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COINCIDENCES AND FIXED POINTS OF NEW MEIR-KEELER TYPE CONTRACTIONS AND APPLICATIONS

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Abstract. The Meir-Keeler contraction, an important generalization of the classical Banach contraction has received enormous attention during the last four decades. In this paper, we present a review of Meir-Keeler type fixed point theorems and obtain some results using general Meir-Keeler type conditions for a sequence of maps in a metric space. Further, a recent result of Meir-Keeler type common fixed point theorem due to M. Kikkawa and T. Suzuki is generalized under tight minimal conditions. Applications regarding the existence of common solutions of certain functional equations are also discussed.
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1. Introduction

One of the most fascinating and classical result of the last century in the field of nonlinear analysis is the celebrated Banach contraction principle (Bcp) which provides a powerful technique for solving a variety of problems in mathematical sciences and engineering. The Bcp states that a selfmap A of complete metric space (X, d) admits a unique fixed point if A is a Banach contraction, i.e. if A satisfies

$$d(Ax, Ay) \le kd(x,y), \quad x, y \in X, \tag{1.1}$$

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where $0 \leq k < 1$. For some fundamental generalizations of the condition (1.1) and their comparison, one may refer to Boyd and Wong [1], Jachymski [2]-[4], Matkowski [5], Rakotch [6] , Rhoades [7, 8], Suzuki [9] and references thereof.

Rakotch [6] generalized the Bcp by replacing the constant k in (1.1) by a real-valued function. Indeed, he considered

$$d(Ax, Ay) \leq \phi(d(x,y))d(x, y) , \quad x, y \in X, \tag{1.2}$$

where $\phi : R^+ \to [0, 1)$ is a monotonically decreasing function, and R^+ denotes the set of non-negative real numbers.

Given a function $\phi : R^+ \to R^+$ such that $\phi(t) < t$ for $t > 0$, and a self map A of X. Then we say that A is ϕ contractive (see, for example, Jachymski [4]) if

$$d(Ax, Ay) \leq \phi(d(x,y)), \quad x, y \in X. \tag{1.3}$$

In general, ϕ is called a contractive gauge function (see [10], [11], [12], [13] and [14]). Various classes of gauge functions have been considered to generalize the result of Rakotch [6]. Browder [15] obtained a result for a complete bounded metric space satisfying the condition (1.3) where $\phi : R^+ \to R^+$ is nondecreasing and continuous from the right. Browder's result was immediately generalized by Boyd and Wong [1]. They relaxed the requirement of boundedness of the space and, instead, assumed $\phi : R^+ \to R^+$ to be upper-semi continuous from the right (not necessarily nondecreasing) such that A is ϕ-contractive (see also Kirk and Sims [16] and Lim [17]). On the other hand, Matkowski [5] generalized Browder's result by taking ϕ to be nondecreasing (not necessarily upper semicontinuous) such that (1.3) and the following condition (1.4) are satisfied:

$$\lim_{n\to\infty} \phi^n(t) = 0, t \in R^+. \tag{1.4}$$

We remark that the classes of contractive gauge functions studied by Boyd and Wong [1] and Matkowski [5] are independent (see Jachymski [3, p. 2328, 2334] and Jachymski [4, p. 151].

A somewhat different approach to generalize the Bcp which received substantial attention was adopted by Meir-Keeler [18]. Precisely, they obtained the following impressive result.

Theorem MK. *Assume that a selfmap A of X satisfies the condition: for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$,*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies } d(Ax, Ay) < \varepsilon. \tag{1.5}$$

Then A possesses a unique fixed point provided that X is complete.

We remark that Theorem MK significantly generalizes the results of Browder [15] and Boyd and Wong [1]. However, it is interesting to note that the result of Matkowski [5, Th. 1.2] is independent of Theorem MK (see [13]). Further, the Meir-Keeler contraction (1.5) is equivalent to the contractive gauge function ϕ defined as $\phi(d(Ax, Ay)) \leq d(x, y)$ for all $x, y \in X$, where ϕ is lower semicontinuous from the right on $[0, \infty)$ such that $\phi(t) > t$ for $t > 0$ (see Wong [19] and Jachymski [13]). The condition (1.5) implies that for all $x, y \in X$ with $x \neq y$,

$$d(Ax, Ay) < d(x, y). \tag{1.6}$$

Edelstein [20] (see also Agarwal et al. [21]) has shown that the map A satisfying the contractive condition (1.6) has a unique fixed point provided that the space X is compact. Meir and Keeler [18] observed that, if X is compact, then (1.5) and (1.6) are equivalent. Notice that A is continuous if it satisfies one of (1.1) or (1.5) or (1.6).

Theorem MK was subsequently generalized, among others, by Bari et al. [22], Chung [23], Ciric [24], [25], Hegedus and Sizilagyi [26], Jachymski [2], [12], [13], Karpagam and Agarwal [27], Leader [28], Maiti and Pal [29], Matkowski [30], Matkowski and Wegrzyk [31], Rao and Rao [32], Suzuki [9], [33], [34], Tan and Minh [35], Tomar [36] and Wlodarczyk et al. [37] for a selfmap. For the comparison of various Meir-Keeler type conditions, one may refer to Jachymski [2] (see also Liu [38] and Park [39]). Some interesting variants of the condition (1.5) have also been discussed by Jachymski [2]. For a detailed study of Meir-Keeler multivalued contractions, one may refer to [40].

The following theorem is due to Ciric [25].

Theorem C. *Let A be a contractive selfmap of a complete metric space (X, d) satisfying the condition:*

for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$,

$$\varepsilon < d(x, y) < \varepsilon + \delta \text{ implies } d(Ax, Ay) \leq \varepsilon. \tag{1.7}$$

Then A has a unique fixed point.

We remark that (1.7) is a weaker condition than the condition (1.5). Further, the contractive requirement (viz. (1.6)) in Theorem C is essential (see Proinov [41] and Suzuki [42]). However, in metrically convex spaces conditions (1.5) and (1.7) are equivalent (see [31]).

Motivated by a novel idea of Goebel [43] and Jungck [44], Park and Bae [45] extended the condition (1.5) to a pair of commuting selfmaps and obtained the following result.

Theorem PB. [45]. *Let A and S be commuting selfmaps of a metric space X such that $AX \subset SX$ and, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all $x, y \in X$,*

$$\varepsilon \leq d(Sx, Sy) < \varepsilon + \delta \text{ implies } d(Ax, Ay) < \varepsilon, \tag{1.8}$$

$$\text{and } Ax = Ay \text{ whenever } Sx = Sy. \tag{1.9}$$

If S is continuous and X is complete, then A and S have a unique common fixed point.

