}(u)A(u)\int_0^u\int_{u-s}^{\infty} P(v)G(z(s))\,\mathrm{d}v\mathrm{d}s\mathrm{d}u\right| \\
\leq\ & \varepsilon+\varepsilon(1+p^*)\int_{t_0}^t |\Phi(t)\Phi^{-1}(u)||A(u)|\,\mathrm{d}u \\
\leq\ & \Big(1+\frac{a}{b}(1+p^*)\Big)\,\varepsilon, \qquad \forall\, t\geq t_2,
\end{aligned}$$

which leads to

$$\lim_{t\to\infty}\int_0^t \Phi(t)\Phi^{-1}(v)A(v)\int_0^u\int_{u-s}^{\infty} P(s)G(z(u))\,\mathrm{d}u\mathrm{d}s\mathrm{d}v=0.$$

Therefore, $\lim_{t\to\infty} T(z)(t)=0$. That is, T maps $\overline{M}$ into itself. Further, from the above discussion, T is contraction on $\overline{M}$. So, there is a unique $\overline{z}(t)\in\overline{M}$ such that $\overline{z}(t)=T(\overline{z})(t)$ and $\overline{z}_0=x_0$. By the existence and uniqueness of the solution, we have $x(t,x_0)=\overline{z}(t)$ for all $t\geq 0$. Thus, $\lim_{t\to\infty} x(t,x_0)=\lim_{t\to\infty}\overline{z}(t)=0$. This shows that the zero solution $x=0$ is globally attractive. This completes the proof. □

Now we state the boundedness of all solutions and global attractivity of the zero solution of Equation (1.11).

Theorem 2.3. *Assume* (A_1) *and* (A_2). *If*

$$\begin{gathered}
\sup_{t\geq 0}\int_{-\infty}^t\int_{-\infty}^{s-t}|Q(u)|\,\mathrm{d}u\mathrm{d}s=p^*<\infty, \quad \Big(1+\frac{a}{b}\Big)kp^*<1 \\
\text{and}\quad \int_{-\infty}^0\int_{-\infty}^s |Q(u)|\,\mathrm{d}u\mathrm{d}s=N \quad \text{exists,}
\end{gathered}\tag{2.12}$$

then every solution of Equation (1.11) is bounded. In addition, if (A_3) *also holds, then every solution tends to zero as time goes to infinity.*

Proof. By the fundamental theory of functional differential equations, we know that for any $\phi\in C^n(\infty,\ 0]$, Equation (1.11) admits a unique solution $x(t,\phi)$ with initial condition $x_0(s)=\phi$ for all $s\in(-\infty,0]$. By the variation of constants, we can get

$$x(t,\phi(t))=\Phi(t)\phi(0)+\Phi(t)\int_0^t\Phi^{-1}(v)\frac{\mathrm{d}}{\mathrm{d}v}\int_{-\infty}^v\int_{-\infty}^{s-v}Q(u)G(x(s))\,\mathrm{d}u\mathrm{d}s\mathrm{d}v.$$

By integration by part and noticing $(\Phi^{-1}(v))'=-\Phi^{-1}(v)A(v)$, we have

$$\begin{aligned}
x(t,\phi)=&\int_{-\infty}^t\int_{-\infty}^{s-t}Q(u)G(x(s))\,\mathrm{d}u\mathrm{d}s-\Phi(t)\Phi^{-1}(0)\int_{-\infty}^0\int_{-\infty}^{s}Q(u)G(x(s))\,\mathrm{d}u\mathrm{d}s \\
&+\Phi(t)\int_0^t\Phi^{-1}(v)A(v)\int_{-\infty}^v\int_{-\infty}^{s-v}Q(u)G(x(s))\,\mathrm{d}u\mathrm{d}s\mathrm{d}v+\Phi(t)\phi(0).
\end{aligned}\tag{2.13}$$

Again we first prove the boundedness of solutions for Equation (1.11). To this end let

$$M = \{z(t) : (-\infty, \infty) \to \mathbb{R}^n | z_0 = \phi \in C^n(-\infty, 0], z(t) \text{ is bonded and continuous}\}.$$

Once again, M is a Banach space with the norm $\|z\| = \sup_{t\in\mathbb{R}} |z(t)|$ for any $z \in M$. According to (2.13), for any $z \in M$, we define a continuous mapping $T : M \to M$ using the Equation (2.13) with x replaced by z, as we what have done before. Now, we prove that T maps M into itself. In fact, by definition, $T(z)(t)$ is defined for any $z \in M$ and $t \in \mathbb{R}$ and $T(z)(0) = \phi$. By (2.6) and $e^{-b\int_0^t |A(s)|\,\mathrm{d}s} < 1$, we have

$$\begin{aligned} |T(z)(t)| \leq \quad & (kz^u + |G(0)|)\left(\int_{-\infty}^t \int_{-\infty}^{s-t} |Q(u)|\,\mathrm{d}u\mathrm{d}s + a\int_{-\infty}^0 \int_{-\infty}^s |Q(u)|\,\mathrm{d}u\mathrm{d}s \right. \\ & \left. + \int_0^t a e^{-b\int_v^t |A(s)|\,\mathrm{d}s} |A(v)| \int_{-\infty}^v \int_{-\infty}^{s-v} |Q(u)|\,\mathrm{d}u\mathrm{d}s\mathrm{d}v\right) + a|\phi(0)|. \end{aligned}$$

where $z^u = \sup_{t\in\mathbb{R}} |z(t)|$. By (2.7) and (2.13), we have $T(z)(t)$ is bounded on $\mathbb{R}$ with the upper bound

$$B = a|\phi(0)| + \left(p^* + aN + \frac{a}{b}p^*\right)(kz^u + |G(0)|).$$

Therefore, T maps M into itself.

