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ON THE EXISTENCE OF CONNECTED SETS OF SOLUTIONS FOR NONLINEAR OPERATORS

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Dedicated to James N. (Jim) Flavin with admiration

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Abstract. In this paper we discuss continua of fixed points and coincidence points.
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1. Introduction

In this paper we investigate the solution set of a map F and in particular we present conditions on F which guarantee that the solution set contains a connected component. These bifurcation results rely on the notion of an essential map [1 ,7]. We refer the reader to [2, 3, 4] for other approaches in the literature.

Let X and Y be Hausdorff topological spaces. Given a class $\mathbf{X}$ of maps, $\mathbf{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathbf{X}$, and $\mathbf{X}_c$ the set of finite compositions of maps in $\mathbf{X}$. We let

$$\mathbf{F}(\mathbf{X}) = \{Z : \ Fix\, F \neq \emptyset \ \text{ for all } \ F \in \mathbf{X}(Z,Z)\}$$

where $Fix\, F$ denotes the set of fixed points of F.

The class $\mathbf{U}$ of maps is defined by the following properties:

(i) $\mathbf{U}$ contains the class $\mathbf{C}$ of single valued continuous functions;

(ii) each $F \in \mathbf{U}_c$ is upper semicontinuous and compact valued; and

(iii) $B^n \in \mathbf{F}(\mathbf{U}_c)$ for all $n \in \{1, 2, ...\}$; here $B^n = \{x \in \mathbf{R}^n : \ \|x\| \leq 1\}$.

We say $F \in \mathbf{U}_c^k(X,Y)$ if for any compact subset K of X there is a $G \in \mathbf{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Recall $\mathbf{U}_c^k$ is closed under compositions. The class $\mathbf{U}_c^k$ contains almost all the well known maps in the literature (see [8] and the references therein). It is also possible to consider more general maps (see [6, 7]) and in this paper we will consider a class of maps which we will call $\mathbf{A}$.

2. Continua of solutions

Let E be a completely regular topological space and U an open subset of E.

We will consider a class $\mathbf{A}$ of maps (see [5]).

Definition 2.1. We say $F \in A(\overline{U}, E)$ if $F \in \mathbf{A}(\overline{U}, E)$ and $F : \overline{U} \to K(E)$ is an upper semicontinuous map; here $\overline{U}$ denotes the closure of U in E and $K(E)$ denotes the family of nonempty compact subsets of E.

Definition 2.2. We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $x \notin F(x)$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

Definition 2.3. Let $F \in A_{\partial U}(\overline{U}, E)$. We say F is essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in J(x)$.

Recall a compact connected set is called a continuum. For our results in this paper we will use Whyburn's lemma from topology which we state here for convenience.

Theorem 2.1. *Let A and B be disjoint closed subsets of a compact Hausdorff topological space K such that no connected component of K intersects both A and B. Then there exists a partition $K = K_1 \cup K_2$ where K_1 and K_2 are disjoint compact sets containing A and B respectively.*

An easy consequence of Theorem 2.1 was established by Martelli in [3].

Theorem 2.2. *Let X be a metric space and K a compact subset of X. Assume that A and B are two disjoint closed subsets of K such that no connected component of K intersects both. Then there exists an open bounded set U such that*

$$A \subset U,\ \overline{U} \cap B = \emptyset \ \text{ and } \ \partial U \cap K = \emptyset.$$

For our next results we assume E is a metric space and U an open subset of $E \times [0,1]$. We will also assume the following condition:

$$\begin{cases} \text{for Hausdorff topogical spaces } X_1 \text{ and } X_2, \text{ if } F \in A(X_1, X_2), \\ v \in \mathbf{C}(X_1, [0,1]) \text{ and if } \Phi(y) = (F(y), v(y)) \text{ for } y \in X_1, \\ \text{then } \Phi \in A(X_1, X_2 \times [0,1]). \end{cases} \tag{2.1}$$

Our first result was motivated by ideas in [7].

Theorem 2.3. *Suppose $N \in A(\overline{U}, E)$ with*

$$x \notin N(x, \lambda) \ \text{ for } \ (x, \lambda) \in \partial U. \tag{2.2}$$

Let $H : \overline{U} \times [0,1] \to K(E \times [0,1])$ be given by $H(x, \lambda, \mu) = (N(x, \lambda), \mu)$ for $(x, \lambda) \in \overline{U}$ and $\mu \in [0,1]$. In addition assume the following two conditions hold:

$$\begin{cases} H_0 \ \text{ is essential in } \ A_{\partial U}(\overline{U}, E \times [0,1]); \ \text{ here} \\ H_0(x, \lambda) = H(x, \lambda, 0) = (N(x, \lambda), 0) \ \text{ for } \ (x, \lambda) \in \overline{U} \end{cases} \tag{2.3}$$

and

$$\Omega = \{(x, \lambda) \in \overline{U} :\ x \in N(x, \lambda)\} \ \text{ is compact and } \ \Omega_1 \neq \emptyset; \tag{2.4}$$

here $\Omega_t = \{x \in E : (x,t) \in \Omega\}$ *for each* $t \in [0,1]$. *Then* Ω *contains a continuum intersecting* $\Omega_0 \times \{0\}$ *and* $\Omega_1 \times \{1\}$.

Remark 2.1. Conditions to guarantee that $\Omega_1 \neq \emptyset$ for maps in $A(\overline{U}, E)$ can be found in [5, Theorem 2.5].