The condition (1.8) was generalized by Park and Rhoades [46]. They obtained a fixed point theorem for a pair of continuous and commuting self-maps A and S satisfying the following condition:

For any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x,\ y \in X$,

$$\varepsilon \leq \max\left\{d(Sx,\ Sy), d(Ax, Sx), d(Ay, Sy), \frac{d(Ax, Sy) + d(Ay, Sx)}{2}\right\} < \varepsilon$$

$$\text{implies } d(Ax,\ Ay) < \varepsilon. \tag{1.10}$$

In [47], Pant considered R-weakly commuting maps and obtained a fixed point theorem analogous to the result of Park and Bae [45]. Pathak et al. [48] obtained a similar result for R-weakly commuting maps of type (A_g) or type (A_f). For a comprehensive comparison of various weaker forms of commuting maps, one may refer to Singh and Tomar [49] (see also [50] and [51]).

In due course of time a number of Meir-Keeler type fixed point theorems for three maps, four maps and a sequence of maps have been obtained with various weaker forms of commuting maps (see [36], [40] and [52]-[68]). The most general Meir-Keeler type condition, involving four maps, which has been studied extensively during the last two decades, is as follows.

For any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$,

$$\varepsilon \leq M(x, y) < \varepsilon + \delta \text{ implies } d(Ax,\ By) < \varepsilon, \tag{1.11}$$

where

$$M(x,\ y) = \max\left\{d(Sx,\ Ty),\ d(Ax,\ Sx),\ d(By,\ Ty),\ \frac{d(Ax,\ Ty) + d(By,\ Sx)}{2}\right\}.$$

The $\phi-$contractive condition analogous to (1.11) is:

$$d(Ax,\ Ay) \leq \phi\, M(x,\ y),\ x, y \in X. \tag{1.12}$$

It is worthwhile to note that conditions (1.11) and (1.12) are independent of each other (see Pant [60]). Moreover, the existence of a common fixed point is not ensured by either of the conditions (1.11) and (1.12) (see Pant [60], [62] and Rao and Rao [64]). The existence of a common fixed point under the condition (1.11) is guaranteed with some additional hypotheses on δ such as δ is lower semi-continuous or nondecreasing. Similarly the existence of a common fixed point under the condition (1.12) is ensured with some extra conditions on ϕ, e.g., ϕ is nondecreasing and continuous from the right; or ϕ is upper semicontinuous and $\phi(t) < t$ for each $t > 0$; or $\phi(t)$ is nondecreasing and (1.4) is satisfied; or $\phi(t)$ is nondecreasing and $\frac{t}{t-\phi(t)}$ is nonincreasing (see also [60, 62]).

Notice that the condition (1.11) with nondecreasing δ implies condition (1.12). This implication is true even if δ is lower semi-continuous in the condition (1.11) (see [12], [62]). This is important to note that the condition (1.12) with any of the above extra conditions on ϕ implies the condition (1.11) (see [12], [62]).

Pant et al. [62] obtained the following fixed point theorem using Meir-Keeler type condition (1.11) without assuming δ to be nondecreasing or lower semi-continuous but required an additional contractive condition with continuity of one of the maps along with the completeness of the space and compatibility of the pair of maps.

Let

$$M_{1i}(x,\, y) := \max\left\{ d(Sx, Ty), d(A_1x, Sx), d(A_iy,\, Ty), \frac{d(A_1x, Ty) + d(A_iy, Sx)}{2} \right\},$$

where $S,\ T,\ A_i : X \to X,\ i = 1,\, 2,\, \ldots$.

Theorem P. *Let $\{A_i, i = 1,\, 2,\, ...\}$, S and T be selfmaps of a complete metric space $(X,\, d)$ such that*

(i): $A_1X \subset TX,\ \ A_iX \subset SX,\ \ i\ >\ 1;$

(ii): *for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x,\, y \in X$,*

$$\varepsilon \le M_{12}(x,\, y) < \varepsilon + \delta \ \textit{ implies } d(A_1x,\, A_2y) < \varepsilon;$$

(iii): $d(A_1x,\, A_iy) \le \phi_i\,(M_{1i}(x,\, y)),\ i > 2,$
where $\phi_i : R^+ \to R^+$ is such that $\phi_i(t) < t$ for each $t > 0$.

Let A_1 and S be compatible and T compatible with A_k for each $k > 1$. If one of the maps is continuous then the maps S, T and A_i, $(i = 1,\ 2, \ldots)$ have a unique common fixed point.

We summarize the above discussion by pointing out that generally, there are three ways of obtaining a common fixed point theorem using a Meir-Keeler type condition (1.11). In first of these approaches the condition (1.11) is used with some additional hypotheses on δ, e. g. Jungck [56], Jungck and Pathak [55] and Jungck et al. [57] required δ to be lower semicontinuous, while Pant [58]-[59] required δ to be non-decreasing. The second approach of obtaining a common fixed point theorem with condition (1.11) involves the assumption of continuity of the maps under consideration (see [56, Cor. 3.1]). The third approach pertaining to the existence of a common fixed point of the maps satisfying the condition (1.11) consists of the use of (1.11) along with an additional contractive condition, without requiring δ to be lower semicontinuous or nondecreasing (see, for instance, [53], [54] and [60]-[62]). However, using entirely a different approach, recently Kikkawa and Suzuki [69] obtained the following important result which is indeed an extension of a recent generalization of the Bcp by Suzuki [9], Theorem MK and Jungck [44].

Theorem KS. *Let A and S be commuting selfmaps of a metric space X such that $AX \subset SX$ and the following conditions are satisfied*

$$\frac{1}{2}d(Ax,\, Sx) < d(Sx,\, Sy) \textit{ implies } d(Ax,\, Ay) < d(Sx,\, Sy) \textit{ for all } x,\, y \in X; \tag{1.13}$$

and for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that, for all $x,\, y \in X$,

$$\frac{1}{2}d(Ax,\, Sx) < d(Sx,\, Sy) \textit{ and } d(Sx,\, Sy) < \varepsilon + \delta(\varepsilon) \textit{ together imply } d(Ax,\, Ay) \le \varepsilon. \tag{1.14}$$

If S is continuous and X is complete then A and S have a unique common fixed point.

The conditions (1.13) and (1.14) were immediately generalized by Popescu [70]. He used the following conditions to obtain a common fixed point theorem for a pair of commuting self-maps on a complete metric space.

$$\frac{1}{2}d(Ax,\, Sx) < d(Sx,\, Sy) \text{ implies}$$

$$d(Ax,\, Ay) < \max\{\, d(Sx,\, Sy),\; [d(Ax,\, Sx) + d(Ay,\, Sy)]/2\} \tag{1.15}$$

for all $x,\, y \in X$; and, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that, for all $x,\, y \in X$,

$$\frac{1}{2}d(Ax,\, Sx) < d(Sx,\, Sy)$$

$$\max\{d(Sx,\, Sy), [d(Ax,\, Sx) + d(Ay,\, Sy)]/2\} < \varepsilon + \delta(\varepsilon)$$

together imply

$$d(Ax,\, Ay) \leq \varepsilon. \tag{1.16}$$

In this paper, we obtain a coincidence theorem (Theorem 2.1 below) for a sequence of maps on a metric space along with two other maps under tight minimal conditions and, using the commuting property of the maps only at their coincidence points, we obtain results guaranteeing the existence of common fixed points of the maps. The condition (2.3) of Theorem 2.1 is very general and includes numerous contractive conditions (cf. [53], [54], [60]-[62] and [64]). Further, we notice a substantial improvement in Theorem 2.1 when the sequence of maps consists of only two maps. This is achieved in the subsequent result (Theorem 2.3 below). Variants of Theorem 2.3 are obtained in Theorems 2.6 & 2.7. Further, we obtain a considerably improved version of Theorem KS. As an application, we show the existence of a common solution of functional equations arising in dynamic programming.