Secondly, we show that T is a contraction mapping on M. Indeed, for any $z_1, z_2 \in M$ and $z_{10} = z_{20} = \phi$, let $T(z_i)(t)$ $(i = 1, 2)$ be defined by Equation (2.13) with x replaced by z_1 and z_2, respectively. By assumptions (A_1), (A_2), (2.7), (2.12) and (2.13), we have

$$\begin{aligned} |T(z_1)(t) - T(z_2)(t)| \leq \quad & k\|z_1 - z_2\| \left(\int_{-\infty}^t \int_{-\infty}^{s-t} |Q(u)|\,\mathrm{d}u\mathrm{d}s \right. \\ & \left. + \int_0^t |\Phi(t)\Phi^{-1}(v)||A(v)| \int_{-\infty}^v \int_{-\infty}^{s-v} |Q(u)|\,\mathrm{d}u\mathrm{d}s\mathrm{d}v\right) \\ \leq \quad & k\left(1 + \frac{a}{b}\right)p^*\|z_1 - z_2\|. \end{aligned}$$

This together with (2.12) shows that T is a contraction on M. By virtue of the contraction mapping principle, there is a unique fixed point which is the bounded solution of Equation (1.11). So all solutions of Equation (1.11) are bounded.

Finally, we prove that the zero solution $x = 0$ is globally attractive. Define

$$\overline{M} = \{z(t) : (-\infty, \infty) \to \mathbb{R}^n | z \in M \text{ and } \lim_{t\to\infty} z(t) = 0\}.$$

$\overline{M}$ is a Banach space with the norm $\|z\| = \sup_{t\in\mathbb{R}} |z(t)|$ for any $z \in \overline{M}$. The mapping T is naturally extended to $\overline{M}$. By assumptions (A_2) and (A_3), it can be easily proved that

$$\lim_{t\to\infty} |\Phi(t)| = 0 \tag{2.14}$$

and $\lim_{t\to\infty} \Phi(t)\phi(0) = 0$.

Since $G(0) = 0$ and $\lim_{t\to\infty} z(t) = 0$, we have $\lim_{t\to\infty} G(z(t)) = 0$. Therefore, for any constant $\varepsilon > 0$, there are positive constants t_1, t_2 and $t_1 < t_2$ such that

$|G(z(t))| < \varepsilon$ for all $|t| \geq t_1$ and $|G(z(v+t))| < \varepsilon/t_1$ for all $|t| \geq t_2$ and $v \in [-t_1,\ 0]$. So

$$
\begin{aligned}
& \left|\int_{-\infty}^{t}\int_{-\infty}^{s-t} Q(u)G(z(s))\,\mathrm{d}u\mathrm{d}s\right| \\
\leq\ & \left(\int_{-\infty}^{-t_1}\int_{-\infty}^{s-t} + \int_{-t_1}^{0}\int_{-\infty}^{s-t} + \int_{0}^{t_1}\int_{-\infty}^{s-t} + \int_{t_1}^{t}\int_{-\infty}^{s-t}\right) |Q(u)||G(z(s))|\,\mathrm{d}u\mathrm{d}s \\
\leq\ & \varepsilon N + \varepsilon p^* + \left(\int_{-t_1}^{0}\int_{-\infty}^{v} + \int_{0}^{t_1}\int_{-\infty}^{v}\right) |Q(u)||G(z(v+t))|\,\mathrm{d}u\mathrm{d}v \\
\leq\ & 2(N+p^*)\varepsilon, \qquad \forall\, t \geq t_2.
\end{aligned}
$$

It then follows that

$$\lim_{t\to\infty}\int_{-\infty}^{t}\int_{-\infty}^{s-t} Q(u)G(z(s))\,\mathrm{d}u\mathrm{d}s = 0.$$

Furthermore, by (2.14), there is a constant $t_3 \geq t_2$ such that for all $t \geq t_3$,

$$\left|\Phi(t)\int_{0}^{t_1}\Phi^{-1}(v)A(v)\int_{-\infty}^{v}\int_{-\infty}^{s-v} Q(u)G(z(s))\,\mathrm{d}u\mathrm{d}s\mathrm{d}v\right| < \varepsilon.$$

Therefore

$$
\begin{aligned}
& \left|\int_{0}^{t}\Phi(t)\Phi^{-1}(v)A(v)\int_{-\infty}^{v}\int_{-\infty}^{s-v} Q(u)G(z(s))\,\mathrm{d}u\mathrm{d}s\mathrm{d}v\right| \\
\leq\ & \varepsilon + 2(N+p^*)\varepsilon\int_{t_1}^{t}|\Phi(t)\Phi^{-1}(v)||A(v)|\,\mathrm{d}v \\
\leq\ & \left(1 + \frac{2a(N+p^*)}{b}\right)\varepsilon, \qquad \forall\, t \geq t_3.
\end{aligned}
$$