Proof. Note $A = \Omega_0 \times \{0\} \subseteq \Omega$ and $B = \Omega_1 \times \{1\} \subseteq \Omega$ are closed (and compact). If there is no continuum intersecting A and B then from Theorem 2.1, Ω can be represented as $\Omega = \Omega^\star \cup \Omega^{\star\star}$ where $\Omega^\star$ and $\Omega^{\star\star}$ are disjoint compact sets with $A \subseteq \Omega^\star$ and $B \subseteq \Omega^{\star\star}$. Notice $\Omega^\star$ and $\Omega^{\star\star} \cup \partial U$ are closed and disjoint (note $\Omega^\star \cap \partial U = \emptyset$ since if there exists a $(x,\lambda) \in \partial U$ and $(x,\lambda) \in \Omega^\star$ then (note $(x,\lambda) \in \Omega^\star \subseteq \Omega$) $x \in N(x,\lambda)$ which contradicts (2.2)). Now there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\Omega^{\star\star} \cup \partial U) = 0$ and $\mu(\Omega^\star) = 1$. Let

$$T(x,\lambda) = (N(x,\lambda), \mu(x,\lambda)) \quad \text{for} \quad (x,\lambda) \in \overline{U}.$$

Notice (2.1) guarantees that $T \in A(\overline{U}, E \times [0,1])$ and in fact $T \in A_{\partial U}(\overline{U}, E \times [0,1])$ since if there exists a $(x,\lambda) \in \partial U$ with $(x,\lambda) \in T(x,\lambda)$ then $(x,\lambda) \in (N(x,\lambda), \mu(x,\lambda)) = (N(x,\lambda), 0)$ so $x \in N(x,0)$ which contradicts (2.2). Notice as well (here $H_0(x,\lambda) = H(x,\lambda,0) = (N(x,\lambda),0)$) that

$$T|_{\partial U} = H_0|_{\partial U}$$

since if $(x,\lambda) \in \partial U$ then $T(x,\lambda) = (N(x,\lambda), \mu(x,\lambda)) = (N(x,\lambda), 0)$ (note $\mu(\Omega^{\star\star} \cup \partial U) = 0$). Now (2.3) guarantees that there exists a $(x,\lambda) \in U$ with $(x,\lambda) \in T(x,\lambda)$ i.e. $x \in N(x,\lambda)$ and $\lambda = \mu(x,\lambda)$. Note $(x,\lambda) \in \Omega$ since $(x,\lambda) \in U$ and $x \in N(x,\lambda)$. Now either $(x,\lambda) \in \Omega^\star$ or $(x,\lambda) \in \Omega^{\star\star}$.

Case 1. Suppose $(x,\lambda) \in \Omega^\star$.

Then $\mu(x,\lambda) = 1$. Thus $\lambda = \mu(x,\lambda) = 1$ and $x \in N(x,\lambda) = N(x,1)$ i.e. $(x,1) \in B \subseteq \Omega^{\star\star}$ which contradicts $(x,1) = (x,\lambda) \in \Omega^\star$.

Case 2. Suppose $(x,\lambda) \in \Omega^{\star\star}$.

Then $\mu(x,\lambda) = 0$. Thus $\lambda = \mu(x,\lambda) = 0$ and $x \in N(x,\lambda) = N(x,0)$ i.e. $(x,0) \in A \subseteq \Omega^\star$ which contradicts $(x,0) = (x,\lambda) \in \Omega^{\star\star}$. □

In our next result (2.2) is not assumed.

Theorem 2.4. *Suppose* $N \in A(\overline{U}, E)$ *with*

$$x \notin N(x,0) \quad \text{for} \quad (x,0) \in \partial U. \tag{2.5}$$

Let $H : \overline{U} \times [0,1] \to K(E \times [0,1])$ *be given by* $H(x,\lambda,\mu) = (N(x,\lambda), \mu)$ *for* $(x,\lambda) \in \overline{U}$ *and* $\mu \in [0,1]$ *and assume* (2.3) *and* (2.4) *hold. In addition for open subsets* W *of* U *with* $\Omega_0 \times \{0\} \subseteq W \subseteq U$ *(so* $x \notin N(x,0)$ *for* $(x,0) \in U \backslash W$*),* $\partial W \cap \Omega = \emptyset$ *and* $\overline{W} \cap (\partial U \cap \Omega) = \emptyset$ *assume* $N \in A(\overline{W}, E)$ *and the following conditions holds:*

$$\begin{cases} H_0 \ \ \text{is essential in} \ \ A_{\partial W}(\overline{W}, E \times [0,1]); \ \ \text{here} \\ H_0(x,\lambda) = H(x,\lambda,0) = (N(x,\lambda),0) \ \ \text{for} \ \ (x,\lambda) \in \overline{W} \end{cases} \tag{2.6}$$

and

$$\Sigma_1 \neq \emptyset; \tag{2.7}$$

here $\Sigma = \{(x,\lambda) \in \overline{W} : \ x \in N(x,\lambda)\}$ *and* $\Sigma_t = \{x \in E : \ (x,t) \in \Sigma\}$ *for each* $t \in [0,1]$. *Then* Ω *contains a continuum intersecting* $\Omega_0 \times \{0\}$ *and* $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$*; here* $\Omega_t = \{x \in E : \ (x,t) \in \Omega\}$ *for each* $t \in [0,1]$.

