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AN INTEGRAL EQUATION RELATED TO AN EPIDEMIC MODEL VIA WEAKLY PICARD OPERATORS TECHNIQUE IN A GAUGE SPACE

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Abstract. In the paper *Qualitative behavior of an integral equation related to some epidemic model* (Demonstratio Mathematica, Vol. XXXVI, No 3/2003, 603-609) the author Eva Brestovanska has considered the integral equation

$$x(t) = \Big[g_1(t) + \int_0^t A_1(t-s)F_1(s,x(s))ds\Big] \cdots \Big[g_p(t) + \int_0^t A_p(t-s)F_p(s,x(s))ds\Big],\ t \geq 0.$$

In this paper we shall study by weakly Picard technique operators in a gauge space: the existence, uniqueness and data dependence such as the continuity, smooth dependence on parameter for the solution of the following integral equation

$$x(t) = \Big[g_1(t) + \int_0^t K_1(t,s,x(s))ds\Big] \cdot \Big[g_2(t) + \int_0^t K_2(t,s,x(s))ds\Big],\ t \in [0,\infty).$$

Our approach are connected with some results due from I.M. Olaru (*An integral equation via weakly Picard operators*, Fixed Point Theory, Vol 11 No1/2010, 97-106 and *Generalization of an integral equation related to some epidemic models*, Carpathian J. Math. Vol 26, No.1(2010), 92-96).
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1. Introduction

The theory of integral equations has many applications in describing numerous events and problems of real world. For example, integral equations are often applicable in mathematical physics, engineering, economics and biology (see [1], [4], [5], [6], [11], [12], [13] and their references for results about existence and uniqueness, continuous dependence of solution and even more specialized topics). The purpose of this paper is to study the following integral equation

$$x(t) = \Big[g_1(t) + \int_0^t K_1(t,s,x(s))ds\Big] \cdot \Big[g_2(t) + \int_0^t K_2(t,s,x(s))ds\Big],\ t \in [0,\infty). \tag{1.1}$$

This equation has, as a particular case, the integral equation studied by Eva Brestovanska in [1]. Also we notice that the survey of equation (1.1) for the case when $t \in [a, b]$ can be found in [11] where I. M. Olaru were obtained the results concerning the existence, uniqueness and data dependence: continuity, monotony, smooth dependence on parameter for the solution of equation (1.1). Also, a generalization of (1.1) with $t \in [a, b]$, is provided by I. M. Olaru in [12]. In order to approach the equation (1.1) we shall use weakly Picard operators technique. Also, weakly Picard operators technique is given in [10] and [14]. The Picard operators are used in various fields, especially in one of the most modern area of mathematics, namely Fractals Theory (see for instance [18]).

2. Basic notions and results of the weakly Picard operators theory

Throughout this paper we shall follow the standard terminologies and notations in nonlinear analysis. For the convenience of the reader we shall recall some of them.

Let X be a nonempty set and $A : X \to X$ an operator. We denote by $A^0 := 1_X$, $A^1 := A$, $A^{n+1} := A^n \circ A$, $n \in \mathbb{N}$, the iterate operators of the operator A. We also have

$$P(X) := \{Y \subset X \mid Y \neq \emptyset\}$$
$$F_A := \{x \in X \mid A(x) = x\}$$
$$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\}$$

By $(X, \to)$ we will denote an L-space. For examples of such spaces one can see [7], [8], [9].

In this paper, we need the following notations, notions and results from weakly Picard operators technique (I.A. Rus [16] and [17]).

Definition 2.1. Let $(X, \to)$ be a L-space. An operator $A : X \to X$ is weakly Picard operator (briefly WPO) if the sequence $(A^n(x))_{n \in N}$ converges, for all $x \in X$, and the limit (which may depending on x) is a fixed point of A.

Definition 2.2. Let $(X, \to)$ be a L-space. An operator $A : X \to X$ is a Picard operator (briefly PO) if the following properties hold:

(i) $F_A = \{x^\star\}$;
(ii) $A^n(x) \to x^\star$ as $n \to \infty$, for all $x \in X$.

