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SOLVABILITY OF A FUNCTIONAL EQUATION ARISING IN DYNAMIC PROGRAMMING

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Abstract. This paper deals with solvability of the following functional equation arising in dynamic programming of multistage decision processes

$$f(x) = \operatorname*{opt}_{y\in D} \{u(x,y)(p(x,y) + f(a(x,y))) + v(x,y) \operatorname{opt}\{q(x,y), f(b(x,y))\}\}, \quad \forall x \in S.$$

Using the Banach fixed point theorem and new iterative techniques, we obtain the existence and uniqueness of solutions for the above equation in the complete metric space $BB(S)$ and the Banach spaces $BC(S)$ and $B(S)$, construct some iterative methods, prove their convergence and provide several error estimates between these iterative sequences generated by the iterative methods and the corresponding solutions, respectively. Four nontrivial examples illustrating applications of the results presented in this paper are provided.

Key Words and Phrases: Dynamic programming, functional equation, Banach fixed point theorem, nonexpansive mapping, iterative methods, error estimates.

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1. Introduction and preliminaries

The existence problems of solutions for some classes of functional equations arising in dynamic programming have been established in [1-12]. In 1984, Bhakta and Mitra [6] studied the following functional equation

$$f(x) = \sup_{y\in D}\{p(x,y) + f(a(x,y))\}, \quad \forall x \in S \tag{1.1}$$

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and gave the existence, uniqueness and iterative approximation of solution for the functional equation (1.1). In 1988, Bhakta and Choudhury [5] proved the existence of solution for the following functional equation

$$f(x) = \inf_{y\in D} \max\{q(x,y), f(b(x,y))\}, \quad \forall x \in S. \tag{1.2}$$

In 2003, Liu and Ume [9] provided sufficient conditions which ensure the existence, uniqueness and iterative approximation of solutions for the functional equation

$$\begin{aligned} f(x) = \operatorname*{opt}_{y\in D} \big\{&u(p(x,y) + f(T(x,y))) \\ &+ (1-u)\,\mathrm{opt}\{q(x,y), f(T(x,y))\}\big\}, \quad \forall x \in S, \end{aligned} \tag{1.3}$$

where $u \in [0,1]$ is a constant.

Motivated and inspired by the research work going on in this field, we introduce the following functional equation arising in dynamic programming of multistage decision processes

$$\begin{aligned} f(x) = \operatorname*{opt}_{y\in D} \big\{&u(x,y)(p(x,y) + f(a(x,y))) \\ &+ v(x,y)\,\mathrm{opt}\{q(x,y), f(b(x,y))\}\big\}, \quad \forall x \in S, \end{aligned} \tag{1.4}$$

where x and y represent the state and decision vectors, respectively, a and b represent the transformations of the processes, and $f(x)$ represents the optimal return function with initial state x, opt denotes sup or inf. It is clear that the functional equation (1.4) includes the functional equations (1.1)-(1.3) as special cases. Utilizing new iterative methods and the Banach fixed point theorem, we establish these conditions which guarantee the existence, uniqueness and iterative approximations of solutions for the functional equation (1.4) in the complete metric space $BB(S)$ and the Banach spaces $BC(S)$ and $B(S)$, and discuss the error estimates between the iterative approximations and the solutions, respectively. Four examples are added to illustrate the results obtained in this paper are more effective than the existing ones in the literature.

Throughout this paper, we assume that

$$\mathbb{R} = (-\infty, +\infty),\ \mathbb{R}^+ = [0, +\infty),\ \mathbb{R}^- = (-\infty, 0],\ \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

and $\mathbb{N}$ denotes the set of all positive integers, $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$ are real Banach spaces, $S \subseteq X$ is the state space and $D \subseteq Y$ is the decision space. Define

$$\begin{aligned} B(S) &= \{f : f : S \to \mathbb{R} \text{ is bounded}\}, \\ BC(S) &= \{f : f \in B(S) \text{ is continuous}\}, \\ BB(S) &= \{f : f : S \to \mathbb{R} \text{ is bounded on bounded subsets of } S\}, \\ \Phi &= \Big\{(\varphi, \psi) : \varphi \text{ and } \psi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ are nondecreasing, } \sum_{n=0}^{\infty} \psi(\varphi^n(t)) < +\infty \\ &\qquad \text{and } \psi(t) > 0 \text{ for all } t > 0\Big\}. \end{aligned}$$

Clearly, $(B(S), \|\cdot\|_1)$ and $(BC(S), \|\cdot\|_1)$ are Banach spaces with the norm

$$\|f\|_1 = \sup_{x \in S} |f(x)|.$$

For each $k \in \mathbb{N}$ and $f, g \in BB(S)$, let

$$d_k(f,g) = \sup\big\{|f(x) - g(x)| : x \in \overline{B}(0,k)\big\},$$

$$d(f,g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(f,g)}{1 + d_k(f,g)},$$

where $\overline{B}(0,k) = \{x : x \in S \text{ and } \|x\| \le k\}$. Clearly $\{d_k\}_{k\in\mathbb{N}}$ is a countable family of pseudometrics on $BB(S)$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in $BB(S)$ is said to be converge to a point $x \in BB(S)$ if for each $k \in \mathbb{N}$, $d_k(x_n, x) \to 0$ as $n \to \infty$, and to be a Cauchy Sequence if for each $k \in \mathbb{N}$, $d_k(x_n, x_m) \to 0$ as $n, m \to \infty$. It is easy to verify that $(BB(S), d)$ is a complete metric space.

Lemma 1.1. ([8]) *Let a, b, c and d be in $\mathbb{R}$. Then*

$$|\operatorname{opt}\{a,b\} - \operatorname{opt}\{c,d\}| \le \max\{|a-c|, |b-d|\}.$$

Lemma 1.2. ([7]) *Let E be a set, p and $q : E \to \mathbb{R}$ be mappings. If $\operatorname{opt}_{y\in E} p(y)$ and $\operatorname{opt}_{y\in E} q(y)$ are bounded, then*

$$\left|\operatorname*{opt}_{y\in E} p(y) - \operatorname*{opt}_{y\in E} q(y)\right| \le \sup_{y\in E} |p(y) - q(y)|.$$

2. Properties of solutions for the functional equation (1.4)

Now we study the solvability of the functional equation (1.4) in the complete metric space $BB(S)$.

Theorem 2.1. *Let $(\varphi, \psi) \in \Phi$, $u, v, p, q : S \times D \to \mathbb{R}$ and $a, b : S \times D \to S$ be mappings such that*

(C1) $\sup_{(x,y)\in S\times D}\{|u(x,y)| + |v(x,y)|\} \le 1$;

(C2) $\sup_{y\in D} \max\{|p(x,y)|, |q(x,y)|\} \le \psi(\|x\|)$, $\forall x \in S$;

(C3) $\sup_{y\in D} \max\{\|a(x,y)\|, \|b(x,y)\|\} \le \varphi(\|x\|)$, $\forall x \in S$.

Then the functional equation (1.4) possesses a solution $z \in BB(S)$ such that

(C4) *For each $z_0 \in BB(S)$ with $|z_0(x)| \le \psi(\|x\|)$, $\forall x \in S$, the sequence $\{z_n\}_{n\in\mathbb{N}_0}$ defined by*

$$z_n(x) = \operatorname*{opt}_{y\in D} \big\{u(x,y)(p(x,y) + z_{n-1}(a(x,y))) + v(x,y) \operatorname{opt}\{q(x,y), z_{n-1}(b(x,y))\}\big\}, \quad \forall (x,n) \in S \times \mathbb{N}$$

converges to z;

(C5) $\lim_{n\to\infty} z(x_n) = 0$ *for any $x_0 \in S$, $\{y_n\}_{n\in\mathbb{N}} \subseteq D$ and $x_n \in \{a(x_{n-1}, y_n), b(x_{n-1}, y_n)\}$, $\forall n \in \mathbb{N}$;*

(C6) *z is unique relative to* (C5);

(C7) *If* u *and* v *are nonnegative and* $u(x,y)+v(x,y)=1$, $\forall(x,y)\in S\times D$, *then for any* $\varepsilon>0$ *and* $x_0\in S$, *there exist* $\{y_n\}_{n\in\mathbb{N}}\subset D$ *and* $x_n\in\{a(x_{n-1},y_n),b(x_{n-1},y_n)\}$, $\forall n\in\mathbb{N}$ *such that*

$$z(x_0)\ge\sum_{n=1}^{\infty}u(x_n,y_{n+1})p(x_n,y_{n+1})-\varepsilon$$

provided that $\text{opt}=\max$ *and*

$$z(x_0)\le\sum_{n=1}^{\infty}u(x_n,y_{n+1})p(x_n,y_{n+1})+\varepsilon$$

provided that $\text{opt}=\min$.