Proof. There are two cases to consider, namely $\Omega \cap \partial U = \emptyset$ or $\Omega \cap \partial U \neq \emptyset$. If $\Omega \cap \partial U = \emptyset$ then (2.2) holds so the result follows from Theorem 2.3. Now suppose $\Omega \cap \partial U \neq \emptyset$. Let $A = \Omega_0 \times \{0\}$, $B = \Omega_1 \times \{1\}$ and $C = \Omega \cap \partial U \, (\neq \emptyset)$. Notice $C \subseteq \Omega$ is closed and (2.5) guarantees that C is disjoint from A. Now from Theorem 2.1 either (1). there exists a continuum of Ω which intersects A and C (and we are finished), or (2). $\Omega = \Omega^\star \cup \Omega^{\star\star}$ where $\Omega^\star$ and $\Omega^{\star\star}$ are disjoint compact sets with $A \subseteq \Omega^\star$ and $B \subseteq \Omega^{\star\star}$. Suppose (2) occurs. Now from Theorem 2.2 there exists an open set V with

$$\Omega^\star \subseteq V, \ \ \overline{V} \cap \Omega^{\star\star} = \emptyset \ \text{ and } \ \partial V \cap \Omega = \emptyset. \tag{2.8}$$

Let $W = U \cap V$. We claim

$$A \subseteq W \subseteq U, \ \ \partial W \cap \Omega = \emptyset \ \text{ and } \ \overline{W} \cap (\partial U \cap \Omega) = \emptyset. \tag{2.9}$$

Note clearly $A \subseteq W$ since $A \subseteq \Omega^\star \subseteq V$ and $A \subseteq U$ from (2.5). To see that $\partial W \cap \Omega = \emptyset$ first notice that

$$\begin{aligned} \partial W &= (\overline{U \cap V}) \backslash (U \cap V) \subseteq (\overline{U} \cap \overline{V}) \backslash (U \cap V) \\ &= ((\overline{U} \backslash U) \cap \overline{V}) \cup ((\overline{V} \backslash V) \cap \overline{U}) \\ &= (\partial U \cap \overline{V}) \cup (\partial V \cap \overline{U}) \subseteq (\partial U \cap \overline{V}) \cup \partial V. \end{aligned}$$

If we show $\partial V \cap \Omega = \emptyset$ and $(\partial U \cap \overline{V}) \cap \Omega = \emptyset$ then $\partial W \cap \Omega = \emptyset$. Clearly $\partial V \cap \Omega = \emptyset$ from (2.8). Also from (2.8) we have $\overline{V} \cap \Omega^{\star\star} = \emptyset$ so since $C = \Omega \cap \partial U \subseteq \Omega^{\star\star}$ we have $\overline{V} \cap \Omega \cap \partial U = \emptyset$. Thus $\partial W \cap \Omega = \emptyset$. Next note $\overline{W} \cap \Omega^{\star\star} = \emptyset$ since $\overline{W} \subseteq \overline{U} \cap \overline{V} \subseteq \overline{V}$ and $\overline{V} \cap \Omega^{\star\star} = \emptyset$ from (2.8). Now $\overline{W} \cap \Omega^{\star\star} = \emptyset$ and $C \subseteq \Omega^{\star\star}$ implies $\overline{W} \cap (\partial U \cap \Omega) = \emptyset$. Consequently (2.9) holds [Note also that $\Omega^\star \subseteq W$ since $\Omega^\star \subseteq V$ from (2.8), $\Omega^\star \subseteq \overline{U}$ and $\partial U \cap \Omega^\star = \emptyset$ since if $x \in \partial U \cap \Omega^\star$ then $x \in \partial U \cap \Omega = C \subseteq \Omega^{\star\star}$ so $x \in \Omega^\star \cap \Omega^{\star\star}$, which is a contradiction since $\Omega^\star \cap \Omega^{\star\star} = \emptyset$. Of course if there exists $(x,0) \in U \backslash W$ with $x \in N(x,0)$ then $(x,0) \in \Omega_0 \times \{0\} = A \subseteq W$, a contradiction since $(x,0) \in U \backslash W$. Thus $x \notin N(x,0)$ for $(x,0) \in U \backslash W$.] Let

$$\Sigma = \left\{(x,\lambda) \in \overline{W} : \ x \in N(x,\lambda)\right\}.$$

Note $\partial W \cap \Sigma = \emptyset$ from (2.9) since $\Sigma \subseteq \Omega$. Now Theorem 2.3 implies that Σ contains a continuum intersecting $\Sigma_0 \times \{0\}$ $(\subseteq \Omega_0 \times \{0\})$ and $\Sigma_1 \times \{1\}$ $(\subseteq \Omega_1 \times \{1\})$ and our result follows. □

Remark 2.2. From the proof above we see that that one could replace (2.4) with the assumption that $\Omega_1 \neq \emptyset$ and $\{(x,\lambda) \in \overline{W} : \ x \in N(x,\lambda)\}$ is compact for open subsets W of U described in the statement of Theorem 2.4. We note also that (2.7) guarantees $\Omega_1 \neq \emptyset$ and (2.6) guarantees (2.3) if we remove $\partial W \cap \Omega = \emptyset$ and $\overline{W} \cap (\partial U \cap \Omega) = \emptyset$ in the statement of Theorem 2.4.

In our next result $\{(x,\lambda) \in \overline{U} : \ x \in N(x,\lambda)\}$ is compact is not assumed. For convenience we assume E is a normed space (basically the same proof below works if E is a metric space), U is an open subset of $E \times [0,1]$ and (2.1) holds.