2. Main Results

Throughout this paper, consistent with [71] let $C(A,\, S) = \{u : Au = Su\}$ denote the collection of all coincidence points of selfmaps A and S of a metric space X.

The following is our main result for a sequence of maps in a metric space.

Theorem 2.1. *Let $\{A_i\}$, $i = 1, 2, \ldots$, S and T be selfmaps of a metric space (X, d) such that*

$$A_1X \subseteq TX \text{ and } A_iX \subseteq SX,\; i > 1; \tag{2.1}$$

given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x,\, y \in X$,

$$\varepsilon \leq M_{12}(x,\, y) < \varepsilon + \delta \text{ implies } d(A_1x,\, A_2y) < \varepsilon; \tag{2.2}$$

$$d(A_1x,\, A_iy) < kd(Sx, Ty) + \alpha[d(A_1x, Sx) + d(A_iy, Ty) + d(A_1x, Ty) + d(A_iy, Sx)], \tag{2.3}$$

whenever the right-hand side is non-zero, where $k \geq 0$, $\alpha < \frac{1}{2}$.

If one of A_iX, SX or TX is a complete subspace of X then:

(I) *$C(A_1,\, S)$ and $C(A_i,\, T)$, $i > 1$, are nonempty.*

Indeed, if $u_1,\, u_2, ..., u_n \in C(A_1,\, S)$, and $v_1,\, v_2,\, ...,\, v_n \in C(A_i,\, T)$, $i > 1$, then $A_1u_j = Su_j = A_i v_j = Tv_j$, $i > 1$, $j = 1, ..., n$.

Further,

(II) *A_1 and S have a common fixed point provided that they commute at some $u \in C(A_1,\, S)$;*

(III) A_i *and* T *have a common fixed point provided that* A_i *and* T *commute at some* $v \in C(A_i, T)$, $i > 1$, *and one of the following holds:*

$$d(A_i x, A_i^2 x)$$
$$\neq \max\{d(Tx, TA_i x), d(A_i x, Tx), d(A_i^2 x, TA_i x), d(A_i x, TA_i x), d(Tx, A_i^2 x)\}, \quad (2.4)$$

whenever the right-hand side of (2.4) *is nonzero for* $x \in C(A_i, T)$;

$$d(Tx, T^2 x)$$
$$\neq \max\{d(A_i x, A_i Tx), d(Tx, A_i x), d(T^2 x, A_i Tx), d(Tx, A_i Tx), d(A_i x, T^2 x)\}, \quad (2.5)$$

whenever the right-hand side of (2.5) *is nonzero for* $x \in C(A_i, T)$;

(IV) A_i, S *and* T, *for each* i, *have a common fixed point provided* (II) *and* (III) *are true.*

Proof. Pick $x_0 \in X$. Construct two sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$y_{2n} = A_1 x_{2n} = Tx_{2n+1},\ y_{2n+1} = A_i x_{2n+1} = Sx_{2n+2},\ i > 1,\ n = 0, 1, 2, \ldots.$$

This can be done by the virtue of (2.1). In view of Jachymski [12, Lemma 2.2] (see also [62, p. 781]), the sequence $\{y_n\}$ is Cauchy. Let TX be complete. The sequence $\{y_{2n}\}$ is contained in TX, and has a limit in TX. Call it w. Let $v \in T^{-1}w$. Then $Tv = w$. The subsequence $\{y_{2n+1}\}$ also converges to w.

Now, we show that $A_i v = Tv = w$. For some $i > 1$, by (2.3),

$$\begin{aligned} d(y_{2n}, A_i v) &= d(A_1 x_{2n}, A_i v) \\ &< kd(Sx_{2n}, Tv) \\ &\quad + \alpha[d(A_1 x_{2n}, Sx_{2n}) + d(A_i v, Tv) + d(A_1 x_{2n}, Tv) + d(A_i v, Sx_{2n})]. \end{aligned}$$

Making $n \to \infty$,

$$\begin{aligned} d(Tv, A_i v) &\leq kd(Tv, Tv) + \alpha[d(Tv, Tv) + d(A_i v, Tv) + d(Tv, Tv) + d(A_i v, Tv)] \\ &= 2\alpha d(A_i v, Tv) < d(A_i v, Tv), \end{aligned}$$

a contradiction. Hence $A_i v = Tv = w$, proving that $C(A_i, T)$, $i > 1$, is nonempty.

Since $A_i X \subset SX$, we can choose a point u in X such that $w = A_i v = Su$. Now we show that $A_i u = Su$. By (2.3),

$$\begin{aligned} d(A_1 u, A_i x_{2n+1}) &< kd(Su, Tx_{2n+1}) \\ &\quad + \alpha\,[d(A_1 u, Su) \\ &\quad + d(A_i x_{2n+1}, Tx_{2n+1}) + d(A_1 u, Tx_{2n+1}) + d(A_i x_{2n+1}, Su)]. \end{aligned}$$

Making $n \to \infty$,

$$\begin{aligned} d(A_1 u, Su) &\leq kd(Su, Su) + \alpha[d(A_1 u, Su) + d(Su, Su) + d(A_1 u, Su) + d(Su, Su)]. \\ &= 2\alpha d(A_1 u, Su) < d(A_1 u, Su), \end{aligned}$$

a contradiction. Hence $A_1 u = Su = w$. So that $A_i v = Tv = w = A_1 u = Su$, $i > 1$. Therefore $C(A_1, S)$ is nonempty. If SX is complete then an analogous argument establishes that $C(A_1, S)$ is nonempty. Further, if A_iX $(i > 1)$ is complete then by (2.1), the limit w belongs to SX. This completes the proof of (I).

By the commutativity of A_1 and S at u,

$$A_1w = A_1Su = SA_1u = SSu = A_1A_1u = Sw. \text{ If } w \neq A_1w.$$

Then by (2.2),

$$d(w, A_1w) = d(A_1A_1u, A_2v)$$
$$< \max\{d(SA_1u, Tv), d(A_1A_1u, SA_1u), d(A_2v, Tv), [d(A_1A_1u, Tv) + d(A_2v, SA_1u)]/2\}$$
$$= d(w, A_1w),$$

a contradiction. Therefore $w = A_1w$ or $A_1u = A_1A_1u = SA_1u$. Hence A_1u is a common fixed point of A_1 and S. This proves (II). Similarly, by the commutativity of A_i and T at v, $A_iw = A_iTv = TA_iv = A_iA_iv = TTv = Tw,\ i > 1$. Taking $x = v$ in (2.4) or (2.5), we immediately see that A_iv is a common fixed point of A_i and T. This proves (III), and (IV) is immediate. □

Recently Jha [53] (see also [54]) obtained a Meir-Keeler type common fixed point theorem for the maps $\{A_i, i = 1, 2, \ldots\}$, S and T in a metric space X satisfying the following additional condition:

$$d(A_1x, A_iy) < \alpha[d(Sx, Ty) + d(A_1x, Sx) + d(A_iy, Ty) + d(A_1x, Ty) + d(A_iy, Sx)], \tag{2.6}$$

$0 \leq \alpha \leq \frac{1}{3}$. The following example demonstrates the generality of Theorem 2.1 over the results of Jha [53], Jungck et al. [57], Pant et al. [62] and Rhoades et al. [66].