Proof. For any $(x,y,h)\in S\times D\times BB(S)$, put

$$H(x,y,h)=u(x,y)(p(x,y)+h(a(x,y)))+v(x,y)\,\text{opt}\{q(x,y),h(b(x,y))\}$$

and

$$Gh(x)=\underset{y\in D}{\text{opt}}\,H(x,y,h).$$

Notice that $(\varphi,\psi)\in\Phi$ implies that

$$\varphi(t)<t,\quad\forall t>0. \tag{2.1}$$

Firstly we assert that $G:BB(S)\to BB(S)$ is nonexpansive.

Let $(k,h)\in\mathbb{N}\times BB(S)$. It follows from (C3) and (2.1) that

$$\sup_{y\in D}\max\{\|a(x,y)\|,\|b(x,y)\|\}\le\varphi(\|x\|)\le\|x\|\le k,\quad\forall x\in\overline{B}(0,k),$$

which yields that there exists a constant $g(k)>0$ satisfying

$$\sup_{y\in D}\max\{|h(a(x,y))|,|h(b(x,y))|\}\le g(k),\quad\forall x\in\overline{B}(0,k).$$

Owing to (C1), (C2), $(\varphi,\psi)\in\Phi$ and Lemma 1.1, we obtain that

$$\begin{aligned}|H(x,y,h)|&=|u(x,y)(p(x,y)+h(a(x,y)))+v(x,y)\,\text{opt}\{q(x,y),h(b(x,y))\}|\\&\le|u(x,y)|(|p(x,y)|+|h(a(x,y))|)+|v(x,y)||\,\text{opt}\{q(x,y),h(b(x,y))\}|\\&\le|u(x,y)|(|p(x,y)|+|h(a(x,y))|)+|v(x,y)|\max\{|q(x,y)|,|h(b(x,y))|\}\\&\le|u(x,y)|(|p(x,y)|+|h(a(x,y))|)+|v(x,y)|(|q(x,y)|+|h(b(x,y))|)\\&\le(|u(x,y)|+|v(x,y)|)(\max\{|p(x,y)|,|q(x,y)|\}\\&\qquad+\max\{|h(a(x,y))|,|h(b(x,y))|\})\\&\le\psi(k)+g(k),\quad\forall(x,y)\in\overline{B}(0,k)\times D,\end{aligned}$$

which together with Lemma 1.2 gives that

$$|Gh(x)|=\left|\underset{y\in D}{\text{opt}}\,H(x,y,h)\right|\le\sup_{y\in D}|H(x,y,h)|\le\psi(k)+g(k),\quad\forall x\in\overline{B}(0,k),$$

which means that Gh is bounded on bounded subsets of S. That is, G is a self mapping in $BB(S)$.

Given $\varepsilon > 0, k \in \mathbb{N}$, $x \in \overline{B}(0,k)$ and $h,t \in BB(S)$. Suppose that $\mathrm{opt}_{y\in D} = \inf_{y\in D}$. It follows that there exist $y, s \in D$ with

$$\begin{aligned} &Gh(x) > H(x,y,h) - \varepsilon, \quad Gt(x) > H(x,s,t) - \varepsilon, \\ &Gh(x) \le H(x,s,h), \qquad\quad Gt(x) \le H(x,y,t). \end{aligned} \tag{2.2}$$

By (C1), (C3), (2.1), (2.2) and Lemma 1.1, we have

$$\begin{aligned}
&|Gh(x) - Gt(x)| \\
&< \max\big\{|H(x,y,h) - H(x,y,t)|, |H(x,s,h) - H(x,s,t)|\big\} + \varepsilon \\
&\le \max\big\{|u(x,y)||h(a(x,y)) - t(a(x,y))| \\
&\qquad\quad + |v(x,y)||\,\mathrm{opt}\{q(x,y), h(b(x,y))\} - \mathrm{opt}\{q(x,y), t(b(x,y))\}|, \\
&\qquad |u(x,s)||h(a(x,s)) - t(a(x,s))| \\
&\qquad\quad + |v(x,s)||\,\mathrm{opt}\{q(x,s), h(b(x,s))\} - \mathrm{opt}\{q(x,s), t(b(x,s))\}|\big\} + \varepsilon \\
&\le \max\big\{|u(x,y)||h(a(x,y)) - t(a(x,y))| + |v(x,y)||h(b(x,y)) - t(b(x,y))|, \\
&\qquad |u(x,s)||h(a(x,s)) - t(a(x,s))| + |v(x,s)||h(b(x,s)) - t(b(x,s))|\big\} + \varepsilon \\
&\le \max\big\{(|u(x,y)| + |v(x,y)|)\max\{|h(a(x,y)) - t(a(x,y))|, |h(b(x,y)) - t(b(x,y))|\}, \\
&\qquad (|u(x,s)| + |v(x,s)|)\max\{|h(a(x,s)) - t(a(x,s))|, |h(b(x,s)) - t(b(x,s))|\}\big\} \\
&\quad + \varepsilon \\
&\le \max\big\{|h(a(x,y)) - t(a(x,y))|, |h(b(x,y)) - t(b(x,y))|, \\
&\qquad |h(a(x,s)) - t(a(x,s))|, |h(b(x,s)) - t(b(x,s))|\big\} + \varepsilon \\
&\le d_k(h,t) + \varepsilon,
\end{aligned}$$

which yields that

$$d_k(Gh, Gt) \le d_k(h,t) + \varepsilon.$$

Letting $\varepsilon \to 0$ in the above inequality, we deduce that

$$d_k(Gh, Gt) \le d_k(h,t),$$

which implies that

$$d(Gh, Gt) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(Gh,Gt)}{1 + d_k(Gh,Gt)} \le \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(h,t)}{1 + d_k(h,t)} = d(h,t). \tag{2.3}$$

Proceeding as above, we infer that (2.3) also holds if $\mathrm{opt}_{y\in D} = \sup_{y\in D}$.

Secondly we claim that for each $n \in \mathbb{N}_0$

$$|z_n(x)| \le \sum_{i=0}^{n} \psi(\varphi^i(\|x\|)), \quad \forall x \in S. \tag{2.4}$$

In view of $|z_0(x)| \le \psi(\|x\|)$, $\forall x \in S$, it is clear that (2.4) holds for $n = 0$. Suppose that (2.4) is true for some $n \in \mathbb{N}_0$. On the basis of (C1), (C2), (C3), Lemmas 1.1 and 1.2, we get that

$$|z_{n+1}(x)| = \left| \mathop{\mathrm{opt}}_{y\in D} \big\{u(x,y)(p(x,y) + z_n(a(x,y))) + v(x,y) \mathop{\mathrm{opt}}_{y\in D}\{q(x,y), z_n(b(x,y))\}\big\} \right|$$

$$
\begin{aligned}
&\le \sup_{y\in D}\{|u(x,y)|(|p(x,y)|+|z_n(a(x,y))|)+|v(x,y)|\max\{|q(x,y)|,|z_n(b(x,y))|\}\\
&\le \sup_{y\in D}\{|u(x,y)|(|p(x,y)|+|z_n(a(x,y))|)+|v(x,y)|(|q(x,y)|+|z_n(b(x,y))|)\}\\
&\le \sup_{y\in D}\left\{|u(x,y)|\left(\psi(\|x\|)+\sum_{i=0}^{n}\psi(\varphi^i(\|a(x,y)\|))\right)\right.\\
&\qquad\left.+|v(x,y)|\left(\psi(\|x\|)+\sum_{i=0}^{n}\psi(\varphi^i(\|b(x,y)\|))\right)\right\}\\
&\le \sup_{y\in D}\left\{|u(x,y)|\left(\psi(\|x\|)+\sum_{i=0}^{n}\psi(\varphi^{i+1}(\|x\|))\right)\right.\\
&\qquad\left.+|v(x,y)|\left(\psi(\|x\|)+\sum_{i=0}^{n}\psi(\varphi^{i+1}(\|x\|))\right)\right\}\\
&= \sup_{y\in D}\left\{(|u(x,y)|+|v(x,y)|)\sum_{i=0}^{n+1}\psi(\varphi^i(\|x\|))\right\}\\
&\le \sum_{i=0}^{n+1}\psi(\varphi^i(\|x\|)),\quad \forall x\in S.
\end{aligned}
$$

Hence (2.4) holds for every $n\in\mathbb{N}_0$.