Theorem 2.5. *Suppose $N \in A(\overline{U}, E)$ with*

$$x \notin N(x,0) \quad \textit{for} \quad (x,0) \in \partial U \tag{2.10}$$

and

$$\Omega_0 \quad \textit{is compact;} \tag{2.11}$$

here $\Omega_0 = \{x \in E : (x,0) \in \Omega\}$ where $\Omega = \{(x,\lambda) \in \overline{U} : x \in N(x,\lambda)\}$. Let $H : \overline{U} \times [0,1] \to K(E \times [0,1])$ be given by $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ for $(x,\lambda) \in \overline{U}$ and $\mu \in [0,1]$. In addition for open bounded subsets W of U with $\Omega_0 \times \{0\} \subseteq W \subseteq U$ (so $x \notin N(x,0)$ for $(x,0) \in U \backslash W$) assume $N \in A(\overline{W}, E)$ and the following conditions hold:

$$\begin{cases} H_0 \quad \textit{is essential in} \quad A_{\partial W}(\overline{W}, E \times [0,1]); \quad \textit{here} \\ H_0(x,\lambda) = H(x,\lambda,0) = (N(x,\lambda),0) \quad \textit{for} \quad (x,\lambda) \in \overline{W} \end{cases} \tag{2.12}$$

and

$$\Sigma = \{(x,\lambda) \in \overline{W} : x \in N(x,\lambda)\} \quad \textit{is compact and} \quad \Sigma_1 \neq \emptyset; \tag{2.13}$$

here $\Sigma_t = \{x \in E : (x,t) \in \Sigma\}$ for each $t \in [0,1]$. Then Ω contains a connected component intersecting $\Omega_0 \times \{0\}$ and which either intersects $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$ or is unbounded; here $\Omega_t = \{x \in E : (x,t) \in \Omega\}$ for each $t \in [0,1]$.

Proof. Since Ω_0 is compact there exists $n_0 \in \mathbf{N}$ with $\Omega_0 \subseteq B(0,n_0)$. For $n \geq n_0$ let

$$U^n = U \cap (B(0,n) \times [0,1]) \quad \text{and} \quad \Omega^n = \{(x,\lambda) \in \overline{U^n} : x \in N(x,\lambda)\}.$$

Now $\Omega_0 \subseteq B(0,n_0)$ and (2.10) implies $\Omega_0 \times \{0\} \subseteq U$ so $\Omega_0 \times \{0\} \subseteq U^n$. Of course if there exists $(x,0) \in U \backslash U^n$ with $x \in N(x,0)$ then $(x,0) \in \Omega_0 \times \{0\} \subseteq U^n$, a contradiction. Thus $x \notin N(x,0)$ for $(x,0) \in U \backslash U^n$. For each $n \geq n_0$, Theorem 2.4 implies there exists $(x_n,0) \in \Omega_0 \times \{0\}$ and a connected component $\mathcal{C}_n$ of Ω^n containing $(x_n,0)$ and intersecting $(\partial U^n \cap \Omega^n) \cup (\Omega_1^n \times \{1\})$ (here $\Omega_1^n = \{x \in E : (x,1) \in \Omega^n\}$). Since Ω_0 is compact the sequence (x_n) has an accumulation point $x_0 \in \Omega_0$. Assume that there is NO connected component of Ω intersecting $\Omega_0 \times \{0\}$ and $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$. Let $\mathcal{C}_0$ be the connected component containing x_0 (and not intersecting $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$). Our result follows if we show $\mathcal{C}_0$ is unbounded. Assume $\mathcal{C}_0$ is bounded. Note $\mathcal{C}_0 \subseteq \overline{U}$ and $\mathcal{C}_0 \cap \partial U = \emptyset$ (since $\mathcal{C}_0$ does not intersect $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$) so $\mathcal{C}_0 \subseteq U$, and note $\mathcal{C}_0$, $\Omega_0 \times \{0\}$ are closed and bounded and as a result we can choose an open bounded set V with

$$\mathcal{C}_0 \cup (\Omega_0 \times \{0\}) \subseteq V \subseteq U.$$

We claim $\partial V \cap \Omega \neq \emptyset$. Suppose $\partial V \cap \Omega = \emptyset$. Of course if there exists $(x,0) \in U \backslash V$ with $x \in N(x,0)$ then $(x,0) \in \Omega_0 \times \{0\} \subseteq V$, a contradiction. Thus $x \notin N(x,0)$ for $(x,0) \in U \backslash V$. Now Theorem 2.3 (note $\tilde{\Omega}_0 \times \{0\} \subseteq V \subseteq U$ and $\partial V \cap \tilde{\Omega} = \emptyset$ since $\tilde{\Omega} \subseteq \Omega$) implies that $\tilde{\Omega} = \{(x,\lambda) \in \overline{V} : x \in N(x,\lambda)\}$ has a connected component intersecting $\tilde{\Omega}_0 \times \{0\}$ ($\subseteq \Omega_0 \times \{0\}$) and $\tilde{\Omega}_1 \times \{1\}$ ($\subseteq \Omega_1 \times \{1\}$), which contradicts the assumption that there is no connected component of Ω intersecting $\Omega_0 \times \{0\}$ and $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$; here $\tilde{\Omega}_t = \{x \in E : (x,t) \in \tilde{\Omega}\}$ for $t \in [0,1]$. Thus $\partial V \cap \Omega \neq \emptyset$. Note $(x_0,0) \in \Omega_0 \times \{0\} \subseteq V$ so $(x_0,0)$ and $\partial V \cap \Omega$ are closed disjoint subsets of the compact set $\tilde{\Omega}$ and the connected component of $\tilde{\Omega}$ containing $(x_0,0)$ does not intersect

$\partial V \cap \Omega$ (since $\mathcal{C}_0 \subseteq V$). Now from Theorem 2.2 there exists an open neighborhood V_0 of $(x_0, 0)$ with

$$(x_0,0) \in V_0, \ \ \overline{V_0} \cap (\Omega \cap \partial V) = \emptyset \quad \text{and} \quad \partial V_0 \cap \tilde{\Omega} = \emptyset. \tag{2.14}$$

Let $W = V \cap V_0$. Now $W \subseteq V$ with

$$(x_0,0) \in W \quad \text{and} \quad \partial W \cap \Omega = \emptyset \tag{2.15}$$

since $\partial W \subseteq (\partial V \cap \overline{V_0}) \cup (\partial V_0 \cap \overline{V})$ and note $(\partial V \cap \overline{V_0}) \cap \Omega = \overline{V_0} \cap (\partial V \cap \Omega) = \emptyset$ from (2.14) and $(\partial V_0 \cap \overline{V}) \cap \Omega = \partial V_0 \cap (\overline{V} \cap \Omega) = \partial V_0 \cap \tilde{\Omega} = \emptyset$ from (2.14).

