Fixed Point Theory, 15(2014), No. 1, 199-212
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED POINT THEOREMS AND APPLICATIONS IN THEORY OF GAMES

MONICA PATRICHE

University of Bucharest
Faculty of Mathematics and Computer Science
14 Academiei Street, 010014 Bucharest, Romania
E-mail: monica.patriche@yahoo.com

Abstract. We introduce the notions of weakly *-concave and weakly naturally quasi-concave correspondence and prove fixed point theorems and continuous selection theorems for these kind of correspondences. As applications in the game theory, by using a tehnique based on a continuous selection, we establish new existence results for the equilibrium of the abstract economies. The constraint correspondences are weakly naturally quasi-concave. We show that the equilibrium exists without continuity assumptions.

Key Words and Phrases: weakly naturally quasi-concave correspondence, fixed point theorem, continuous selection, abstract economy, equilibrium.
2010 Mathematics Subject Classification: 91B52, 91B50, 91A80, 47H10.

1. Introduction

It is known that the theory of correspondences has very widely developed and produced many applications, especially during the last few decades. Most of these applications concern fixed point theory and game theory. The fixed point theorems are closely connected with convexity. A considerable number of papers devotes to correspondences on nonconvex and noncompact domains (see e.g. [16], [17], [18]) or to correspondences without convex values and continuity ([5]).

The aim of this paper is to introduce the notions of weakly *-concave and weakly naturally quasi-concave correspondence and prove fixed point theorems and continuous selection theorems for these kind of correspondences. We also define the correspondences with WNQS and e-WNQS property.

The applications concern the equilibrium theory: we establish new existence results for the equilibrium of the abstract economies. The constraint correspondences are weakly concave-like or have the WNQS, respectively the e-WNQS property.

For the reader's convenience, we review the main results in the equilibrium theory, emphasizing that most authors have studied the existence of equilibrium for abstract economies with preferences represented as correspondences which have continuity properties. We mention here the results obtained by W. Shafer and H. Sonnenschein [14], which concern economies with finite dimensional commodity space and preference correspondences having an open graph. N. C. Yannelis and N. D. Prahbakar [21] used selection theorems and fixed-point theorems for correspondences with open lower sections defined on infinite dimensional strategy spaces. Some authors developed the theory of continuous selections of correspondences and gave numerous applications in game theory. Michael's selection theorem [11] is well-known and basic in many applications. In [3,4], F. Browder firstly used a continuous selection theorem to prove Fan-Browder fixed point theorem. Later, N. C. Yannelis and N. D. Prabhakar [21], H. Ben-El-Mechaiekh [1], X. Ding, W. Kim and K.Tan [6], C.Horvath [9], T. Husain and E. Taradfar [10], S.Park [12],[13], X. Wu [19], X. Wu and S. Shen [20], Z. Yu and L. Lin [22] and many others established several continuous selection theorems with applications.

In this paper, we show that an equilibrium for an abstract economy exists without continuity assumptions. By using a tehnique based on a continuous selection, we prove the new equilibrium existence theorem for an abstract economy.

The paper is organized in the following way: Section 2 contains preliminaries and notations. The fixed point and the selection theorem are presented in Section 3. The equilibrium theorems are stated in Section 4.

2. Preliminaries and notations

Throughout this paper, we shall use the following notations and definitions:

Let A be a subset of a topological space X.

1. 2^A denotes the family of all subsets of A.
2. cl A denotes the closure of A in X.
3. If A is a subset of a vector space, co$A$ denotes the convex hull of A.
4. If F, $T : A \to 2^X$ are correspondences, then co$T$, cl T, $T \cap F : A \to 2^X$ are correspondences defined by $(\mathrm{co}T)(x) = \mathrm{co}T(x)$, $(\mathrm{cl}T)(x) = \mathrm{cl}T(x)$ and $(T \cap F)(x) = T(x) \cap F(x)$ for each $x \in A$, respectively.
5. The graph of $T : X \to 2^Y$ is the set $\mathrm{Gr}(T) = \{(x, y) \in X \times Y \mid y \in T(x)\}$
6. The correspondence $\overline{T}$ is defined by $\overline{T}(x) = \{y \in Y : (x, y) \in \mathrm{cl}_{X\times Y}\mathrm{Gr}T\}$ (the set $\mathrm{cl}_{X\times Y}\mathrm{Gr}(T)$ is called the adherence of the graph of T).

It is easy to see that $\mathrm{cl}T(x) \subset \overline{T}(x)$ for each $x \in X$.

Lemma 2.1. (see [23]) *Let X be a topological space, Y be a non-empty subset of a topological vector space E, ß be a base of the neighborhoods of 0 in E and $A : X \to 2^Y$. For each $V \in$ß, let $A_V : X \to 2^Y$ be defined by $A_V(x) = (A(x) + V) \cap Y$ for each $x \in X$. If $\widehat{x} \in X$ and $\widehat{y} \in Y$ are such that $\widehat{y} \in \cap_{V\in ß}\overline{A_V}(\widehat{x})$, then $\widehat{y} \in \overline{A}(\widehat{x})$.*

Definition 2.2. Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence

1. T is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$.

2. T is said to be *lower semicontinuous* if for each x$\in X$ and each open set V in Y with $T(x)\cap V\neq\emptyset$, there exists an open neighborhood U of x in X such that $T(y)\cap V\neq\emptyset$ for each $y\in U$.

3. T is said to have *open lower sections* if $T^{-1}(y):=\{x\in X : y\in T(x)\}$ is open in X for each $y\in Y$.

Lemma 2.3. (see [24]).*Let X be a topological space, Y be a topological linear space, and let $A : X\to 2^Y$ be an upper semicontinuous correspondence with compact values. Assume that the sets $C\subset Y$ and $K\subset Y$ are closed and respectively compact. Then $T : X\to 2^Y$ defined by $T(x)=(A(x)+C)\cap K$ for all $x\in X$ is upper semicontinuous.*

We present the following types of generalized convex functions and correspondences.

Definition 2.4. (see [15]) Let X be a convex set in a real vector space, and let Z be an ordered t.v.s, with a pointed convex cone C. A vector-valued $f : X\to Z$ is said to be *natural quasi $C-$convex* on X if $f(\lambda x_1+(1-\lambda)x_2)\in$co$\{f(x_1), f(x_2)\}-C$ for every $x_1, x_2\in X$ and $\lambda\in[0,1]$. This condition is equivalent with the following condition: there exists $\mu\in[0,1]$ such that $f(\lambda x_1+(1-\lambda)x_2)\leq_C \mu f(x_1)+(1-\mu)f(x_2)$, where $x\leq_C y\Leftrightarrow y-x\in C$.

A vector-valued function f is said to be *natural quasi $C-$concave* on X if $-f$ is natural quasi $C-$convex on X.

Definition 2.5. (see [26]) Let E_1, E_2 and Z be real Hausdorff topological vector spaces, $C\subset Z$ be a closed convex pointed cone with int$S\neq\emptyset$; let X be a nonempty convex subset of E_1, $T : X\to 2^Z$ be a correspondence. T is said to be *naturally C-quasi-concave* on X, if for any x_1,$x_2\in X$ and $\lambda\in[0,1]$, co$(T(x_1), T(x_2))\subset T(\lambda x_1+(1-\lambda)x_2)-C$.

