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EXISTENCE OF STRONG VIABLE SOLUTIONS OF BACKWARD STOCHASTIC DIFFERENTIAL INCLUSIONS

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Abstract. Existence of strong viable solutions for backward stochastic differential inclusions is considered. The paper contains the basic notions dealing with backward stochastic differential inclusions, some viable approximation theorem and existence viable theorem for backward stochastic differential inclusions.
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1. Introduction

Given measurable set-valued mappings $F : [0,T] \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ and $H : \mathbb{R}^m \to Cl(\mathbb{R}^m)$ by a backward stochastic differential inclusion $BSDI(F,H)$ we mean relations of the form

$$\begin{cases} x_s \in E\left[x_t + \int_s^t F(\tau, x_\tau)d\tau|\mathcal{F}_s\right] \quad \text{a.s.} \quad \text{for} \quad 0 \le t \le T \\ x_T \in H(x_T) \quad \text{a.s.} \end{cases} \tag{1.1}$$

that have to be satisfied by a càdlág process $x = (x_t)_{0\le t\le T}$ defined on a complete filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, P, \mathbb{F})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual hypothesis (see [9]). $E[x_t + \int_s^t F(\tau, x_\tau)d\tau|\mathcal{F}_s]$ denotes the set-valued conditional expectation (see [3], [4]) of the set-valued mapping $\Omega \ni \omega \longrightarrow x_t(\omega) + \int_s^t F(\tau, x_\tau(\omega))d\tau \subset \mathbb{R}^m$ relative to $\mathcal{F}_s$. A pair $(x, \mathcal{P}_{\mathbb{F}})$ satisfying conditions (1.1) is said to be a weak solutions of $BSDI(F,H)$. If $\mathcal{P}_{\mathbb{F}}$ is given then x, satisfying conditions presented above, is said to be a strong solution of $BSDI(F,H)$. Existence of strong solutions of $BSDI(F,H)$ has been considered in the author's paper [6]. In particular case, $BSDI(F,H)$ generalizes a backward stochastic differential equation considered in [2]. If a filtered probability space $\mathcal{P}_{\mathbb{F}}$ has a "constant" filtration $\mathbb{F} = (\mathcal{F})$ then a strong solution x for such $BSDI(F,H)$ is a solution to a backward random inclusion $-x'_t \in \text{co}F(t,x_t)$ with a terminal condition $x_T \in H(x_T)$.

The present paper is devoted to the existence of strong solutions of the following viability problem $BSDI(F,K)$:

$$\begin{cases} x_s \in E\left[x_t + \int_s^t F(\tau, x_\tau)d\tau | \mathcal{F}_s\right] \quad \text{a.s.} \quad \text{for} \quad 0 \le t \le T \\ x_t \in K(t) \quad \text{a.s.} \quad \text{for} \quad 0 \le t \le T, \end{cases} \tag{1.2}$$

where $K : [0,T] \times \Omega \to \mathrm{Cl}(\mathbb{R}^m)$ is a given set-valued process. Throughout the paper we assume that $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, P, \mathbb{F})$ is a given complete filtered probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual hypotheses. By $\mathbb{D}(\mathbb{F}, \mathbb{R}^d)$ we denote the space of all m - dimensional $\mathbb{F}$ - adapted càdlág processes on $\mathcal{P}_{\mathbb{F}}$ and by $\mathcal{S}(\mathbb{F}, \mathbb{R}^m)$ the set of all m - dimensional $\mathbb{F}$ - semimartingales x such that $||x||_{\mathcal{S}} = E[\sup_{s \in [0,T]} |x_s|^2] < \infty$. We have $\mathcal{S}(\mathbb{F}, \mathbb{R}^d) \subset \mathbb{D}(\mathbb{F}, \mathbb{R}^m)$. It can be proved (see [9], Th.IV2.1.,Th.V.2.2.) that $(\mathcal{S}(\mathbb{F}, \mathbb{R}^m), || \cdot ||_{\mathcal{S}})$ is a Banach space.

The paper is organized as follows. Section 2 contains some properties of set-valued conditional expectation of Aumann's set-valued integrals. In Section 3 some measurable selection theorem is given. Section 4 contains some viable approximation theorem. Existence of strong viable solutions for $BSDI(F,K)$ is proved in Section 5.