Now V is bounded and W is an open neighborhood of $(x_0, 0)$ so there exists a $n_1 \geq n_0$ with

$$(x_{n_1}, 0) \in W \quad \text{and} \quad V \subseteq B(0, n_1) \times [0,1].$$

Note $(x_{n_1}, 0) \in W \cap \mathcal{C}_{n_1}$ so $W \cap \mathcal{C}_{n_1} \neq \emptyset$. Also note that $\mathcal{C}_{n_1}$ meets $(E \times [0,1]) \backslash W$ since $\mathcal{C}_{n_1}$ intersects $(\partial U^{n_1} \cap \Omega^{n_1}) \cup (\Omega_1^{n_1} \times \{1\})$ (and does not intersect $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$). Now $\mathcal{C}_{n_1}$ is connected so $\mathcal{C}_{n_1} \cap \partial W \neq \emptyset$. This is a contradiction since $\mathcal{C}_{n_1} \cap \partial W \subseteq \Omega^{n_1} \cap \partial W \subseteq \Omega \cap \partial W = \emptyset$ from (2.15). □

We now show that the ideas in this section can be applied to other natural situations. First let E be a completely regular topological vector space, Y a topological vector space, and U an open subset of E. Also let $L : dom\, L \subseteq E \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\, L$ is a vector subspace of E. Finally $T : E \to Y$ will be a linear, continuous single valued map with $L + T : dom\, L \to Y$ an isomorphism (i.e. a linear homeomorphism); for convenience we say $T \in H_L(E, Y)$.

Definition 2.4. Let $F : \overline{U} \to 2^Y$. We say $F \in A(\overline{U}, Y; L, T)$ if $(L+T)^{-1}(F+T) \in A(\overline{U}, E)$.

Definition 2.5. We say $F \in A_{\partial U}(\overline{U}, Y; L, T)$ if $F \in A(\overline{U}, Y; L, T)$ with $L\, x \notin F(x)$ for $x \in \partial U \cap dom\, L$.

Definition 2.6. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$. We say F is essential in $A_{\partial U}(\overline{U}, Y; L, T)$ if for every map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J|_{\partial U} = F|_{\partial U}$ there exists $x \in U \cap dom\, L$ with $L\, x \in J(x)$.

For our next results we assume E is a metric vector space, Y a topological vector space, and U an open subset of $E \times [0,1]$. Also let $L : dom\, L \subseteq E \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\, L$ is a vector subspace of E. Now let $\mathcal{L} : dom\, \mathcal{L} = dom\, L \times [0,1] \to Y \times [0,1]$ be given by $\mathcal{L}(y, \lambda) = (L\, y, \lambda)$. Let $T : E \to Y$ be a linear, continuous single valued map with $L + T : dom\, L \to Y$ an isomorphism (i.e. a linear homeomorphism) and let $\mathcal{T} : E \times [0,1] \to Y \times [0,1]$ be given by $\mathcal{T}(y, \lambda) = (T\, y, 0)$. Notice $(\mathcal{L} + \mathcal{T})^{-1}(y, \lambda) = ((L+T)^{-1} y, \lambda)$ for $(y, \lambda) \in Y \times [0,1]$.

We will also assume

$$\begin{cases} \text{if } \ F \in A(\overline{U}, Y; L, T), v \in \mathbf{C}(\overline{U}, [0,1]) \\ \text{and if } \ \Phi(y) = (F(y), v(y)) \ \text{ for } \ y \in \overline{U}, \\ \text{then } \ \Phi \in A(\overline{U}, Y \times [0,1]; \mathcal{L}, \mathcal{T}). \end{cases} \tag{2.16}$$

Theorem 2.6. *Suppose $N \in A(\overline{U}, Y; L, T)$ with*

$$L\,x \notin N(x,\lambda) \quad \text{for} \quad (x,\lambda) \in \partial U \cap dom\,\mathcal{L}. \tag{2.17}$$

Let $H : \overline{U} \times [0,1] \to 2^{Y \times [0,1]}$ be given by $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ for $(x,\lambda) \in \overline{U}$ and $\mu \in [0,1]$. In addition assume the following two conditions hold:

$$\begin{cases} H_0 \;\; \text{is essential in} \;\; A_{\partial U}(\overline{U}, Y \times [0,1]; \mathcal{L}, \mathcal{T}); \;\; \text{here} \\ H_0(x,\lambda) = H(x,\lambda,0) = (N(x,\lambda),0) \;\; \text{for} \;\; (x,\lambda) \in \overline{U} \end{cases} \tag{2.18}$$

and

$$\Omega = \{(x,\lambda) \in \overline{U} \cap dom\,\mathcal{L} : \; L\,x \in N(x,\lambda)\} \;\; \text{is compact and} \;\; \Omega_1 \neq \emptyset; \tag{2.19}$$

here $\Omega_t = \{x \in E : (x,t) \in \Omega\}$ for each $t \in [0,1]$. Then Ω contains a continuum intersecting $\Omega_0 \times \{0\}$ and $\Omega_1 \times \{1\}$.

