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POSITIVE SOLUTIONS FOR A COUPLED SYSTEM OF MIXED HIGHER-ORDER NONLINEAR SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate the existence of positive solutions for a coupled system of mixed higher-order nonlinear singular fractional differential equations with integral boundary conditions. By applying the fixed point theorem on cones and some new general type conditions, some results on the existence of at least one or two positive solutions are obtained.

Key Words and Phrases: Coupled system, fractional differential equations, positive solution, fixed-point theorem.

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1. Introduction

The purpose of this paper is to study the existence of positive solutions for the following coupled system of mixed higher-order nonlinear singular fractional differential equations with integral boundary conditions

$$\begin{cases} D_{0^+}^{\alpha_1}u(t) + a_1(t)f_1(t, u(t), v(t)) = 0, \quad t \in (0,1), \\ D_{0^+}^{\alpha_2}v(t) + a_2(t)f_2(t, u(t)) = 0, \quad t \in (0,1), \\ u^{(j)}(0) = v^{(k)}(0) = 0, 0 \le j \le n_1 - 2, \quad 0 \le k \le n_2 - 2 \\ u(1) = \displaystyle\int_0^1 h_1(t)u(t)dt, \quad v(1) = \int_0^1 h_2(t)v(t)\mathrm{d}t, \end{cases} \tag{1.1}$$

where $n_i - 1 < \alpha_i \le n_i, n_i \ge 3, D_{0^+}^{\alpha_i}$ are the standard Riemann-Liouville fractional derivative, $a_i(t) \in C((0,1),\ [0,+\infty))$ and $a_i(t)$ may be singular at $t = 0$ and/or $t = 1, h_i(t) \in L^1[0,1]$ are nonnegative ($i = 1,\ 2$).

In the last few decades, fractional-order models are found to be more adequate than integer-order for real world problems. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, biology, control theory, and so forth, involves derivatives of fractional order. For the basic theory and recent development of subject, see [1-3]. There are also a large number of papers dealing with the existence of solutions of nonlinear fractional differential equations by the use of techniques of nonlinear analysis, see [4-15] and the references therein.

On the other hand, the study of coupled systems involving fractional differential equations is also important as such systems occur in various problems, see [10-14] and the references therein.

Bai and Fang [10] considered the existence of positive solutions of singular coupled system

$$\begin{cases} D^{\alpha}u(t) = f_1(t, v(t)),\ t \in (0,1), \\ D^{\beta}v(t) = f_2(t, u(t)),\ t \in (0,1), \end{cases} \tag{1.2}$$

where $0 < \alpha, \beta < 1, D^{\alpha}, D^{\beta}$ are the standard Riemann-Liouville fractional derivative and $f_i \in C([0,1] \times [0,+\infty), [0,+\infty)), \lim_{t\to 0^+} f_i(t,\cdot) = +\infty, i = 1,2$. They established the existence results by a nonlinear alternative of Leray-Schauder type and Krasnosel'skii fixed point theorem on a cone.

Ahmad and J.Nieto[12] discussed a coupled system of nonlinear fractional differential equations with three-point boundary conditions

$$\begin{cases} D^{\alpha}_{0+}u(t) = f_1(t, v(t), D^{p}_{0+}v(t)), \quad t \in (0,1), \\ D^{\beta}_{0+}v(t) = f_2(t, u(t), D^{q}_{0+}u(t)), \quad t \in (0,1), \\ u(0) = 0, u(1) = \gamma u(\eta), v(0) = 0, v(1) = \gamma v(\eta), \end{cases} \tag{1.3}$$

where $1 < \alpha, \beta < 2$ and p, q, γ, η satisfy certain conditions. By applying the Schauder fixed point theorem, an existence result is proved.

Yuan [13] investigated the existence of positive solutions for a coupled system of nonlinear differential equations of fractional orders

$$\begin{cases} D^{\alpha}_{0+}u(t) + \lambda f(t, v(t)) = 0, \quad t \in (0,1), \lambda > 0 \\ D^{\alpha}_{0+}v(t) + \lambda g(t, u(t)) = 0, \quad t \in (0,1), \lambda > 0 \\ u^{(i)}(0) = v^{(i)}(0) = 0,\ 0 \le i \le n-2, \\ u(1) = v(1) = 0, \end{cases} \tag{1.4}$$

where $\alpha \in (n-1, n]$ is a real number and $n \ge 3, \lambda$ is a parameter, and D^{α}_{0+} is the standard Riemann-Liouville fractional derivative, and f, g are continuous and semipositone. By applying nonlinear alternative of Leray-Schauder type and Krasnosel'skii fixed-point theorems, Some sufficient conditions for the existence of positive solutions for the boundary value problem (1.4) are established.