2. Conditional expectation of set-valued integrals

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$ and $\Phi : \Omega \to Cl(\mathbb{R}^m)$ be an $\mathcal{F}$-measurable set-valued mapping with a nonempty subtrajectory integrals $S(\Phi)$ containing all its integrable selectors. By properties of $S(\Phi)$ there exists (see [4]) a unique (in the a.s. sense) $\mathcal{G}$-measurable set-valued mapping $E[\Phi|\mathcal{G}]$ satisfying

$$S(E[\Phi|\mathcal{G}]) = cl_L\{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\} \tag{2.1}$$

where cl_L denotes the closure operation in $\mathbb{L}(\Omega, \mathcal{G}, \mathbb{R}^m)$. We call $E[\Phi|\mathcal{G}]$ the multivalued conditional expectation of Φ relative to $\mathcal{G}$. The multivalued conditional expectation possesses properties similar to those of the usual ones. For example, we have $\int_A E[\Phi|\mathcal{G}]dP = \int_A \Phi dP$ for every $A \in \mathcal{G}$, where integrals are understood in the Aumann's sense (see [4], Prop.6.8). It can be proved (see [4], Prop. 6.2.) that for given measurable and integrably bounded set-valued mappings $\Phi, \Psi : \Omega \to Cl(\mathbb{R}^m)$ one has $Eh\{E[\Phi|\mathcal{G}], E[\Psi|\mathcal{G}]\} \le E[h(\Phi, \Psi)]$, where h is the Hausdorff metric on $Cl(\mathbb{R}^m)$.

Let $G : [0,T] \times \Omega \to Cl(\mathbb{R}^m)$ be measurable and integrably bounded, i.e., such that there is $m \in \mathbb{L}([0,T] \times \Omega, \beta_T \otimes \mathcal{F}_T, \mathbb{R}_+)$ satisfying an inequality $\|G(t,x)\| \le m(t,\omega)$ a.e. In what follows we shall denote such set-valued mappings as measurable set-valued processes $F = (F_t)_{0 \le t \le T}$ with $F_t = G(t, \cdot)$. The space of all such defined set-valued processes satisfying conditions mentioned above will be denoted by $\mathcal{L}(T, \Omega, \mathbb{R}^m)$. As usual by $S(G)$ we denote subtrajectory integrals of G, i.e., a set of all integrable selectors of G. It is easy to verify (see [5]) that $S(G)$ is nonempty closed and decomposable, i.e., that for every $f, g \in S(G)$ and $E \in \beta_T \otimes \mathcal{F}_T$ one has $\mathbf{1}_E f + \mathbf{1}_{E^\sim} g \in S(G)$, where β_T denotes the Borel σ-algebra of $[0,T]$ and $E^\sim$ is the complement of E. In particular, if $G(t,\omega)$ are convex subsets of $\mathbb{R}^m$ for $(t,\omega) \in [0,T] \times \Omega$, then $S(G)$ is a convex weakly compact subset of $\mathbb{L}([0,T] \times \Omega, \beta_T \otimes \mathcal{F}_T, \mathbb{R}^m)$. For a given above G we can define an Aumann integral $\Phi(\omega) = \int_0^T G(t,\omega)dt$ depending on a parameter $\omega \in \Omega$.

Proposition 2.1. *For every $F \in \mathcal{L}(T, \Omega, \mathbb{R}^m)$ a set-valued mapping $\int_0^T F_t(\cdot)dt$ defined by $\Omega \ni \omega \to \int_0^T F_t(\omega)dt \in \mathrm{Cl}(\mathbb{R}^m)$ is $\mathcal{F}_T$-measurable with compact convex values.*

Proof. By virtue of Aumann theorem (see [5],Th.II.3.20) $\int_0^T F_t(\omega)dt$ is a nonempty compact convex subset of $\mathbb{R}^m$ for every $\omega \in \Omega$ and $\int_0^T F_t(\omega)dt = \int_0^T \mathrm{co}\, F_t(\omega)dt$. Therefore, to verify that the set-valued mapping $\Omega \ni \omega \to \int_0^T F_t(\omega)dt \in \mathrm{Cl}(\mathbb{R}^d)$ is $\mathcal{F}_T$-measurability (see [5], Th.II.3.8) it is enough to show that the function $\Omega \ni \omega \to \sigma(p, \int_0^T F_t(\omega)dt) \in \mathbb{R}$ is $\mathcal{F}_T$-measurable for every $p \in \mathbb{R}^d$, where $\sigma(\cdot, A)$ is a support function of a set $A \in \mathrm{Cl}(\mathbb{R}^m)$. By measurability of F and its integrably boundedness the function $[0,T] \times \Omega \ni (t, \omega) \to \sigma(p, \mathrm{co}F_t(\omega)) \subset \mathbb{R}$ is measurable for every $p \in \mathbb{R}^d$. By virtue of ([5], Th II.3.21) for every $p \in \mathbb{R}^m$ one has $\sigma(p, \int_0^T F_t(\omega)dt)) = \int_0^T \sigma(p, \mathrm{co}F_t(\omega))dt$ for every $\omega \in \Omega$. Hence, by Fubini's theorem, $\mathcal{F}_T$-measurability of the function $\Omega \ni \omega \to \sigma(p, \int_0^T F_t(\omega)dt) \in \mathbb{R}$ follows for every $p \in \mathbb{R}^d$. Therefore, $\int_0^T F_t(\cdot)dt$ is $\mathcal{F}_T$-measurable. □