Thirdly we verify that $\{z_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in $(BB(S),d)$. Let $k\in\mathbb{N}$ and $x_0\in\overline{B}(0,k)$. Given $\varepsilon>0$ and $n,m\in\mathbb{N}$.

Assume that $\mathrm{opt}_{y\in D}=\inf_{y\in D}$. Obviously there exist $s,t\in D$ satisfying

$$
\begin{aligned}
&z_n(x_0)>H(x_0,s,z_{n-1})-2^{-1}\varepsilon, && z_{n+m}(x_0)>H(x_0,t,z_{n+m-1})-2^{-1}\varepsilon,\\
&z_n(x_0)\le H(x_0,t,z_{n-1}), && z_{n+m}(x_0)\le H(x_0,s,z_{n+m-1}),
\end{aligned}
$$

which together with (C1) and Lemma 1.1 mean that there exist $y_1\in\{s,t\}$ and $x_1\in\{a(x_0,y_1),b(x_0,y_1)\}$ satisfying

$$
\begin{aligned}
&|z_{n+m}(x_0)-z_n(x_0)|\\
&<\max\big\{|H(x_0,s,z_{n+m-1})-H(x_0,s,z_{n-1})|,\\
&\qquad|H(x_0,t,z_{n+m-1})-H(x_0,t,z_{n-1})|\big\}+2^{-1}\varepsilon\\
&\le\max\big\{|u(x_0,s)||z_{n+m-1}(a(x_0,s))-z_{n-1}(a(x_0,s))|\\
&\qquad+|v(x_0,s)||\,\mathrm{opt}\{q(x_0,s),z_{n+m-1}(b(x_0,s))\}\\
&\qquad-\mathrm{opt}\{q(x_0,s),z_{n-1}(b(x_0,s))\}|,\\
&\qquad|u(x_0,t)||z_{n+m-1}(a(x_0,t))-z_{n-1}(a(x_0,t))|\\
&\qquad+|v(x_0,t)||\,\mathrm{opt}\{q(x_0,t),z_{n+m-1}(b(x_0,t))\}\\
&\qquad-\mathrm{opt}\{q(x_0,t),z_{n-1}(b(x_0,t))\}|\big\}+2^{-1}\varepsilon\\
&\le\max\big\{|u(x_0,s)||z_{n+m-1}(a(x_0,s))-z_{n-1}(a(x_0,s))|\\
&\qquad+|v(x_0,s)||z_{n+m-1}(b(x_0,s))-z_{n-1}(b(x_0,s))|,\\
&\qquad|u(x_0,t)||z_{n+m-1}(a(x_0,t))-z_{n-1}(a(x_0,t))|
\end{aligned}
$$

$$
\begin{aligned}
&+|v(x_0,t)||z_{n+m-1}(b(x_0,t))-z_{n-1}(b(x_0,t))|\big\}+2^{-1}\varepsilon\\
&\le \max\big\{(|u(x_0,s)|+|v(x_0,s)|)\max\{|z_{n+m-1}(a(x_0,s))-z_{n-1}(a(x_0,s))|,\\
&\qquad |z_{n+m-1}(b(x_0,s))-z_{n-1}(b(x_0,s))|\},\\
&\qquad (|u(x_0,t)|+|v(x_0,t)|)\max\{|z_{n+m-1}(a(x_0,t))-z_{n-1}(a(x_0,t))|,\\
&\qquad |z_{n+m-1}(b(x_0,t))-z_{n-1}(b(x_0,t))|\}\big\}+2^{-1}\varepsilon\\
&\le |z_{n+m-1}(x_1)-z_{n-1}(x_1)|+2^{-1}\varepsilon,
\end{aligned}
$$

that is,

$$|z_{n+m}(x_0)-z_n(x_0)|<|z_{n+m-1}(x_1)-z_{n-1}(x_1)|+2^{-1}\varepsilon. \tag{2.5}$$

Similarly we deduce that (2.5) holds also for $\text{opt}_{y\in D}=\sup_{y\in D}$. Proceeding in this way, we infer that there exist $y_i\in D$ and $x_i\in\{a(x_{i-1},y_i),b(x_{i-1},y_i)\}$ for $i\in\{2,3,\cdots,n\}$ satisfying

$$
\begin{aligned}
&|z_{n+m-1}(x_1)-z_{n-1}(x_1)|<|z_{n+m-2}(x_2)-z_{n-2}(x_2)|+2^{-2}\varepsilon,\\
&|z_{n+m-2}(x_2)-z_{n-2}(x_2)|<|z_{n+m-3}(x_3)-z_{n-3}(x_3)|+2^{-3}\varepsilon,\\
&\qquad\cdots\cdots\\
&|z_{m+1}(x_{n-1})-z_1(x_{n-1})|<|z_m(x_n)-z_0(x_n)|+2^{-n}\varepsilon.
\end{aligned} \tag{2.6}
$$

In view of (C3), (2.1) and (2.4)-(2.6), we conclude that

$$
\begin{aligned}
&|z_{n+m}(x_0)-z_n(x_0)|\\
&<|z_m(x_n)|+|z_0(x_n)|+\varepsilon\le\sum_{i=0}^{m}\psi(\varphi^i(\|x_n\|))+\psi(\|x_n\|)+\varepsilon\\
&\le\sum_{i=0}^{m}\psi(\varphi^{i+1}(\|x_{n-1}\|))+\psi(\varphi(\|x_{n-1}\|))+\varepsilon\\
&\le\sum_{i=0}^{m}\psi(\varphi^{i+n}(\|x_0\|))+\psi(\varphi^n(\|x_0\|))+\varepsilon\\
&\le\sum_{i=n-1}^{\infty}\psi(\varphi^i(\|x_0\|))+\varepsilon\le\sum_{i=n-1}^{\infty}\psi(\varphi^i(k))+\varepsilon,
\end{aligned}
$$

which yields that

$$d_k(z_{n+m},z_n)\le\sum_{i=n-1}^{\infty}\psi(\varphi^i(k))+\varepsilon.$$

Letting $\varepsilon\to 0$ in the above inequality, we get that

$$d_k(z_{n+m},z_n)\le\sum_{i=n-1}^{\infty}\psi(\varphi^i(k)). \tag{2.7}$$

Note that $\sum_{n=0}^{\infty}\psi(\varphi^n(t))<+\infty$ for each $t>0$. Hence (2.7) ensures that $\{z_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in $(BB(S),d)$ and it converges to some $z\in BB(S)$. Due to (2.3), we arrive at

$$
\begin{aligned}
d(Gz,z)&\le d(Gz,Gz_n)+d(z_{n+1},z)\\
&\le d(z,z_n)+d(z_{n+1},z)\to 0\quad\text{as } n\to\infty,
\end{aligned}
$$

which means that $Gz = z$. Thus the functional equation (1.4) possesses a solution $z \in BB(S)$.

Given $x_0 \in S$, $\{y_n\}_{n\in\mathbb{N}} \subset D$ and $x_n \in \{a(x_{n-1}, y_n), b(x_{n-1}, y_n)\}$ for each $n \in \mathbb{N}$. Put $k = [\|x_0\|] + 1$, where $[t]$ denotes the largest integer not exceeding t. For each $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$d_k(z, z_n) + \sum_{i=n}^{\infty} \psi(\varphi^i(k)) < \varepsilon, \quad \forall n > m. \tag{2.8}$$

It is clear that (C3) implies that

$$\|x_n\| \le \varphi(\|x_{n-1}\|) \le \cdots \le \varphi^n(\|x_0\|) \le \varphi^n(k) < k, \quad \forall n \in \mathbb{N}. \tag{2.9}$$

It follows from (2.4), (2.8) and (2.9) that

$$\begin{aligned} |z(x_n)| &\le |z(x_n) - z_n(x_n)| + |z_n(x_n)| \le d_k(z, z_n) + \sum_{i=0}^{n} \psi(\varphi^i(\|x_n\|)) \\ &\le d_k(z, z_n) + \sum_{i=n}^{2n} \psi(\varphi^i(k)) < \varepsilon, \quad \forall n > m, \end{aligned}$$

which yields that $\lim_{n\to\infty} z(x_n) = 0$.