Example 2.2. Let $X = [2, 15]$ be endowed with the usual metric and consider the following discontinuous maps on X:

$$A_1x = \begin{cases} 2 & \text{if } x < 10, \\ \dfrac{10+x}{8} & \text{if } x \geq 10; \end{cases} \qquad Sx = \begin{cases} 2 & \text{if } x \leq 4, \\ x & \text{if } 4 < x \leq 10, \\ 8 & \text{if } x > 10; \end{cases}$$

$$Tx = \begin{cases} 2 & \text{if } x = 2, \\ 11 + x & \text{if } 2 < x < 5, \\ 11 & \text{if } 5 \leq x \leq 10, \\ \dfrac{x+1}{5} & \text{if } x > 10; \end{cases} \qquad A_2x = \begin{cases} 2 & \text{if } x < 4 \text{ or } x \geq 5, \\ 2 + x & \text{if } 4 \leq x < 5; \end{cases}$$

and, for each $i > 2$,

$$A_ix = \begin{cases} 2 & \text{if } x = 2 \\ \dfrac{30+x}{4} & \text{if } x < 2 < 4, \\ 10 & \text{if } 4 \leq x \leq 15. \end{cases}$$

Notice that $\{A_i\}$, S and T have a common fixed point at $x = 2$. It can be verified that the conditions (2.1) and (2.3) are satisfied. Further, maps A_1, A_2, S and T satisfy the condition (2.2) when $\delta(\varepsilon) = 14 - \varepsilon$ if $\varepsilon \geq 5$ and, $\delta(\varepsilon) = 5 - \varepsilon$ if $\varepsilon < 5$. Now, it is easy to see that $\delta(\varepsilon)$ is neither nondecreasing nor lower semicontinuous. Moreover, the condition (2.6) is not satisfied for $x = 2$ and $y \in (10, 15]$.

The following common fixed point theorem for four maps in a metric space generalizes, among others, the results of Jha et al. [54], Jungck and Pathak [55], Jungck [56], Pant [60]-[61], Rao and Rao [64] and others. Indeed, if $\{A_i\} = \{A,\ B\}$, the requirements (2.4) and (2.5) of Theorem 2.1 are not needed. We do this below and give only a sketch of the proof.

Theorem 2.3. *Let A, B, S and T be selfmaps of a metric space (X, d) such that*

$$AX \subseteq TX \text{ and } BX \subseteq SX. \tag{2.7}$$

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x,\ y \in X$;

$$\varepsilon \leq M(x,\ y) < \varepsilon + \delta \text{ implies } d(Ax,\ By) < \varepsilon; \tag{2.8}$$

$$d(Ax,\ By) < k\, d(Sx,\ Ty) + \alpha[d(Ax,\ Sx) + d(By,\ Ty) + d(Ax,\ Ty) + d(By,\ Sx)], \tag{2.9}$$

whenever the right-hand side of (2.9) *is nonzero, where $k \geq 0$, $\alpha < \frac{1}{2}$.*

If one of AX, BX, SX or TX is a complete subspace of X then:

(I) *$C(A,\ S)$ and $C(B,\ T)$, are nonempty. Indeed, if $u_1,\ u_2, \dots,\ u_n \in C(A,\ S)$, and $v_1,\ v_2, \dots,\ v_n \in C(B,\ T)$, then $Au_j = Su_j = Bv_j = Tv_j$, $j = 1, \dots, n$.*

Further,

(II) *A and S have a common fixed point provided that they commute at some $u \in C(A,\ S)$;*

(III) *B and T have a common fixed point provided that B and T commute at some $v \in C(B,\ T)$;*

(IV) *A, B, S and T have a unique common fixed point provided* (II) *and* (III) *are true.*

Proof. Pick $x_0 \in X$. Construct two sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$y_{2n} = Ax_{2n} = Tx_{2n+1},\ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2},\ n = 0,\ 1,\ 2,\ \dots.$$

Following the proof of Theorem 2.1, it can be seen that the maps A and S have a coincidence at u, and B and T have a coincidence at v. This proves (I).

Further, commutativity of A, S and B, T at their coincidences implies $AAu = ASu = SAu = SSu$ and $BTv = TBv = TTv = BBv$. By (2.8), we have

$$\begin{aligned} d(Au,\ AAu) = d(AAu,\ Bv) &< \max\{\ d(SAu,\ Tv), d(AAu,\ SAu), d(Bv,\ Tv), \\ &\quad [d(AAu,\ Tv) + d(Bv,\ SAu)]/2\ \} = d(AAu,\ Bv), \end{aligned}$$

a contradiction. This proves (II). Similarly, using (2.8), we have

$$\begin{aligned} d(Bv,\ BBv) = d(Au,\ BBv) &< \max\{\ d(Su,\ TBv), d(Au,\ Su), d(BBv,\ TBv), \\ &\quad [d(Au,\ TBv) + d(BBv,\ Su)]/2\ \} = d(Au,\ BBv), \end{aligned}$$

a contradiction, proving (III), now (IV) is immediate. □

For a variant of Theorem 2.3, refer to Singh and Kumar [72, Theorem 1]. The following example shows that the maps A, B, S and T of Theorem 2.3 need not have a common coincidence.

Example 2.4. Let $X = [0, \infty)$ be endowed with the usual metric and

$$AX = x^2 + 4/9,\ BX = x^3 + 4/9,\ SX = 5x^2,\ TX = 5x^3,\ x \in X.$$

Then

$$d(Ax, By) = \left|x^2 - y^3\right| < 5\left|x^2 - y^3\right| = d(Sx, Ty) \text{ for all } x, y \in X.$$

So, (2.9) and other hypotheses of Theorem 2.3 are satisfied. We see that $A(1/3) = S(1/3) = 5/9$ and $B(1/9)^{1/3} = T(1/9)^{1/3} = 5/9$, that is, A and S have a coincidence at $x = 1/3$ and B and T have a (different) coincidence at $x = (1/9)^{1/3}$.

Now a natural question is that if $A = B$ in Theorem 2.3, can we have a common coincidence point of the maps A, S and T under the hypotheses of Theorem 2.3. The following example answers it negatively.

Example 2.5. Let $X = \{1, 2, 3\}$ with the usual metric and $A1 = A2 = A3 = 1$, $S1 = 2$, $S2 = 3$, $S3 = 1$ and $T1 = 3$, $T2 = 1$, $T3 = 2$. It is easy to see that A, S and T satisfy all the conditions of Theorem 2.3 with $A = B$. Notice that $A3 = S3 = 1 = A2 = T2$, that is, A and S have a coincidence at $x = 3$ and A and T have a (different) coincidence at $x = 2$.