Let $\Delta_{n-1}=\left\{(\lambda_1,\lambda_2,...,\lambda_n)\in\mathbb{R}^n : \sum_{i=1}^{n}\lambda_i=1 \text{ and } \lambda_i\geqslant 0, i=1,2,...,n\right\}$ be the standard (n-1)-dimensional simplex in $\mathbb{R}^n$.

Definition 2.6. (see [5]) Let X be a non-empty convex subset of a topological vector space E and Y be a non-empty subset of E. The correspondence $T : X\longrightarrow 2^Y$ is said to have *weakly convex graph* (in short, it is a WCG correspondence) if for each finite set $\{x_1, x_2, ..., x_n\}\subset X$, there exists $y_i\in T(x_i)$, $(i=1,2,...,n)$ such that

$$\text{co}(\{(x_1,y_1),(x_2,y_2),...,(x_n,y_n)\})\subset \text{Gr}(T) \tag{2.1}$$

The relation (2.1) is equivalent to

$$\sum_{i=1}^{n}\lambda_i y_i\in T(\sum_{i=1}^{n}\lambda_i x_i)\ (\forall(\lambda_1,\lambda_2,...,\lambda_n)\in\Delta_{n-1}). \tag{2.2}$$

We introduce the concept of weakly naturally quasi-concave correspondence.

Definition 2.7. Let X be a nonempty convex subset of a topological vector space E and Y be a nonempty subset of a topological vector space F. The correspondence $T : X\longrightarrow 2^Y$ is said to be *weakly naturally quasi-concave (WNQ)* if for each n and for each finite set $\{x_1, x_2, ..., x_n\}\subset X$, there exists $y_i\in T(x_i)$, $(i=1,2,...,n)$ and $g=(g_1, g_2, ..., g_n) : \Delta_{n-1}\to\Delta_{n-1}$ a function with g_i continuous, $g_i(1)=1$ and

$g_i(0) = 0$ for each $i = 1, 2, ..., n$, such that for every $(\lambda_1, \lambda_2, ..., \lambda_n) \in \Delta_{n-1}$, there exists $y = \sum\limits_{i=1}^{n} g_i(\lambda_i) y_i \in T(\sum\limits_{i=1}^{n} \lambda_i x_i)$.

Remark 2.8. If $g_i(\lambda_i) = \lambda_i$ for each $i \in (1, 2, ..., n)$ and $(\lambda_1, \lambda_2, ..., \lambda_n) \in \Delta_{n-1}$, we get a correspondence with weakly convex graph, as it is defined by Ding and He Yiran in [5]. In the same time, the weakly naturally quasi-concavity is a weakening of the notion of naturally C-quasi-concavity with $C = \{0\}$.

Remark 2.9. If T is a single valued mapping, then it must be natural quasi C-concave for $C = \{0\}$.

Example 2.10. Let $T : [0, 4] \to 2^{[-2,2]}$ be defined by $T(x) = \begin{cases} [0, 2] \text{ if } x \in [0, 2); \\ [-2, 0] \quad \text{if} \quad x = 2; \\ (0, 2] \text{ if } x \in (2, 4]. \end{cases}$

T is neither upper semicontinuous, nor lower semicontinuous in 2. T also has not weakly convex graph, since if we consider $n = 2$, $x_1 = 1$ and $x_2 = 3$, we have that $\text{co}\{(1, y_1), (3, y_2)\} \not\subseteq \text{Gr}T$ for every $y_1 \in T(x_1), y_2 \in T(x_2)$.

We shall prove that T is a weakly naturally quasi-concave correspondence.

1) Let's consider first $n = 2$.

a) If $x_1, x_2 \in [0, 2)$ and $x_1, x_2 \in (2, 4]$, there exists $y_1 = 2 \in T(x_1)$, $y_2 = 2 \in T(x_2)$ and $g_i(\lambda_i) = \lambda_i$, $i = 1, 2$ such that for each (λ_1, λ_2) with the property that $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, there exists $y = \sum\limits_{i=1}^{2} g_i(\lambda_i) y_i \in T(\sum\limits_{i=1}^{2} \lambda_i x_i)$.

b) If $x_1 \in [0, 2)$ and $x_2 \in (2, 4]$, there exists $\lambda_1^* \neq 0$ such that $\lambda_1^* x_1 + (1 - \lambda_2^*) x_2 = 2$.

Let's consider $g_i : [0, 1] \to [0, 1]$ continuous functions such that $g_i(1) = 1$, $g_i(0) = 0$ for each $i = 1, 2$ and $g_1(\lambda_1) + g_2(\lambda_2) = 1$ if $\lambda_1 + \lambda_2 = 1$, defined by

$$g_1(\lambda_1) = \begin{cases} \frac{1}{\lambda_1^*} \lambda_1 \text{ if } \lambda_1 \in [0, \lambda_1^*); \\ 1 \quad \text{if} \quad \lambda_1 \in [\lambda_1^*, 1] \end{cases}$$

and

$$g_2(\lambda_2) = \begin{cases} 0 \qquad \text{if} \qquad \lambda_2 \in [0, 1 - \lambda_1^*]; \\ 1 - \frac{1}{\lambda_1^*} + \frac{1}{\lambda_1^*} \lambda_2 \text{ if } \lambda_2 \in (1 - \lambda_1^*, 1]. \end{cases}$$

There exists $y_1 = 0$ and $y_2 = 2$ such that

b1) for $\lambda_1 \in [0, \lambda_1^*)$ and $\lambda_2 = 1 - \lambda_1$, $x = \lambda_1 x_1 + \lambda_2 x_2 \in (2, x_2]$, then $T(x) = (0, 2]$ and

$$y = g_1(\lambda_1) y_1 + g_2(\lambda_2) y_2 = \frac{1}{\lambda_1^*} \lambda_1 y_1 + (1 - \frac{1}{\lambda_1^*} \lambda_1) y_2$$

$$= (1 - \frac{1}{\lambda_1^*} \lambda_1) 2 \in (0, 2] = T(\lambda_1 x_1 + \lambda_2 x_2);$$

b2) for $\lambda_1 \in (\lambda_1^*, 1]$, and $\lambda_2 = 1 - \lambda_1$, $x = \lambda_1 x_1 + \lambda_2 x_2 \in [x_1, 2)$, then $T(x) = [0, 2]$ and

$$y = g_1(\lambda_1) y_1 + g_2(\lambda_2) y_2 = 1 \times 0 + 0 \times 2 = 0 \in\in T(\lambda_1 x_1 + \lambda_2 x_2);$$

b3) If $\lambda_1 = \lambda_1^*$, $\lambda_2 = 1 - \lambda_1^*$, $x = \lambda_1 x_1 + \lambda_2 x_2 = 2$, then $T(x) = [-2, 0]$ and

$$y = g_1(\lambda_1) y_1 + g_2(\lambda_2) y_2 = 1 \times 0 + 0 \times 2 = 0 \in T(2);$$