Suppose that g is another solution of the functional equation (1.4) relative to condition (C5). Given $\varepsilon > 0$ and $x_0 \in S$. Assume that $\operatorname{opt}_{y\in D} = \inf_{y\in D}$. Clearly there exist $s, t \in D$ such that

$$\begin{aligned} &z(x_0) > H(x_0, s, z) - 2^{-1}\varepsilon, \quad &&g(x_0) > H(x_0, t, g) - 2^{-1}\varepsilon, \\ &z(x_0) \le H(x_0, t, z), &&g(x_0) \le H(x_0, s, g). \end{aligned} \tag{2.10}$$

On account of Lemma 1.1, (C1) and (2.10), we know that there exist $y_1 \in \{s, t\} \subset D$ and $x_1 \in \{a(x_0, y_1),\ b(x_0, y_1)\}$ satisfying

$$\begin{aligned} &|z(x_0) - g(x_0)| \\ &< \max\big\{|H(x_0, s, z) - H(x_0, s, g)|, |H(x_0, t, z) - H(x_0, t, g)|\big\} + 2^{-1}\varepsilon \\ &\le \max\big\{|u(x_0, s)||z(a(x_0, s)) - g(a(x_0, s))| \\ &\qquad + |v(x_0, s)||\operatorname{opt}\{q(x_0, s), z(b(x_0, s))\} - \operatorname{opt}\{q(x_0, s), g(b(x_0, s))\}|, \\ &\qquad |u(x_0, t)||z(a(x_0, t)) - g(a(x_0, t))| \\ &\qquad + |v(x_0, t)||\operatorname{opt}\{q(x_0, t), z(b(x_0, t))\} - \operatorname{opt}\{q(x_0, t), g(b(x_0, t))\}|\big\} + 2^{-1}\varepsilon \\ &\le \max\big\{(|u(x_0, s)| + |v(x_0, s)|) \max\{|z(a(x_0, s)) - g(a(x_0, s))|, \\ &\qquad |z(b(x_0, s)) - g(b(x_0, s))|\}, \\ &\qquad (|u(x_0, t)| + |v(x_0, t)|) \max\{|z(a(x_0, t)) - g(a(x_0, t))|, \\ &\qquad |z(b(x_0, t)) - g(b(x_0, t))|\}\big\} + 2^{-1}\varepsilon \\ &\le |z(x_1) - g(x_1)| + 2^{-1}\varepsilon, \end{aligned}$$

that is,

$$|z(x_0) - g(x_0)| < |z(x_1) - g(x_1)| + 2^{-1}\varepsilon. \tag{2.11}$$

Similarly we deduce that (2.11) holds for $\mathrm{opt}_{y\in D} = \sup_{y\in D}$. Proceeding in this way, we know that there exist $y_i \in D$ and $x_i \in \{a(x_{i-1}, y_i), b(x_{i-1}, y_i)\}$ for $i \in \{2, 3, \ldots, n\}$ with

$$\begin{aligned}
|z(x_1) - g(x_1)| &< |z(x_2) - g(x_2)| + 2^{-2}\varepsilon,\\
|z(x_2) - g(x_2)| &< |z(x_3) - g(x_3)| + 2^{-3}\varepsilon,\\
&\cdots\cdots\\
|z(x_{n-1}) - g(x_{n-1})| &< |z(x_n) - g(x_n)| + 2^{-n}\varepsilon.
\end{aligned} \tag{2.12}$$

It follows from (C5), (2.11) and (2.12) that

$$|z(x_0) - g(x_0)| < |z(x_n) - g(x_n)| + \varepsilon \to \varepsilon \ \text{ as } \ n \to \infty.$$

Since ε is arbitrary, we conclude immediately that $z(x_0) = g(x_0)$.

Finally we prove that (C7) holds. Given $\varepsilon > 0$ and $x_0 \in S$. We consider two possible cases as follows:

Case 1. opt = max. It follows that there exist $y_1 \in D$ and $x_1 \in \{a(x_0, y_1), b(x_0, y_1)\}$ with

$$\begin{aligned}
z(x_0) &> u(x_0, y_1)(p(x_0, y_1) + z(a(x_0, y_1)))\\
&\quad + v(x_0, y_1)\max\{q(x_0, y_1), z(b(x_0, y_1))\} - 2^{-1}\varepsilon\\
&\geq u(x_0, y_1)p(x_0, y_1) + u(x_0, y_1)z(a(x_0, y_1))\\
&\quad + v(x_0, y_1)z(b(x_0, y_1)) - 2^{-1}\varepsilon\\
&\geq u(x_0, y_1)p(x_0, y_1)\\
&\quad + (u(x_0, y_1) + v(x_0, y_1))\min\{z(a(x_0, y_1)), z(b(x_0, y_1))\} - 2^{-1}\varepsilon\\
&= u(x_0, y_1)p(x_0, y_1) + z(x_1) - 2^{-1}\varepsilon.
\end{aligned} \tag{2.13}$$

Similarly we conclude that for each $n \in \mathbb{N}$ and $i \in \{2, 3, \ldots, n\}$, there exist $y_i \in D$ and $x_i \in \{a(x_{i-1}, y_i),\ b(x_{i-1}, y_i)\}$ such that

$$\begin{aligned}
z(x_1) &> u(x_1, y_2)p(x_1, y_2) + z(x_2) - 2^{-2}\varepsilon,\\
z(x_2) &> u(x_2, y_3)p(x_2, y_3) + z(x_3) - 2^{-3}\varepsilon,\\
&\cdots\cdots\\
z(x_{n-1}) &> u(x_{n-1}, y_n)p(x_{n-1}, y_n) + z(x_n) - 2^{-n}\varepsilon.
\end{aligned} \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$z(x_0) > \sum_{i=1}^{n} u(x_{i-1}, y_i)p(x_{i-1}, y_i) + z(x_n) - \varepsilon, \quad \forall n \in \mathbb{N}. \tag{2.15}$$

Note that (C2) and (C3) ensure that

$$\begin{aligned}
|u(x_{n-1}, y_n)p(x_{n-1}, y_n)| &= u(x_{n-1}, y_n)|p(x_{n-1}, y_n)| \leq \psi(\|x_{n-1}\|)\\
&\leq \psi(\varphi(\|x_{n-2}\|)) \leq \cdots \leq \psi(\varphi^{n-1}(\|x_0\|)), \quad \forall n \in \mathbb{N}
\end{aligned}$$

and $\sum_{n=1}^{\infty} \psi(\varphi^{n-1}(\|x_0\|))$ is convergent.