From the above works, we can see a fact, although the coupled systems of fractional boundary value problems have been investigated by some authors, coupled systems due to mixed higher-order nonlinear singular fractional differential equations with integral boundary conditions are seldom considered. Motivated by the above mentioned works, our aim in this paper is to study the existence of positive solutions for the coupled system (1.1). Further, we present some general type conditions $(H_4)-(H_7)$ instead of the sublinear or superlinear conditions are used in [5,6,10,14,15]. Our conditions are applicable for more general functions.

The rest of the paper is organized as follows. In Section 2, we present preliminaries and lemmas. The existence of at least one and two positive solutions for the system (1.1) is formulated and proved in Section 3. Finally, in section 4, we give some examples to illustrate our results.

2. Preliminaries and lemmas

For the convenience of readers, in this Section, we present preliminaries and lemmas. Let

$$\mu_i=\int_0^1 K_i(s)a_i(s)\mathrm{d}s,\quad \delta_i=\int_\theta^{1-\theta} K_i(s)a_i(s)\mathrm{d}s,\quad i=1,2,\quad \theta\in(0,\frac{1}{2}).$$

We make the following assumptions in what follows:

(H_1) $a_i(t)\in C((0,1),[0,+\infty)), a_i(t)$ do not vanish identically for $t\in(0,1)$ and $0<\int_0^1 K_i(s)a_i(s)\mathrm{d}s<+\infty$, where $K_i(s)$ are defined later by Lemma 2.9.

(H_2) $f_1\in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty)), f_2\in C([0,1]\times[0,+\infty),[0,+\infty))$ and $f_2(t,0)\equiv 0$ uniformly with respect to t on $[0,1]$.

(H_3) $h_i\in[0,1)$, where $h_i=\int_0^1 h_i(t)t^{\alpha_i-1}\mathrm{d}t$.

(H_4) There exist $\alpha\in(0,1], \lambda_1>0$ and a sufficiently large $M_1>0$ such that

(1) $f_1(t,u,v)\geq\lambda_1 v^\alpha,\ \forall(t,u,v)\in[0,1]\times[0,+\infty)\times[M_1,+\infty)$;

(2) $f_2(t,u)\geq C_1 u^{\frac{1}{\alpha}},\ \forall(t,u)\in[0,1]\times[M_1,+\infty)$,

where $C_1=\max\Big\{(\gamma\delta_2)^{-1},(\gamma\delta_2)^{-1}(\gamma^2\lambda_1\delta_1)^{-\frac{1}{\alpha}}\Big\}$.

(H_5) There exist $\beta>0, \lambda_2>0$ and a sufficiently small $\rho_2\in(0,1)$ such that

(1) $f_1(t,u,v)\leq\lambda_2 v^\beta,\ \forall(t,u,v)\in[0,1]\times[0,+\infty)\times[0,\rho_2]$;

(2) $f_2(t,u)\leq C_2 u^{\frac{1}{\beta}},\ \forall(t,u)\in[0,1]\times[0,\rho_2]$,

where $C_2=\min\{\rho_2\mu_2^{-1},\mu_2^{-1}(\mu_1\lambda_2)^{-\frac{1}{\beta}}\}$.

(H_6) There exist $p>0, \lambda_3>0$ and $M_2>0$ such that

(1) $f_1(t,u,v)\leq\lambda_3 v^p+M_2,\ \forall(t,u,v)\in[0,1]\times[0,+\infty)\times[0,+\infty)$;

(2) $f_2(t,u)\leq C_3 u^{\frac{1}{p}}+M_2,\ \forall(t,u)\in[0,1]\times[0,+\infty)$, where $C_3=(2\mu_1\lambda_3)^{-\frac{1}{p}}\mu_2^{-1}$.

(H_7) There exist $q\in(0,1], \lambda_4>0$ and a sufficiently small $\varepsilon\in(0,1)$ such that

(1) $f_1(t,u,v)\geq\lambda_4 v^q,\ \forall(t,u,v)\in[0,1]\times[0,+\infty)\times(0,\varepsilon)$;

(2) $f_2(t,u)\geq C_4 u^{\frac{1}{q}},\ \forall(t,u)\in[0,1]\times(0,\varepsilon)$, where $C_4=\gamma^{-\frac{1}{q}(2+q)}(\lambda_4\delta_1)^{-\frac{1}{q}}\delta_2^{-1}$.

(H_8) $f_1(t,u,v)$ and $f_2(t,u)$ are increasing in u and v and there exists $R>0$ such that

$$f_1\left(s, R, \int_0^1 K_2(r)a_2(r)f_2(r,R)\mathrm{d}r\right) < \mu_1^{-1}R, \quad s\in[0,1].$$

Definition 2.1 (see [2]) The Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $f:[0,+\infty)\to R$ is given by

$$D_{0+}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\mathrm{d}t,$$

where $n=[\alpha]+1, [\alpha]$ denotes the integer part of number α, provied that the right side is pointwise defined on $(0,+\infty)$.