Proposition 2.2. *Let $F \in \mathcal{L}(T, \Omega, \mathbb{R}^m)$. Subtrajectory integrals $S[\int_0^T F_t(\cdot) dt]$ of $\int_0^T F_t(\cdot)dt$ is a nonempty convex weakly compact subset of the space $\mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^m)$ and $S[\int_0^T F_t(\cdot)dt] = J[S(\mathrm{co}F)]$, where $J : \mathbb{L}([0,T] \times \Omega, \beta_T \otimes \mathcal{F}_T, \mathbb{R}^m) \to \mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^m)$ is defined by $J(f) = \int_0^T f(t, \cdot)dt$ for $f \in \mathbb{L}([0,T] \times \Omega, \beta_T \otimes \mathcal{F}_T, \mathbb{R}^m)$.*

Proof. By the properties of the mapping $\Omega \ni \omega \to \int_0^T \mathrm{co}F_t(\omega)dt \in \mathrm{Cl}(\mathbb{R}^m)$ it follows that $S[\int_0^T \mathrm{co}F_t(\cdot)dt]$ is a nonempty convex weakly compact subset of the space $\mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^m)$. Hence, by the equality $\int_0^T F_t(\omega)dt = \int_0^T \mathrm{co}F_t(\omega)dt$ for a.e. $\omega \in \Omega$ it follows that $S[\int_0^T F_t(\cdot)dt]$ is also a nonempty convex weakly compact subset of the space $\mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^m)$. By the definition of $J[S(\mathrm{co}F)]$ it follows that the set $J[S(\mathrm{co}F)]$ is a nonempty convex weakly compact subset of $\mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^m)$ such that $J[S(\mathrm{co}F)] \subset S[\int_0^T \mathrm{co}F_t(\cdot)dt] = S_T[\int_0^T F_t(\cdot)dt]$.

Assume $\varphi \in S[\int_0^T F(t, \cdot)dt]$. Then for every $A \in \mathcal{F}_T$ one has $E_A\varphi \in E_A\Phi$, where $\Phi = \int_0^T F_t(\cdot)dt$, $E_A\varphi = \int_A \varphi dP$ and $E_A\Phi = \int_A \Phi dP$. Let $\varepsilon > 0$ be given and select an $\mathcal{F}_T$-measurable partition $(A_n^\varepsilon)_{n=1}^{N_\varepsilon}$ of Ω such that $E_{A_n^\varepsilon} \int_0^T ||F_t(\cdot)||dt < \varepsilon/2^{n+1}$. For every $n = 1, ..., N_\varepsilon$ there is an $f_n^\varepsilon \in S(F)$ such that $E_{A_n^\varepsilon}\varphi = E_{A_n^\varepsilon} \int_0^T f_n^\varepsilon(t, \cdot)dt$. Let $f^\varepsilon = \sum_{n=1}^{N_\varepsilon} \mathbb{I}_{A_n^\varepsilon} f_n^\varepsilon$. By decomposability of $S(F)$ one has $f^\varepsilon \in S(F)$. We have $f^\varepsilon \in S(\mathrm{co}F)$ because $S(F) \subset S(\mathrm{co}\, F)$. Taking a sequence $(\varepsilon_k)_{k=1}^\infty$ of positive numbers $\varepsilon_k > 0$ such that $\varepsilon_k \to 0$ as $k \to \infty$ we can select a subsequence, denoted again by $(f^{\varepsilon_k})_{k=1}^\infty$, of $(f^{\varepsilon_k})_{k=1}^\infty$ weakly converging to $f \in S(\mathrm{co}F)$, because $S(\mathrm{co}F)$ is a weakly compact subset of $\mathbb{L}([0,T] \times \Omega, \beta_T \otimes \mathcal{F}_T, \mathbb{R}^m)$. For every $A \in \mathcal{F}$ and $k = 1, 2, \ldots$ there is a subset $\{n_1, ..., n_p\}$ of $\{1, ..., N_{\varepsilon_k}\}$ such that $A \cap A_{n_i}^{\varepsilon_k} \neq \emptyset$ for $i = 1, 2, ..., p$ and $A \cap A_r = \emptyset$ for $r \in \{1, 2, ..., N_{\varepsilon_k}\} \setminus \{n_1, ..., n_p\}$. Therefore,