It follows that the series $\sum_{n=1}^{\infty} |u(x_{n-1}, y_n)p(x_{n-1}, y_n)|$ is convergent. Letting $n \to \infty$ in (2.15), by (C5) we get that

$$z(x_0) \geq \sum_{n=1}^{\infty} u(x_{n-1}, y_n)p(x_{n-1}, y_n) - \varepsilon;$$

Case 2. opt = min. Obviously there exist $y_1 \in D$ and $x_1 \in \{a(x_0, y_1), b(x_0, y_1)\}$ satisfying

$$\begin{aligned} z(x_0) &< u(x_0, y_1)(p(x_0, y_1) + z(a(x_0, y_1))) \\ &\quad + v(x_0, y_1) \min\{q(x_0, y_1), z(b(x_0, y_1))\} + 2^{-1}\varepsilon \\ &\leq u(x_0, y_1)p(x_0, y_1) + u(x_0, y_1)z(a(x_0, y_1)) \\ &\quad + v(x_0, y_1)z(b(x_0, y_1)) + 2^{-1}\varepsilon \\ &\leq u(x_0, y_1)p(x_0, y_1) \\ &\quad + (u(x_0, y_1) + v(x_0, y_1)) \max\{z(a(x_0, y_1)), z(b(x_0, y_1))\} + 2^{-1}\varepsilon \\ &= u(x_0, y_1)p(x_0, y_1) + z(x_1) + 2^{-1}\varepsilon. \end{aligned} \tag{2.16}$$

Similarly we know that for each $n \in \mathbb{N}$ and $i \in \{2, 3, \ldots, n\}$, there exist $y_i \in D$ and $x_i \in \{a(x_{i-1}, y_i),\ b(x_{i-1}, y_i)\}$ with

$$\begin{aligned} z(x_1) &< u(x_1, y_2)p(x_1, y_2) + z(x_2) + 2^{-2}\varepsilon, \\ z(x_2) &< u(x_2, y_3)p(x_2, y_3) + z(x_3) + 2^{-3}\varepsilon, \\ &\cdots\cdots \\ z(x_{n-1}) &< u(x_{n-1}, y_n)p(x_{n-1}, y_n) + z(x_n) + 2^{-n}\varepsilon. \end{aligned} \tag{2.17}$$

According to (2.16) and (2.17), we get that

$$z(x_0) < \sum_{i=1}^{n} u(x_{i-1}, y_i)p(x_{i-1}, y_i) + z(x_n) + \varepsilon, \quad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$ in the above inequality, we infer that by (C5)

$$z(x_0) \leq \sum_{n=1}^{\infty} u(x_{n-1}, y_n)p(x_{n-1}, y_n) + \varepsilon.$$

This completes the proof.

Remark 2.2. If $u(x, y) = u_0$, $v(x, y) = v_0$ and $a(x, y) = b(x, y)$ for each $(x, y) \in S \times D$, where u_0 and v_0 are nonnegative constants with $u_0 + v_0 = 1$, then Theorem 2.1 reduces to Theorem 3.1 in [9], which is an extension of Theorem 3.5 in [5] and Theorem 2.4 in [6]. The following example shows that Theorem 2.1 is an indeed generalization of the corresponding results in [5,6,9].

Example 2.3. Consider the following functional equation

$$\begin{aligned} f(x) = \operatorname*{opt}_{y\in\mathbb{R}_-} \Big\{ & \sin^2\big(x^3y^2 - \sqrt{|xy-1|}\big)\Big[x^2\sin(x^2\sqrt{|y+3|}) \\ & + f\Big(\frac{x^2|y|^3}{1+x^2+y^6}\Big)\Big] + \cos^2\big(x^3y^2 - \sqrt{|xy-1|}\big) \\ & \times \operatorname{opt}\Big\{\frac{3x^3y}{x^2+4y^2+2}, f\Big(\frac{x}{3+\cos(x^4y^3)}\Big)\Big\}\Big\}, \quad \forall x\in\mathbb{R}^+. \end{aligned} \tag{2.18}$$

Let $X = Y = \mathbb{R}$, $S = \mathbb{R}^+$, $D = \mathbb{R}_-$ and define $u, v, p, q : S \times D \to \mathbb{R}$, $a, b : S \times D \to S$ and $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\begin{aligned} u(x,y) &= \sin^2\big(x^3y^2 - \sqrt{|xy-1|}\big), \\ v(x,y) &= \cos^2\big(x^3y^2 - \sqrt{|xy-1|}\big), \\ p(x,y) &= x^2\sin(x^2\sqrt{|y+3|}), \\ q(x,y) &= \frac{3x^3y}{x^2+4y^2+2}, \\ a(x,y) &= \frac{x^2|y|^3}{1+x^2+y^6}, \\ b(x,y) &= \frac{x}{3+\cos(x^4y^3)}, \\ \psi(x) &= x^2, \\ \varphi(x) &= \frac{1}{2}x, \ \forall (x,y)\in S\times D. \end{aligned}$$

Choose $z_0 \in BB(S)$ with $|z_0(x)| \le \psi(\|x\|)$ for each $x \in S$. Obviously, the assumptions of Theorem 2.1 are fulfilled. Thus Theorem 2.1 guarantees that the functional equation (2.18) possesses a solution $z \in BB(S)$ satisfying (C4)-(C7). But Theorem 3.5 in [5], Theorem 2.4 in [6] and Theorem 3.1 in [9] are useless for the functional equation (2.18).

Theorem 2.4. *Let $\alpha \in (0,1), u, v, p, q : S \times D \to \mathbb{R}$ and $a, b : S \times D \to S$ be mappings such that*

(C8) *p and q are bounded on $\overline{B}(0,k) \times D$, $\forall k \in \mathbb{N}$;*

(C9) $\sup_{(x,y)\in\overline{B}(0,k)\times D} \max\{\|a(x,y)\|, \|b(x,y)\|\} \le k, \ \forall k \in \mathbb{N}$;

(C10) $\sup_{(x,y)\in S\times D}\{|u(x,y)| + |v(x,y)|\} \le \alpha$.

Then for each $h_0 \in BB(S)$, the sequence $\{h_n\}_{n\in\mathbb{N}_0}$ defined by

$$\begin{aligned} h_n(x) = \operatorname*{opt}_{y\in D} \big\{ & u(x,y)(p(x,y) + h_{n-1}(a(x,y))) \\ & + v(x,y)\operatorname{opt}\{q(x,y), h_{n-1}(b(x,y))\}\big\}, \quad \forall (x,n) \in S\times\mathbb{N} \end{aligned} \tag{2.19}$$

converges to a unique solution $z \in BB(S)$ *of the functional equation* (1.4) *and has the following error estimate:*

$$d_k(h_n, z) \le \frac{\alpha^n}{1-\alpha} d_k(h_0, h_1), \quad \forall n, k \in \mathbb{N}. \tag{2.20}$$

Proof. Define a mapping H in $BB(S)$ by

$$\begin{aligned} Hh(x) = \operatorname*{opt}_{y \in D} \big\{ & u(x,y)(p(x,y) + h(a(x,y))) \\ & + v(x,y) \operatorname{opt}\{q(x,y), h(b(x,y))\}\big\}, \quad \forall (x,h) \in S \times BB(S). \end{aligned} \tag{2.21}$$

Given $k \in \mathbb{N}$ and $h \in BB(S)$. It follows from (C8) and (C9) that there exist $\beta(k) > 0$ and $\eta(k,h) > 0$ satisfying

$$\begin{aligned} &\sup_{(x,y)\in \overline{B}(0,k)\times D} \{|p(x,y)|, |q(x,y)|\} \le \beta(k), \\ &\sup_{(x,y)\in \overline{B}(0,k)\times D} \{|h(a(x,y))|, |h(b(x,y))|\} \le \eta(k,h). \end{aligned} \tag{2.22}$$

In view of (C10), (2.21), (2.22), Lemma 1.1 and 1.2, we have

$$\begin{aligned} |Hh(x)| &= \left| \operatorname*{opt}_{y \in D} \{u(x,y)(p(x,y) + h(a(x,y))) + v(x,y) \operatorname{opt}\{q(x,y), h(b(x,y))\}\} \right| \\ &\le \sup_{y \in D} \big| u(x,y)(p(x,y) + h(a(x,y))) + v(x,y) \operatorname{opt}\{q(x,y), h(b(x,y))\} \big| \\ &\le \sup_{y \in D} \big\{ |u(x,y)|(|p(x,y)| + |h(a(x,y))|) + |v(x,y)| \max\{|q(x,y)|, |h(b(x,y))|\} \\ &\le \sup_{y \in D} \big\{ |u(x,y)||p(x,y)| + |u(x,y)||h(a(x,y))| + |v(x,y)||q(x,y)| \\ &\qquad + |v(x,y)||h(b(x,y))| \big\} \\ &\le \sup_{y \in D} \big\{ (|u(x,y)| + |v(x,y)|) \max\{|p(x,y)|, |q(x,y)|\} \\ &\qquad + (|u(x,y)| + |v(x,y)|) \max\{|h(a(x,y))|, |h(b(x,y))|\} \big\} \\ &\le \alpha(\beta(k) + \eta(k,h)), \quad \forall x \in \overline{B}(0,k), \end{aligned}$$

which gives that H is a mapping from $BB(S)$ into itself.