If $A : X \to X$ is weakly Picard operator, then we may define the operator $A^\infty : X \to X$ by $A^\infty(x) = \lim\limits_{n\to\infty} A^n(x)$. Moreover, if A is PO and we denote by $x^\star$ its unique fixed point, then $A^\infty(x) = x^\star$, for each $x \in X$.

We have (see [15], [16], [17] and [19]):

Theorem 2.1. *(existence and uniqueness) Let $(X, (d_i)_{i \in I})$ be a sequentially complete Hausdorff gauge space and let $T : X \to X$ be such that, for every $i \in I$, there exists $\alpha_i \in I$ such that*

$$d_i(T(x), T(y)) \leq \alpha_i \cdot d_i(x, y),$$

for each $x, y \in X$. Then T is PO.

Definition 2.3. Let X be a nonempty. By definition $(X, \to, \leq)$ is an ordered L-space if and only if:

(i) $(X, \to)$ is an L-space;
(ii) $(X, \leq)$ is a partially ordered set;
(iii) $x_n \to x$, $y_n \to y$ and $x_n \leq y_n$ for each $n \in \mathbb{N}$ imply $x \leq y$.

Theorem 2.2. *(abstract Gronwall's Lemma) Let $(X, \to, \leq)$ be an ordered L-space and $A : X \to X$ be an operator. We suppose that:*

(i) *operator A is Picard and $F_A = \{x_A^\star\}$;*
(ii) *A is increasing.*

Then

(a) *$x \leq A(x)$ implies $x \leq x_A^\star$;*
(b) *$x \geq A(x)$ implies $x \geq x_A^\star$.*

Theorem 2.3. *(data dependence) Let $(X, (d_\lambda)_{\lambda \in \Lambda})$ be a gauge space and $A, B : X \to X$ be two $c_\lambda-$ WPOs. We suppose that, for each $\lambda \in \Lambda$, there exists $\eta_\lambda > 0$ such that*

$$d_\lambda(A(x)), B(x) \leq \eta_\lambda, \ for \ all \ x \in X.$$

Then

$$H_{d_\lambda}(F_A, F_B) \leq c_\lambda \cdot \eta_\lambda, \ for \ all \ \lambda \in \Lambda.$$

Other results about data dependence on gauge spaces can be found in [2].

For the study of the smooth dependence of parameter we shall use the following result

Theorem 2.4. *Let $(X, \to)$ be a L-space and $(Y, (d_i)_{i \in I})$ be a sequentially complete Hausdorff gauge space. Let $B : X \to X$ and $C : X \times Y \to Y$ be two operators. We suppose that:*

(i) *B is a Picard operator(PO) (we denote by $x^\star$ its unique fixed point);*
(ii) *for every $i \in I$ there exists $\alpha_i \in (0, 1)$ such that*

$$d_i\big(C(x, y_1), C(x, y_2)\big) \leq \alpha_i d_i(y_1, y_2),$$

for all $x \in X$ and $y_1, y_2 \in Y$ (we denote by $y^\star$ the unique fixed point of the operator $C(x^\star, \cdot)$);

(iii) *the operator $C(\cdot, y^\star)$ is continuous in $x^\star$.*

Then $A : X \times Y \to X \times Y$, $A(x, y) := \big(B(x), C(x, y)\big)$ is a Picard operator. Moreover, $F_A = \big\{(x^\star, y^\star)\big\}$.

3. Existence and uniqueness results

In this section we shall prove that the equation (1.1) has a unique solution in $C\big([0, \infty), \mathbb{R}\big)$. For this, in what follows we consider the gauge space $X := \big(C([0, \infty), \mathbb{R}), (d_m)_{m \in \mathbb{N}^\star}\big)$, where

$$d_m(x, y) := \max_{t \in [0, m]} \big|x(t) - y(t)\big| e^{-\tau t}, \tau > 0.$$

Our first main result is the following.