Remark 2.3. Conditions to guarantee that $\Omega_1 \neq \emptyset$ for maps in $A(\overline{U}, Y; L, T)$ can be found in [5, Theorem 2.12].

Proof. Note $A = \Omega_0 \times \{0\} \subseteq \Omega$ and $B = \Omega_1 \times \{1\} \subseteq \Omega$ are closed (and compact). If there is no continuum intersecting A and B then from Theorem 2.1, Ω can be represented as $\Omega = \Omega^\star \cup \Omega^{\star\star}$ where $\Omega^\star$ and $\Omega^{\star\star}$ are disjoint compact sets with $A \subseteq \Omega^\star$ and $B \subseteq \Omega^{\star\star}$. Notice $\Omega^\star$ and $\Omega^{\star\star} \cup \partial U$ are closed and disjoint (note $\Omega^\star \cap \partial U = \emptyset$ since if there exists a $(x,\lambda) \in \partial U$ and $(x,\lambda) \in \Omega^\star$ then (note $(x,\lambda) \in \Omega^\star \subseteq \Omega$) $L\,x \in N(x,\lambda)$ which contradicts (2.17)). Now there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\Omega^{\star\star} \cup \partial U) = 0$ and $\mu(\Omega^\star) = 1$. Let

$$T_0(x,\lambda) = (N(x,\lambda), \mu(x,\lambda)) \quad \text{for} \quad (x,\lambda) \in \overline{U}.$$

Notice (2.16) guarantees that $T_0 \in A(\overline{U}, Y \times [0,1]; \mathcal{L}, \mathcal{T})$ and in fact $T_0 \in A_{\partial U}(\overline{U}, Y \times [0,1]; \mathcal{L}, \mathcal{T})$ since if there exists a $(x,\lambda) \in \partial U$ with $\mathcal{L}(x,\lambda) = (L\,x,\lambda) \in T_0(x,\lambda)$ then $(L\,x,\lambda) \in (N(x,\lambda), \mu(x,\lambda)) = (N(x,\lambda),0)$ so $L\,x \in N(x,0)$ which contradicts (2.17). Notice as well (here $H_0(x,\lambda) = H(x,\lambda,0) = (N(x,\lambda),0)$) that

$$T_0|_{\partial U} = H_0|_{\partial U}$$

since if $(x,\lambda) \in \partial U$ then $T_0(x,\lambda) = (N(x,\lambda), \mu(x,\lambda)) = (N(x,\lambda),0)$ (note $\mu(\Omega^{\star\star} \cup \partial U) = 0$). Now (2.18) guarantees that there exists a $(x,\lambda) \in U \cap dom\,\mathcal{L}$ with $\mathcal{L}(x,\lambda) \in T_0(x,\lambda)$ i.e. $L\,x \in N(x,\lambda)$ and $\lambda = \mu(x,\lambda)$. Note $(x,\lambda) \in \Omega$ since $(x,\lambda) \in U \cap dom\,\mathcal{L}$ and $L\,x \in N(x,\lambda)$. Now either $(x,\lambda) \in \Omega^\star$ or $(x,\lambda) \in \Omega^{\star\star}$.

Case 1. Suppose $(x,\lambda) \in \Omega^\star$.

Then $\mu(x,\lambda) = 1$. Thus $\lambda = \mu(x,\lambda) = 1$ and $L\,x \in N(x,\lambda) = N(x,1)$ i.e. $(x,1) \in B \subseteq \Omega^{\star\star}$ which contradicts $(x,1) = (x,\lambda) \in \Omega^\star$.

Case 2. Suppose $(x,\lambda) \in \Omega^{\star\star}$.

Then $\mu(x,\lambda) = 0$. Thus $\lambda = \mu(x,\lambda) = 0$ and $L\,x \in N(x,\lambda) = N(x,0)$ i.e. $(x,0) \in A \subseteq \Omega^\star$ which contradicts $(x,0) = (x,\lambda) \in \Omega^{\star\star}$. □

In our next result (2.17) is not assumed.

Theorem 2.7. *Suppose $N \in A(\overline{U}, Y; L, T)$ with*

$$L\,x \notin N(x,0) \quad \text{for} \quad (x,\lambda) \in \partial U \cap dom\,\mathcal{L}. \tag{2.20}$$

Let $H : \overline{U} \times [0,1] \to 2^{Y\times[0,1]}$ be given by $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ for $(x,\lambda) \in \overline{U}$ and $\mu \in [0,1]$ and assume (2.18) *and* (2.19) *hold. In addition for open subsets W of U with $\Omega_0 \times \{0\} \subseteq W \subseteq U$, $\partial W \cap \Omega = \emptyset$, and $\overline{W} \cap (\partial U \cap \Omega) = \emptyset$ assume $N \in A(\overline{W}, Y; L, T)$ and the following conditions hold:*

$$\begin{cases} \text{if } \ F \in A(\overline{W}, Y; L, T), v \in \mathbf{C}(\overline{W}, [0,1]) \\ \text{and if } \ \Phi(y) = (F(y), v(y)) \ \text{ for } y \in \overline{W}, \\ \text{then } \ \Phi \in A(\overline{W}, Y \times [0,1]; \mathcal{L}, \mathcal{T}) \end{cases} \tag{2.21}$$

$$\begin{cases} H_0 \ \text{ is essential in } \ A_{\partial W}(\overline{W}, Y \times [0,1]; \mathcal{L}, \mathcal{T}); \ \text{ here} \\ H_0(x,\lambda) = H(x,\lambda,0) = (N(x,\lambda),0) \ \text{ for } \ (x,\lambda) \in \overline{W} \end{cases} \tag{2.22}$$

and

$$\Sigma_1 \neq \emptyset; \tag{2.23}$$

here $\Sigma = \big\{(x,\lambda) \in \overline{W} \cap dom\, \mathcal{L} : \ L\, x \in N(x,\lambda)\big\}$ and $\Sigma_t = \{x \in E : \ (x,t) \in \Sigma\}$ for each $t \in [0,1]$. Then Ω contains a continuum intersecting $\Omega_0 \times \{0\}$ and $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$; here $\Omega_t = \{x \in E : \ (x,t) \in \Omega\}$ for each $t \in [0,1]$.