Definition 2.2 (see [2]) The Riemann-Liouville fractional integral of order $\alpha>0$ of a continuous function $f:[0,+\infty)\to R$ is given by

$$I_{0+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\mathrm{d}t,$$

proved that the right side is pointwise defined on $(0,+\infty)$.

Lemma 2.3 (see [2]) Assume that $u\in C(0,1)\cap L(0,1)$ with a fractional derivative of order $\alpha>0$ that belongs to $C(0,1)\cap L(0,1)$. Then

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},$$

for some $c_i\in R, i=1,2,\cdots,n$, where n is the smallest integer greater than or equal to α.

Lemma 2.4 (see [15]) Assume that $h_i\neq 1$. Then for any $y\in C[0,1]$ and $n_i-1<\alpha_i\le n_i, n_i\ge 3, i=1,2$, the unique solution of boundary value problem

$$\begin{cases} D_{0+}^{\alpha_i}u(t)+y(t)=0, \ t\in(0,1),\\ u(0)=u'(0)=\cdots=u^{(n_i-2)}(0)=0,\\ u(1)=\displaystyle\int_0^1 h_i(t)u(t)\mathrm{d}t, \end{cases} \tag{2.1}$$

is given by

$$u(t)=\int_0^1 K_i(t,s)y(s)\mathrm{d}s, \tag{2.2}$$

where

$$K_i(t,s)=G_{i1}(t,s)+G_{i2}(t,s), \tag{2.3}$$

$$G_{i1}(t,s)=\frac{1}{\Gamma(\alpha_i)}\begin{cases} t^{\alpha_i-1}(1-s)^{\alpha_i-1}-(t-s)^{\alpha_i-1}, & 0\le s\le t\le 1,\\ t^{\alpha_i-1}(1-s)^{\alpha_i-1}, & 0\le t\le s\le 1, \end{cases} \tag{2.4}$$

$$G_{i2}(t,s)=\frac{t^{\alpha_i-1}}{1-h_i}\int_0^1 h_i(t)G_{i1}(t,s)\mathrm{d}s. \tag{2.5}$$

Remark 2.5 It is easy to know that the function $G_{i1}(t,s) \geq 0$ are continuous from proposition 2.2 in [15].

Lemma 2.6 The function $G_{i1}(t,s)(i=1,2)$ defined by (2.4) satisfy

$$c_i(t)G_{i1}(s) \leq G_{i1}(t,s) \leq G_{i1}(s), \quad \text{for} \quad t,\ s \in [0,1], \tag{2.6}$$

where

$$\tau(s) = \frac{s}{\left(1-(1-s)^{\frac{\alpha_i-1}{\alpha_i-2}}\right)},$$

$$G_{i1}(s) = G_{i1}(\tau(s),s) = \frac{\tau(s)^{\alpha_i-2}s(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)},$$

and

$$c_i(t) = \min\left\{\frac{(\alpha_i-1)^{\alpha_i-1}t^{\alpha_i-2}(1-t)}{(\alpha_i-2)^{\alpha_i-2}}, t^{\alpha_i-1}\right\}.$$

Proof. We write

$$G_{i1}(t,s) = \begin{cases} G_{i11}(t,s) = \frac{t^{\alpha_i-1}(1-s)^{\alpha_i-1}-(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}, & 0 \leq s \leq t \leq 1, \\ G_{i12}(t,s) = \frac{t^{\alpha_i-1}(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

We consider s as fixed, from

$$\frac{\partial G_{i11}(t,s)}{\partial t} = \frac{1}{\Gamma(\alpha_i-1)}\{t^{\alpha_i-2}(1-s)^{\alpha_i-1}-(t-s)^{\alpha_i-2}\}$$

and there is a critical point when

$$(1-\frac{s}{t})^{\alpha_i-2} = (1-s)^{\alpha_i-1}.$$

Thus the critical point is at

$$\tau(s) = \frac{s}{1-(1-s)^{\frac{\alpha_i-1}{\alpha_i-2}}}$$

and $G_{i11}(t,s)$ arrive at maximum at $(\tau(s),s)$ when $s<t$. Therefore we have

$$G_{i11}(t,s) \leq \Phi_{i1}(s) := \frac{\tau(s)^{\alpha_i-2}s(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}.$$

Defining

$$\tau(0) := \lim_{s\to 0}\tau(s) = \frac{\alpha_i-2}{\alpha_i-1}$$

and noting that $\tau(s)$ is an increasing function of s, we see that

$$\frac{\alpha_i-2}{\alpha_i-1} \leq \tau(s) \leq 1.$$

Also from

$$\frac{\partial G_{i12}(t,s)}{\partial t} = \frac{t^{\alpha_i-2}(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i-1)} > 0,$$

we have

$$G_{i12}(t,s) \leq \Phi_{i2}(s) := \frac{s^{\alpha_i-1}(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}.$$

By writing

$$\Phi_{i1}(s) := \frac{(\tau(s)/s)^{\alpha_i-2} s^{\alpha_i-1}(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}.$$

we obtain $\Phi_{i2}(s) \leq \Phi_{i1}(s)$.
This proves the second inequality in (2.6) with $\Phi_{i1}(s) = G_{i1}(s)$.