Theorem 3.1. *We suppose that*

(i) $g_1, g_2 \in C(\mathbb{R}_+, \mathbb{R})$, $K_1, K_2 \in C({\mathbb{R}_+}^2 \times \mathbb{R}, \mathbb{R})$;

(ii) *for each* $m \in \mathbb{N}^\star$ *and* $i = 1, 2$, *there exists* $M(K_i, m) > 0$ *such that*

$$|K_i(t,s,u)| \leq M(K_i, m),$$

for all $t, s \in [0, m]$, $u \in \mathbb{R}$;

(iii) *there exists* $L > 0$ *such that*

$$|K_i(t,s,u) - K_i(t,s,v)| \leq L|u - v|,$$

for all $t, s \in [0, +\infty)$, $u, v \in \mathbb{R}$, $i = 1, 2$;

Then the equation (1.1) has a unique solution $x^\star$ *in* $C(\mathbb{R}_+, \mathbb{R})$.

Proof. We consider the operator $A : X \to X$ defined by:

$$A(x)(t) = \Big[g_1(t) + \int_0^t K_1(t,s,x(s))ds\Big] \cdot \Big[g_2(t) + \int_0^t K_2(t,s,x(s))ds\Big].$$

We set

$$\alpha_m := L \cdot \sum_{i=1}^{2} \big(M(g_i, m) + m \cdot M(K_i, m)\big),$$

where

$$M(g_i, m) := \max_{t \in [0,m]} |g_i(t)|,$$

Notice that, for all $x, y \in X$, one has

$$\begin{aligned}
&\big|A(x)(t) - A(y)(t)\big| \\
&= \Big|\Big(g_1(t) + \int_0^t K_1\big(t,s,x(s)\big)ds\Big) \cdot \Big(g_2(t) + \int_0^t K_2\big(t,s,x(s)\big)ds\Big) \\
&\quad -\Big(g_1(t) + \int_0^t K_1\big(t,s,y(s)\big)ds\Big) \cdot \Big(g_2(t) + \int_0^t K_2\big(t,s,y(s)\big)ds\Big)\Big| \\
&= \Big|\Big(g_1(t) + \int_0^t K_1\big(t,s,x(s)\big)ds\Big) \cdot \Big(g_2(t) + \int_0^t K_2\big(t,s,x(s)\big)ds\Big) \\
&\quad -\Big(g_1(t) + \int_0^t K_1\big(t,s,x(s)\big)ds\Big) \cdot \Big(g_2(t) + \int_0^t K_2\big(t,s,y(s)\big)ds\Big) \\
&\quad +\Big(g_1(t) + \int_0^t K_1\big(t,s,x(s)\big)ds\Big) \cdot \Big(g_2(t) + \int_0^t K_2\big(t,s,y(s)\big)ds\Big)
\end{aligned}$$

$$-\Big(g_1(t)+\int_0^t K_1\big(t,s,y(s)\big)ds\Big)\cdot\Big(g_2(t)+\int_0^t K_2\big(t,s,y(s)\big)ds\Big)\Big|$$

$$\leq L\cdot(M(g_1,m)+mM(K_1,m))\|x-y\|_m\int_0^t e^{\tau\cdot s}ds$$

$$+L\cdot\big(M(g_2,m)+mM(K_2,m)\big)\|x-y\|_m\int_0^t e^{\tau\cdot s}ds$$

$$\leq\frac{\alpha_m}{\tau}\|x-y\|_m\cdot e^{\tau t}.$$

It follows that

$$\|A(x)-A(y)\|_m\leq\frac{\alpha_m}{\tau}\|x-y\|_m.$$

For a suitable choice of τ, according to Theorem 2.1, we obtain the conclusion. □

Example 3.1. Let us consider the following integral equation

$$x(t)=\Big(g_1(t)+\int_0^t 2t\cdot s\cdot\sin\frac{a_1x(s)}{1+(t+s)^2}ds\Big)\cdot\Big(g_2(t)+\int_0^t 2t\cdot s\cdot\sin\frac{a_2x(s)}{1+(t+s)^2}ds\Big),$$

where the functions $g_i\in C(\mathbb{R}_+,\mathbb{R})$ are arbitrarily chosen and $a_i\in\mathbb{R}$, $i=1,2$. Then one can apply Theorem 3.1.