Proof. There are two cases to consider, namely $\Omega \cap \partial U = \emptyset$ or $\Omega \cap \partial U \neq \emptyset$. If $\Omega \cap \partial U = \emptyset$ then (2.17) holds so the result follows from Theorem 2.6. Now suppose $\Omega \cap \partial U \neq \emptyset$. Let $A = \Omega_0 \times \{0\}$, $B = \Omega_1 \times \{1\}$ and $C = \Omega \cap \partial U$ $(\neq \emptyset)$. Notice $C \subseteq \Omega$ is closed and (2.20) guarantees that C is disjoint from A. Now from Theorem 2.1 either (1). there exists a continuum of Ω which intersects A and C (and we are finished), or (2). $\Omega = \Omega^\star \cup \Omega^{\star\star}$ where $\Omega^\star$ and $\Omega^{\star\star}$ are disjoint compact sets with $A \subseteq \Omega^\star$ and $B \subseteq \Omega^{\star\star}$. Suppose (2) occurs. Now from Theorem 2.2 there exists an open set V with

$$\Omega^\star \subseteq V, \ \ \overline{V} \cap \Omega^{\star\star} = \emptyset \ \text{ and } \ \partial V \cap \Omega = \emptyset.$$

Let $W = U \cap V$ and the same reasoning as in Theorem 2.4 establishes that

$$A \subseteq W \subseteq U, \ \ \partial W \cap \Omega = \emptyset \ \text{ and } \ \overline{W} \cap (\partial U \cap \Omega) = \emptyset. \tag{2.24}$$

Let

$$\Sigma = \big\{(x,\lambda) \in \overline{W} \cap dom\, \mathcal{L} : \ L\, x \in N(x,\lambda)\big\}.$$

Note $\partial W \cap \Sigma = \emptyset$ from (2.24) since $\Sigma \subseteq \Omega$. Now Theorem 2.6 implies that Σ contains a continuum intersecting $\Sigma_0 \times \{0\}$ $(\subseteq \Omega_0 \times \{0\})$ and $\Sigma_1 \times \{1\}$ $(\subseteq \Omega_1 \times \{1\})$ and our result follows. □

Remark 2.4. From the proof above we see that that one could replace (2.19) with the assumption that $\Omega_1 \neq \emptyset$ and $\{(x,\lambda) \in \overline{W} \cap dom\, \mathcal{L} : \ L\, x \in N(x,\lambda)\}$ is compact for open subsets W of U described in the statement of Theorem 2.4. We note also that (2.23) guarantees $\Omega_1 \neq \emptyset$ and (2.22) guarantees (2.18) if we remove $\partial W \cap \Omega = \emptyset$ and $\overline{W} \cap (\partial U \cap \Omega) = \emptyset$ in the statement of Theorem 2.7.

In our next result $\{(x,\lambda) \in \overline{U} \cap dom\, \mathcal{L} : \ L\, x \in N(x,\lambda)\}$ is compact is not assumed. For convenience we assume E is a normed space, U is an open subset of $E \times [0,1]$ and (2.16) holds.

Theorem 2.8. *Suppose $N \in A(\overline{U}, Y; L, T)$ with*

$$L\, x \notin N(x,0) \ \ \text{for} \ \ (x,\lambda) \in \partial U \cap dom\, \mathcal{L} \tag{2.25}$$

and and

$$\Omega_0 \quad \textit{is compact;} \tag{2.26}$$

here $\Omega_0 = \{x \in E : \ (x,0) \in \Omega\}$ *where* $\Omega = \{(x,\lambda) \in \overline{U} \cap dom\,\mathcal{L} : \ L\,x \in N(x,\lambda)\}$. *Let* $H : \overline{U} \times [0,1] \to 2^{Y \times [0,1]}$ *be given by* $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ *for* $(x,\lambda) \in \overline{U}$ *and* $\mu \in [0,1]$. *In addition for open bounded subsets* W *of* U *with* $\Omega_0 \times \{0\} \subseteq W \subseteq U$ *assume* $N \in A(\overline{W}, Y; L, T)$ *and the following conditions hold:*

$$\begin{cases} \textit{if } \ F \in A(\overline{W}, Y; L, T), v \in \mathbf{C}(\overline{W}, [0,1]) \\ \textit{and if } \ \Phi(y) = (F(y), v(y)) \ \textit{ for } \ y \in \overline{W}, \\ \textit{then } \ \Phi \in A(\overline{W}, Y \times [0,1]; \mathcal{L}, \mathcal{T}) \end{cases} \tag{2.27}$$

$$\begin{cases} H_0 \ \ \textit{is essential in } \ A_{\partial W}(\overline{W}, Y \times [0,1]; \mathcal{L}, \mathcal{T}); \ \ \textit{here} \\ H_0(x,\lambda) = H(x,\lambda,0) = (N(x,\lambda),0) \ \textit{ for } \ (x,\lambda) \in \overline{W} \end{cases} \tag{2.28}$$

and

$$\Sigma = \{(x,\lambda) \in \overline{W} \cap dom\,\mathcal{L} : \ L\,x \in N(x,\lambda)\} \quad \textit{is compact and} \ \ \Sigma_1 \neq \emptyset; \tag{2.29}$$

here $\Sigma_t = \{x \in E : \ (x,t) \in \Sigma\}$ *for each* $t \in [0,1]$. *Then* Ω *contains a connected component intersecting* $\Omega_0 \times \{0\}$ *and which either intersects* $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$ *or is unbounded; here* $\Omega_t = \{x \in E : \ (x,t) \in \Omega\}$ *for each* $t \in [0,1]$.