c) If $x_1 \in [0,2)$ and $x_2 = 2$, there exists $y_1 = 2$, $y_2 = 0$ and the continuous functions $g_i : [0,1] \to [0,1]$ with $g_i(1) = 1$, $g_i(0) = 0$ for each $i = 1,2$ and $g_1(\lambda_1) + g_2(\lambda_2) = 1$ if $\lambda_1 + \lambda_2 = 1$ such that

c1) for $\lambda_1 \in (0,1]$ and $\lambda_2 = 1 - \lambda_1$, $x = \lambda_1 x_1 + \lambda_2 x_2 \in [x_1, x_2)$, then $T(x) = [-2,0]$ and

$$y = g_1(\lambda_1)y_1 + g_2(\lambda_2)y_2 = g_1(\lambda_1) \times 2 + g_2(\lambda_2) \times 0 = g_1(\lambda_1) \times 2 \in T(x);$$

c2) for $\lambda_1 = 0$ and $\lambda_2 = 1$, $x = 2$, then $T(2) = [-2.0]$ and

$$y = g_1(0)y_1 + g_2(1)y_2 = 0 \times 2 + 1 \times 0 = 0 \in T(2);$$

d) If $x_1 = 2$ and $x_2 \in (2,4]$, there exists $y_1 = 0$, $y_2 = 2$ and the continuous functions $g_i : [0,1] \to [0,1]$ with $g_i(1) = 1$, $g_i(0) = 0$ for each $i = 1,2$ and $g_1(\lambda_1) + g_2(\lambda_2) = 1$ if $\lambda_1 + \lambda_2 = 1$ such that

d1) for $\lambda_1 = 1$ and $\lambda_2 = 0$, $x = 2$, then $T(2) = [-2.0]$ and

$$y = g_1(1)y_1 + g_2(0)y_2 = 1 \times 0 + 0 \times 2 = 0 \in T(2);$$

d2) for $\lambda_1 \in [0,1)$ and $\lambda_2 = 1 - \lambda_1$, $x = \lambda_1 x_1 + \lambda_2 x_2 \in (x_1, x_2]$, then $T(x) = (0,2]$ and

$$y = g_1(\lambda_1)y_1 + g_2(\lambda_2)y_2 = g_1(\lambda_1) \times 0 + g_2(\lambda_2) \times 2 = g_2(\lambda_2) \times 2 \in (0,2] = T(x).$$

2) The case $n > 2$ can be reduced to the case 1).

Now, we introduce the following definitions.

Let I be an index set. For each $i \in I$, let X_i be a non-empty convex subset of a topological linear space E_i and denote $X = \prod_{i \in I} X_i$.

Definition 2.11. Let K_i be a subset of X. The correspondence $A_i : X \to 2^{X_i}$ is said to have the WNQS *property* on K_i, if there is a weakly naturally quasi-concave correspondence $T_i : K_i \to 2^{X_i}$ such that $x_i \notin T_i(x)$ and $T_i(x) \subset A_i(x)$ for all $x \in K_i$.

Definition 2.12. Let K_i be a subset of X. The correspondence $A_i : X \to 2^{X_i}$ is said to have the e-WNQS *property* on K_i if for each convex neighborhood V of 0 in X_i, there is a weakly naturally quasi-concave correspondence $T_i^V : K_i \to 2^{X_i}$ such that $x_i \notin T_i^V(x)$ and $T_i^V(x) \subset A_i(x) + V$ for all $x \in K_i$.

Definition 2.13. Let X be a nonempty convex subset of a topological vector space E and Y be a nonempty subset of a topological vector space F. The correspondence $T : X \longrightarrow 2^Y$ is said to be *weakly *-concave* if for each n and for each finite set $\{x_1, x_2, ..., x_n\} \subset X$, there exists $y_i \in T(x_i)$, $(i = 1,2,...,n)$, such that for every $(\lambda_1, \lambda_2, ..., \lambda_n) \in \Delta_{n-1}$, $\sum_{i=1}^{n} \lambda_i y_i \subset T(x)$, for each $x \in X$.

To prove our theorems of equilibrium existence, we need the following:

Theorem 2.14. (Wu's fixed point theorem [19]) *Let I be an index set. For each $i \in I$, let X_i be a nonempty convex subset of a Hausdorff locally convex topological vector space E_i, D_i a non-empty compact metrizable subset of X_i and $S_i, T_i : X := \prod_{i \in I} X_i \to 2^{D_i}$ two correspondences with the following conditions:*

(i) *for each $x \in X$, $\mathrm{clco}S_i(x) \subset T_i(x)$ and $S_i(x) \neq \emptyset$,*

(ii) S_i *is lower semicontinuous.*

Then, there exists a point $\overline{x} = \prod_{i\in I} x_i \in D = \prod_{i\in I} D_i$ *such that* $\overline{x}_i \in T_i(\overline{x})$ *for each* $i \in I$.

The extension of Kakutani's theorem on locally convex spaces is due to Ky Fan.

Theorem 2.15. (Ky-Fan, [7]) *Let* Y *be a locally convex space,* $X \subset Y$ *be a compact and convex subset and* $T : X \to 2^X$ *be an upper semicontinuous correspondence with non-empty compact convex values. Then,* T *has a fixed point.*

For the case when X is not compact, Himmelberg got the following result.

Theorem 2.16. (Himmelberg, [8]) *Let* X *be a non-empty convex subset of a separated locally convex space* Y*. Let* $T : X \to 2^X$ *be an upper semicontinuous correspondence such that* $T(x)$ *is closed and convex for each* $x \in X$*, and* $T(X)$ *is contained in a compact subset* C *of* X*. Then,* T *has a fixed point.*

3. Fixed point theorems

We formulate the following fixed point theorem for weakly naturally quasi-concave correspondences.

Theorem 3.1. (selection theorem) *Let* Y *be a non-empty subset of a topological vector space* E *and* K *be a* $(n-1)$*- dimensional simplex in a topological vector space* F*. Let* $T : K \to 2^Y$ *be a weakly naturally quasi-concave correspondence. Then,* T *has a continuous selection on* K*.*

Proof. Let $a_1, a_2, ..., a_n$ be the vertices of K. Since T is weakly naturally quasi-concave, there exist $b_i \in T(a_i)$, $(i = 1, 2, ..., n)$ and $g = (g_1, g_2, ..., g_n) : \Delta_{n-1} \to \Delta_{n-1}$ a function with g_i continuous, $g_i(1) = 1$ and $g_i(0) = 0$ for each $i = 1, 2, ...n$, such that for every $(\lambda_1, \lambda_2, ..., \lambda_n) \in \Delta_{n-1}$, there exists $y \in T(\sum_{i=1}^{n} \lambda_i a_i)$ with $y = \sum_{i=1}^{n} g_i(\lambda_i) y_i$.

Since K is a $(n-1)$-dimensional simplex with the vertices $a_1, ..., a_n$, there exists unique continuous functions $\lambda_i : K \to \mathbb{R}$, $i = 1, 2, ..., n$ such that for each $x \in K$, we have $(\lambda_1(x), \lambda_2(x), ..., \lambda_n(x)) \in \Delta_{n-1}$ and $x = \sum_{i=1}^{n} \lambda_i(x) a_i$.