In case $S = T$ in Theorem 2.3, we obtain a slightly improved version which we state below. Indeed, a slight modification in the proof of Theorem 2.3 yields the following result.

Theorem 2.6. *Let A, B and S be selfmaps of a metric space (X, d) such that* (2.8) *and* (2.9) *with $S = T$, and $AX \cup BX \subset SX$. If one of AX, BX or SX is a complete subspace of X. Then, A, B and S have a common coincidence. Further, if S commutes with each of A and B at their coincidences, then A, B and S have a unique common fixed point.*

As an immediate consequence of Theorem 2.3, we have the following result under a slightly weaker condition than (2.9).

Theorem 2.7. *Let A, B, S and T be selfmaps of a metric space (X, d) such that conditions* (2.7) *and* (2.8) *of Theorem* 2.3 *and the following are satisfied:*

$$d(Ax, By) \le k\, d(Sx, Ty) + \alpha\,[d(Ax, Sx) + d(By, Ty) + d(Ax, Ty) + d(By, Sx)], \tag{2.10}$$

where $k \ge 0, \alpha < \frac{1}{2}$. If one of AX, BX, SX or TX is a complete subspace of X. Then all the conclusions of the Theorem 2.3 *are true.*

Proof. It follows from the proof of Theorem 2.3 by noting that (2.9) implies (2.10). □

Substituting $\alpha = 0$ in the Theorem 2.7 we have the following result.

Corollary 2.8. [61, Theorem 1] *Let A, B, S and T be selfmaps of a metric space (X, d) such that conditions* (2.7) *and* (2.8) *of Theorem* 2.3 *and the following are satisfied:*

$$d(Ax, By) \le kd(Sx, Ty).$$

If one of AX, BX, SX or TX is a complete subspace of X then all the conclusions of Theorem 2.3 *are true.*

In case $A = B$ and $S = T = I_X$(identity map) in Theorem 2.3, we obtain the following result which generalizes, among others, the results of Jachymski [2], Pant et al. [62] and Rao and Rao [32].

Corollary 2.9. *Let A be a self map of a metric space (X, d) such that given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$,*

$$\varepsilon \leq \max\left\{d(x, y), d(x, Ax), d(y, Ay), \frac{d(x, Ay) + d(y, Ax)}{2}\right\} < \varepsilon + \delta$$

implies $d(Ax, Ay) < \varepsilon$, and

$$d(Ax, Ay) < k\, d(x, y) + \alpha\, [d(x, Ax) + d(y, Ay) + d(x, Ay) + d(y, Ax)],$$

whenever the right-hand side is non-zero, where $k \geq 0$, $\alpha < \frac{1}{2}$. If X is complete or AX is a complete subspace of X, then A has a unique fixed point.

Now, we present a variant and a generalization of Theorems MK & KS (see also [40]).

Theorem 2.10. *Let A and S be selfmaps of a metric space X such that*

$$AX \subset SX; \tag{2.11}$$

$$\frac{1}{2}d(Ax, Sx) < d(Sx, Sy) \text{ implies } d(Ax, Ay) < d(Sx, Sy) \text{ for all } x, y \in X; \tag{2.12}$$

for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for all $x, y \in X$,

$$\frac{1}{2}d(Ax, Sx) < d(Sx, Sy) \text{ and } d(Sx, Sy) < \varepsilon + \delta(\varepsilon) \text{ imply } d(Ax, Ay) \leq \varepsilon. \tag{2.13}$$

If one of AX or SX is a complete subspace of X then $C(A, S)$ is nonempty. Further, A and S have a unique common fixed point provided that A and S commute at a point $u \in C(A, S)$.

Proof. Pick $x_0 \in X$. Construct two sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$y_{n+1} = Ax_n = Sx_{n+1}, \; n = 0, 1, 2, ...$$

This can be done by the virtue of (2.11). Kikkawa and Suzuki [69] have shown that the sequence $\{Sx_n\}$ is Cauchy. Let SX be complete. The sequence $\{Sx_n\}$ is contained in SX, and has a limit in SX. Call it z. If AX is complete then, by (2.11), $z \in SX$. Let $u \in S^{-1}z$. Then $Su = z$.

Now, we show that $Au = Su = z$.

Define F: $SX \to SX$ by $Fa = A(S^{-1}a)$ for each $a \in SX$. First we show that F is well defined. Observe by (2.11) that for $x \in S^{-1}a$.

$$Fa = Ax, \; Fa \subset SX. \tag{2.14}$$

Take $x, y \in S^{-1}a$ such that $b = Ax$, $c = Ay$. Then, since $SX = SY$, we have $b = c$. Therefore F is well defined map.

Now, for $a \neq b$ and $a, b \in SX$, $S^{-1}a \cap S^{-1}b = \phi$. Therefore, for distinct $a, b \in SX$, we suppose $\frac{1}{2}d(a, Fa) < d(a, b)$. Then for $x \in S^{-1}a$, and $y \in S^{-1}b$, we have

$$\frac{1}{2}d(Ax, Sx) = \frac{1}{2}d(a, Fa) < d(a, b) = d(Sx, Sy). \tag{2.15}$$

From (2.12), this inequality implies that $d(Ax, Ay) < d(Sx, Sy)$. Let $x = u$ and $y = x_n$ then $d(Au, Ax_n) < d(Su, Sx_n)$. Making $n \to \infty$,

$$d(Au, z) \leq d(Su, Su), \text{ i.e., } Au = Su = z.$$

This proves that $C(A, S)$, is nonempty. The commutativity of A and S at u implies $AAu = ASu = SAu = SSu$.

Further, $\frac{1}{2}d(Au, Su) = 0 < d(Su, SAu)$.

If $Su \neq SSu = SAu$, then $\frac{1}{2}d(Au, Su) = 0 < d(Su, SAu)$.

So, by (2.12),

$$d(Au, AAu) < d(Su, SSu) = d(Au, AAu),$$

a contradiction. This yields that Au is a common fixed point of A and S. The uniqueness of the common fixed point follows easily. □

Theorem Bis. *Theorem* 2.10 *with* (2.12) *replaced by* (2.16)*:*

$$\frac{1}{2}d(Ax, Sx) < d(Sx, Sy) \text{ implies } d(Ax, Ay) \leq d(Sx, Sy) \text{ for all } x, y \in X. \tag{2.16}$$

Proof. It may be completed following the proofs of Theorem 2.10 and Jachymski [12, Theorem 4.3]. □

Finally, we ask whether in Theorem 2.10, one can replace the conditions (2.12) and (2.13) respectively by (2.17) and (2.18). In particular, we have the following conjecture:

Let A and S be selfmaps of a metric space X such that $AX \subset SX$;

$$\frac{1}{2}d(Ax, Sx) < d(Sx, Sy)$$

implies

$$d(Ax, Ay) < \max\{d(Sx, Sy), [d(Ax, Sx) + d(Ay, Sy)]/2\}, \tag{2.17}$$

$x, y \in X$; for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that, for all $x, y \in X$,

$$\frac{1}{2}d(Ax, Sx) < d(Sx, Sy)$$

and

$$\max\{d(Sx, Sy), [d(Ax, Sx) + d(Ay, Sy)]/2\} < \varepsilon + \delta(\varepsilon) \tag{2.18}$$

together imply $d(Ax, Ay) \leq \varepsilon$. If one of AX or SX is a complete subspace of X then $C(A, S)$ is nonempty. Further, A and S have a unique common fixed point provided that A and S commute at a point $u \in C(A, S)$.