To prove the first inequality satisfied,we first have to show $G_{i11}(t,s) \geq c_{i1}(s)G_{i1}(s)$, that is,

$$\frac{c_{i1}(t)}{t^{\alpha_i-1}} \leq \frac{1-(\frac{1-s/t}{1-s})^{\alpha_i-1}}{\tau(s)^{\alpha_i-2}s}.$$

The minimum of the right-hand side with respect to s for fixed t, occurs either when $s = t$ or when $s \to 0^+$. This gives the largest possible $c_{i1}(t)$ to be

$$c_{i1}(t) = \min\{\frac{(\alpha_i-1)^{\alpha_i-1}}{(\alpha_i-2)^{\alpha_i-2}} t^{\alpha_i-2}(1-t), (1-(1-t)^{\frac{\alpha_i-1}{\alpha_i-2}})^{\alpha_i-2}\}.$$

On the other hand, to show $G_{i12}(t,s) \geq c_{i2}(s)G_{i1}(s)$, that is,

$$\frac{c_{i2}(t)}{t^{\alpha_i-1}} \leq \frac{1}{s\tau(s)^{\alpha_i-2}},\ for\ t \leq s \leq 1.$$

Since s and $\tau(s)$ are strictly increasing this requires $c_{i2}(t) \leq t^{\alpha_i-1}$.
Thus $c_i(t)$ are the minimum of the values of $c_{i1}(t), c_{i2}(t)$ which finally gives

$$c_i(t) = \min\left\{\frac{(\alpha_i-1)^{\alpha_i-1}t^{\alpha_i-2}(1-t)}{(\alpha_i-2)^{\alpha_i-2}}, t^{\alpha_i-1}\right\}.$$

Remark 2.7 A stronger inequality namely $\min_{t\in[\theta,1-\theta]} G_{i1}(t,s) \geq \gamma_i G_{i1}(s)$ have been used in [15], but γ_i are not found explicitly and no explicit lower bound is given. So, it is not clear how one can obtain the existence of positive solutions. similar to the proof of Theorem 3.1 in [16], we obtain Lemma 2.6 and find the explicit lower bound $c_i(t)$.

Remark 2.8 Combining Lemma 2.6 and Remark 2.7, we can easily see

$$\min_{t\in[\theta,1-\theta]} G_{i1}(t,s) \geq \gamma_i G_{i1}(s) \geq \gamma' G_{i1}(s), \quad \forall s \in [0,1], \tag{2.7}$$

where $\gamma_i = \min\{c_i(t) :\ t \in [\theta, 1-\theta]\}, i = 1,2,\ \gamma' = \min\{\gamma_1,\ \gamma_2\}$.

Lemma 2.9 (see [15]) If $h_i \in [0,1)$, the function $K_i(t,s)$ $(i = 1,2)$ defined by (2.3) satisfies

(i) $K_i(t,s) \geq 0$ are continuous for all $t,s \in [0,1]$, $K_i(t,s) > 0$ for all $t,s \in (0,1)$;

(ii) $K_i(t,s) \leq K_i(s)$ for each $t,s \in [0,1]$, and

$$\min_{t\in[\theta,1-\theta]} K_i(t,s) \geq \gamma K_i(s), \forall s \in [0,1].$$

where

$$\gamma = \min\{\gamma', \theta^{\alpha_1-1}, \theta^{\alpha_2-1}\},\ K_i(s) = G_{i1}(s) + G_{i2}(1,s),$$

γ' is defined by Remark 2.8, $G_{i1}(s), G_{i2}(1,s)$ are defined by (2.6), (2.5).

Lemma 2.10 Suppose that the assumptions $(H_1)-(H_3)$ are satisfied. Then the coupled system (1.1) has a positive solution (u,v) if and only if the following coupled system has a positive solution (u,v)

$$\begin{cases} u(t)=\displaystyle\int_0^1 K_1(t,s)a_1(s)f_1(s,u(s),v(s))\mathrm{d}s, \\ \\ v(t)=\displaystyle\int_0^1 K_2(t,s)a_2(s)f_2(s,u(s))\mathrm{d}s. \end{cases} \tag{2.8}$$

Proof. From Lemma 2.4, we can prove easily the result of Lemma 2.10.