Proof. We establish that the requirement of Theorem 3.1 are verified.

Indeed, we have

$$K_1,K_2:\mathbb{R}_+^2\times\mathbb{R}\to\mathbb{R},\ K_i(t,s,u)=2t\cdot s\cdot\sin\frac{a_i\cdot u}{1+(t+s)^2},\ i=1,2.$$

We notice that, for each $m\in\mathbb{N}$, one has

$$|K_i(t,s,u)|=2t\cdot s\cdot\Big|\sin\frac{u}{1+(t+s)^2}\Big|\leq 2m^2,$$

for all $t,s\in[0,m]$ and $u\in\mathbb{R}$ $(i=1,2)$. So, $M(K_i,m)=2m^2$. Also, we observe that

$$\Big|\frac{\partial K_i}{\partial u}(t,s,u)\Big|=\frac{2t\cdot s\cdot|a_i|}{1+(t+s)^2}\cdot\Big|\cos\frac{a_i\cdot u}{1+(t+s)^2}\Big|\leq|a_i|\leq\max\{|a_1|,|a_2|\}=:L,$$

for all $t,s\in\mathbb{R}_+$ and $u\in\mathbb{R}$. Therefore

$$\big|K_i(t,s,u)-K_i(t,s,v)\big|\leq L|u-v|,$$

for any $t,s\in[0,+\infty)$, $u,v\in\mathbb{R}$, $i=1,2$, as required. □

4. Data dependence results

We consider the following integral equations

$$x(t)=\Big(g_1^i(t)+\int_0^t K_1^i\big(t,s,x(s)\big)ds\Big)\cdot\Big(g_2^i(t)+\int_0^t K_2^i\big(t,s,x(s)\big)ds\Big),\ i=1,2. \quad (4.1)$$

Assume that we are in the conditions of Theorem 3.1. Let $x_i^\star$, $i=1,2$ be the unique solution of equation (4.1). Then we have:

Theorem 4.1. *Let $g_1^i, g_2^i, K_1^i, K_2^i$, $i=1,2$ be as in the statement of Theorem 3.1. We suppose that*

(a) *there exists η_1 such that $\left|g_i^1(t)-g_i^2(t)\right|\le\eta_1$, for every $t\ge 0$, $i=1,2$;*

(b) *there exists $\eta_2>0$ such that*

$$\left|K_i^1(t,s,u)-K_i^2(t,s,u)\right|\le\eta_2,$$

for all $t,s\in\mathbb{R}_+$ and $u\in\mathbb{R}$, $i=1,2$;

(c) *$\tau>\max\{\alpha_m^1,\alpha_m^2\}$, where $\alpha_m^i:=L\cdot\sum\limits_{j=1}^{2}\big(M(g_j^i,m)+m\cdot M(K_j^i,m)\big)$, $i=1,2$.*

Then

$$\|x_1^*-x_2^*\|_m\le(\eta_1+m\cdot\eta_2)(\alpha_m^1+\alpha_m^2)\cdot\max\Big\{\frac{1}{1-\frac{\alpha_m^1}{\tau}},\frac{1}{1-\frac{\alpha_m^2}{\tau}}\Big\},$$

for all $m\in\mathbb{N}^\star$

Proof. Let $A_i:X\to X$ be defined as

$$A_i(x)(t)=\Big(g_1^i(t)+\int_0^t K_1^i(t,s,x(s))ds\Big)\cdot\Big(g_2^i(t)+\int_0^t K_2^i(t,s,x(s))ds\Big),\ i=1,2.$$

Since

$$\|A_i(x)-A_i(y)\|_m\le\frac{\alpha_m^i}{\tau}\|x-y\|_m,\ i=1,2,$$

deduce that A_1, A_2 are c_m-Picard operators, where $c_m=\max\Big\{\frac{1}{1-\frac{\alpha_m^1}{\tau}},\frac{1}{1-\frac{\alpha_m^2}{\tau}}\Big\}$.