This results in

$$\lim_{t\to\infty}\int_{0}^{t}\Phi(t)\Phi^{-1}(v)A(v)\int_{-\infty}^{v}\int_{-\infty}^{s-v} Q(u)G(z(s))\,\mathrm{d}u\mathrm{d}s\mathrm{d}v = 0.$$

Therefore, $\lim_{t\to\infty} T(z)(t) = 0$. That is, T maps $\overline{M}$ into itself. Moreover, the above discussion shows that T is a contraction on $\overline{M}$. So, there is a unique $\overline{z}(t) \in \overline{M}$ such that $\overline{z}(t) = T(\overline{z})(t)$ and $\overline{z}_0 = x_0$. By the existence and uniqueness of the solution, we have $x(t,\phi) = \overline{z}(t)$ for all $t \geq 0$. Thus, $\lim_{t\to\infty} x(t,\phi) = \lim_{t\to\infty}\overline{z}(t) = 0$. This shows that the zero solution $x = 0$ is globally attractive. This completes the proof. □

On the boundedness of all solutions and global attractability of the zero solution for Equation (1.12), we have the following result.

Theorem 2.4. *Suppose that assumptions* (A_1) *and* (A_2) *hold. If*

$$\sup_{t\geq 0}\int_{t}^{h(t)}|B(u)|\,\mathrm{d}u = p^* < \infty, \quad \left(1 + a + \frac{a}{b}\right)kp^* < 1, \tag{2.15}$$

then every solution of Equation (1.12) is bounded. In addition, if (A_3) *also holds, then every solution tends to zero as times goes to infinity.*

Proof. For any $\phi \in C^n[q(0),\ 0]$, by the fundamental theory of functional differential equations, we know that Equation (1.12) admits a unique solution $x(t,\phi)$ with initial condition $x_0(s) = \phi$ for all $s \in [q(0), 0]$. By variation of constants, we have

$$x(t,\phi(t)) = \Phi(t)\phi(0) - \Phi(t)\int_0^t \Phi^{-1}(v)\frac{\mathrm{d}}{\mathrm{d}v}\int_{h(v)}^v B(s)G(x(q(s)))\,\mathrm{d}s\mathrm{d}v.$$

By integration by parts and $(\Phi^{-1}(v))' = -\Phi^{-1}(v)A(v)$, we have

$$\begin{aligned} x(t,\phi) = \quad & -\int_{h(t)}^t B(s)G(x(q(s)))\,\mathrm{d}s + \Phi(t)\Phi^{-1}(0)\int_{h(0)}^0 B(s)G(x(q(s)))\,\mathrm{d}s \\ & +\Phi(t)\int_0^t (\Phi^{-1}(v)A(v))\int_{h(v)}^v B(s)G(x(q(s)))\,\mathrm{d}s\mathrm{d}v + \Phi(t)\phi(0). \end{aligned} \tag{2.16}$$

Firstly, we prove the boundedness of solutions of Equation (1.12). To this purpose let

$$M = \{z(t) : [q(0),\infty) \to \mathbb{R}^n | z_0 = \phi \in C^n[q(0),0], z(t) \text{ is bonded and continuous}\}.$$

With the supremum norm $\|z\| = \sup_{t\in[q(0),\infty)} |z(t)|$ for any $z \in M$, M becomes a Banach space. For any $z \in M$, we define a continuous mapping $T : M \to M$ using the Equation (2.16) with x replaced by z. Now, we prove that T maps M into itself. In fact, by the definition, $T(z)(t)$ is defined for any $z \in M$ and $t \in \mathbb{R}^+$ and $T(z)(0) = \phi$. By (2.6), we obtain

$$\begin{aligned} & |T(z)(t)| \\ \le\ & (kz^u + |G(0)|)\left(\int_t^{h(t)} |B(s)|\,\mathrm{d}s + ae^{-b\int_0^t |A(s)|\,\mathrm{d}s}\int_0^{h(0)} |B(s)|\,\mathrm{d}s\right. \\ & \left. + \int_0^t ae^{-b\int_v^t |A(s)|\,\mathrm{d}s}|A(v)|\int_v^{h(v)} |B(s)|\,\mathrm{d}s\mathrm{d}v\right) + ae^{-b\int_0^t |A(s)|\,\mathrm{d}s}|\phi(0)|, \end{aligned} \tag{2.17}$$

where $z^u = \sup_{t\in[q(0),\infty)} |z(t)|$. From (2.7), (2.15) and (2.17), it follows that $T(z)(t)$ is bounded on $[0,\infty)$ with the upper bound

$$K = a|\phi(0)| + \left(1 + a + \frac{a}{b}\right)(kz^u + |G(0)|)p^*.$$

Therefore, T maps M into itself.