3. Applications

Throughout this section, we assume that U and V are Banach spaces, $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space and $\Re$ denote the field of real numbers. Let $\tau : W \times D \to W,\ g, g' : W \times D \to \Re$ and $G, F : W \times D \times \Re \to \Re$.

As studied by Bellman [73], Bellman and Lee [74] (see also [75]-[79] and references thereof), the basic form of the functional equations of the dynamic programming is the following:

$$p(x) = \underset{y \in D}{opt}\{G(x, y, p(\tau(x, y)))\}, \quad x \in W,$$

where x and y are the state and decision vectors respectively, and $p(x)$ represents the optimal return function with initial state x.

In this section, we study the existence and uniqueness of the common solution of the following functional equations arising in dynamic programming.

$$p := \sup_{y \in D}\{g(x, y) + G(x, y, p(\tau(x, y)))\}, \quad x \in W, \tag{3.1}$$

$$q := \sup_{y \in D}\{g'(x, y) + F(x, y, q(\tau(x, y)))\}, \quad x \in W. \tag{3.2}$$

Let $B(W)$ denote the set of all bounded real-valued functions on W. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{y \in D} |h(x)|$, then $(B(W), \|\,.\,\|)$ is a Banach space. Suppose that the following conditions hold:

$$G, F, g \text{ and } g' \text{ are bounded.} \tag{DP.1}$$

$$\text{For every } (x, y) \in W \times D,\ h, k \in B(W) \text{ and } t \in W \tag{DP.2}$$

$$\frac{1}{2}|Kh(t) - Jh(t)| < |Jh(t) - Jk(t)| \tag{3.3}$$

implies

$$|G(x, y, h(t) - G(x, y, k(t)| \leq |Jh(t) - Jk(t)|, \tag{3.4}$$

where K and J are defined as follows:

$$Kh(x) = \sup_{y \in D}\{g(x, y) + G(x, y, h(\tau(x, y)))\}, \quad x \in W,\ h \in B(W), \tag{$*$}$$

$$Jh(x) = \sup_{y \in D}\{g'(x, y) + F(x, y, h(\tau(x, y)))\}, \quad x \in W,\ h \in B(W) \tag{$**$}$$

For any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$$\text{for all } (x, y) \in W \times D,\ h, k \in B(W) \text{ and } t \in W, \tag{DP.3}$$

$$\frac{1}{2}|Kh(t) - Jh(t)| < |Jh(t) - Jk(t)| \tag{3.5}$$

and

$$|Jh(t) - Jk(t)| < \varepsilon + \delta(\varepsilon) \tag{3.6}$$

implies

$$|G(x, y, h(t) - G(x, y, k(t)| \leq \varepsilon. \tag{3.7}$$

$$\text{For any } h \in B(W), \text{ there exists } k \in B(W) \text{ such that} \tag{DP.4}$$

$$Kh(x) = Jk(x),\ x \in W. \tag{3.8}$$

$$\text{There exists } h \in B(W) \text{ such that} \tag{DP.5}$$

$$Jh(x) = Kh(x) \text{ implies } JKh(x) = KJh(x). \tag{3.9}$$

Theorem 3.1. *Let conditions* (DP.1), (DP.2), (DP.3), (DP.4) *and* (DP.5) *be satisfied. If* $K(B(W))$ *or* $J(B(W))$ *is a closed convex subspace of* $B(W)$*, then the functional equations* (3.1) *and* (3.2) *have a unique bounded solution.*

Proof. Notice that $(B(W), d)$ is a complete metric space, where d is a metric induced by supremum norm on $B(W)$. By (DP.1), J and K are self-maps of $B(W)$. The condition (DP.4) implies that $K(B(W)) \subseteq J(B(W))$. It follows from (DP.5) that J and K commute at their coincidence points.

Let λ be an arbitrary positive number and $h_1, h_2 \in B(W)$. Pick $x \in W$ and choose $y_1, y_2 \in D$ such that

$$Kh_j < g(x, y_j) + G(x, y_j, h_j(x_j)) + \lambda, \tag{3.10}$$

where $x_j = \tau(x, y_j)$, $j = 1, 2$.
Further,

$$Kh_1(x) \geq g(x, y_2) + G(x, y_2, h_1(x_2)) \tag{3.11}$$

and

$$Kh_2(x) \geq g(x, y_1) + G(x, y_1, h_2(x_1)). \tag{3.12}$$

Therefore, (3.3) of (DP.2) becomes

$$\frac{1}{2}\left|Kh_1(x) - Jh_1(x)\right| < \left|Jh_1(x) - Jh_2(x)\right|, \tag{3.13}$$

and this together with (3.10) and (3.12) implies

$$\begin{aligned} Kh_1(x) - Kh_2(x) &< G(x, y_1, h_1(x_1)) - G(x, y_1, h_2(x_1)) + \lambda \\ &\leq |G(x, y_1, h_1(x_1)) - G(x, y_1, h_2(x_1))| + \lambda \\ &< d(Jh_1, Jh_2) + \lambda. \end{aligned} \tag{3.14}$$

Similarly, (3.10), (3.11) and (3.13) imply

$$Kh_2(x) - Kh_1(x) < d(Jh_1, Jh_2) + \lambda. \tag{3.15}$$

So, from (3.14) and (3.15), we have

$$|Kh_1(x) - Kh_2(x)| < d(Jh_1, Jh_2) + \lambda \tag{3.16}$$

Since $x \in W$, and $\lambda > 0$ is arbitrary, we find from (3.13) that

$$\frac{1}{2}d(Kh_1, Jh_1) < d(Jh_1, Jh_2) \tag{3.17}$$

implies

$$d(Kh_1, Kh_2) \leq d(Jh_1, Jh_2). \tag{3.18}$$

Analogously, (3.5) and (3.6) become respectively

$$\frac{1}{2}\left|Kh_1(x) - Jh_1(x)\right| < \left|Jh_1(x) - Jh_2(x)\right|, \tag{3.19}$$

and

$$|Jh_1(x) - Jh_2(x)| < \varepsilon + \delta(\varepsilon). \tag{3.20}$$

These together with (3.10), (3.11) and (3.12) imply

$$|Kh_1(x) - Kh_2(x)| \leq \varepsilon + \lambda. \tag{3.21}$$

Since $x \in W$, and $\lambda > 0$ is arbitrary, we find from (3.19) and (3.20) that

$$\frac{1}{2}d(Kh_1, Jh_1) < d(Jh_1, Jh_2) \tag{3.22}$$

and

$$d(Jh_1, Jh_2) < \varepsilon + \delta(\varepsilon). \tag{3.23}$$

Together imply

$$d(Kh_1, Kh_2) \leq \varepsilon. \tag{3.24}$$

Inequalities (3.17) and (3.18) are the same as inequality (2.16) and inequalities (3.22), (3.23) and (3.24) are the same as the inequality (2.13), wherein K and J correspond, respectively, to the maps A and S. Therefore, by Theorem Bis, K and J have a unique common fixed point h^*, that is $h^*(x)$ is the unique bounded common solution of functional equations (3.1) and (3.2). □