$$\left| E_A\varphi - E_A \int_0^T f^{\varepsilon_k}(t, \cdot)dt \right| \leq \sum_{n=1}^{N_{\varepsilon_k}} \left| E_{A \cap A_n^{\varepsilon_k}}\varphi - E_{A \cap A_n^{\varepsilon_k}} \int_0^T f_n^{\varepsilon_k}(t, \cdot)dt \right|$$

$$= \sum_{i=1}^{p} \left| E_{A\cap A_{n_i}^{\varepsilon_k}}\varphi - E_{A\cap A_{n_i}^{\varepsilon_k}} \int_0^T f_n^{\varepsilon_k}(t,\cdot)dt \right| \le 2\sum_{i=1}^{p} E_{A_{n_i}^{\varepsilon_k}} \int_0^T ||F_t(\cdot)||dt \le \varepsilon_k$$

for every $k = 1, 2, ...$ On the other hand for every $A \in \mathcal{F}$ we also have

$$\left| E_A\varphi - E_A \int_0^T f(t,\cdot)dt \right| \le \left| E_A\varphi - E_A \int_0^T f^{\varepsilon_k}(t,\cdot)dt \right|$$

$$+ \left| E_A \int_0^T f^{\varepsilon_k}(t,\cdot)dt - E_A \int_0^T f(t,\cdot)dt \right|$$

$$\le \varepsilon_k + \left| E_A \int_0^T f^{\varepsilon_k}(t,\cdot)dt - E_A \int_0^T f(t,\cdot)dt \right|$$

for $k = 1, 2, ...$ Hence it follows that $E_A\varphi = E_A \int_0^T f(t,\cdot)dt$ for every $A \in \mathcal{F}$, because $\varepsilon_k \to 0$ and $|E_A \int_0^T f^{\varepsilon_k}(t,\cdot)dt - E_A \int_0^T f(t,\cdot)dt| \to 0$ as $k \to \infty$. Therefore, $\varphi(\omega) = \int_0^T f(t,\omega)dt$ for a.e. $\omega \in \Omega$. Then $\varphi \in J[S(\text{co}F)]$ and $S[\int_0^T F_t(\cdot)dt] = J[S(\text{co}F)]$. □

Corollary 2.1. *If $G : [0,T]\times\Omega \to Cl(\mathbb{R}^m)$ is measurable and integrably bounded and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$ then*

$$S\left(E\left[\int_0^T G(t,\cdot)dt|\mathcal{G}\right]\right) = \left\{E\left[\int_0^T g(t,\cdot)dt|\mathcal{G}\right] : g \in S(\text{co}\, G)\right\}.$$

Proof. It is enough only to see that the set $\mathcal{H} = \{E[\int_0^T g(t,\cdot)dt|\mathcal{G}] : g \in S(\text{co } G)\}$ is a closed subset of $\mathbb{L}(\Omega,\mathcal{G},\mathbb{R}^m)$. By properties of the conditional expectations and properties of the set $S(\text{co}\, G)$ it follows that $\mathcal{H}$ is a convex weakly compact subset of $\mathbb{L}(\Omega,\mathcal{G},\mathbb{R}^m)$. Therefore, $\mathcal{H}$ is a closed subset of $\mathbb{L}(\Omega,\mathcal{G},\mathbb{R}^m)$. □

3. Measurable selection theorems

Let $x = (x_t)_{0\le t\le T}$ be an measurable m-dimensional cádlág process on $\mathcal{P}_{\mathbb{F}}$. Given a measurable and uniformly integrably bounded multivalued mapping $F : [0,T]\times\mathbb{R}^m \to Cl(\mathbb{R}^m)$ let $F \circ x$ be a set-valued process defined by $(F \circ x)(t,\omega) = F(t, x_t(\omega))$ for $(t,\omega) \in [0,T]\times\Omega$. It is clear that $F \circ x$ is measurable. In what follows by $S(F \circ x)$ we denote subtrajectory integrals of $F \circ x$. Immediately from Kuratowski and Ryll-Nardzewski measurable selection theorem (see [7], Th.1) it follows that for a given above F and x the set $S(\text{co}F \circ x)$ is a nonempty convex weakly compact subset of $\mathbb{L}([0,T]\times\Omega, \beta_T \otimes \mathcal{F}_T, \mathbb{R}^m)$.