Secondly, we prove that T is a contraction on M. For any $z_1, z_2 \in M$ and $z_{10} = z_{20} = \phi$, let $T(z_i(t))$ $(i = 1,2)$ be defined by Equation (2.16) with x replaced by z_1

and z_2, respectively. By assumptions (A_1), (A_2), (2.7) and (2.16), we have

$$\begin{aligned}
&|T(z_1)(t) - T(z_2)(t)| \\
\leq\ & k\|z_1 - z_2\| \left(\int_t^{h(t)} |B(s)|\,\mathrm{d}s + |\Phi(t)\Phi^{-1}(0)| \int_0^{h(0)} |B(s)|\,\mathrm{d}s \right. \\
&\left. + \int_0^t |\Phi(t)\Phi^{-1}(v)||A(v)| \int_v^{h(v)} |B(s)|\,\mathrm{d}s\mathrm{d}v \right) \\
\leq\ & k\left(1 + a + \frac{a}{b}\right) p^* \|z_1 - z_2\|.
\end{aligned}$$

So, T is contraction on M. By virtue of the contraction mapping principle, there is a unique fixed point which is the bounded solution of Equation (1.12). So all solutions of (1.12) are bounded.

Finally, we prove that the zero solution $x = 0$ is globally attractive. To this end, define

$$\overline{M} = \{z(t) : [q(0), \infty) \to \mathbb{R}^n | z \in M \text{ and } \lim_{t\to\infty} z(t) = 0\}.$$

Again, with the supremum defined by $\|z\| = \sup_{t\in\mathbb{R}^+} |z(t)|$, for any $z \in \overline{M}$, $\overline{M}$ becomes a Banach space. We extend naturally T from M onto $\overline{M}$. Now, we show that T maps $\overline{M}$ into itself. In fact, by assumptions (A_2) and (A_3), it can be easily proved that

$$\lim_{t\to\infty} |\Phi(t)| = 0, \tag{2.18}$$

and $\lim_{t\to\infty} \Phi(t)\phi(0) = 0$.

Since $G(0) = 0$ and $\lim_{t\to\infty} z(t) = 0$, we have $\lim_{t\to\infty} G(z(t)) = 0$. Therefore, given $\varepsilon > 0$, there exists a $t_1 > 0$ such that

$$|G(z(t))| < \varepsilon, \ \left| \int_{h(t)}^t B(s)G(x(q(s)))\,\mathrm{d}s \right| \leq \varepsilon p^*, \ \left| \int_{h(0)}^0 B(s)G(x(q(s)))\,\mathrm{d}s \right| \leq \varepsilon p^*$$

for all $t \geq t_1$. Therefore,

$$\lim_{t\to\infty} \int_{h(t)}^t B(s)G(x(q(s)))\,\mathrm{d}s = 0, \quad \lim_{t\to\infty} \Phi(t) \int_{h(0)}^0 B(s)G(x(q(s)))\,\mathrm{d}s = 0.$$

Furthermore, by (2.18), there is a constant $t_2 \geq t_1$ such that for all $t \geq t_2$,

$$\left| \Phi(t) \int_0^{t_1} \Phi^{-1}(v)A(v) \int_{h(v)}^v B(s)G(x(q(s)))\,\mathrm{d}s\mathrm{d}v \right| < \varepsilon.$$

This together with (2.7) yields

$$\begin{aligned}
&\left|\int_0^t \Phi(t)\Phi^{-1}(v)A(v)\int_{h(v)}^v B(s)G(x(q(s)))\,\mathrm{d}s\mathrm{d}v\right| \\
\leq\ &\left|\int_0^{t_1} \Phi(t)\Phi^{-1}(v)A(v)\int_{h(v)}^v B(s)G(x(q(s)))\,\mathrm{d}s\mathrm{d}v\right| \\
&+\left|\int_{t_1}^t \Phi(t)\Phi^{-1}(v)A(v)\int_{h(v)}^v B(s)G(x(q(s)))\,\mathrm{d}s\mathrm{d}v\right| \\
\leq\ &\varepsilon + p^*\varepsilon\int_{t_1}^t |\Phi(t)\Phi^{-1}(v)||A(v)|\,\mathrm{d}v \leq \left(1+\frac{ap^*}{b}\right)\varepsilon, \qquad \forall\, t\geq t_2.
\end{aligned}$$

This leads to

$$\lim_{t\to\infty}\int_0^t \Phi(t)\Phi^{-1}(v)A(v)\int_{h(v)}^v B(s)G(x(q(s)))\,\mathrm{d}s\mathrm{d}v = 0.$$

Therefore, $\lim_{t\to\infty} T(z)(t) = 0$. That is, T maps $\overline{M}$ into itself. In addition, the above discussion shows that T is a contraction on $\overline{M}$. So, there is a unique $\overline{z}(t)\in\overline{M}$ such that $\overline{z}(t) = T(\overline{z})(t)$ and $\overline{z}_0 = \phi$. By the existence and uniqueness of the solution, we have $x(t,\phi) = \overline{z}(t)$ for all $t\in\mathbb{R}^+$. Thus, $\lim_{t\to\infty} x(t,\phi) = \lim_{t\to\infty}\overline{z}(t) = 0$. So the zero solution $x = 0$ is globally attractive. This completes the proof. □

3. Some special cases

In this paper, we consider the boundedness and global attractivity of some vector functional differential equations by fixed point technique instead of the usual Lyapunov direct method, some quite general criteria for boundedness and attractivity are established. However, we note that the conditional of (H_2) is not easy to test. So, as some consequence of Theorem 2.1-2.4, we will discuss some special case of coefficient matrix $A(t)$ of equation (2.1).