From (2.8), we can obtain the following integral equations

$$u(t)=\int_0^1 K_1(t,s)a_1(s)f_1\Big(s,u(s),\int_0^1 K_2(s,r)a_2(r)f_2(r,u(r))\mathrm{d}r\Big)\mathrm{d}s. \tag{2.9}$$

Let $E=C[0,1]$ be a Banach space endowed with the norm

$$\| u \|=\max_{t\in[0,1]} \mid u(t) \mid .$$

Define the cone $P\subset E$ by

$$P=\Big\{u\in E:\min_{t\in[\theta,1-\theta]} u(t)\geq\gamma \parallel u \parallel, t\in[0,1]\Big\}. \tag{2.10}$$

We define the operator $T:P\to E$ by

$$Tu(t)=\int_0^1 K_1(t,s)a_1(s)f_1\Big(s,u(s),\int_0^1 K_2(s,r)a_2(r)f_2(r,u(r))\mathrm{d}r\Big)\mathrm{d}s. \tag{2.11}$$

Lemma 2.11 Suppose that the assumptions $(H_1)-(H_3)$ are satisfied. Then the operator $T:\ P\to P$ is completely continuous.

Proof. For $u\in P$, consider (2.11), from Lemma 2.9, we have

$$\begin{aligned} \|Tu(t)\| &= \max\nolimits_{0\leq t\leq 1}|Tu(t)| \\ &\leq \textstyle\int_0^1 K_1(s)a_1(s)f_1\Big(s,u(s),\int_0^1 K_2(s,r)a_2(r)f_2(r,u(r))\mathrm{d}r\Big)\mathrm{d}s. \end{aligned}$$

$$\begin{aligned} \min\nolimits_{t\in[\theta,1-\theta]}(Tu)(t) &\geq \gamma\textstyle\int_0^1 K_1(s)a_1(s)f_1\Big(s,u(s),\int_0^1 K_2(s,r)a_2(r)f_2(r,u(r))\mathrm{d}r\Big)\mathrm{d}s \\ &\geq\gamma\|Tu(t)\|. \end{aligned}$$

Therefore $T:P\to P$. Next by similar proof of Lemma 3.1 in [11] and Ascoli-Arzela theorem one can prove $T:P\to P$ is completely continuous.

Lemma 2.12 (see [17]) Suppose E is a real Banach space and P is cone in E, and let Ω_1,Ω_2 be bounded open sets in E such that $\theta\in\Omega_1,\ \overline{\Omega}_1\subset\Omega_2$. Let operator $T:\ P\cap(\overline{\Omega}_2\backslash\Omega_1)\to P$ be completely continuous. Suppose that one of two conditions holds

(1) $\parallel Tu \parallel\leq\parallel u \parallel,\ \forall u\in P\cap\partial\Omega_1;\ \parallel Tu \parallel\geq\parallel u \parallel,\ \forall u\in P\cap\partial\Omega_2;$

(2) $\parallel Tu \parallel\geq\parallel u \parallel,\ \forall u\in P\cap\partial\Omega_1;\ \parallel Tu \parallel\leq\parallel u \parallel,\ \forall u\in P\cap\partial\Omega_2.$

Then T has at least one fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

Lemma 2.13 (see [18]) Suppose E is a real Banach space and P is cone in E, and let Ω_1, Ω_2 and Ω_3 be bounded open sets in E such that $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2, \overline{\Omega}_2 \subset \Omega_3$. Let operator $T: \ P \cap (\overline{\Omega}_3 \backslash \Omega_1) \to P$ be completely continuous. such that

(1) $\| Tu \| \geq \| u \|, \ \forall u \in P \cap \partial\Omega_1$;

(2) $\| Tu \| \leq \| u \|, \ Tu \neq u, \forall u \in P \cap \partial\Omega_2$;

(3) $\| Tu \| \geq \| u \|, \ \forall u \in P \cap \partial\Omega_3$.

Then T has at least two fixed point u_1 and u_2 in $P \cap (\overline{\Omega}_3 \backslash \Omega_1)$ with $u_1 \in (\Omega_2 \backslash \Omega_1)$ and $u_2 \in (\overline{\Omega}_3 \backslash \overline{\Omega}_2)$.

3. Main results

Theorem 3.1 Suppose that the assumptions $(H_1) - (H_5)$ hold. Then the system (1.1) has at least one positive solution (u, v).