Let $\varepsilon > 0$, $(k,x) \in \mathbb{N} \times \overline{B}(0,k)$ and $g, h \in BB(S)$. Suppose that $\operatorname{opt}_{y \in D} = \sup_{y \in D}$. It follows that there exist $s, t \in D$ with

$$\begin{aligned} Hg(x) &< u(x,s)(p(x,s) + g(a(x,s))) + v(x,s) \operatorname{opt}\{q(x,s), g(b(x,s))\} + \varepsilon, \\ Hh(x) &< u(x,t)(p(x,t) + h(a(x,t))) + v(x,t) \operatorname{opt}\{q(x,t), h(b(x,t))\} + \varepsilon, \\ Hg(x) &\ge u(x,t)(p(x,t) + g(a(x,t))) + v(x,t) \operatorname{opt}\{q(x,t), g(b(x,t))\}, \\ Hh(x) &\ge u(x,s)(p(x,s) + h(a(x,s))) + v(x,s) \operatorname{opt}\{q(x,s), h(b(x,s))\}. \end{aligned} \tag{2.23}$$

On account of (2.23), Lemma 1.1 and (C10), we get that

$$\begin{aligned}
&Hg(x) - Hh(x)\\
&> u(x,t)(g(a(x,t)) - h(a(x,t)))\\
&\quad + v(x,t)(\text{opt}\{q(x,t), g(b(x,t))\} - \text{opt}\{q(x,t), h(b(x,t))\}) - \varepsilon\\
&\geq -|u(x,t)||g(a(x,t)) - h(a(x,t))| - |v(x,t)||g(b(x,t)) - h(b(x,t))| - \varepsilon\\
&\geq -(|u(x,t)| + |v(x,t)|)\max\{|g(a(x,t)) - h(a(x,t))|, |g(b(x,t)) - h(b(x,t))|\} - \varepsilon\\
&\geq -\alpha \max\{|g(a(x,t)) - h(a(x,t))|, |g(b(x,t)) - h(b(x,t))|\} - \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&Hg(x) - Hh(x)\\
&< u(x,s)(g(a(x,s)) - h(a(x,s)))\\
&\quad + v(x,s)(\text{opt}\{q(x,s), g(b(x,s))\} - \text{opt}\{q(x,s), h(b(x,s))\}) + \varepsilon\\
&\leq |u(x,s)||g(a(x,s)) - h(a(x,s))| + |v(x,s)||g(b(x,s)) - h(b(x,s))| + \varepsilon\\
&\leq (|u(x,s)| + |v(x,s)|)\max\{|g(a(x,s)) - h(a(x,s))|, |g(b(x,s)) - h(b(x,s))|\} + \varepsilon\\
&\leq \alpha \max\{|g(a(x,s)) - h(a(x,s))|, |g(b(x,s)) - h(b(x,s))|\} + \varepsilon,
\end{aligned}$$

which yield that

$$\begin{aligned}
|Hg(x) - Hh(x)| &< \alpha \max\{|g(a(x,t)) - h(a(x,t))|, |g(b(x,t)) - h(b(x,t))|,\\
&\qquad |g(a(x,s)) - h(a(x,s))|, |g(b(x,s)) - h(b(x,s))|\} + \varepsilon\\
&\leq \alpha d_k(g,h) + \varepsilon,
\end{aligned}$$

that is,

$$d_k(Hg, Hh) \leq \alpha d_k(g,h) + \varepsilon,$$

letting $\varepsilon \to 0$ in the above inequality, we deduce that

$$d_k(Hg, Hh) \leq \alpha d_k(g,h). \tag{2.24}$$

Similarly, we infer that (2.24) holds also for $\text{opt}_{y\in D} = \inf_{y\in D}$. On account of (2.19), (2.21) and (2.24), we get that

$$\begin{aligned}
d_k(h_n, h_{n+1}) &= d_k(Hh_{n-1}, Hh_n) \leq \alpha d_k(h_{n-1}, h_n)\\
&\leq \alpha^2 d_k(h_{n-2}, h_{n-1}) \leq \cdots \leq \alpha^n d_k(h_0, h_1), \quad \forall n, k \in \mathbb{N},
\end{aligned}$$

which implies that

$$\begin{aligned}
d_k(h_n, h_{n+m}) &\leq \sum_{i=n}^{n+m-1} d_k(h_i, h_{i+1}) \leq \sum_{i=n}^{n+m-1} \alpha^i d_k(h_0, h_1)\\
&\leq \frac{\alpha^n}{1-\alpha} d_k(h_0, h_1), \quad \forall n, m, k \in \mathbb{N},
\end{aligned} \tag{2.25}$$

which yields that $\{h_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in $BB(S)$. It follows that $\{h_n\}_{n\in\mathbb{N}_0}$ converges to $z \in BB(S)$. In view of (2.24), we deduce that for any $k \in \mathbb{N}$

$$\begin{aligned} d_k(z, Hz) &\le d_k(z, h_n) + d_k(Hh_{n-1}, Hz) \\ &\le d_k(z, h_n) + \alpha d_k(h_{n-1}, z) \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which guarantees that

$$d_k(z, Hz) = 0, \quad \forall k \in \mathbb{N},$$

which yields that

$$d(z, Hz) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(z, Hz)}{1 + d_k(z, Hz)} = 0.$$

Hence $z = Hz$, that is, $z \in BB(S)$ is a fixed point of H.

Suppose that H has another fixed point $w \in BB(S)$. For each $k \in \mathbb{N}$, we have

$$d_k(z, w) = d_k(H^n z, H^n w) \le \alpha^n d_k(z, w) \to 0 \quad \text{as } n \to \infty,$$

which implies that

$$d(z, w) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(z, w)}{1 + d_k(z, w)} = 0,$$

that is, $z = w$. Thus z is the unique fixed point of H. Obviously z is also a unique solution of the functional equation (1.4). (2.20) follows from (2.25) by taking $m \to \infty$. This completes the proof.

Remark 2.5. If $u(x, y) = 0$ for each $(x, y) \in S \times D$, opt $= \max$ and $\text{opt}_{y\in D} = \inf_{y\in D}$, then Theorem 2.2 reduces to Theorem 3.4 in [5]; If $v(x, y) = 0$ for each $(x, y) \in S \times D$ and $\text{opt}_{y\in D} = \sup_{y\in D}$, then Theorem 2.4 reduces to Theorem 2.5 in [10] and Theorem 3.3 in [12]. The following example demonstrates that Theorem 2.4 extends properly the corresponding results in [5,10,12].

Example 2.6. Consider the following functional equation

$$\begin{aligned} f(x) = \operatorname*{opt}_{y\in\mathbb{R}^+} \bigg\{ &\frac{x + y^2}{1 + 3x + 3y^2} \cos^5 \sqrt{1 + x^3 y^4} \Big[x^4 \sin^3 \big(x + e^{x^2 - 3y^2}\big) \\ &+ f\Big(\frac{8x^3 y}{5x^4 + 4y^2 + 2}\Big)\Big] + \frac{1}{\pi} \arctan\big(x + 3x^3 y^2\big) \\ &\times \operatorname{opt}\Big\{ \frac{3x^3 y}{1 + x^2 y^2} \cos\big(x^2 - y^2\big), f\big(xe^{-5x-3y}\big)\Big\}\bigg\}, \quad \forall x \in \mathbb{R}^+. \end{aligned} \tag{2.26}$$

Let $X = Y = \mathbb{R}$, $S = D = \mathbb{R}^+$ and $\alpha = \frac{5}{6}$. Define $u, v, p, q : S \times D \to \mathbb{R}$ and $a, b : S \times D \to S$ by

$$u(x, y) = \frac{x + y^2}{1 + 3x + 3y^2} \cos^5 \sqrt{1 + x^3 y^4},$$

$$v(x, y) = \frac{1}{\pi} \arctan\big(x + 3x^3 y^2\big),$$

$$p(x, y) = x^4 \sin^3(x + e^{x^2 - 3y^2}),$$

$$q(x,y)=\frac{3x^3y}{1+x^2y^2}\cos\left(x^2-y^2\right),$$

$$a(x,y)=\frac{8x^3y}{5x^4+4y^2+2},$$

$$b(x,y)=xe^{-5x-3y},\ \forall(x,y)\in S\times D.$$

It is obvious that the assumptions of Theorem 2.4 hold. It follows from Theorem 2.4 that the functional equation (2.26) possesses a unique solution $z\in BB(S)$ satisfying (2.20). However, Theorem 3.4 in [5], Theorem 2.5 in [10] and Theorem 3.3 in [12] are not valid for the functional equation (2.26).