Case I. *The coefficient matrix $A(t)$ of equation (2.1) is a diagonal matrix.*

Suppose that

$$A(t) = -\mathrm{diag}(a_1(t), a_2(t), \cdots, a_n(t)),$$

where $a_i(t)\,(i = 1, 2, \cdots, n)$ are nonnegative continuous functions defined over $\mathbb{R}^+$. For this special case, the fundamental matrix solution of equation (2.1) has the following form

$$\Phi(t) = \mathrm{diag}\left(e^{-\int_0^t a_1(u)\,\mathrm{d}u}, e^{-\int_0^t a_2(u)\,\mathrm{d}u}, \cdots, e^{-\int_0^t a_n(u)\,\mathrm{d}u}\right).$$

Hence,

$$\Phi(t)\Phi^{-1}(s) = \mathrm{diag}\left(e^{-\int_s^t a_1(u)\,\mathrm{d}u}, e^{-\int_s^t a_2(u)\,\mathrm{d}u}, \cdots, e^{-\int_s^t a_n(u)\,\mathrm{d}u}\right)$$

for all $t, s\in\mathbb{R}^+$ with $t > s$. Obviously, we have

$$|\Phi(t)\Phi^{-1}(s)| = \max_{1\leq i\leq n}\left\{e^{-\int_s^t a_i(u)\,\mathrm{d}u}\right\}.$$

We introduce the following assumptions.

(B_1) There exists a constant $d > 0$ such that for any $t \in \mathbb{R}^+$

$$d \max_{1\le i\le n}\{|a_i(t)|\} \le \min_{1\le i\le n}\{|a_i(t)|\}.$$

(B_2) $\int_0^\infty \max_{1\le i\le n}\{|a_i(t)|\}\,\mathrm{d}t = \infty$.

Since for this special case, $|A(t)| = \max_{1\le i\le n}\{|a_i(t)|\}$, we can easily show that if assumption (B_1) holds, then assumption (A_2) also holds with $a = 1$ and $b = d$, and if assumption (B_2) holds, then assumption (A_3) also holds.

For the special case that in Equation (1.9), we have the following Corollary 3.1 that is a direct consequence of Theorem 2.1.

Corollary 3.1. *Suppose that* $A(t) = -\mathrm{diag}(a_1(t), a_2(t), \cdots, a_n(t))$*, where* $a_i(t)$ $(i = 1, 2, \cdots, n)$ *are nonnegative continuous functions on* $t \in \mathbb{R}^+$*. Assume* (A_1) *and* (B_1)*, and*

$$k\left(1+\frac{1}{d}\right)\int_{-L}^{0} |P(s)s|\,\mathrm{d}s < 1.$$

Then every solution of Equation (1.9) is bounded. In addition, if (B_2) *also holds, then every solution tends to zero as time goes to infinity.*

Similar to Corollary 3.1, we have the following consequence of Theorem 2.2.

Corollary 3.2. *Suppose that* $A(t) = -\mathrm{diag}(a_1(t), a_2(t), \cdots, a_n(t))$*, where* $a_i(t)$ $(i = 1, 2, \cdots, n)$ *are nonnegative continuous functions for* $t \ge 0$*. If assumptions* (A_1) *and* (B_1) *hold and*

$$\sup_{t\ge 0}\int_0^t\int_{t-s}^{\infty} |P(v)|\,\mathrm{d}v\mathrm{d}s = p^* < \infty, \quad \left(1+\frac{1}{d}\right)kp^* < 1,$$

then every solution of Equation (1.10) is bounded. In addition, (B_2) *also holds, then every solution tends to zero as time goes to infinity.*

Again, we have the consequence of Theorem 2.3 as Corollary 3.3 following.

Corollary 3.3. *Suppose that* $A(t) = -\mathrm{diag}(a_1(t), a_2(t), \cdots, a_n(t))$*, where* $a_i(t)$ $(i = 1, 2, \cdots, n)$ *are nonnegative continuous functions on* $t \ge 0$*. If assumptions* (A_1) *and* (B_1) *hold and*

$$\sup_{t\ge 0}\int_0^t\int_{-\infty}^{s-t} |Q(u)|\,\mathrm{d}u\mathrm{d}s = p^* < \infty, \quad \left(1+\frac{1}{d}\right)kp^* < 1,$$

$$\int_{-\infty}^{0}\int_{-\infty}^{s} |Q(u)|\,\mathrm{d}u\mathrm{d}s \quad \textit{exists},$$

then every solution of equation (1.11) is bounded. In addition, if (B_2) *also holds, then every solution tends to zero as times goes to infinity.*

As a consequence of Theorem 2.4, we have the following Corollary 2.4.