On the other hand, for al $x\in X$ we have

$$\begin{aligned}&|A_1(x)(t)-A_2(x)(t)|\\ &=\Big|\Big(g_1^1(t)+\int_0^t K_1^1(t,s,x(s))ds\Big)\cdot\Big(g_2^1(t)+\int_0^t K_2^1(t,s,x(s))ds\Big)\\ &-\Big(g_1^2(t)+\int_0^t K_1^2(t,s,x(s))ds\Big)\cdot\Big(g_2^2(t)+\int_0^t K_2^2(t,s,x(s))ds\Big)\Big|\\ &\le\Big|\Big(g_1^1(t)+\int_0^t K_1^1(t,s,x(s))ds\Big)\cdot\Big(g_2^1(t)+\int_0^t K_2^1(t,s,x(s))ds\Big)\end{aligned}$$

$$-\Big(g_1^1(t)+\int_0^t K_1^1(t,s,x(s))ds\Big)\cdot\Big(g_2^2(t)+\int_0^t K_2^2(t,s,x(s))ds\Big)\Big|$$

$$+\Big|\Big(g_1^1(t)+\int_0^t K_1^1(t,s,x(s))ds\Big)\cdot\Big(g_2^2(t)+\int_0^t K_2^2(t,s,x(s))ds\Big)$$

$$-\Big(g_1^2(t)+\int_0^t K_1^2(t,s,x(s))ds\Big)\cdot\Big(g_2^2(t)+\int_0^t K_2^2(t,s,x(s))ds\Big)\Big|$$

$$\leq (\eta_1+m\eta_2)\cdot\Big(M(g_1^1,m)+m\cdot M(K_1^1,m)+M(g_2^2,m)+m\cdot M(K_2^2,m)\Big)$$

$$\leq (\eta_1+m\eta_2)\cdot\frac{\alpha_m^1+\alpha_m^2}{L}=:\eta_m.$$

The conclusion follows from Theorem 2.3. □

Next we consider the inequalities

$$x(t)\leq\Big(g_1(t)+\int_0^t K_1(t,s,x(s))ds\Big)\cdot\Big(g_2(t)+\int_0^t K_2(t,s,x(s))ds\Big) \tag{4.2}$$

$$x(t)\geq\Big(g_1(t)+\int_0^t K_1(t,s,x(s))ds\Big)\cdot\Big(g_2(t)+\int_0^t K_2(t,s,x(s))ds\Big) \tag{4.3}$$

Theorem 4.2. *We suppose that:*

(a) $g_i(\mathbb{R}_+)\subseteq\mathbb{R}_+$ *and* $K_i(\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R})\subseteq\mathbb{R}_+$, $i=1,2$*;*
(b) g_1,g_2,K_1,K_2 *verify the hypothesis of Theorem 3.1;*
(c) g_i *and* $K_i(t,s,\cdot)$ *are increasing;*

Then

(i) *for each solution x of inequality (4.2) we have $x\leq x^*$;*
(ii) *for each solution x of inequality (4.3) we have $x\leq x^*$,*

where x^* is the unique solution of equation (1.1).

Proof. Consider on $C(\mathbb{R}_+,\mathbb{R})$ the partial order defined by

$$x\leq y \text{ if and only if } x(t)\leq y(t) \text{ for any } t\in\mathbb{R}_+.$$

Also, we consider the L-space $(C(\mathbb{R}_+,\mathbb{R}),\to)$ where "$\to$" stands for uniform convergence on each compact subset of $\mathbb{R}_+$. Then $(C(\mathbb{R}_+,\mathbb{R}),\to,\leq)$ is an ordered L-space.

