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AN APPLICATION OF A FIXED POINT THEOREM FOR NONEXPANSIVE OPERATORS

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Abstract. In this note, we present an application of a recent fixed point theorem by Ricceri to a two-point boundary value problem.
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Very recently, in [1], B. Ricceri established the following fixed point theorem:
Theorem 1. *Let X be a real Hilbert space and let $J : X \to \mathbf{R}$ be a C^1 functional whose Gâteaux derivative is nonexpansive. Assume also that there exists $r > 0$ such that*

$$\sup_{\|x\|=r} J(x) < r^2 \sup_{\|x\|>r} \frac{J(x)}{\|x\|^2} . \qquad (*)$$

Under such hypotheses, there exists a fixed point of J' whose norm is less than r.

The aim of the present short note is to give an application of this result to the two-point boundary value problem

$$\begin{cases} -u'' = f(u) & \text{in} \quad [0,1] \\ 0 < u < \rho & \text{in} \quad]0,1[\\ u(0) = u(1) = 0 \end{cases} \qquad (P)$$

where $\rho > 0$ and $f : [0, \rho] \to [0, +\infty[$.

Before stating our main result, we introduce the following notation. For each $\rho > 0$ and $a \geq 0$, we denote by $\mathcal{A}_{\rho,a}$ the family of all Lipschitzian functions $g : [\rho, +\infty[\to [0, +\infty[$, with Lipschitz constant not greater than π^2, such that $g(\rho) = a$. Moreover, we put

$$\theta_{\rho,a} = \rho^2 \sup \left\{ \frac{\int_\rho^\sigma g(t)dt}{4\sigma^2 - \rho^2} : (g, \sigma) \in \mathcal{A}_{\rho,a} \times]\rho, +\infty[\right\} .$$

Our main result reads as follows:

Theorem 2. *Let $\rho > 0$ and let $f : [0, \rho] \to [0, +\infty[$ be a Lipschitzian function, with Lipschitz constant not greater than π^2 and with $f(0) > 0$. Assume also that*

$$\int_0^\rho f(t)dt < \theta_{\rho, f(\rho)} \ . \tag{1}$$

Under such hypotheses, problem (P) has a classical solution u such that

$$\int_0^1 |u'(t)|^2 dt < 4\rho^2 \ .$$

Proof. Let $H_0^1(0,1)$ be the usual Sobolev space with the scalar product

$$\langle u, v \rangle = \int_0^1 u'(t)v'(t)dt$$

and the induced norm

$$\|u\| = \left(\int_0^1 |u'(t)|^2 dt \right)^{\frac{1}{2}} \ .$$

Let us recall that

$$\max_{[0,1]} |u| \le \frac{1}{2}\|u\| \tag{2}$$

and

$$\left(\int_0^1 |u(t)|^2 dt \right)^{\frac{1}{2}} \le \frac{1}{\pi}\|u\| \tag{3}$$

for all $u \in H_0^1(0,1)$. In view of (1), there exist $g \in \mathcal{A}_{\rho, f(\rho)}$ and $\sigma > \rho$ such that

$$\int_0^\rho f(t)dt < \rho^2 \frac{\int_\rho^\sigma g(t)dt}{4\sigma^2 - \rho^2} \ . \tag{4}$$

Now, consider the function $\varphi : \mathbf{R} \to [0, +\infty[$ defined by

$$\varphi(\xi) = \begin{cases} f(0) & \text{if} \quad \xi < 0 \\ f(\xi) & \text{if} \quad \xi \in [0, \rho] \\ g(\xi) & \text{if} \quad \xi > \rho \end{cases}$$