Let's define $f : K \to 2^Y$ by $f(a_i) = b_i$ $(i = 1, ..., n)$ and

$$f(\sum_{i=1}^{n} \lambda_i a_i) = \sum_{i=1}^{n} g_i(\lambda_i) b_i \in T(x).$$

We show that f is continuous.

Let $(x_m)_{m\in N}$ be a sequence which converges to $x_0 \in K$, where $x_m = \sum_{i=1}^{n} \lambda_i(x_m) a_i$ and $x_0 = \sum_{i=1}^{n} \lambda_i(x_0) a_i$. By the continuity of λ_i, it follows that for each $i = 1, 2, ..., n$, $\lambda_i(x_m) \to \lambda_i(x_0)$ as $m \to \infty$. Since $g_1, ..., g_n$ are continuous, we have $g_i(\lambda_i(x_m)) \to g_i(\lambda_i(x_0))$ as $m \to \infty$. Hence $f(x_m) \to f(x_0)$ as $m \to \infty$, i.e. f is continuous. □

Theorem 3.2. *Let* Y *be a non-empty subset of a topological vector space* E *and* K *be a* $(n-1)$*- dimensional simplex in* E*. Let* $T : K \to 2^Y$ *be an weakly naturally* *quasi-concave correspondence and $s : Y \to K$ be a continuous function. Then, there exists $x^* \in K$ such that $x^* \in s \circ T(x^*)$.*

Proof. By Theorem 3.1, T has a continuous selection theorem on K. Since $s : Y \to K$ is continuous, we obtain that $s \circ f : K \to K$ is continuous. By Brouwer's fixed point theorem, there exists a point $x^* \in K$ such that $x^* = s \circ f(x^*)$ and then, $x^* \in s \circ T(x^*)$. □

Theorem 3.3. (selection theorem). *Let Y be a non-empty subset of a topological vector space E and K be a $(n-1)$- dimensional simplex in a topological vector space F. Let $T : K \to 2^Y$ be a weakly *-concave correspondence. Then, T has a continuous selection on K.*

Proof. Let $a_1, a_2, ..., a_n$ be the vertices of K. Since T is weakly *-concave, there exist $b_i \in T(a_i)$, $(i = 1, 2, ..., n)$ such that for every $(\lambda_1, \lambda_2, ..., \lambda_n) \in \Delta_{n-1}$, $\sum_{i=1}^{n} \lambda_i b_i \subset T(x)$, for each $x \in X$.

Since K is a $(n-1)$-dimensional simplex with the vertices $a_1, ..., a_n$, there exists unique continuous functions $\lambda_i : K \to \mathbb{R}$, $i = 1, 2, ..., n$ such that for each $x \in K$, we have $(\lambda_1(x), \lambda_2(x), ..., \lambda_n(x)) \in \Delta_{n-1}$ and $x = \sum_{i=1}^{n} \lambda_i(x) a_i$.

Let's define $f : K \to 2^Y$ by
$f(a_i) = b_i$ $(i = 1, ..., n)$ and
$f(\sum_{i=1}^{n} \lambda_i a_i) = \sum_{i=1}^{n} \lambda_i b_i \in T(x)$.
We show that f is continuous.

Let $(x_m)_{m \in N}$ be a sequence which converges to $x_0 \in K$ where $x_m = \sum_{i=1}^{n} \lambda_i(x_m) a_i$ and $x_0 = \sum_{i=1}^{n} \lambda_i(x_0) a_i$. By the continuity of λ_i, it follows that for each $i = 1, 2, ..., n$, $\lambda_i(x_m) \to \lambda_i(x_0)$ as $m \to \infty$. Hence we must have $f(x_m) \to f(x_0)$ as $m \to \infty$, i.e. f is continuous. □

Theorem 3.4. *Let Y be a non-empty subset of a topological vector space E and K be a $(n-1)$- dimensional simplex in E. Let $T : K \to 2^Y$ be a weakly *-concave correspondence and $s : Y \to K$ be a continuous function. Then, there exists $x^* \in K$ such that $x^* \in s \circ T(x^*)$.*

Proof. By Theorem 3.3, T has a continuous selection theorem on K. Since $s : Y \to K$ is continuous, we obtain that $s \circ f : K \to K$ is continuous. By Brouwer's fixed point theorem, there exists a point $x^* \in K$ such that $x^* = s \circ f(x^*)$ and then, $x^* \in s \circ T(x^*)$. □

4. Equilibrium theorems

First, we present the model of an abstract economy and the definition of an equilibrium.

Let I be a non-empty set (the set of agents). For each $i \in I$, let X_i be a non-empty topological vector space representing the set of actions and define $X := \prod_{i \in I} X_i$; let A_i,

$B_i : X \to 2^{X_i}$ be the constraint correspondences and P_i the preference correspondence.

Definition 4.1. The family $\Gamma = (X_i, A_i, P_i, B_i)_{i\in I}$ is said to be an *abstract economy.*

Definition 4.2. An *equilibrium* for Γ is defined as a point $\overline{x} \in X$ such that for each $i \in I$, $\overline{x}_i \in \overline{B}_i(\overline{x})$ and $A_i(\overline{x},) \cap P_i(\overline{x}) = \emptyset$.

Remark 4.3. When for each $i \in I$, $A_i(x) = B_i(x)$ for all $x \in X$, this abstract economy model coincides with the classical one introduced by Borglin and Keiding in [2]. If in addition, $\overline{B}_i(\overline{x}) = \mathrm{cl}_{X_i} B_i(\overline{x})$ for each $x \in X$, which is the case if B_i has a closed graph in $X \times X_i$, the definition of an equilibrium coincides with the one used by Yannelis and Prabhakar [21].

To prove the following theorems we use the selection theorem mentioned in Section 3. We show the existence of equilibrium for an abstract economy without assuming the continuity of the constraint and the preference correspondences A_i and P_i.

First, we prove a new equilibrium existence theorem for a noncompact abstract economy with constraint and preference correspondences A_i and P_i, which have the property that their intersection $A_i \cap P_i$ contains a WNQ selector on the domain W_i of $A_i \cap P_i$ and W_i must be a simplex. To find the equilibrium point, we use Wu's fixed point theorem [19].

Since the constraint correspondence B_i is lower semicontinuous for each $i \in I$, the next theorem can be compared with Theorem 5 of Wu [19]. The proofs of these results are based on similar methods.

Theorem 4.4. *Let $\Gamma = (X_i, A_i, P_i, B_i)_{i\in I}$ be an abstract economy, where I is a (possibly uncountable) set of agents such that for each $i \in I$:*

(1) *X_i is a non-empty convex set in a locally convex space E_i and there exists a compact subset D_i of X_i containing all the values of the correspondences A_i, P_i and B_i such that $D = \prod_{i\in I} D_i$ is metrizable;*

(2) $\mathrm{cl}B_i$ *is lower semicontinuous, has non-empty convex values and for each $x \in X$, $A_i(x) \subset B_i(x)$;*

(3) *$W_i = \{x \in X \ / \ (A_i \cap P_i)(x) \neq \emptyset\}$ is a $(n_i - 1)$-dimensional simplex in X such that $W_i \subset \mathrm{co}D$;*

(4) *there exists a weakly naturally quasi-concave correspondence $S_i : W_i \to 2^{D_i}$ such that $S_i(x) \subset (A_i \cap P_i)(x)$* for each $x \in W_i$;

(5) *for each $x \in W_i$, $x_i \notin (A_i \cap P_i)(x)$.*

Then, there exists an equilibrium point $\overline{x} \in D$ for Γ, i.e., for each $i \in I$, $\overline{x}_i \in \mathrm{cl}B_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$.