Now we define, $A: C(\mathbb{R}_+,\mathbb{R})\to C(\mathbb{R}_+,\mathbb{R})$ by

$$A(x)(t)=\Big(g_1(t)+\int_0^t K_1\big(t,s,x(s)\big)ds\Big)\cdot\Big(g_2(t)+\int_0^t K_2\big(t,s,x(s)\big)ds\Big)$$

First observe that from (b) A is PO. From (a) and (c) it follows that A is an increasing operator. The conclusion follows from Theorem 2.2. □

5. Smooth dependence on parameter

Next we are going to apply fiber principle contraction to study smooth dependence on parameter for the equation (5.1). We remark that fiber principle contraction technique, can be found in many papers(see for instance [13], [15], [16], [17]). Throughout of this section we consider gauge space $X := (C(\mathbb{R}_+ \times J, \mathbb{R}), d_m)$, where

$$d_m(x,y) = \max_{(t,\lambda)\in[0,m]\times J} |x(t,\lambda) - y(t,\lambda)|e^{-\tau t}, m \in \mathbb{N}^\star.$$

Let us consider the integral equation

$$x(t,\lambda) = \Big(g_1(t,\lambda) + \int_0^t K_1\big(t,s,x(s,\lambda),\lambda\big)ds\Big) \cdot \Big(g_2(t,\lambda) + \int_0^t K_2\big(t,s,x(s,\lambda),\lambda\big)ds\Big), \tag{5.1}$$

for all $t \in [0,\infty), \lambda \in J \subset \mathbb{R}$. We assume that

(H_1) $J \subset \mathbb{R}$ an compact interval;

(H_2) $g_1, g_2 \in C^1(\mathbb{R}_+ \times J, \mathbb{R})$, $K_1, K_2 \in C^1({\mathbb{R}_+}^2 \times \mathbb{R} \times J, \mathbb{R})$;

(H_3) there exists $L > 0$ such that

$$\Big|\frac{\partial K_i}{\partial u}(t,s,u,\lambda)\Big| \le L,$$

for all $t, s \in \mathbb{R}_+$, $u \in \mathbb{R}$, $\lambda \in J$;

(H_4) for each $m \in \mathbb{N}^\star$ and $i = 1, 2$, there exists $M(K_i, m) > 0$ such that

$$|K_i(t,s,u,\lambda)| \le M(K_i, m),$$

for all $t, s \in [0,m]$, $u \in \mathbb{R}$ and $\lambda \in J$;

We set $\alpha_m := L \cdot \sum_{i=1}^{2} [M(g_i, m) + m \cdot M(K_i, m)]$, where

$$M(g_i, m) := \max_{(t\times\lambda)\in[0,m]\times J} |g_i(t,\lambda)|.$$

We define the operator $B : X \to X$, given by

$$B(x)(t,\lambda) = \Big(g_1(t,\lambda) + \int_0^t K_1(t,s,x(s),\lambda)ds\Big) \cdot \Big(g_2(t,\lambda) + \int_0^t K_2(t,s,x(s),\lambda)ds\Big)$$

It is clear that, in the conditions $(H_1) - (H_4)$, B is Picard operator. Let $x^\star(\cdot,\lambda)$ the unique fixed point of operator B. Then

$$x^\star(t,\lambda)$$

$$= \Big(g_1(t,\lambda) + \int_0^t K_1(t,s,x^\star(s,\lambda),\lambda)ds\Big) \cdot \Big(g_2(t,\lambda) + \int_0^t K_2(t,s,x^\star(s,\lambda),\lambda)ds\Big), \tag{5.2}$$

for all $t \in [a,b], \lambda \in J \subset \mathbb{R}$.