Proof. At first, let
$$\rho_1 = \max\{M_1 \gamma^{-1}, 1\},$$
set $\Omega_1 = \{u \in E : \| u \| < \rho_1\}$ and for $u \in P \cap \partial\Omega_1$, then
$$\min_{t \in [\theta, 1-\theta]} u(t) \geq \gamma \| u \| = M_1.$$
From (2.8), (2.11), (H_4) and Lemma 2.9, we have
$$\begin{aligned} v(t) &= \int_0^1 K_2(t,s) a_2(s) f_2(t, u(s)) \mathrm{d}s \geq C_1 \int_0^1 K_2(t,s) a_2(s) u^{\frac{1}{\alpha}}(s) \mathrm{d}s \\ &\geq \gamma C_1 \int_\theta^{1-\theta} K_2(s) a_2(s) u^{\frac{1}{\alpha}}(s) \mathrm{d}s \geq \gamma C_1 \int_\theta^{1-\theta} K_2(s) a_2(s) \mathrm{d}s (\gamma \|u\|)^{\frac{1}{\alpha}} \\ &= \gamma C_1 \delta_2 M_1^{\frac{1}{\alpha}} \geq M_1, t \in [\theta, 1-\theta], \end{aligned}$$
$$\begin{aligned} \min_{t \in [\theta, 1-\theta]} (Tu)(t) &\geq \gamma \int_0^1 K_1(s) a_1(s) f_1(s, u(s), v(s)) \mathrm{d}s \\ &\geq \gamma \lambda_1 \textstyle\int_\theta^{1-\theta} K_1(s) a_1(s) v^\alpha(s) \mathrm{d}s \\ &\geq \gamma \lambda_1 \delta_1 (\gamma C_1 \delta_2)^\alpha (\gamma \|u\|) \geq \|u\|. \end{aligned}$$
Therefore, we have
$$\| Tu \| \geq \| u \|, \ for \ u \in P \cap \partial\Omega_1. \tag{3.1}$$
Further, Set $\Omega_2 = \{u \in E : \| u \| < \rho_2 < 1\}$, for $u \in P \cap \partial\Omega_2$, by (2.8),(2.10) and (H_5), we have
$$v(t) \leq \ C_2 \int_0^1 K_2(s) a_2(s) u^{\frac{1}{\beta}}(s) \mathrm{d}s \leq C_2 \mu_2 \|u\|^{\frac{1}{\beta}} \leq \rho_2^{1+\frac{1}{\beta}} \leq \rho_2, \quad t \in [0,1],$$
$$\begin{aligned} (Tu)(t) &\leq \int_0^1 K_1(s) a_1(s) f_1(s, u(s), v(s)) \mathrm{d}s \leq \mu_1 \lambda_2 \|v\|^\beta \leq \mu_1 \lambda_2 (C_2 \mu_2)^\beta \|u\| \\ &\leq \|u\|. \end{aligned}$$
Therefore, we have
$$\| Tu \| \leq \| u \|, \quad for \ u \in P \cap \partial\Omega_2. \tag{3.2}$$

Thus from (3.1),(3.2), Lemma 2.11 and Lemma 2.12, T has a fixed point $u \in P\cap(\overline{\Omega}_1 \setminus \Omega_2)$. This means that the system (1.1) has at least one positive solutions $(u(t), v(t))$.

Theorem 3.2 Suppose that the assumptions $(H_1)-(H_3)$ and $(H_6)(H_7)$ hold. Then the system (1.1) has at least one positive solution (u, v).

Proof. At first, it follows from the assumption (H_6) and (2.11), we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 K_1(t,s)a_1(s)f_1(t,u(s),v(s))\mathrm{d}s \le \int_0^1 K_1(s)a_1(s)\left(\lambda_3 v^p(s)+M_2\right)\mathrm{d}s \\ &\le \int_0^1 K_1(s)a_1(s)\left[\lambda_3\left(\int_0^1 K_2(s)a_2(s)f_2(s,u(s))\mathrm{d}s\right)^p + M_2\right]\mathrm{d}s \\ &\le \mu_1\lambda_3\mu_2^p\left(C_3 u^{\frac{1}{p}}+M_2\right)^p+\mu_1 M_2 \qquad (3.3) \\ &\le \mu_1\lambda_3\mu_2^p\left(C_3\|u\|^{\frac{1}{p}}+M_2\right)^p+\mu_1 M_2.\end{aligned}$$

By means of simple calculation, we have

$$\lim_{\|u\|\to+\infty}\frac{\mu_1\lambda_3\mu_2^p(C_3\|u\|^{\frac{1}{p}}+M_2)^p+\mu_1M_2}{\|u\|}=\frac{1}{2}.$$

Then there exists a sufficiently large $M>1$ such that

$$\mu_1\lambda_3\mu_2^p(C_3\|u\|^{\frac{1}{p}}+M_2)^p+\mu_1M_2 \le \frac{3}{4}\|u\|, \quad \|u\|\ge M. \qquad (3.4)$$

Set $\Omega_3=\{u\in C[0,1] :\| u \|< M\}$, for $u\in P\cap\partial\Omega_3$, by (3.3) and (3.4), we obtain that

$$\|Tu\|\le\|u\|, \quad for\ u\in P\cap\partial\Omega_3. \qquad (3.5)$$

Further, Since the continuity of $f_2(t,u)$ and $f_2(t,0)\equiv 0$, there exists $\rho\in(0,\varepsilon)$ such that

$$f_2(t,u)<\mu_2^{-1}\rho, \quad \text{for} \quad (t,u)\in[0,1]\times(0,\rho). \qquad (3.6)$$