Corollary 3.4. *Suppose that* $A(t) = -\mathrm{diag}(a_1(t), a_2(t), \cdots, a_n(t))$*, where* $a_i(t)$ $(i = 1, 2, \cdots, n)$ *are nonnegative continuous functions on* $t \in \mathbb{R}^+$*. If assumptions* (A_1) *and* (B_1) *hold and*

$$\sup_{t\ge 0}\int_t^{h(t)} |B(u)|\,\mathrm{d}u = p^* < \infty, \quad \left(2+\frac{1}{d}\right)kp^* < 1,$$

then every solution of equation (1.12) is bounded. In addition, if (B_2) *also holds, then every solution tends to zero as time goes to infinity.*

Remark. *Particularly, Corollaries 3.1-3.4 are just the theorems 3.2, 3.4, 3.6 and 3.8 of* [1] *when our equations are reduced to scalar ones considered in* [1]. *So, our results have generalized the existing ones in literature.*

Case II. *The coefficient matrix* $A(t)$ *of equation (2.1) is a constant matrix.*

We assume that all eigenvalues of A have negative real part, then for any real symmetric positive matrix C, there is a positive matrix P such that $A^TP+PA=-C$. Let $V(t)=x^T(t)Px(t)$, calculating the upper right derivative of $V(t)$ along solutions of (2.1), it follows that

$$V'(t)=-x^T(t)Cx(t)\leq-\alpha_c x^T(t)x(t)\leq-\frac{\alpha_c}{\beta_p}V(t), \tag{3.1}$$

where α_c is the smallest eigenvalue of C and β_p is the largest eigenvalue of P. Further, using the comparison principle, we can get

$$\alpha_p x^T(t)x(t)\leq V(t)\leq V(0)e^{-\alpha_c t/\beta_p},$$

where α_p is the smallest eigenvalue of P. Let $\Phi(t)$ be a fundamental matrix solution of (2.1), from the above, one have

$$|\Phi(t)\Phi^{-1}(s)|=|\Phi(t-s)|\leq\frac{1}{\alpha_p}e^{-\alpha_c(t-s)/\beta_p}.$$

(C_1) There exists a constant $d>0$ such that for any $t\in\mathbb{R}^+$

$$d|A|\leq\alpha_c/\beta_p.$$

(C_2) $\int_0^\infty|A|\,\mathrm{d}t=\infty$.

It is easy to prove that assumption (A_2) holds with $a=1/\alpha_p$ and $b=d$. Further, assumption (A_3) also holds if (C_2) holds.

For the special case that in Equations (1.9)-(1.12), we have the following Corollaries 3.5-3.8, these are direct consequences of Theorems 2.1-2.4.

Corollary 3.5. *Suppose that the coefficient matrix* $A(t)$ *of (2.1) is a constant matrix and the eigenvalues of* A *have negative real part. Assume* (A_1) *and* (C_1)*, and*

$$k\left(1+\frac{1}{d\alpha_p}\right)\int_{-L}^0|P(s)s|\,\mathrm{d}s<1.$$

Then every solution of Equation (1.9) is bounded. In addition, if (C_2) *also holds, then every solution tends to zero as time goes to infinity.*

Corollary 3.6. *Suppose that the coefficient matrix* $A(t)$ *of (2.1) is a constant matrix and the eigenvalues of* A *have negative real part. If assumptions* (A_1) *and* (C_1) *hold and*

$$\sup_{t\geq0}\int_0^t\int_{t-s}^\infty|P(v)|\,\mathrm{d}v\mathrm{d}s=p^*<\infty,\quad\left(1+\frac{1}{d\alpha_p}\right)kp^*<1,$$

then every solution of Equation (1.10) is bounded. In addition, if (C_2) *also holds, then every solution tends to zero as time goes to infinity.*

Corollary 3.7. *Suppose that the coefficient matrix $A(t)$ of (2.1) is a constant matrix and the eigenvalues of A have negative real part. If assumptions (A_1) and (C_1) hold and*

$$\sup_{t\geq 0}\int_0^t\int_{-\infty}^{s-t}|Q(u)|\,\mathrm{d}u\mathrm{d}s = p^* < \infty, \quad \left(1+\frac{1}{d\alpha_p}\right)kp^* < 1,$$

$$\int_{-\infty}^0\int_{-\infty}^{s}|Q(u)|\,\mathrm{d}u\mathrm{d}s \quad \textit{exists},$$

then every solution of equation (1.11) is bounded. In addition, if (C_2) also holds, then every solution tends to zero as times goes to infinity.

Corollary 3.8. *Suppose that the coefficient matrix $A(t)$ of (2.1) is a constant matrix and the eigenvalues of A have negative real part. If assumptions (A_1) and (C_1) hold and*

$$\sup_{t\geq 0}\int_t^{h(t)}|B(u)|\,\mathrm{d}u = p^* < \infty, \quad \left(1+\frac{1}{\alpha_p}+\frac{1}{d\alpha_p}\right)kp^* < 1,$$

then every solution of equation (1.12) is bounded. In addition, if (C_2) also holds, then every solution tends to zero as time goes to infinity.

Case III. *The coefficient matrix $A(t)$ is a row strictly diagonally dominant matrix.*

We introduce the following assumption for this special case.