Proof. Let be $i \in I$. From the assumption (4) and the selection theorem (Theorem 3.1), it follows that there exists a continuous function $f_i : W_i \to D_i$ such that for each $x \in W_i$, $f_i(x) \in S_i(x) \subset A_i(x) \cap P_i(x) \subset B_i(x)$.

Define the correspondence $T_i : X \to 2^{D_i}$, by $T_i(x) := \begin{cases} \{f_i(x)\}, \text{ if } x \in W_i, \\ \mathrm{cl}B_i(x), \text{ if } x \notin W_i. \end{cases}$

T_i is lower semicontinuous on X.

Let V be a closed subset of X_i, then

$U := \{x \in X \mid T_i(x) \subset V\} = \{x \in W_i \mid T_i(x) \subset V\} \cup \{x \in X \setminus W_i \mid T_i(x) \subset V\}$

$=\{x \in W_i \mid f_i(x) \in V\} \cup \{x \in X \mid \mathrm{cl}B_i(x) \subset V\}$
$=(f_i^{-1}(V) \cap W_i) \cup \{x \in X \mid \mathrm{cl}B_i(x) \subset V\}$.

U is a closed set, because W_i is closed, f_i is a continuous function on $\mathrm{int}_X K_i$ and the set $\{x \in X \mid \mathrm{cl}B_i(x) \subset V\}$ is closed since $\mathrm{cl}B_i$ is l.s.c. Let $D = \prod_{i \in I} D_i$. Then, by Tychonoff's Theorem, D is compact in the convex set X.

By Theorem 2.14 (Wu's fixed-point theorem), applied for the correspondences $S_i = T_i$ and $T_i : X \to 2^{D_i}$, there exists $\overline{x} \in D$ such that for each $i \in I$, $\overline{x}_i \in T_i(\overline{x})$. If $\overline{x} \in W_i$ for some $i \in I$, then $\overline{x}_i = f_i(\overline{x})$, which is a contradiction.

Therefore, $\overline{x} \notin W_i$, and hence, $(A_i \cap P_i)(\overline{x}) = \emptyset$. Also, for each $i \in I$, we have $\overline{x}_i \in T_i(\overline{x})$, and then, $\overline{x}_i \in \mathrm{cl}B_i(\overline{x})$. □

Remark 4.5. In this theorem, the correspondences $A_i \cap P_i$, $i \in I$, may not verify continuity assumptions and may not have convex or compact values.

Remark 4.6. In assumption (3), W_i must be a proper subset of X. In fact, if $W_i = X_i$, then, by applying Himmelberg's fixed point theorem ([8]) to $\prod_{i \in I} f_i(x)$, where f_i is a continuous selection of $S_i \subset A_i \cap P_i$, we can get a fixed point $\overline{x} \in \prod_{i \in I} (A_i \cap P_i)(\overline{x})$, which contradicts assumption (5).

Since a correspondence $T : X \to 2^Y$ having the property that $\cap\{T(x) : x \in X\}$ is nonempty and convex, is a WNQ correspondence, we obtain the following corollary.

Corollary 4.7. *Let $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ be an abstract economy, where I is a (possibly uncountable) set of agents such that for each $i \in I$:*

(1) *X_i is a non-empty convex set in a locally convex space E_i and there exists a compact subset D_i of X_i containing all the values of the correspondences A_i, P_i and B_i such that $D = \prod_{i \in I} D_i$ is metrizable;*

(2) $\mathrm{cl}B_i$ *is lower semicontinuous, has non-empty convex values and for each $x \in X$, $A_i(x) \subset B_i(x)$;*

(3) *$W_i = \{x \in X \ / \ (A_i \cap P_i)(x) \neq \emptyset\}$ is a $(n_i - 1)$-dimensional simplex in X such that $W_i \subset \mathrm{co}D$;*

(4) *there exists a correspondence $S_i : W_i \to 2^{D_i}$ such that S_i has the property that $\cap\{T(x) : x \in X\}$ is nonempty and convex, and $S_i(x) \subset (A_i \cap P_i)(x)$ for each $x \in W_i$;*

(5) *for each $x \in W_i$, $x_i \notin (A_i \cap P_i)(x)$.*

Then there exists an equilibrium point $\overline{x} \in D$ for Γ, i.e., for each $i \in I$, $\overline{x}_i \in \mathrm{cl}B_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$.

A correspondence $T : X \to 2^Y$ with convex graph is a WNQ correspondence, and then we have:

Corollary 4.8. *Let $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ be an abstract economy, where I is a (possibly uncountable) set of agents such that for each $i \in I$:*

(1) *X_i is a non-empty compact convex set in a locally convex space E_i;*

(2) $\mathrm{cl}B_i$ *is lower semicontinuous, has non-empty convex values and for each $x \in X$, $A_i(x) \subset B_i(x)$;*

(3) *$W_i = \{x \in X \ / \ (A_i \cap P_i)(x) \neq \emptyset\}$ is a $(n_i - 1)$-dimensional simplex in X;*

(4) *there exists a correspondence $S_i : W_i \to 2^{X_i}$ with convex graph such that $S_i(x)$*

$\subset (A_i \cap P_i)(x)$ *for each* $x \in W_i$;

(5) *for each* $x \in W_i$, $x_i \notin (A_i \cap P_i)(x)$.

Then, there exists an equilibrium point $\overline{x} \in X$ *for* Γ, *i.e., for each* $i \in I$, $\overline{x}_i \in \mathrm{cl}B_i(\overline{x})$ *and* $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$.

For Theorem 4.9, we use an approximation method, in the meaning that we obtain, for each $i \in I$, a continuous selection $f_i^{V_i}$ of $(A_i + V_i) \cap P_i$, where V_i is a convex neighborhood of 0 in X_i. For every $V = \prod_{i\in I} V_i$, we obtain an equilibrium point for the associated approximate abstract economy $\Gamma_V = (X_i, A_i, P_i, B_{V_i})_{i\in I}$, i.e., a point $\overline{x} \in X$ such that $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$ and $\overline{x}_i \in B_{V_i}(\overline{x})$, where the correspondence $B_{V_i} : X \to 2^{X_i}$ is defined by $B_{V_i}(x) = \mathrm{cl}(B_i(x) + V_i) \cap X_i$ for each $x \in X$ and for each $i \in I$. Finally, we use Lemma 2.1 to get an equilibrium point for Γ in X. The compactness assumption for X_i is essential in the proof.

Examples of results which use an approximation method are Theorem 3.1 pg. 37 or Theorem 1.2, pg. 41 in [23]. This method is usually used in relation with abstract economies which have lower semicontinuous constraint correspondences.