Set $\Omega_4=\{u\in E :\| u \|< \rho\}$, for $u\in P\cap\partial\Omega_4$, we have

$$v(t)=\int_0^1 K_2(t,s)a_2(s)f_2(t,u(s))\mathrm{d}s<\mu_2^{-1}\rho\int_0^1 K_2(s)a_2(s)\mathrm{d}s=\rho. \qquad (3.7)$$

It follows from the assumption (H_7) and Lemma 2.9, we have

$$\begin{aligned}\min_{t\in[\theta,1-\theta]}(Tu)(t) &\ge \gamma\int_0^1 K_1(s)a_1(s)f_1(s,u(s),v(s))\mathrm{d}s \\ &\ge \gamma\lambda_4\int_\theta^{1-\theta}K_1(s)a_1(s)\mathrm{d}s\ \left(\int_\theta^{1-\theta}K_2(s,r)a_2(r)f_2(r,u(r))dr\right)^q \\ &\ge \gamma\lambda_4\delta_1\left(\gamma\int_\theta^{1-\theta}K_2(r)a_2(r)C_4u^{\frac{1}{q}}(r)\mathrm{d}r\right)^q \\ &\ge \gamma^{2+q}\lambda_4\delta_1(C_4\delta_2)^q\|u\|\ge\|u\|, \text{ for } u\in P\cap\partial\Omega_4.\end{aligned}$$

Hence, we have

$$\| Tu \|\ge\| u \|, \text{ for } u\in P\cap\partial\Omega_4. \qquad (3.8)$$

Thus from (3.5),(3.8), Lemma 2.11 and Lemma 2.12, T has a fixed point $u \in P \cap (\overline{\Omega}_3 \setminus \Omega_4)$. This means that the system (1.1) has at least one positive solutions $(u(t), v(t))$.

Theorem 3.3 Suppose that the assumptions $(H_1)-(H_4)$ and $(H_7)(H_8)$ hold. Then the system (1.1) has at least two positive solution (u_1, v_1) and (u_2, v_2).
Proof. From (H_8), for $u \in P \cap \partial\Omega_5$, we obtain that

$$\begin{aligned}(Tu)(t) \quad &\leq \int_0^1 K_1(s)a_1(s)f_1\left(s, R, \int_0^1 K_2(r)a_2(r)f_2(r, R)\mathrm{d}r\right)\mathrm{d}s \\ &< \mu_1^{-1} R \int_0^1 K_1(s)a_1(s)\mathrm{d}s = R.\end{aligned}$$

Thus, $\|Tu\| < \|u\|,\ for\ u \in P \cap \partial\Omega_5$. By (H_4) and (H_7), we can get

$$\| Tu \| \geq \| u \|,\ for\ \ u \in P \cap \partial\Omega_1;\ \| Tu \| \geq \| u \|,\ for\ u \in P \cap \partial\Omega_4.$$

So, we can choose ρ, R and ρ_1 such that $\rho < R < \rho_1$ and satisfy the above three inequalities. By lemma 2.11 and lemma 2.13, we guarantee that T has two fixed points $u_1 \in P \cap (\Omega_5 \backslash \Omega_4)$ and $u_2 \in P \cap (\overline{\Omega}_1 \backslash \overline{\Omega}_5)$. Then then the system (1.1) has at least two positive solutions (u_1, v_1) and (u_2, v_2).

In the end, in order to illustrate our results, we consider the following two examples.

Example 3.4 Consider the system (1.1), let

$$\alpha_1 = \frac{5}{2},\ \alpha_2 = \frac{7}{2},\ n_1 = 3,\ n_2 = 4,$$

$$a_1(t) = \frac{\Gamma(\frac{5}{2})}{(1-t)^{\frac{3}{2}}},\ a_2(t) = \frac{\Gamma(\frac{7}{2})}{(1-t)^{\frac{5}{2}}},$$

$$f_1(t, u, v) = (1 + e^{-u})v^{\frac{1}{2}},\ f_2(t, u) = u^3,$$

$$h_1(t) = t^{-\frac{2}{3}},\ h_2(t) = t^{-\frac{3}{2}}.$$

By simple computation,

$$0 < \int_0^1 K_i(s)a_i(s)\mathrm{d}s \leq 1 < +\infty, \quad 0 < \int_0^1 h_i(t)t^{\alpha_i}\mathrm{d}s < 1,\ i = 1,\ 2.$$

So the assumptions $(H_1) - (H_3)$ are satisfied. Let $\alpha = \frac{1}{2}, \beta = \frac{1}{2}$, Clearly,

$$\lim_{u\to+\infty} \inf_{t\in[0,1]} \frac{f_2(t,u)}{u^{\frac{1}{\alpha}}} = +\infty,\ \lim_{v\to+\infty} \inf_{(t,u)\in[0,1]\times[0,+\infty)} \frac{f_1(t,u,v)}{v^{\alpha}} > 0;$$

$$\lim_{u\to0^+} \sup_{t\in[0,1]} \frac{f_2(t,u)}{u^{\frac{1}{\beta}}} = 0,\ \lim_{v\to0^+} \sup_{(t,u)\in[0,1]\times[0,+\infty)} \frac{f_1(t,u,v)}{v^{\beta}} < +\infty.$$

The assumptions (H_4) and (H_5) hold. Thus it follows that the system (1.1) has at least one positive solution by Theorem 3.1.