(D_1) There are positive constants λ and λ_i $(i = 1, 2, \cdots, n)$ such that for any $t \in \mathbb{R}^+$

$$\lambda_i a_{ii}(t) + \sum_{j\neq i}^{n}\lambda_j|a_{ij}(t)| \leq -\lambda. \tag{3.2}$$

Let $(t_0, x_0) \in R_+ \times R^n$ be any initial point and

$$x(t, t_0, x_0) = (x_1(t, t_0, x_0), x_2(t, t_0, x_0), \cdots, x_n(t, t_0, x_0))$$

be the solution of equation (2.1) satisfying initial condition $x(t_0) = x_0$. Then, we have

$$x(t, t_0, x_0) = \Phi(t)\Phi^{-1}(t_0)x_0, \quad t \geq t_0.$$

Further, let

$$V(t) = \max_{1\leq i\leq n}\left\{\frac{|x_i(t, t_0, x_0)|}{\lambda_i}\right\}.$$

For any $t \geq t_0$, we assume $V(t) = |x_i(t, t_0, x_0)|/\lambda_i$. Hence, we have

$$\frac{|x_j(t, t_0, x_0)|}{\lambda_j} \leq \frac{|x_i(t, t_0, x_0)|}{\lambda_i}, \quad j = 1, 2, \cdots, n.$$

Calculating the upper right derivative of $V(t)$, it follows that

$$\begin{aligned} D^+V(t) \le \quad & a_{ii}(t)\frac{|x_i(t,t_0,x_0)|}{\lambda_i} + \sum_{j\ne i}^{n} |a_{ij}(t)|\frac{|x_j(t,t_0,x_0)|}{\lambda_i} \\ = \quad & \frac{1}{\lambda_i}(\lambda_i a_{ii}(t)\frac{|x_i(t,t_0,x_0)|}{\lambda_i} + \sum_{j\ne i}^{n} \lambda_j |a_{ij}(t)|\frac{|x_j(t,t_0,x_0)|}{\lambda_j}) \\ \le \quad & \frac{1}{\lambda_i}(\lambda_i a_{ii}(t)\frac{|x_i(t,t_0,x_0)|}{\lambda_i} + \sum_{j\ne i}^{n} \lambda_j |a_{ij}(t)|\frac{|x_i(t,t_0,x_0)|}{\lambda_i}) \le -\frac{\lambda}{\lambda_{\max}}V(t). \end{aligned}$$

where $\lambda_{\max} = \max\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Integrating from t_0 to t, we obtain

$$V(t) \le V(t_0)e^{-\frac{\lambda}{\lambda_{\max}}(t-t_0)}. \tag{3.3}$$

Since $V(t) \ge \frac{|x(t,t_0,x_0)|}{\lambda_{\max}}$, $V(t_0) \le \frac{|x_0|}{\lambda_{\min}}$, from (3.3) we further obtain for any $t \ge t_0$

$$|x(t,t_0,x_0)| \le \frac{\lambda_{\max}}{\lambda_{\min}}|x_0|e^{-\frac{\lambda}{\lambda_{\max}}(t-t_0)}.$$

Hence, for any $t \ge t_0$ we have

$$|\Phi(t)\Phi^{-1}(t_0)x_0| \le \frac{\lambda_{\max}}{\lambda_{\min}}|x_0|e^{-\frac{\lambda}{\lambda_{\max}}(t-t_0)}.$$

Since $\Phi(t)\Phi^{-1}(t_0) = (\Phi(t)\Phi^{-1}(t_0)e_1, \Phi(t)\Phi^{-1}(t_0)e_2, \cdots, \Phi(t)\Phi^{-1}(t_0)e_n)$, where $e_i = (0,\cdots,0,1,0,\cdots,0)$ $(i = 1,2,\cdots,n)$, we further obtain

$$|\Phi(t)\Phi^{-1}(t_0)| \le \max_{1\le i\le n} |\Phi(t)\Phi^{-1}(t_0)e_i| \le \frac{\lambda_{\max}}{\lambda_{\min}}e^{-\frac{\lambda}{\lambda_{\max}}(t-t_0)}.$$

Therefore, we finally obtain for any $t, s \in R_+$ and $t \ge s$

$$|\Phi(t)\Phi^{-1}(s)| \le \frac{\lambda_{\max}}{\lambda_{\min}}e^{-\frac{\lambda}{\lambda_{\max}}(t-s)}. \tag{3.4}$$

Further, we introduce the assumption

(D_2) There exists a constant $d > 0$ such that for any $t \in \mathbb{R}^+$, $d|A(t)| \le \frac{\lambda}{\lambda_{\max}}$.

Then, from (3.4) we can obtain that assumption (A_2) holds with $a = \lambda_{\max}/\lambda_{\min}$ and $b = d$, and assumption (A_3) also holds due to (3.2).

For the special case that in Equations (1.9)-(1.12), we have the following Corollaries 3.9-3.12, these are direct consequences of Theorems 2.1-2.4.

Corollary 3.9. *Assume (A_1), (D_1) and (D_2), and*

$$k\left(1 + \frac{\lambda_{\max}}{b\lambda_{\min}}\right)\int_{-L}^{0} |P(s)s|\,\mathrm{d}s < 1.$$

Then every solution of Equation (1.9) is bounded and every solution tends to zero as time goes to infinity.

Corollary 3.10. *If assumptions (A_1), (D_1) and (D_2) hold and*

$$\sup_{t\ge 0}\int_0^t \int_{t-s}^{\infty} |P(v)|\,\mathrm{d}v\mathrm{d}s = p^* < \infty, \qquad \left(1 + \frac{\lambda_{\max}}{b\lambda_{\min}}\right)kp^* < 1,$$

then every solution of Equation (1.10) is bounded and every solution tends to zero as time goes to infinity.