Theorem 3.1. *Assume $F : [0,T]\times\mathbb{R}^m \to Cl(\mathbb{R}^m)$ is measurable and uniformly integrably bounded and let $x = (x_t)_{0\le t\le T}$ and $z = (z_t)_{0\le t\le T}$ be m-dimensional measurable stochastic processes on a filtred probability space $\mathcal{P}_{\mathbb{F}} = (\Omega,\mathcal{F},\mathbb{F},P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\ge0}$ satisfying the usual conditions and let $E|x_T| < \infty$. If x is $\mathbb{F}$-adapted then*

$$x_s \in E\left[x_t + \int_s^t F(\tau, z_\tau)d\tau|\mathcal{F}_s\right] \qquad a.s. \tag{3.1}$$

for every $0 \le s \le t \le T$ *if and only if there is* $f \in S(\mathrm{co}F \circ z)$ *such that*

$$x_t = E\left[x_T + \int_t^T f(\tau,\cdot)d\tau|\mathcal{F}_t\right] \qquad a.s. \tag{3.2}$$

for every $0 \le t \le T$.

Proof. Suppose there is $f \in S(\mathrm{co}F \circ z)$ such that (3.2) is satisfied. For every $0 \le s \le t \le T$ one has

$$x_s = E\left[x_T + \int_s^T f(\tau,\cdot)d\tau|\mathcal{F}_s\right]$$
$$= E\left[\int_s^t f(\tau,\cdot)d\tau|\mathcal{F}_s\right] + E\left[x_T + \int_t^T f(\tau,\cdot)d\tau|\mathcal{F}_s\right]$$

and $E[x_t|\mathcal{F}_s] = E[x_T + \int_t^T f(\tau,\cdot)d\tau|\mathcal{F}_s]$ a.s. Then $x_s = E[x_t + \int_s^t f(\tau,\cdot)d\tau|\mathcal{F}_s]$, a.s. for $0 \le s \le t \le T$. Hence by Corollary 2.1 it follows that $x_s \in S(E[x_t + \int_s^t F(\tau, z_\tau)d\tau|\mathcal{F}_s])$ for $0 \le s \le t \le T$. Therefore (3.1) is satisfied a.s. for $0 \le s \le t \le T$.

Assume (3.1) is satisfied a.s. for every $0 \le s \le t \le T$ and let $m \in \mathbb{L}([0,T],\mathbb{R}_+)$ be such that $\|F(t,x\| \le m(t)$ for a.e. $t \in [0,T]$ and $x \in \mathbb{R}^m$. For every $0 \le t \le T$ one has $E|x_t| \le E|x_T| + E\int_0^T m(t)dt < \infty$. Let $\eta > 0$ be fixed and select $\delta \in (0,T)$ such that $\sup_{0\le t\le T-\delta}\int_t^{t+\delta} m(\tau)d\tau < \eta/2$. For fixed $t \in [0, T-\delta]$ and $t \le \tau \le t+\delta$ we have $x_t \in E[x_\tau + \int_t^\tau F(s,z_s)ds|\mathcal{F}_t]$ a.s. Therefore, for every $A \in \mathcal{F}_t$ we get $E_A(x_t - x_\tau) \in E_A\int_t^\tau F(s,z_s)ds$, where $E_A(x_t - x_\tau) = E[\mathbb{I}_A(x_t - x_\tau)]$ and $E_A\int_t^\tau F(s,z_s)ds = E[\mathbb{I}_A\int_t^\tau F(s,z_s)ds]$ for $A \in \mathcal{F}_t$. Then

$$|E_A(x_t - x_\tau)| \le E_A\int_t^\tau \|F(s,z_s)\|ds \le E\int_t^{t+\delta} m(s)ds < \eta/2$$

for every $0 \le t \le T-\delta$ and $A \in \mathcal{F}_t$. Therefore, $\sup_{t\le\tau\le t+\delta}|E_A(x_t - x_\tau)| \le \eta/2$ for every $A \in \mathcal{F}_t$ and fixed $0 \le t \le T-\delta$.

Let $\tau_0 = 0$, $\tau_1 = \delta, ..., \tau_{N-1} = (N-1)\delta < T \le N\delta$. Immediately from (3.1) and Corollary 2.1 it follows that for every $i = 1,2,...,N-1$ there is $f_i^\eta \in S(\mathrm{co}F \circ z)$ such that

$$E\left|x_{\tau_{i-1}} - E\left[x_{\tau_i} + \int_{\tau_{i-1}}^{\tau_i} f_i^\eta(s,\cdot)ds|\mathcal{F}_{\tau_{i-1}}\right]\right| = 0.$$

Furthermore, there is $f_N^\eta \in S(\mathrm{co}F \circ z)$ such that

$$E\left|x_{\tau_{N-1}} - E\left[x_T + \int_{\tau_{N-1}}^T f_N^\eta(s,\cdot)ds|\mathcal{F}_{\tau_{N-1}}\right]\right| = 0.$$