Next we study the solvability of the functional equation (1.4) in the Banach spaces $BC(S)$ and $B(S)$, respectively.

Theorem 2.7. *Let* $\alpha\in(0,1), S$ *be compact,* $u,v,p,q:S\times D\to\mathbb{R}$ *and* $a,b:S\times D\to S$ *be mappings satisfying* (C10) *and*
(C11) p *and* q *are bounded on* $S\times D$;
(C12) *for each* $x_0\in S$ *and each* $A\in\{u,v,p,q,a,b\}$,

$$\lim_{x\to x_0}A(x,y)=A(x_0,y)$$

uniformly for $y\in D$, *respectively.*
Then for any $h_0\in BC(S)$, *the Mann iterative sequence* $\{h_n\}_{n\in\mathbb{N}_0}$ *defined by*

$$\begin{aligned}h_{n+1}(x)=(1-\lambda_n)h_n(x)+\lambda_n\operatorname*{opt}_{y\in D}&\left\{u(x,y)(p(x,y)+h_n(a(x,y)))\right.\\&\left.+v(x,y)\operatorname{opt}\{q(x,y),h_n(b(x,y))\}\right\},\quad\forall(n,x)\in\mathbb{N}_0\times S\end{aligned}\tag{2.27}$$

converges to a unique solution $z\in BC(S)$ *of the functional equation* (1.4) *and has the following error estimate:*

$$\|h_{n+1}-z\|_1\le e^{-(1-\alpha)\sum_{i=0}^{n}\lambda_i}\|h_0-z\|_1,\quad\forall n\in\mathbb{N}_0,\tag{2.28}$$

where

$$\{\lambda_n\}_{n\in\mathbb{N}_0}\in[0,1]\quad\text{and}\quad\sum_{n=0}^{\infty}\lambda_n=+\infty.\tag{2.29}$$

Proof. Define a mapping H in $BC(S)$ by

$$\begin{aligned}Hh(x)=\operatorname*{opt}_{y\in D}&\left\{u(x,y)(p(x,y)+h(a(x,y)))+v(x,y)\operatorname{opt}\{q(x,y),h(b(x,y))\}\right\},\\&\forall(x,h)\in S\times BC(S).\end{aligned}\tag{2.30}$$

Let $x_0\in S$ and $h\in BC(S)$. It follows from (C10) and (C11) that there exists a constant $M>1$ such that

$$\begin{aligned}\sup_{(x,y)\in S\times D}\max\{&|u(x,y)|,|v(x,y)|,|p(x,y)|,|q(x,y)|,|h(a(x,y))|,\\&|h(b(x,y))|\}\le M.\end{aligned}\tag{2.31}$$

(C12) ensures that for given $\varepsilon > 0$, there exists $\delta > 0$ satisfying

$$\begin{aligned}&\max\big\{|A(x,y)-A(x_0,y)|, |h(B(x,y))-h(B(x_0,y))| : A \in \{u,v,p,q\},\\ &\qquad B \in \{a,b\}\big\} < \frac{\varepsilon}{7M}, \quad \forall (x,y) \in S \times D \text{ with } \|x-x_0\| < \delta.\end{aligned} \tag{2.32}$$

By virtue of (2.31), Lemmas 1.1 and 1.2, we have

$$\begin{aligned}|Hh(x)| &= \left|\operatorname*{opt}_{y\in D}\big\{u(x,y)(p(x,y)+h(a(x,y)))+v(x,y)\operatorname{opt}\{q(x,y),h(b(x,y))\}\big\}\right|\\ &\le \sup_{y\in D}\big\{|u(x,y)|(|p(x,y)|+|h(a(x,y))|)+|v(x,y)||\operatorname{opt}\{q(x,y),h(b(x,y))\}|\big\}\\ &\le \sup_{y\in D}\big\{|u(x,y)|(|p(x,y)|+|h(a(x,y))|)\\ &\qquad + |v(x,y)|\max\{|q(x,y)|,|h(b(x,y))|\}\big\}\\ &\le 3M^2,\end{aligned}$$

that is,

$$\|Hh\|_1 \le 3M^2,$$

which implies that Hh is bounded in S.

By (2.31), (2.32), Lemmas 1.1 and 1.2, we obtain that for all $x \in S$ with $\|x-x_0\| < \delta$,

$$\begin{aligned}&|Hh(x)-Hh(x_0)|\\ &= \Big|\operatorname*{opt}_{y\in D}\big\{u(x,y)(p(x,y)+h(a(x,y)))+v(x,y)\operatorname{opt}\{q(x,y),h(b(x,y))\}\big\}\\ &\qquad - \operatorname*{opt}_{y\in D}\big\{u(x_0,y)(p(x_0,y)+h(a(x_0,y)))+v(x_0,y)\operatorname{opt}\{q(x_0,y),h(b(x_0,y))\}\big\}\Big|\\ &\le \sup_{y\in D}\big|u(x,y)p(x,y)-u(x_0,y)p(x_0,y)+u(x,y)h(a(x,y))-u(x_0,y)h(a(x_0,y))\\ &\qquad + v(x,y)\operatorname{opt}\{q(x,y),h(b(x,y))\}-v(x_0,y)\operatorname{opt}\{q(x_0,y),h(b(x_0,y))\}\big|\\ &= \sup_{y\in D}\big|u(x,y)(p(x,y)-p(x_0,y))+(u(x,y)-u(x_0,y))p(x_0,y)\\ &\qquad + u(x,y)(h(a(x,y))-h(a(x_0,y)))+(u(x,y)-u(x_0,y))h(a(x_0,y))\\ &\qquad + v(x,y)(\operatorname{opt}\{q(x,y),h(b(x,y))\}-\operatorname{opt}\{q(x_0,y),h(b(x_0,y))\})\\ &\qquad + (v(x,y)-v(x_0,y))\operatorname{opt}\{q(x_0,y),h(b(x_0,y))\}\big|\\ &\le \sup_{y\in D}\big\{|u(x,y)||p(x,y)-p(x_0,y)|+|u(x,y)-u(x_0,y)||p(x_0,y)|\\ &\qquad + |u(x,y)||h(a(x,y))-h(a(x_0,y))|+|u(x,y)-u(x_0,y)||h(a(x_0,y))|\\ &\qquad + |v(x,y)|\max\{|q(x,y)-q(x_0,y)|,|h(b(x,y))-h(b(x_0,y))|\}\\ &\qquad + |v(x,y)-v(x_0,y)|\sup\{|q(x_0,y)|,|h(b(x_0,y))|\}\big\}\\ &\le M\cdot\frac{\varepsilon}{7M}+\frac{\varepsilon}{7M}\cdot M+M\cdot\frac{\varepsilon}{7M}+\frac{\varepsilon}{7M}\cdot M+M\cdot\frac{\varepsilon}{7M}+\frac{\varepsilon}{7M}\cdot M\\ &< \varepsilon.\end{aligned}$$

Hence Hh is continuous at x_0. Since x_0 is arbitrary in S, it follows that Hh is continuous in S. Thus H is a mapping from $BC(S)$ into itself.