Corollary 3.11. *If assumptions (A_1), (D_1) and (D_2) hold and*

$$\sup_{t\geq 0}\int_0^t\int_{-\infty}^{s-t}|Q(u)|\,\mathrm{d}u\mathrm{d}s = p^* < \infty, \quad \left(1+\frac{\lambda_{\max}}{b\lambda_{\min}}\right)kp^* < 1,$$

$$\int_{-\infty}^0\int_{-\infty}^s|Q(u)|\,\mathrm{d}u\mathrm{d}s \quad \textit{exists},$$

then every solution of equation (1.10) is bounded and every solution tends to zero as times goes to infinity.

Corollary 3.12. *If assumptions (A_1), (D_1) and (D_2) hold and*

$$\sup_{t\geq 0}\int_t^{h(t)}|B(u)|\,\mathrm{d}u = p^* < \infty, \quad \left(1+\frac{\lambda_{\max}}{\lambda_{\min}}+\frac{\lambda_{\max}}{b\lambda_{\min}}\right)kp^* < 1,$$

then every solution of equation (1.10) is bounded and every solution tends to zero as time goes to infinity.

Case IV. *The coefficient matrix $A(t)$ of equation (2.1) is a lower triangular matrix.*

In this case, for the convenience we only give the discussion for $n = 2$. Using the formula of the variation of constants, we can get

$$\Phi(t) = \begin{pmatrix} e^{\int_0^t a_{11}(u)\,\mathrm{d}u} & 0 \\ e^{\int_0^t a_{22}(u)\,\mathrm{d}u}\displaystyle\int_0^t a_{21}(u)e^{-\int_0^u[a_{11}(v)+a_{22}(v)]\,\mathrm{d}v} & e^{\int_0^t a_{22}(u)\,\mathrm{d}u} \end{pmatrix}$$

and hence,

$$\Phi(t)\Phi^{-1}(s) = \begin{pmatrix} e^{\int_s^t a_{11}(u)\,\mathrm{d}u} & 0 \\ \Psi(s,t)e^{\int_s^t a_{22}(u)\,\mathrm{d}u}e^{\int_0^s [a_{22}(u)-a_{11}(u)]\,\mathrm{d}u} & e^{\int_s^t a_{22}(u)\,\mathrm{d}u} \end{pmatrix},$$

where $\Psi(s,t) = \int_s^t a_{21}(u)e^{-\int_s^u[a_{11}(v)+a_{22}(v)]\,\mathrm{d}v}\,\mathrm{d}u$.

We introduce the following assumptions for this special case.

(E_1) There exists a constant $d > 0$ such that for any $t \in \mathbb{R}^+$

$$d\max_{1\leq i,j\leq 2}\{|a_{ij}(t)|\} \leq \min\{|a_{11}(t)|, |a_{22}(t)|\}.$$

(E_2) $\int_0^\infty |A(t)|\,\mathrm{d}t = \infty$.

Similarly, we can easily show that if assumption (E_1) holds, then assumption (A_2) also holds with

$$a = \max\left\{1, \max_{0<s<t} e^{\int_0^s[a_{22}(u)-a_{11}(u)]\,\mathrm{d}u}\int_s^t a_{21}(u)e^{-\int_s^u[a_{11}(v)+a_{22}(v)]\,\mathrm{d}v}\,\mathrm{d}u\right\}$$

and $b = d$, and if assumption (E_2) holds, then assumption (A_3) also holds.

Directly applying Theorems 2.1-2.4 we can obtained the boundedness and global attractivity of (1.9)-(1.12).

Remark. *Obviously, the results of this special case can be generalized to the more general case that the coefficient matrix $A(t) = (a_{ij}(t))_{n\times n}$ of (2.1) is lower triangular matrix and $a_{ii}(t) < 0$ $(i = 1, 2, \cdots, n)$, where $n \geq 3$.*

Remark. *Similarly, if $A(t)$ is upper triangular matrix, we also obtained the boundedness and global attractivity of (1.9)-(1.12).*

Case V. *The coefficient matrix $A(t)$ of equation (2.1) is periodic matrix.*

If the matrix $A(t)$ is a periodic matrix function with minimum positive period ω, it follows the Floquet's Theory there is a differentiable ω periodic nonsingular matrix function $P(t)$ and a constant matrix R such that $\Phi(t) = P(t)e^{Rt}$, where $\Phi(t)$ be a fundamental matrix solution of (2.1). Further, let $x = P(t)y$, then Equation (2.1) equivalent to the following linear differential system of equations with constant coefficients $y'(t) = Ry(t)$. Therefore, similar to the Case II, we also obtained the boundedness and global attractivity of (1.9)-(1.12) if the eigenvalues of R have negative real part.

Acknowledgment. This work was supported by the Scientific Research Programmes of Colleges in Xinjiang (Grant Nos. XJEDU2011S08), the Natural Science Foundation of Xinjiang (Grant Nos. 2011211B08), the National Natural Science Foundation of China (Grant Nos. 10961022, 11001235), the China Postdoctoral Science Foundation (Grant Nos. 20110491750) and the Natural Science Foundation of Xinjiang University (Grant Nos. BS100104). We also would like to thank anonymous referee for their careful reading the original manuscript, and giving many valuable comments and suggestions that greatly improve the presentation of this paper.

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Received: April 25, 2012; Accepted: June 31, 2012.