Theorem 4.9. *Let* $\Gamma = (X_i, A_i, P_i, B_i)_{i\in I}$ *be an abstract economy, where* I *is a (possibly uncountable) set of agents such that for each* $i \in I$:

(1) X_i *is a non-empty compact convex set in a locally convex space* E_i;

(2) $\mathrm{cl}B_i$ *is upper semicontinuous, has non-empty convex values and for each* $x \in X$, $A_i(x) \subset B_i(x)$;

(3) *the set* $W_i := \{x \in X / (A_i \cap P_i)(x) \neq \emptyset\}$ *is non-empty, open and* $K_i = \mathrm{cl}W_i$ *is a* $(n_i - 1)$*-dimensional simplex in* X;

(4) *For each convex neighbourhood* V *of* 0 *in* X_i, $(A_i + V) \cap P_i : K_i \to 2^{X_i}$ *is a weakly naturally quasi-concave correspondence*;

(5) *for each* $x \in K_i$, $x_i \notin P_i(x)$.

Then there exists an equilibrium point $\overline{x} \in X$ *for* Γ, *i.e., for each* $i \in I$, $\overline{x}_i \in \overline{B}_i(\overline{x})$ *and* $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$.

Proof. For each $i \in I$, let ß$_i$ denote the family of all open convex neighborhoods of zero in E_i. Let $V = (V_i)_{i\in I} \in \prod_{i\in I}$ß$_i$. Since $(A_i + V_i) \cap P_i$ is a weakly concave like correspondence on K_i, then, from the selection theorem (Theorem 3.1), there exists a continuous function $f_i^{V_i} : K_i \to X_i$ such that for each $x \in K_i$,

$f_i^{V_i}(x) \in (A_i(x) + V_i) \cap P_i(x) \subset (A_i(x) + V_i) \cap X_i$.

It follows that $f_i^{V_i}(x) \in \mathrm{cl}(B_i(x) + V_i)$ for $x \in K_i$. Since X_i is compact, we have that $\mathrm{cl}B_i(x)$ is compact for every $x \in X$ and $\mathrm{cl}(B_i(x) + V_i) = \mathrm{cl}(B_i(x)) + \mathrm{cl}V_i$ for every $V_i \subset E_i$.

Define the correspondence $T_i^{V_i} : X \to 2^{X_i}$, by

$$T_i^{V_i}(x) := \begin{cases} \{f_i^{V_i}(x)\}, & \text{if } x \in \mathrm{int}_X K = W_i, \\ \mathrm{cl}(B_i(x) + V_i) \cap X_i, & \text{if } x \in X \setminus \mathrm{int}_X K_i; \end{cases}$$

The correspondence $B_{V_i} : X \to 2^{X_i}$, defined by $B_{V_i}(x) := \mathrm{cl}(B_i(x) + V_i) \cap X_i$ is u.s.c. by Lemma 2.3. Then following the same line as in Theorem 4.4, we can prove that $T_i^{V_i}$ is upper semicontinuous on X and has closed convex values.

Define $T^V : X \to 2^X$ by $T^V(x) := \prod_{i\in I} T_i^{V_i}(x)$ for each $x \in X$.

T^V is an upper semicontinuous correspondence and it also has non-empty convex closed values.

Since X is a compact convex set, by Fan's fixed-point theorem [7], there exists $\overline{x}_V \in X$ such that $\overline{x}_V \in T^V(\overline{x}_V)$, i.e., for each $i \in I$, $(\overline{x}_V)_i \in T_i^{V_i}(\overline{x}_V)$.

We state that $\overline{x}_V \in X \setminus \bigcup_{i\in I} \text{int}_X K_i$.

If $\overline{x}_V \in \text{int}_X K_i$, $(\overline{x}_V)_i \in T_i^{V_i}(\overline{x}_V) = f_i(\overline{x}_V) \in ((A_i(\overline{x}_V) + V_i) \cap P_i)(\overline{x}_V) \subset P_i(\overline{x}_V)$, which contradicts assumption (5).

Hence $(\overline{x}_V)_i \in \text{cl}(B_i(\overline{x}_V) + V_i) \cap X_i$ and $(A_i \cap P_i)(\overline{x}_V) = \emptyset$, i.e. $\overline{x}_V \in Q_V$ where $Q_V = \cap_{i\in I}\{x \in X : x_i \in \text{cl}(B_i(x) + V_i) \cap X_i \text{ and } (A_i \cap P_i)(x) = \emptyset\}$.

Since W_i is open, Q_V is the intersection of non-empty closed sets, then it is non-empty, closed in X.

We prove that the family $\{Q_V : V \in \prod_{i\in I} ß_i\}$ has the finite intersection property.

Let $\{V^{(1)}, V^{(2)}, ..., V^{(n)}\}$ be any finite set of $\prod_{i\in I} ß_i$ and let $V^{(k)} = (V_i^{(k)})_{i\in I}$, $k = 1, ..., n$. For each $i \in I$, let $V_i = \bigcap_{k=1}^{n} V_i^{(k)}$, then $V_i \in ß_i$; thus $V = (V_i)_{i\in I} \in \prod_{i\in I} ß_i$. Clearly $Q_V \subset \bigcap_{k=1}^{n} Q_{V^{(k)}}$ so that $\bigcap_{k=1}^{n} Q_{V^{(k)}} \neq \emptyset$.

Since X is compact and the family $\{Q_V : V \in \prod_{i\in I} ß_i\}$ has the finite intersection property, we have that $\cap\{Q_V : V \in \prod_{i\in I} ß_i\} \neq \emptyset$. Take any $\overline{x} \in \cap\{Q_V : V \in \prod_{i\in I} ß_i\}$, then for each $i \in I$ and each $V_i \in ß_i$, $\overline{x}_i \in \text{cl}(B_i(\overline{x}) + V_i) \cap X_i$ and $(A_i \cap P_i)(\overline{x}) = \emptyset$; but then $\overline{x_i} \in \text{cl}(B_i(\overline{x}))$ by Lemma 2.1 and $(A_i \cap P_i)(\overline{x}) = \emptyset$ for each $i \in I$ so that $\overline{x}$ is an equilibrium point of Γ in X. □

The last two theorems can be compared with Zheng's theorems 3.1 and 3.2 in [24] and Zhou's theorems 5 and 6 in [25] where the constraint correspondences have continuous selections on a closed subset $C_i \subset X$ which contains the set $\{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$.

To find the equilibrium point in Theorem 4.10, we use Wu's fixed point theorem for correspondences $\text{cl}B_i$ which are lower semicontinuous and we need a non-empty compact metrizable set D_i in X_i for each $i \in I$. The spaces X_i are not compact.

Theorem 4.10. *Let $\Gamma = (X_i, A_i, P_i, B_i)_{i\in I}$ be an abstract economy, where I is a (possibly uncountable) set of agents such that for each $i \in I$:*

(1) *X_i is a non-empty convex set in a Hausdorff locally convex space E_i and there exists a nonempty compact metrizable subset D_i of X_i containing all values of the correspondences A_i, P_i and B_i;*

(2) *$\text{cl}B_i$ is lower semicontinuous with non-empty convex values;*

(3) *there exists a $(n_i - 1)$-dimensional simplex K_i in X and*
$W_i := \{x \in X \ / \ (A_i \cap P_i)(x) \neq \emptyset\} \subset \text{int}_X(K_i)$;

(4) *$\text{cl}B_i$ has the (WNQS)-property on K_i;*

Then there exists an equilibrium point $\overline{x} \in D$ for Γ, i.e., for each $i \in I$, $\overline{x}_i \in \text{cl}B_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$.