Corollary 3.2. *Assume that the following conditions hold:*

$$G \text{ and } g \text{ are bounded.} \tag{I'}$$

$$\text{For every } (x, y) \in W \times D, \ h, k \in B(W) \text{ and } t \in W \tag{II'}$$

$$\frac{1}{2}|h(t) - Kh(t)| < |h(t) - k(t)|, \tag{3.25}$$

implies

$$|G(x, y, h(t) - G(x, y, k(t)| \leq |h(t) - k(t)|, \tag{3.26}$$

where K is defined by (*)

$$\text{For any } \varepsilon > 0, \text{ there exists a } \delta(\varepsilon) > 0 \text{ such that,}$$

$$\text{for all } (x, y) \in W \times D, \ h, k \in B(W) \text{ and } t \in W, \tag{III'}$$

(3.25) *and*

$$|h(t) - k(t)| < \varepsilon + \delta(\varepsilon), \tag{3.27}$$

implies

$$|G(x, y, h(t) - G(x, y, k(t)| \leq \varepsilon. \tag{3.28}$$

where K is defined by (*). Then the functional equation (3.1) has a unique bounded solution in W.

Proof. It comes from Theorem 3.1 where $g' = 0$, $\tau(x, y) = x$ and $F(x, y, t) = t$ as the assumptions (DP.4) and (DP.5) become redundant. □

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References

[1] D.W. Boyd, J.S. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc., **20**(1969), 458-464.

[2] J. Jachymski, *Equivalent conditions and the Meir-Keeler type theorems*, J. Math. Anal. Appl., **194**(1995), 293-303.

[3] J. Jachymski, *Equivalence of some contractivity properties over metrical structures*, Proc. Amer. Math. Soc., **125**(1997), 2327-2335.

[4] J. Jachymski, *On the iterative equivalence of some classes of mappings*, Ann. Math. Sil., **13**(1999), 149-165.

[5] J. Matkowski, *Integrable solutions of functional equations*, Diss. Math., **127**(1975).

[6] E. Rakotch, *A note on contractive mappings*, Proc. Amer. Math. Soc., **13**(1962), 459-463.

[7] B.E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., **226**(1977), 257-290.

[8] B.E. Rhoades, *Contractive definitions*, Nonlinear Analysis (ed. T.M. Rassias), World Scientific Publ. Company, New Jersey, 1987, 513-526.

[9] T. Suzuki, *A generalized Banach Contraction principle that characterizes metric completeness*, Proc. Amer. Math. Soc., **130**(5)(2008), 1861-1869.

[10] R.M. Bianchini, M. Grandolfi, *Trasformazioni di tipo contrattivo generalizzato in uno spazio metrico* (Italian), Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **45**(8)(1968), 212–216.

[11] M. Furi, *Un teorema di punto fisso per trasformazioni di uno spazio metrico completo in sé* (Italian), Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **45**(8)(1968), 207–211.

[12] J. Jachymski, *Common fixed point theorems for some families of maps*, Indian J. Pure. Appl. Math., **25**(1994), 925-937.

[13] J. Jachymski, *An iff fixed point criterion for continuous self-mappings on a complete metric space*, Aequations Math., **48**(1994), 163-170.

[14] A. Zitarosa, *Una generalizzazione del teorema di Banach sulle contrazioni*, Le Mathematiche, **23**(1968), 417-424.

[15] F.E. Browder, *On the convergence of successive approximations for nonlinear functional equations*, Indag. Math. (N.S.), **30**(1968), 27-38.

[16] W.A. Kirk, B. Sims, *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publ., Dordrecht, 2001.

[17] T.C. Lim, *On Characterizations of Meir-Keeler contractive maps*, Nonlinear Anal., **46**(2001), 113-120.

[18] A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl., **28**(1969), 326-329.

[19] C.S. Wong, *Characterization of certain maps of contractive type*, Pacific J. Math., **68**(1977), 293-296.

[20] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc., **37**(1962), 74-79.

[21] R.P. Agarwal, M. Meehan, D. O'Regan, *Fixed Point Theory and Applications*, Vol. 141, Cambridge University Press, Cambridge, 2001.

[22] C. Di Bari, T. Suzuki, C. Vetro, *Best proximity points for cyclic Meir-Keeler contractions*, Nonlinear Anal., **69**(11)(2008), 3790-3794.

[23] K.J. Chung, *On fixed point theorems of Meir and Keeler*, Math. Japon., **23**(1978), 381-383.

[24] Lj. B. Ciric, *Fixed points of weakly contraction mappings*, Publ. de l'Institut Math. Nouvelle Ser., **20**(34)(1976), 79-84.

[25] Lj. B. Ciric, *A new fixed point theorem for contractive mapping*, Publ. de l'Institut Math. Nouvelle Ser., **30**(44)(1981), 25-27.

[26] M. Hegedus, T. Szilagyi, *Equivalent conditions and a new fixed point theorem in the theory of contractive type mappings*, Math. Japon., **25**(1980), 147-157.

[27] S. Karpagam, S. Agarwal, *Best proximity point theorems for p-cyclic Meir-Keeler contractions*, Fixed Point Theory Applications, 2009, Article Id. 197308, 9 pages.

[28] S. Leader, *Equivalent Cauchy sequences and contractive fixed points in metric spaces*, Studia Math., **66**(1983), 63-67.

[29] M. Maiti, T.K. Pal, *Generalization of two fixed point theorems*, Bull. Cal. Math. Soc., **70**(1978), 57-61.

[30] J. Matkowski, *Fixed point theorems for contractive mappings in metric spaces*, Casopis pro pestování matematiky, **105**(4)(1980), 341–344.

[31] J. Matkowski, R. Wegrzyk, *On equivalence of some fixed point theorems for self mappings of metrically convex spaces*, Bull. Un. Mat. Ital., A(5), **15**(1978), 359-369.

[32] I.H.N. Rao, K.P.R. Rao, *On some fixed point theorems*, Indian J. Pure Appl. Math., **15**(1984), 459-462.

[33] T. Suzuki, *Some notes on Meir-Keeler contractions and L-function*, Bull. Kyushu Inst. Tech., **53**(2006), 1-13.

[34] T. Suzuki, *Fixed Point Theorem for asymptotic contractions of Meir-Keeler type in complete metric spaces*, Nonlinear Anal., **64**(2006), 971-978.

[35] D.H. Tan, N.A. Minh, *Some fixed point theorems for mappings of contractive type*, Acta Math. Vietnam., **3**(1978), 24-42.