Define $f^\eta(t,\omega) = \sum_{i=1}^{N-1}\mathbb{I}_{[\tau_{i-1},\tau_i)}(t)f_i^\eta(t,\omega) + \mathbb{I}_{[\tau_{N-1},T]}(t)f_N^\eta(t,\omega)$ for $(t,\omega) \in [0,T]\times\Omega$. By decomposability of $S(\mathrm{co}F \circ z)$ we have $f^\eta \in S(\mathrm{co}F \circ z)$. For fixed $t \in [0,T]$ there is $p \in \{1,2,...,N-1\}$ or $p = N$ such that $t \in [\tau_{p-1},\tau_p)$ or $t \in [\tau_{N-1},T]$. Let $t \in [\tau_{p-1},\tau_p)$ with $1 \le p \le N-1$. For every $A \in \mathcal{F}_t$ one has

$$\left|E_A\left(x_t - E\left[x_T + \int_t^T f^\eta(s,\cdot)ds|\mathcal{F}_t\right]\right)\right| \le |E_A(x_t - x_{\tau_p})| + E\left|x_{\tau_p} - E\left[x_{\tau_{p+1}}\right.\right.$$

$$+\int_{\tau_p}^{\tau_{p+1}} f^\eta(s,\cdot)d\tau|\mathcal{F}_{\tau_p}\Big]\Big| + \ |E_A(E[x_{\tau_{p+1}}|\mathcal{F}_{\tau_p}] - x_{\tau_{p+1}})| + E\Big|\int_t^{\tau_p} f^\eta(s,\cdot)ds\Big|$$

$$+\Big|E_A\Big(E\Big[\int_{\tau_p}^{\tau_{p+1}} f^\eta(s,\cdot)ds|\mathcal{F}_{\tau_p}\Big] - E\Big[\int_{\tau_p}^{\tau_{p+1}} f^\eta(s,\cdot)d\tau|\mathcal{F}_t\Big]\Big)\Big| + \ldots$$

$$+E\Big|x_{\tau_{N-1}} - E\Big[x_T + \int_{\tau_{N-1}}^{T} f^\eta(s,\cdot)d\tau|\mathcal{F}_{\tau_{N-1}}\Big]\Big| + \Big|E_A(E[x_{\tau_{N-1}}|\mathcal{F}_{\tau_{N-1}}] - x_{\tau_{N-1}})\Big|$$

$$+E_A\Big(E\Big[\int_{\tau_{N-1}}^{T} f^\eta(s,\cdot)ds|\mathcal{F}_{\tau_{N-1}}\Big] - E\Big[\int_{\tau_{N-1}}^{T} f^\eta(s,\cdot)ds|\mathcal{F}_t\Big]\Big)\Big|$$

$$\le \sup_{t\le\tau\le t+\delta}|E_A(x_t - x_\tau)| + \int_t^{t+\delta} m(s)ds + \sum_{i=p}^{N-2} E\Big|x_{\tau_i} - E\Big[x_{\tau_{i+1}} + \int_{\tau_i}^{\tau_{i+1}} f^\eta(s,\cdot)ds|\mathcal{F}_{\tau_i}\Big]\Big|$$

$$+E\Big|x_{\tau_{N-1}} - E\Big[x_T + \int_{\tau_{N-1}}^{T} f^\eta(s,\cdot)d\tau|\mathcal{F}_{\tau_{N-1}}\Big]\Big| + \sum_{i=p}^{N-2} |E_A(E[x_{\tau_{i+1}}|\mathcal{F}_{\tau_i}] - x_{\tau_{i+1}})|$$

$$+\sum_{i=p}^{N-2}\Big|E_A\Big(E\Big[\int_{\tau_i}^{\tau_{i+1}} f^\eta(s,\cdot)ds|\mathcal{F}_{\tau_i}\Big] - E\Big[\int_{\tau_i}^{\tau_{i+1}} f^\eta(s,\cdot)ds|\mathcal{F}_t\Big]\Big)\Big|$$

$$+\Big|E_A\Big(E\Big[\int_{\tau_{N-1}}^{T} f^\eta(s,\cdot)ds|\mathcal{F}_{\tau_{N-1}}\Big] - E\Big[\int_{\tau_{N-1}}^{T} f^\eta(s,\cdot)ds|\mathcal{F}_t\Big]\Big)\Big|.$$

But $\mathcal{F}_t \subset \mathcal{F}_{\tau_i}$ for $i = p, p+1, ..., N-1$. Then for $A \in \mathcal{F}_t$ one has

$$\sum_{i=p}^{N-2} |E_A(E[x_{\tau_{i+1}}|\mathcal{F}_{\tau_i}] - x_{\tau_{i+1}})| = 0,$$

$$\sum_{i=p}^{N-2}\Big|E_A\Big(E\Big[\int_{\tau_i}^{\tau_{i+1}} f^\eta(s,\cdot)ds|\mathcal{F}_{\tau_i}\Big] - E\Big[\int_{\tau_i}^{\tau_{i+1}} f^\eta(s,\cdot)ds|\mathcal{F}_t\Big]\Big)\Big| = 0$$

and

$$\Big|E_A\Big(E\Big[\int_{\tau_{N-1}}^{T} f^\eta(s,\cdot)ds|\mathcal{F}_{\tau_{N-1}}\Big] - E\Big[\int_{\tau_{N-1}}^{T} f^\eta(s,\cdot)ds|\mathcal{F}_t\Big]\Big)\Big| = 0.$$