Proof. Since clB_i has the WNQS property on K_i, it follows that there exists a weakly concave like correspondence $F_i : X \to 2^{D_i}$ such that $F_i(x) \subset$cl$B_i(x)$ and $x_i \notin F_i(x)$ for each $x \in K_i$.

K_i is *a* (n_i-1)- dimensional simplex, then, from the selection theorem, there exists a continuous function $f_i : K_i \to D_i$ such that $f_i(x) \in F_i(x)$ for each $x \in K_i$. Because $x_i \notin F_i(x)$ for each $x \in K_i$, we have that $x_i \neq f_i(x)$ for each $x \in K_i$.

Define the correspondence $T_i : X \to 2^{D_i}$, by $T_i(x) := \begin{cases} \{f_i(x)\}, \text{ if } x \in K_i, \\ \text{cl}B_i(x), \text{ if } x \notin K_i. \end{cases}$

T_i is lower semicontinuous on X and has closed convex values.

Let U be a closed subset of X_i, then

$U^{'} := \{x \in X \mid T_i(x) \subset U\} =\{x \in K_i \mid T_i(x) \subset U\} \cup \{x \in X \setminus K_i \mid T_i(x) \subset U\}$
$=\{x \in K_i \mid f_i(x,y) \in U\} \cup \{x \in X \mid \text{cl}B_i(x) \subset U\}$
$=((f_i)^{-1}(U) \cap K_i) \cup .\{x \in X \mid \text{cl}B_i(x) \subset U\}$.

$U^{'}$ is a closed set, because K_i is closed, f_i is a continuous function on K_i and the set $\{x \in X \mid \text{cl}B_i(x) \subset U\}$ is closed since cl$B_i(x)$ is l.s.c. Then T_i is lower semicontinuous on X and has non-empty closed convex values.

By Theorem 2.14 (Wu's fixed-point theorem) applied for the correspondences $S_i = T_i$ and $T_i : X \to 2^{D_i}$, there exists $\overline{x} \in D$ such that for each $i \in I$, $\overline{x}_i \in T_i(\overline{x})$. If $\overline{x} \in W_i$ for some $i \in I$, then $\overline{x}_i = f_i(\overline{x})$, which is a contradiction.

Therefore, $\overline{x} \notin W_i$, and hence $(A_i \cap P_i)(\overline{x}) = \emptyset$. Also, for each $i \in I$, we have $\overline{x}_i \in T_i(\overline{x})$, and then $\overline{x}_i \in$cl$B_i(\overline{x})$. □

In Theorem 4.11 the sets X_i are non-empty compact convex in locally convex spaces E_i. As in Theorem 4.9, we first obtain equilibria for Γ_V, and then, the proof coincides with the proof of Theorem 4.9.

Theorem 4.11. *Let* $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ *be an abstract economy, where* I *is a (possibly uncountable) set of agents such that for each* $i \in I$:

(1) X_i *is a non-empty compact convex set in a locally convex space* E_i*;*

(2) clB_i *is upper semicontinuous with non-empty convex values;*

(3) *the set* $W_i := \{x \in X \ / \ (A_i \cap P_i)(x) \neq \emptyset\}$ *is open and there exists a* $(n_i - 1)$*-dimensional simplex* K_i *in* X *such that* $W_i \subset \text{int}_X(K_i)$.

(3) clB_i *has the (e-WNQS) property on* K_i.

Then there exists an equilibrium point $\overline{x} \in X$ *for* Γ*, i.e., for each* $i \in I$*,* $\overline{x}_i \in \overline{B}_i(\overline{x})$ *and* $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset$.

Proof. For each $i \in I$, let ß$_i$ denote the family of all open convex neighborhoods of zero in E_i.Let $V = (V_i)_{i \in I} \in \prod_{i \in I}$ß$_i$. Since cl$B_i$ has the e-WNQS property on K_i, it follows that there exists a weakly concave like correspondence $F_i^{V_i} : X \to 2^{X_i}$ such that $F_i^{V_i}(x) \subset clB_i(x) + V_i$ and $x_i \notin F_i^{V_i}(x)$ for each $x \in K_i$.

K_i is a $(n_i - 1)$- dimensional simplex, then, from the selection theorem, there exists a continuous function $f_i^{V_i} : K_i \to X_i$ such that $f_i^{V_i}(x) \in F_i^{V_i}(x)$ for each $x \in K_i$. Because $x_i \notin F_i^{V_i}(x)$ for each $x \in K_i$, we have that $x_i \neq f_i^{V_i}(x)$ for each $x \in K_i$.

Define the correspondence $T_i^{V_i} : X \to 2^{X_i}$, by

$$T_i^{V_i}(x) := \begin{cases} \{f_i^{V_i}(x)\}, & \text{if } x \in \text{int}_X K_i, \\ \text{cl}(B_i(x) + V_i) \cap X_i, & \text{if } x \in X \setminus \text{int}_X K_i; \end{cases}$$

$B_{V_i} : X \to 2^{X_i}$, $B_{V_i}(x)$ =cl$(B_i(x) + V_i) \cap X_i$ = (cl$B_i(x)$+cl$V_i) \cap X_i$ is upper semicontinuous by Lemma 2.3.

Let U be an open subset of X_i, then

$U' := \{x \in X \mid T_i^{V_i}(x) \subset U\}$

$=\{x \in \text{int}_X K_i \mid T_i^{V_i}(x) \subset U\} \cup \{x \in X \backslash \text{int}_X K_i \mid T_i^{V_i}(x) \subset U\}$

$=\left\{x \in \text{int}_X K_i \mid f_i^{V_i}(x,y) \in U\right\} \cup \left\{x \in X \mid (\text{cl}B_i(x) + \overline{V_i}) \cap X_i \subset U\right\}$

$=((f_i^{V_i})^{-1}(U) \cap \text{int}_K K_i) \cup . \left\{x \in X \mid (\text{cl}B_i(x) + \overline{V_i}) \cap X_i \subset U\right\}.$

U' is an open set, because $\text{int}_X K_i$ is open, $f_i^{V_i}$ is a continuous function on K_i and the set $\{x \in X \mid (\text{cl}B_i(x) + \text{cl}V_i) \cap X_i \subset U\}$ is open since $(\text{cl}B_i(x)+\text{cl}V_i) \cap X_i$ is u.s.c. Then, $T_i^{V_i}$ is upper semicontinuous on X and has closed convex values.

Define $T^V : X \to 2^X$ by $T^V(x) := \prod_{i \in I} T_i^{V_i}(x)$ for each $x \in X$.

T^V is an upper semicontinuous correspondence and it has also non-empty convex closed values.