[36] A. Tomar, *Coincidence and fixed points of single-valued and multivalued maps*, Ph. D. Thesis, Gurukula Kangri Univ., Hardwar. India, 2002.

[37] K. Wlodarczyk, R. Plebaniak, C. Obczynski, *The uniqueness of endpoints for set-valued dynamical systems of contractions of Meir-Keeler type in uniform spaces*, Nonlinear Anal., **67**(2007), 3373-3383.

[38] Z. Liu, *On Park's open questions and some fixed point theorems for general contractive type mappings*, J. Math. Anal. Appl., **234**(1999), 165-182.

[39] S. Park, *On general contractive type conditions*, J. Korean Math. Soc., **17**(1980), 131-140.

[40] S.L. Singh, A. Kumar, Y.J. Cho, *Fixed points of Meir-Keeler type hybrid contractions*, Pan Amer. Math. J., **16**(2006), 35-54.

[41] P.D. Proinov, *Fixed point theorems in metric spaces*, Nonlinear Anal., **64**(2006), 546-557.

[42] T. Suzuki, *Meir-Keeler contractions of integral type are still Meir-Keeler contractions*, Internat. J. Math. Math. Sci., Article ID. 39281, 6 pages, 2007.

[43] K. Goebel, *A coincidence theorem*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., **16**(1968), 733-735.

[44] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly, **83**(1976), 261-263.

[45] S. Park, J.S. Bae, *Extensions of a fixed point theorem of Meir and Keeler*, Arkiv Math., **19**(1981), 223-228.

[46] S. Park, B.E. Rhoades, *Meir-Keeler type contractive conditions*, Math. Japon., **26**(1981), 13-20.

[47] R.P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl., **188**(1994), 436-440.

[48] H.K. Pathak, Y.J. Cho, S.M. Kang, *Remarks on R-weakly commuting mappings and common fixed point theorems*, Bull. Korean Math. Soc., **34**(1997), 247-257.

[49] S.L. Singh, A. Tomar, *Weaker forms of commuting maps and existence of fixed points*, J. Korea Soc. Math. Edu. Ser. B, Pure Appl. Math., **10**(2003), 145-162.

[50] P.P. Murthy, *Important tools and possible applications of metric fixed point theory*, Nonlinear Anal., **47**(2001), 3479-3490.

[51] H.K. Pathak, M.S. Khan, *A comparison of various types of compatible maps and common fixed points*, Indian J. Pure Appl. Math., **28**(1997), 477-485.

[52] Y.J. Cho, P.P. Murthy, G. Jungck, *A common fixed point theorem of Meir and Keeler type*, Internat. J. Math. Math. Sci., **16**(1993), 669-674.

[53] K. Jha, *Common fixed point theorem for weakly compatible non-continuous mappings*, Thai J. Math., **5**(2007), 191-197.

[54] K. Jha, R.P. Pant, S.L. Singh, *Common fixed points for compatible mappings in metric spaces*, Radovi Math., **12**(2003), 107-114.

[55] G. Jungck, H.K. Pathak, *Fixed Points via "Biased maps"*, Proc. Amer. Math. Soc., **123**(1995), 2049-2060.

[56] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci., **16**(1986), 771-779.

[57] G. Jungck, K.B. Moon, S. Park, B.E. Rhoades, *On generalizations of Meir-Keeler type contraction maps: Corrections*, J. Math. Anal. Appl., **180**(1993), 221-222.
[58] R.P. Pant, *Common fixed points of two pairs of commuting mappings*, Indian J. Pure Appl. Math., **17**(1986), 187-192.
[59] R.P. Pant, *Common fixed points of weakly commuting mappings*, Math. Student, **62**(1993), 97-102.
[60] R.P. Pant, *A new common fixed point principle*, Soochow J. Math., **27**(2001), 287-297.
[61] R.P. Pant, *Meir-Keeler type fixed point theorems and dynamics of functions*, Demonstratio Math., **35**(2003), 199-206.
[62] R.P. Pant, P.C. Joshi, V. Gupta, *A Meir-Keeler type fixed point theorem*, Indian J. Pure Appl. Math., **32**(2001), 779-787.
[63] H.K. Pathak, S.M. Kang, Y.J. Cho, *A Meir-Keeler type common fixed point theorem*, Math. Japon., **44**(1996), 323-330.
[64] I.H.N. Rao, K.P.R. Rao, *Generalizations of fixed point theorems of Meir and Keeler type*, Indian J. Pure Appl. Math., **16**(1985), 1249-1262.
[65] B.E. Rhoades, *Fixed point theorems for some families of maps*, Indian J. Pure Appl. Math., **21**(1990), 10-20.
[66] B.E. Rhoades, S. Park, K.B. Moon, *On generalizations of Meir-Keeler type contraction maps*, J. Math. Anal. Appl., **146**(1990), 482-494.
[67] S.L. Singh, V. Chadha, S.N. Mishra, *Remarks on recent fixed point theorems for compatible maps*, Internat. J. Math. Math. Sci., **19**(1996), 801-804.
[68] D.H. Tan, *Some fixed point theorems of mappings of contractive type*, Univ. Nov. Sadu. Ser. Mat., **25**(1993), 9-22.
[69] M. Kikkawa, T. Suzuki, *Three fixed point theorems for generalized contractions with constants in complete metric spaces*, Nonlinear Anal., **69**(2008), 2942-2949.
[70] O. Popescu, *Two fixed point theorems for generalized contractions with constants in complete metric space*, Cent. Eur. J. Math., **7**(3)(2009), 529-538.
[71] S.L. Singh, B.D. Pant, *Coincidence and fixed point theorems for a family of mappings on menger spaces and extension to uniform spaces*, Math Japon., **33**(6)(1988), 957-973.
[72] S.L. Singh, A. Kumar, *Fixed point theorems for Lipschtiz type maps*, Riv. Mat. Univ. Parma, **4**(7)(2004), 25-34.
[73] R. Bellman, *Methods of Nonlinear Analysis*, Vol II, Academic Press, New York, NY, USA, 1973.
[74] R. Bellman, E.S. Lee, *Functional equations in dynamic programming*, Aequationes Math., **17**(1)(1978), 1-18.
[75] R. Baskaran, P.V. Subrahmanyam, *A note on the solution of a class of functional equations*, Applicable Anal., **22**(1986), 235-241.
[76] P.C. Bhakta, S. Mitra, *Some existence for functional equations arising in dynamic programming*, J. Math. Anal. Appl., **98**(2)(1984), 348-462.
[77] H.K. Pathak, Y.J. Cho, S.M. Kang, B.S. Lee, *Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming*, Le Matematiche, **50**(1995), 15-33.
[78] S.L. Singh, S.N. Mishra, *Remarks on recent fixed point theorems*, Fixed Point Theory Appl., Article Id. 452905, 18 pages, 2010.
[79] S.L. Singh, S.N. Mishra, *Fixed point theorems for single-valued and multi-valued maps*, Nonlinear Anal., **74**(2011), 2243-2248.

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