With this and the definition of f^η it follows

$$\left|E_A\left(x_t - E\left[x_T + \int_t^T f^\eta(s,\cdot)ds|\mathcal{F}_t\right]\right)\right| \le \eta \tag{3.3}$$

for fixed $0 \le t \le T$ and $A \in \mathcal{F}_t$. Let $(\eta_j)_{j=1}^\infty$ be a sequence of positive numbers converging to zero. For every $j = 1, 2, ...$ we can select $f^{\eta_j} \in S(\text{co}F \circ z)$ such that (3.3) is satisfied with $\eta = \eta_j$. By weak compactness of $S(\text{co}F \circ z)$ there is a subsequence $(f^{\eta_k})_{k=1}^\infty$ of $(f^{\eta_j})_{j=1}^\infty$ weakly converging to $f \in S(\text{co}F \circ z)$. Then for every $A \in \mathcal{F}_t \subset \mathcal{F}_T$ one has

$$\lim_{k\to\infty} E_A \int_t^T f^{\eta_k}(s,\cdot)ds = E_A \int_t^T f(s,\cdot)ds.$$

On the other hand for every fixed $t \in [0,T]$ and $A \in \mathcal{F}_t$ we have

$$\left|E_A\left(x_t - E\left[x_T + \int_t^T f(s,\cdot)ds|\mathcal{F}_t\right]\right)\right| \leq \left|E_A\left(x_t - E\left[x_T + \int_t^T f^{\eta_k}(s,\cdot)ds|\mathcal{F}_t\right]\right)\right|$$
$$+\left|E_A\left(E\left[\int_t^T f^{\eta_k}(s,\cdot)ds|\mathcal{F}_t\right] - E\left[\int_t^T f(s,\cdot)ds|\mathcal{F}_t\right]\right)\right|$$
$$\leq \eta_k + \left|E_A\int_t^T f^{\eta_k}(s,\cdot)ds - E_A\int_t^T f(s,\cdot)ds\right|$$

for $k = 1,2,...$ Therefore, $E_A(x_t - E[x_T + \int_t^T f(s,\cdot)ds|\mathcal{F}_t]) = 0$ for every $A \in \mathcal{F}_t$ and fixed $0 \leq t \leq T$. But x_t and $E[x_T + \int_t^T f(s,\cdot)ds|\mathcal{F}_t]$ are $\mathcal{F}_t$-measurable. Then $x_t = E[x_T + \int_t^T f(s,\cdot)ds|\mathcal{F}_t]$ a.s. for $0 \leq t \leq T$. Then there exists $f \in S(\mathrm{co}F \circ z)$ such that (3.2) is satisfied. □

For a measurable process Z on $\mathcal{P}_{\mathbb{F}}$ by $[Z]^{\mathbb{F}}$ we shall denote the "conditional expectation" with respect to a measure $\mu \otimes P$ and an $\mathbb{F}$-optional σ-algerbra $\mathcal{O}$, i.e., $[Z]^{\mathbb{F}} = E_{\mu\otimes P}[Z|\mathcal{O}]$, where μ denotes the Lebesgue measure on $[0,T]$.

Corollary 3.1. *If the assumptions of Theorem 3.1 are satisfied then a process $x = (x_t)_{0\leq t\leq T}$ defined by $x_t = E[x_T + \int_t^T f(\tau,\cdot)d\tau|\mathcal{F}_t]$ a.s. for $0 \leq t \leq T$ with $f \in S(\mathrm{co}F \circ z)$ belongs to $\mathcal{S}(\mathbb{F},\mathbb{R}^m)$ and has a supermartingale representation $x_t = x_0 + M_t + A_t$, where $x_0 = E[x_T + \int_0^T f_\tau d\tau|\mathcal{F}_0]$, $A_t = -\int_0^t [f]_\tau^{\mathbb{F}} d_\tau$ and $M_t = E[x_T + \int_0^T f_\tau d\tau|\mathcal{F}_t] - E[x_T + \int_0^T f(_\tau d\tau|\mathcal{F}_0] - E[\int_0^t \{f(\tau,\cdot) - [f]_\tau^{\mathbb{F}}\}d\tau|\mathcal{F}_t]$. Process x is continuous if and only if $(M_t)_{0\leq t\leq T}$ is a continuous martingale.*