Since X is a compact convex set, by Fan's fixed-point theorem [7], there exists $\overline{x}_V \in X$ such that $\overline{x}_V \in T^V(\overline{x}_V)$, i.e., for each $i \in I$, $(\overline{x}_V)_i \in T_i^{V_i}(\overline{x}_V)$. If $\overline{x}_V \in \text{int}_X K_i$, $(\overline{x}_V)_i = f_i^{V_i}(\overline{x}_V)$, which is a contradiction.

Hence $(\overline{x}_V)_i \in \text{cl}(B_i(\overline{x}_V) + V_i) \cap X_i$ and $(A_i \cap P_i)(\overline{x}_V) = \emptyset$, i.e. $\overline{x}_V \in Q_V$ where

$Q_V = \cap_{i \in I}\{x \in X : x_i \in \text{cl}(B_i(x) + V_i) \cap X_i$ and $(A_i \cap P_i)(x) = \emptyset\}$.

Since W_i is open, Q_V is the intersection of non-empty closed sets, then it is non-empty, closed in X.

We prove that the family $\{Q_V : V \in \prod_{i \in I} ß_i\}$ has the finite intersection property.

Let $\{V^{(1)}, V^{(2)}, ..., V^{(n)}\}$ be any finite set of $\prod_{i \in I} ß_i$ and let $V^{(k)} = (V_i^{(k)})_{i \in I}$, $k = 1, ..., n$. For each $i \in I$, let $V_i = \bigcap_{k=1}^{n} V_i^{(k)}$, then $V_i \in ß_i$; thus $V = (V_i)_{i \in I} \in \prod_{i \in I} ß_i$. Clearly $Q_V \subset \bigcap_{k=1}^{n} Q_{V^{(k)}}$ so that $\bigcap_{k=1}^{n} Q_{V^{(k)}} \neq \emptyset$.Since X is compact and the family $\{Q_V : V \in \prod_{i \in I} ß_i\}$ has the finite intersection property, we have that $\cap\{Q_V : V \in \prod_{i \in I} ß_i\} \neq \emptyset$. Take any $\overline{x} \in \cap\{Q_V : V \in \prod_{i \in I} ß_i\}$, then for each $i \in I$ and each $V_i \in ß_i$, $\overline{x}_i \in \text{cl}(B_i(\overline{x}) + V_i) \cap X_i$ and $(A_i \cap P_i)(\overline{x}) = \emptyset$; but then $\overline{x}_i \in \text{cl}(B_i(\overline{x}))$ from Lemma 2.1 and $(A_i \cap P_i)(\overline{x}) = \emptyset$ for each $i \in I$ so that $\overline{x}$ is an equilibrium point of Γ in X. □

Acknowledgement. This work was supported by the strategic grant POSDRU/89/1.5/S/58852, Project "Postdoctoral programme for training scientific researchers" cofinanced by the European Social Found within the Sectorial Operational Program Human Resources Development 2007-2013.

References

[1] H. Ben-El-Mechaiekh, *Fixed points for compact set-valued maps,* Questions Answers Gen. Topology, **10**(1992), 153-156.

[2] A. Borglin, H. Keiding, *Existence of equilibrium action and of equilibrium:A note on the 'new' existence theorem,* J. Math. Econom., **3**(1976), 313-316.
[3] F.E. Browder, *A new generation of the Schauder fixed point theorems,* Math. Ann., **174**(1967), 285-290.
[4] F.E. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces,* Math. Ann., **177**(1968), 283-301.
[5] X. Ding, H. Yiran, *Best Approximation Theorem for Set-valued Mappings without Convex Values and Continuity,* Appl Math. Mech. English Edition, **19**(9)(1998), 831-836.
[6] X.P. Ding, W.K. Kim, K.K. Tan, *A selection theorem and its applications,* Bull. Austral. Math. Soc., **46**(1992), 205-212.
[7] K. Fan, *Fixed-point and minimax theorems in locally convex topological linear spaces,* Proc. Nat. Acad. Sci. U.S.A., **38**(1952), 121-126.
[8] C.J. Himmelberg, *Fixed points of compact multifunctions,* J. Math. Anal. Appl., **38**(1972), 205-207.
[9] C.D. Horvath, *Extension and selection theorems in topological vector spaces with a generalized convexity structure,* Ann. Fac. Sci., Toulouse, **2**(1993), 253-269.
[10] T. Husain, E. Taradfar, *A selection theorem and a fixed point theorem and an equilibrium point of an abstract economy*, Internat. J. Math. Math. Sci., **18**(1995), 179-184.
[11] E. Michael, *Continuous selection,* Annals of Mathematics, **63**(2)(1956), 361-382.
[12] S. Park, *Continuous selection theorems in generalized convex spaces,* Numer. Funct. Anal. Optim., **25**(1999), 567-583.
[13] S. Park, *The Knaster-Kuratowski-Mazurkiewicz Theorem and almost fixed points,* Topol. Meth. Nonlinear Anal., **16**(2000), 195-200.
[14] W. Shafer, H. Sonnenschein, *Equilibrium in abstract economies without ordered preferences,* J. Math. Economics, **2**(1975), 345-348.
[15] T. Tanaka, *Generalized Quasiconvexities, Cone Saddle Points, and Minimax Theorem for Vector-Valued Functions,* J. Optim. Theory Appl., **81**(2)(1994), 355-377.
[16] G. Tian, *Fixed points theorems for mappings with noncompact and nonconvex domains,* J. Math. Anal. Appl., **158**(1991), 161-167.
[17] K. Wlodarczyk, D. Klim, *Fixed point and coincidence theorems for set-valued maps with nonconvex or noncompact domains in topological vector spaces,* Abstr. Appl. Anal., **1**(2003), 1-18.
[18] K. Wlodarczyk, D. Klim, *Equilibria and fixed points of sets-valued maps with nonconvex and noncompact domains and ranges,* Nonlinear Anal., **65**(2006), 918-932.
[19] X. Wu, *A new fixed point theorem and its applications,* Proc. Amer. Math. Soc., **125**(1997), 1779-1783.
[20] X. Wu, S. Shen, *A further generalization of Yannelis-Prabhakar's continuous selection theorem and its applications,* J. Math. Anal. Appl., **197**(1996), 61-74.
[21] N.C. Yannelis and N.D. Prabhakar, *Existence of maximal elements and equilibrium in linear topological spaces,* J. Math. Econom., **12**(1983), 233-245.
[22] Z.T. Yu, L.J. Lin, *Continuous selection and fixed point theorems,* Nonlinear Anal., **52**(2003), 445-455.
[23] X.Z. Yuan, *The Study of Minimax Inequalities and Applications to Economies and Variational Inequalities,* Memoirs Amer. Math. Soc., **625**(1988).
[24] X. Zheng, *Approximate selection theorems and their applications,* J. Math. Anal. Appl., **212**(1997), 88-97.
[25] J. Zhou, *On the Existence of equilibrium for abstract economies.* J. Math. Anal. Appl., **193**(1992), 839-857.
[26] Q. Zhang, C. Cheng, X. Li, *Generalized minimax theorems for two set-valued mappings,* J. Indust. Managem. Optimization, **9**(1)(2013), 1-12.

Received: January 19, 2012; Accepted: January 17, 2013.