Clearly, observing that $g(\rho) = f(\rho)$, the function φ is Lipschitzian in $\mathbf{R}$ with Lipschitz constant not greater than π^2. Now, put

$$J(u) = \int_0^1 \Phi(u(t))dt$$

for all $u \in H_0^1(0,1)$, where $\Phi(\xi) = \int_0^\xi \varphi(t)dt$. The functional J is C^1 with derivative given by

$$\langle J'(u), v \rangle = \int_0^1 \varphi(u(t))v(t)dt$$

for all $u, v \in H_0^1(0,1)$. So, the fixed points of J' are exactly the classical solutions of the problem

$$\begin{cases} -u'' = \varphi(u) & \text{in } [0,1] \\ u(0) = u(1) = 0 \end{cases} \tag{P_1}$$

Since $f(0) > 0$ and φ is non-negative, each classical solution of (P_1) is positive in $]0,1[$. On the other hand, if $u \in H_0^1(0,1)$ is a fixed point of J' with $\|u\| < 2\rho$, then, in view of (2), we have

$$u(t) < \rho$$

for all $t \in [0,1]$, and so u turns out to be a solution of problem (P). As a consequence, to reach our conclusion it is enough to show that it is possible to apply Theorem 1 to the functional J (with $X = H_0^1(0,1)$ of course), with $r = 2\rho$. In this connection, let us show that J' is non-expansive. So, let $u, v \in H_0^1(0,1)$. We have

$$\|J'(u) - J'(v)\| = \sup_{\|w\|=1} |\langle J'(u) - J'(v), w\rangle| \ . \tag{5}$$

Fix $w \in H_0^1(0,1)$ with $\|w\| = 1$. Taking (3) into account, we have

$$|\langle J'(u) - J'(v), w\rangle| \leq \int_0^1 |\varphi(u(t)) - \varphi(v(t))||w(t)|dt \leq \pi^2 \int_0^1 |u(t) - v(t)||w(t)|dt$$

$$\leq \pi^2 \left(\int_0^1 |u(t) - v(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_0^1 |w(t)|^2 dt\right)^{\frac{1}{2}} \leq \pi^2 \frac{1}{\pi}\|u - v\| \frac{1}{\pi}\|w\| = \|u - v\| \ .$$

Hence, from (5), we get

$$\|J'(u) - J'(v)\| \leq \|u - v\| \ ,$$

as claimed. The final step consists in checking that condition $(*)$ holds. To this end, consider the function $\hat{u} : [0,1] \to [0, +\infty[$ defined by

$$\hat{u}(t) = \begin{cases} 4\sigma t & \text{if } \ t \in \left[0, \frac{1}{4}\right[\\ \sigma & \text{if } \ t \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ 4\sigma(1-t) & \text{if } \ t \in \left]\frac{3}{4}, 1\right] \end{cases}$$

Clearly, $\hat{u} \in H_0^1(0,1)$ and

$$\|\hat{u}\| = \sqrt{8}\sigma \ . \tag{6}$$

Since Φ is non-decreasing (recall that $\varphi \geq 0$), in view of (2) again, we have

$$\sup_{\|u\|=2\rho} J(u) \leq \Phi(\rho) \ . \tag{7}$$

Further, by the positivity of Φ in $]0, +\infty[$, we clearly have

$$J(\hat{u}) > \frac{\Phi(\sigma)}{2} \ . \tag{8}$$

From (4), we get

$$\left(1 - \frac{\rho^2}{4\sigma^2}\right) \int_0^\rho f(t)dt < \frac{\rho^2}{4\sigma^2} \int_\rho^\sigma g(t)dt$$

and so

$$\Phi(\rho) < \frac{\rho^2}{4\sigma^2} \Phi(\sigma) \ . \tag{9}$$

Finally, from (6) − (9), we obtain

$$\sup_{\|u\|=2\rho} J(u) < 4\rho^2 \frac{J(\hat{u})}{\|\hat{u}\|^2}$$

which gives ($*$). The proof is complete. □

References

[1] B. Ricceri, *Another fixed point theorem for nonexpansive potential operators*, Studia Math., 211 (2012), 147-151.

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