Example 3.5 Let the system (1.1) be as in Example 3.4,

$$f_1(t, u, v) = (1 + u^{-2})v^{\frac{1}{2}},\ f_2(t, u) = u^{\frac{1}{2}},$$

so the assumptions $(H_1) - (H_3)$ are satisfied. Let $p = q = \frac{1}{2}$, by simple computation,

$$\lim_{u\to+\infty} \sup_{t\in[0,1]} \frac{f_2(t,u)}{u^{\frac{1}{p}}} = 0, \quad \lim_{v\to+\infty} \sup_{(t,u)\in[0,1]\times[0,+\infty)} \frac{f_1(t,u,v)}{v^p} < +\infty;$$

$$\lim_{u\to 0^+} \inf_{t\in[0,1]} \frac{f_2(t,u)}{u^{\frac{1}{q}}} = +\infty, \quad \lim_{v\to 0^+} \sup_{(t,u)\in[0,1]\times[0,+\infty)} \frac{f_1(t,u,v)}{v^q} > 0.$$

The assumptions (H_6) and (H_7) hold. Thus it follows that the system (1.1) has at least one positive solution by Theorem 3.2.

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References

[1] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*, Vol. 198, Academic Press, New York, London, Toronto, 1999.

[2] A.A. Kilbas, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Vol. 204, North-Holland Mathematics Studies, Elsevier, Amsterdam, The Netherlands, 2006.

[3] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Academic, Cambridge, UK, 2009.

[4] Z. Bai, H. La, *Positive solutions for boundary value problem of nonlinear fractional differential equation,* J. Math. Anal. Appl., **311**(2005), no. 2, 495-505.

[5] S. Zhang, *Positive solutions for boundary-value problems of nonlinear fractional differential equations,* Electronic J. Diff. Eq., **2006**(2006), no. 36, 1-12.

[6] X. Xu, D. Jiang, C. Yuan, *Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation,* Nonlinear Anal., **71**(2009), no. 10, 4676-4688.

[7] R.P. Agarwal, D. O'Regan, K.S. Stane, *Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations,* J. Math. Anal. Appl., **371**(2010), no. 1, 57-68.

[8] X. Yang, Z. Wei, D. Wei, *Existence of positive solutions for the boundary value problem of nonlinear fractional differential equations,* Commun Nonlinear Sci. Numer Simul., **17**(2012), no. 1, 85-92.

[9] Z. Wei, Q. Li, J. Che, *Initial value problems for fractional differential equations involving Rieman-Liouville sequential fractional derivative,* J. Math. Anal. Appl., **367**(2010), no. 1, 260-272.

[10] C. Bai, J. Fang, *The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations,* Appl. Math. Comput., **150**(2004), no. 2, 611-621.

[11] X. Su, *Boundary value problem for a coupled system of nonlinear fractional differential equations,* Appl. Math. Lett., **22**(2009), no. 1, 64-69.

[12] B. Ahmad, J.J. Nieto, *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions,* Comput. Math. Appl., **58**(2009), no. 9, 1838-1843.

[13] C. Yuan, *Multiple positive solutions for (n-1,1) type semipositone conjugate boundary value problems for coupled systems of nonlinear fractional differential equations,* Electronic J. Qualitative Th. Differ. Eq., **2011**(2011), no. 13, 1-12.

[14] Y. Zhao, S. Sun, Z. Han, W. Feng, *Positive solutions for for a coupled system of nonlinear differential equations of mixed fractional orders,* Advance in Difference Eq., **2011**(2011), 64-69.

[15] M. Feng, X. Zhang, W. Ge, *New existence results for higher-order nonlinear fractional differential equations with integral boundary conditions,* Boundary Value Problems, **2011**, Article ID 720702, 11 pages.

[16] J.R.L. Webb, *Nonlocal conjugate type boundary value problem of higher order,* Nonlinear Anal., **71**(2009), no. 5-6, 1933-1940.
[17] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations,* Noordhoff Gronigen, Netherland, 1964.
[18] P. Kang, Z. Wei, *Three positive solutions of singular nonlocal boundary value problems for systems of nonlinear second-order ordinary differential Equations,* Nonlinear Anal., **70**(2009), no. 1, 444-451.

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