Proof. Since Ω_0 is compact there exists $n_0 \in \mathbf{N}$ with $\Omega_0 \subseteq B(0,n_0)$. For $n \geq n_0$ let

$$U^n = U \cap (B(0,n) \times [0,1]) \quad \text{and} \quad \Omega^n = \{(x,\lambda) \in \overline{U^n} \cap dom\,\mathcal{L} : \ L\,x \in N(x,\lambda)\}.$$

Now $\Omega_0 \subseteq B(0,n_0)$ and $\Omega_0 \times \{0\} \subseteq U$ so $\Omega_0 \times \{0\} \subseteq U^n$. For each $n \geq n_0$, Theorem 2.7 implies there exists $(x_n,0) \in \Omega_0 \times \{0\}$ and a connected component $\mathcal{C}_n$ of Ω^n containing $(x_n,0)$ and intersecting $(\partial U^n \cap \Omega^n) \cup (\Omega_1^n \times \{1\})$ (here $\Omega_1^n = \{x \in E : (x,1) \in \Omega^n\}$). Since Ω_0 is compact the sequence (x_n) has an accumulation point $x_0 \in \Omega_0$. Assume that there is NO connected component of Ω intersecting $\Omega_0 \times \{0\}$ and $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$. Let $\mathcal{C}_0$ be the connected component containing x_0 (and not intersecting $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$). Our result follows if we show $\mathcal{C}_0$ is unbounded. Assume $\mathcal{C}_0$ is bounded. Note $\mathcal{C}_0 \subseteq \overline{U}$ and $\mathcal{C}_0 \cap \partial U = \emptyset$ (since $\mathcal{C}_0$ does not intersect $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$) so $\mathcal{C}_0 \subseteq U$, and note $\mathcal{C}_0$, $\Omega_0 \times \{0\}$ are closed and bounded and as a result we can choose an open bounded set V with

$$\mathcal{C}_0 \cup (\Omega_0 \times \{0\}) \subseteq V \subseteq U.$$

We claim $\partial V \cap \Omega \neq \emptyset$. Suppose $\partial V \cap \Omega = \emptyset$. Now Theorem 2.6 (note $\tilde{\Omega}_0 \times \{0\} \subseteq V \subseteq U$ and $\partial V \cap \tilde{\Omega} = \emptyset$ since $\tilde{\Omega} \subseteq \Omega$) implies that $\tilde{\Omega} = \{(x,\lambda) \in \overline{V} \cap dom\,\mathcal{L} : \ L\,x \in N(x,\lambda)\}$ has a connected component intersecting $\tilde{\Omega}_0 \times \{0\}$ $(\subseteq \Omega_0 \times \{0\})$ and $\tilde{\Omega}_1 \times \{1\}$ $(\subseteq \Omega_1 \times \{1\})$, which contradicts the assumption that there is no connected component of Ω intersecting $\Omega_0 \times \{0\}$ and $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$; here $\tilde{\Omega}_t = \{x \in E : \ (x,t) \in \tilde{\Omega}\}$ for $t \in [0,1]$. Thus $\partial V \cap \Omega \neq \emptyset$. Note $(x_0,0) \in \Omega_0 \times \{0\} \subseteq V$ so $(x_0,0)$ and $\partial V \cap \Omega$ are closed disjoint subsets of the compact set $\tilde{\Omega}$ and the connected component of $\tilde{\Omega}$ containing $(x_0,0)$ does not intersect $\partial V \cap \Omega$ (since $\mathcal{C}_0 \subseteq V$). Now from Theorem 2.2 there exists an open neighborhood V_0 of $(x_0,0)$ with

$$(x_0,0) \in V_0, \ \ \overline{V_0} \cap (\Omega \cap \partial V) = \emptyset \quad \text{and} \quad \partial V_0 \cap \tilde{\Omega} = \emptyset.$$

Let $W = U \cap V$ and the same reasoning as in Theorem 2.5 establishes that

$$(x_0, 0) \in W \quad \text{and} \quad \partial W \cap \Omega = \emptyset. \tag{2.30}$$

Now V is bounded and W is an open neighborhood of $(x_0, 0)$ so there exists a $n_1 \geq n_0$ with

$$(x_{n_1}, 0) \in W \quad \text{and} \quad V \subseteq B(0, n_1) \times [0, 1].$$

Note $(x_{n_1}, 0) \in W \cap \mathcal{C}_{n_1}$ so $W \cap \mathcal{C}_{n_1} \neq \emptyset$. Also note that $\mathcal{C}_{n_1}$ meets $(E \times [0,1]) \backslash W$ since $\mathcal{C}_{n_1}$ intersects $(\partial U^{n_1} \cap \Omega^{n_1}) \cup (\Omega_1^{n_1} \times \{1\})$ (and does not intersect $(\partial U \cap \Omega) \cup (\Omega_1 \times \{1\})$). Now $\mathcal{C}_{n_1}$ is connected so $\mathcal{C}_{n_1} \cap \partial W \neq \emptyset$. This is a contradiction since $\mathcal{C}_{n_1} \cap \partial W \subseteq \Omega^{n_1} \cap \partial W \subseteq \Omega \cap \partial W = \emptyset$ from (2.30). $\square$

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