Let $\eta > 0$, $x \in S$ and $h, g \in BC(S)$. Assume that $\operatorname{opt}_{y\in D} = \inf_{y\in D}$. It follows that there exist $s, t \in D$ with

$$\begin{aligned}
&Hh(x) > u(x,s)(p(x,s) + h(a(x,s))) + v(x,s)\operatorname{opt}\{q(x,s), h(b(x,s))\} - \eta,\\
&Hg(x) > u(x,t)(p(x,t) + g(a(x,t))) + v(x,t)\operatorname{opt}\{q(x,t), g(b(x,t))\} - \eta,\\
&Hh(x) \le u(x,t)(p(x,t) + h(a(x,t))) + v(x,t)\operatorname{opt}\{q(x,t), h(b(x,t))\},\\
&Hg(x) \le u(x,s)(p(x,s) + g(a(x,s))) + v(x,s)\operatorname{opt}\{q(x,s), g(b(x,s))\}.
\end{aligned} \tag{2.33}$$

On account of (2.33), (C10) and Lemma 1.1, we get that

$$\begin{aligned}
&Hh(x) - Hg(x)\\
&< u(x,t)(h(a(x,t)) - g(a(x,t)))\\
&\quad + v(x,t)(\operatorname{opt}\{q(x,t), h(b(x,t))\} - \operatorname{opt}\{q(x,t), g(b(x,t))\}) + \eta\\
&\le |u(x,t)||h(a(x,t)) - g(a(x,t))| + |v(x,t)||h(b(x,t)) - g(b(x,t))| + \eta\\
&\le (|u(x,t)| + |v(x,t)|)\max\{|h(a(x,t)) - g(a(x,t))|, |h(b(x,t)) - g(b(x,t))|\} + \eta\\
&\le \alpha\|h - g\|_1 + \eta
\end{aligned}$$

and

$$\begin{aligned}
&Hh(x) - Hg(x)\\
&> u(x,s)(h(a(x,s)) - g(a(x,s)))\\
&\quad + v(x,s)(\operatorname{opt}\{q(x,s), h(b(x,s))\} - \operatorname{opt}\{q(x,s), g(b(x,s))\}) - \eta\\
&\ge -|u(x,s)||h(a(x,s)) - g(a(x,s))| - |v(x,s)||h(b(x,s)) - g(b(x,s))| - \eta\\
&\ge -(|u(x,s)| + |v(x,s)|)\max\{|h(a(x,s)) - g(a(x,s))|, |h(b(x,s)) - g(b(x,s))|\} - \eta\\
&\ge -\alpha\|h - g\|_1 - \eta,
\end{aligned}$$

which yield that

$$|Hh(x) - Hg(x)| < \alpha\|h - g\|_1 + \eta,$$

that is,

$$\|Hg - Hh\|_1 \le \alpha\|h - g\|_1 + \eta.$$

Letting $\eta \to 0^+$ in the above inequality, we infer that

$$\|Hg - Hh\|_1 \le \alpha\|h - g\|_1. \tag{2.34}$$

Similarly, (2.34) holds also for $\operatorname{opt}_{y\in D} = \sup_{y\in D}$. By the Banach fixed point theorem, we know that the contraction mapping H has a unique fixed point $z \in BC(S)$. Clearly, z is also a unique solution of the functional equation (1.4). Notice that

$$\begin{aligned}
z(x) = (1 - \lambda_n)z(x) + \lambda_n \operatorname*{opt}_{y\in D}\big\{&u(x,y)(p(x,y) + z(a(x,y)))\\
&+ v(x,y)\operatorname{opt}\{q(x,y), z(b(x,y))\}\big\}, \quad \forall (n,x) \in \mathbb{N}_0 \times S.
\end{aligned} \tag{2.35}$$

Due to (2.27), (2.29), (2.30), (2.34) and (2.35), we obtain that

$$\begin{aligned}|h_{n+1}(x) - z(x)| &\le (1-\lambda_n)|h_n(x) - z(x)| + \lambda_n|Hh_n(x) - Hz(x)|\\ &\le (1-(1-\alpha)\lambda_n)|h_n(x) - z(x)|\\ &\le e^{-(1-\alpha)\lambda_n}|h_n(x) - z(x)|\\ &\le e^{-(1-\alpha)\sum_{i=0}^{n}\lambda_i}|h_0(x) - z(x)|\\ &\le e^{-(1-\alpha)\sum_{i=0}^{n}\lambda_i}\|h_0 - z\|_1, \quad \forall (n,x) \in \mathbb{N}_0 \times S,\end{aligned}$$

which gives (2.28), which together with (2.29) implies that $\{h_n\}_{n\in\mathbb{N}_0}$ converges to z. This completes the proof.

Example 2.8. Consider the following functional equation

$$\begin{aligned}f(x) = \operatorname*{opt}_{y\in\mathbb{R}^+}\bigg\{&\frac{1}{2}\sin(x+2y)\bigg[\frac{5x^2}{8x^2+3y^2+2}\\ &+ f\big(\sqrt{-x^2+10x+|\sin y^2|+1}\big)\bigg]\\ &+ \frac{x}{3x+y^2}\operatorname{opt}\bigg\{\frac{3x^2}{2x^2+y}, f\bigg(\frac{x^2+y^2}{x+y^2}\bigg)\bigg\}\bigg\}, \quad \forall x \in [1,10].\end{aligned} \tag{2.36}$$

Let $X = Y = \mathbb{R}$, $S = [1,10]$, $D = \mathbb{R}^+$ and $\alpha = \frac{5}{6}$. Define $u, v, p, q : S \times D \to \mathbb{R}$ and $a, b : S \times D \to S$ by

$$\begin{aligned}u(x,y) &= \frac{1}{2}\sin(x+2y),\\ v(x,y) &= \frac{x}{3x+y^2},\\ p(x,y) &= \frac{5x^2}{8x^2+3y^2+2},\\ q(x,y) &= \frac{3x^2}{2x^2+y},\\ a(x,y) &= \sqrt{-x^2+10x+|\sin y^2|+1},\\ b(x,y) &= \frac{x^2+y^2}{x+y^2}, \quad \forall (x,y) \in S \times D.\end{aligned}$$

It is clear that the assumptions of Theorem 2.7 are fulfilled. Thus Theorem 2.7 yields that the functional equation (2.36) possesses a unique solution $z \in BC(S)$.

As in the proof of Theorem 2.7, we conclude similarly the following

Theorem 2.9. *Let $\alpha \in (0,1), u, v, p, q : S \times D \to \mathbb{R}$ and $a, b : S \times D \to S$ be mappings satisfying* (C10) *and* (C11). *Then for any $h_0 \in B(S)$, the sequence $\{h_n\}_{n\in\mathbb{N}_0}$ defined by* (2.27) *and* (2.29) *converges to a unique solution $z \in B(S)$ of the functional equation* (1.4) *and satisfies* (2.28).

Remark 2.10. In case $\operatorname{opt}_{y\in D} = \sup_{y\in D}$ and $v(x,y) = 0$ for all $(x,y) \in S \times D$, then Theorem 2.9 reduces to Corollary 2.2.1 in [6]. The following example reveals that Theorem 2.9 is a real generalization of the result in [6].

Example 2.11. Consider the following functional equation

$$\begin{aligned} f(x) = \operatorname*{opt}_{y\in\mathbb{R}^+} \Bigg\{ & \frac{xy\sin(x-y)}{4x^2+y^2+1}\Bigg[\frac{x^3+y^3-1}{x^3+y^3+1} \\ & + f\bigg(\frac{x^2y^2}{x^2y^3+(x-y)^2+1}\bigg)\Bigg] + \frac{x-3y^2}{(x-3y^2)^2+5} \\ & \times \operatorname{opt}\Big\{\cos^5\big(x^8y^3\big), f\big(\ln\big(1+x^2y^2\big)\big)\Big\}\Bigg\}, \quad \forall x\in\mathbb{R}^+. \end{aligned} \tag{2.37}$$

Let $X = Y = \mathbb{R}$, $S = D = \mathbb{R}^+$ and $\alpha = \frac{1}{2}$. Define $u, v, p, q : S\times D \to \mathbb{R}$ and $a, b : S\times D\to S$ by

$$\begin{aligned} u(x,y) &= \frac{xy\sin(x-y)}{4x^2+y^2+1}, \\ v(x,y) &= \frac{x-3y^2}{(x-3y^2)^2+5}, \\ p(x,y) &= \frac{x^3+y^3-1}{x^3+y^3+1}, \\ q(x,y) &= \cos^5\big(x^8y^3\big), \\ a(x,y) &= \frac{x^2y^2}{x^2y^3+(x-y)^2+1}, \\ b(x,y) &= \ln\big(1+x^2y^2\big), \quad \forall (x,y)\in S\times D. \end{aligned}$$

It is easy to check that the assumptions of Theorem 2.9 are fulfilled. Consequently, Theorem 2.9 guarantees that the functional equation (2.37) has a unique solution $z\in B(S)$. But Corollary 2.2.1 in [6] is invalid for the functional equation (2.37).

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