We suppose that there exists $\frac{\partial x^\star}{\partial \lambda}$. Then from relation (5.2) we obtain that

$$\frac{\partial x^\star}{\partial \lambda} = \Big(\frac{\partial g_1}{\partial \lambda}(t,\lambda)+\int\limits_0^t \frac{\partial K_1}{\partial u}(t,s,x^\star(s,\lambda),\lambda)\cdot\frac{\partial x^\star}{\partial \lambda}(s,\lambda)ds+\int\limits_0^t \frac{\partial K_1}{\partial \lambda}(t,s,x^\star(s,\lambda),\lambda)ds\Big)\cdot$$

$$\Big(g_2(t,\lambda)+\int\limits_0^t K_2(t,s,x^\star(s,\lambda),\lambda)ds)+(g_1(t,\lambda)+\int\limits_0^t K_1(t,s,x^\star(s,\lambda),\lambda)ds\Big)\cdot$$

$$\Big(\frac{\partial g_2}{\partial \lambda}(t,\lambda)+\int\limits_0^t \frac{\partial K_2}{\partial u}(t,s,x^\star(s,\lambda),\lambda)\cdot\frac{\partial x^\star}{\partial \lambda}(s,\lambda)ds+\int\limits_0^t \frac{\partial K_2}{\partial \lambda}(t,s,x^\star(s,\lambda),\lambda)ds\Big).$$

This relation suggest us to consider the following operator $C : X \times X \to X$,

$$C(x,y)(t,\lambda)$$

$$= \Big(\frac{\partial g_1}{\partial \lambda}(t,\lambda)+\int\limits_0^t \frac{\partial K_1}{\partial u}(t,s,x(s,\lambda),\lambda)\cdot y(s,\lambda)ds+\int\limits_0^t \frac{\partial K_1}{\partial \lambda}(t,s,x(s,\lambda),\lambda)ds\Big)\cdot$$

$$\Big(g_2(t,\lambda)+\int\limits_0^t K_2(t,s,x(s,\lambda),\lambda)ds\Big)+\Big(g_1(t,\lambda)+\int\limits_0^t K_1(t,s,x(s,\lambda),\lambda)ds\Big)\cdot$$

$$\Big(\frac{\partial g_2}{\partial \lambda}(t,\lambda)+\int\limits_0^t \frac{\partial K_2}{\partial u}(t,s,x(s,\lambda),\lambda)\cdot y(s,\lambda)ds+\int\limits_0^t \frac{\partial K_2}{\partial \lambda}(t,s,x(s,\lambda),\lambda)ds\Big).$$

Let be $m \in \mathbb{N}$ and $x \in X$. Then for all $y, z \in X$ we have

$$\|C(x,y)-C(x,z)\|_m \le \frac{\alpha_m}{\tau}\cdot\|y-z\|_m.$$

For a suitable choice of τ, the operator $C(x,\cdot)$ is contraction. In this way we have the triangular operator

$$A : X \times X \to X \times X,$$

$$A(x,y)(t,\lambda) = \big(B(x)(t,\lambda), C(x,y)(t,\lambda)\big).$$

Using Theorem 2.4 we conclude that A is a Picard operator. So, the sequences

$$x_{n+1} = B(x_n), n \in \mathbb{N}$$

$$y_{n+1} = C(x_n, y_n)$$

converges uniformly on each compact of $\mathbb{R}_+ \times J$ to $(x^\star, y^\star) \in F_A$, for all $x_0, y_0 \in X$.

If we take $x_0 = 0$, $y_0 = \dfrac{\partial x_0}{\partial \lambda} = 0$ then $y_1 = \dfrac{\partial x_1}{\partial \lambda}$ and thus by induction we can prove that $y_n = \frac{\partial x_n}{\partial \lambda}$, for all $n \in \mathbb{N}^\star$.

Hence $x_n \to x^\star$ and, also, $\dfrac{\partial x_n}{\partial \lambda} \to y^\star$ as $n \to \infty$, uniformly on each interval $[0,m]$, $m \in \mathbb{N}^\star$.

Therefore, there exists $\dfrac{\partial x^\star}{\partial \lambda}$ and $\dfrac{\partial x^\star}{\partial \lambda} = y^\star$.

From the above considerations, we have that

Theorem 5.1. *We consider the integral equation (5.1) in the hypothesis* $(H_1)-(H_4)$. *Then*

(i) *the equation (5.2) has a unique solution* $x^{\star}(t,\cdot) \in X$;
(ii) $x^{\star}(t,\cdot) \in C^1(J)$, *for all* $t \in [a,b]$.

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