Proof. It is clear that $x_t = x_0 + M_t + A_t$ a.s. for $0 \leq t \leq T$, where x_0, M_t and A_t are for every $0 \leq t \leq T$ such as above. To see that $(A_t)_{0\leq t\leq T}$ is $\mathbb{F}$-adapted absolutely continuous process and $(M_t)_{0\leq t\leq T}$ is $\mathbb{F}$-martingale let us observe that $[f]_t^{\mathbb{F}}$ is $\mathcal{F}_t$ -measurable for every $f \in S(\mathrm{co}F \circ z)$ and $t \in [0,T]$, which implies that also A_t is $\mathcal{F}_t$-measurable for every $f \in S(\mathrm{co}F \circ z)$ and $t \in [0,T]$. Furthermore, the process $(A_t)_{0\leq t\leq T}$ is absolutely continuous because $|[f]_t^{\mathbb{F}}| \leq |f_t| \leq \|F(t,z_t)\|$ a.s. for a.e. $t \in [0,T]$. To verify that $(M_t)_{0\leq t\leq T}$ is an $\mathbb{F}$- martingale let us observe first that $E[\int_s^t f_\tau d\tau|\mathcal{F}_t] = \int_s^t E[f_\tau|\mathcal{F}_t]d\tau$ a.s. for every $s \leq< t \leq T$. Indeed, for every $C \in \mathcal{F}_t$ and $0 \leq s < t \leq T$ one has

$$\int_C \left\{E\left[\int_s^t f_\tau d\tau|\mathcal{F}_t\right]\right\}dP = \int_C \left\{\int_s^t f_\tau d\tau\right\}\}dP = \int_C\int_s^t f_\tau dPd\tau$$
$$= \int_s^t \int_C \{E[f_\tau|\mathcal{F}_t]\}\, dPd\tau = \int_C \left\{\int_s^t E[f_\tau|\mathcal{F}_t]d\tau\right\}dP.$$

Then $E[\int_s^t f_\tau d\tau|\mathcal{F}_t] = \int_s^t E[f_\tau|\mathcal{F}_t]d\tau$ a.s. for every $s \leq< t \leq T$. Let $N_t = E[\int_0^t(f_\tau - [f]_\tau^{\mathbb{F}})d\tau|\mathcal{F}_t]$ a.s. for $0 \leq s < t \leq T$. It is clear that $(M_t)_{0\leq t\leq T}$ is an $\mathbb{F}$-martingale if and only if the process $(N_t)_{0\leq t\leq T}$ is an $\mathbb{F}$- martingale. We have $E|N_t| < \infty$ for every $0 \leq t \leq T$. Furthermore, for every $0 \leq s < t \leq T$ one has

$$E[N_t - N_s|\mathcal{F}_s]$$
$$= E\left[\left(E\left[\int_0^t (f_\tau - [f]_\tau^{\mathbb{F}})d\tau|\mathcal{F}_t\right] - E\left[\int_0^s (f_\tau - [f]_\tau^{\mathbb{F}})d\tau|\mathcal{F}_s\right]\right)\Big|\mathcal{F}_s\right]$$

$$= E\left[\int_0^t E[(f_\tau - [f]_\tau^{\mathbb{F}})|\mathcal{F}_t]\bigg|\mathcal{F}_s\right] - E\left[\int_0^s E[(f_\tau - [f]_\tau^{\mathbb{F}})|\mathcal{F}_s]\bigg|\mathcal{F}_s\right]$$

$$= \int_0^t E[(f_\tau - [f]_\tau^{\mathbb{F}})|\mathcal{F}_s]d\tau - \int_0^s E[(f_\tau - [f]_\tau^{\mathbb{F}})|\mathcal{F}_s]d\tau = \int_s^t E[(f_\tau - [f]_\tau^{\mathbb{F}})|\mathcal{F}_s]d\tau.$$

But for every $C \in \mathcal{F}_s$ one has $(s,t] \times C \in \mathcal{O}$. Therefore, for every $C \in \mathcal{F}_s$ one gets

$$\int_C \left[\int_s^t E[(f_\tau - [f]_\tau^{\mathbb{F}})|\mathcal{F}_s]d\tau\right] = \int\int_{(s,t]\times C} f_\tau d\tau dP - \int\int_{(s,t]\times C} [f]_\tau^{\mathbb{F}} d\tau dP$$

$$= \int\int_{(s,t]\times C} f_\tau d\tau dP - \int\int_{(s,t]\times C} f_\tau d\tau dP = 0.$$

Hence it follows $\int_s^t E[(f_\tau - [f]_\tau^{\mathbb{F}})|\mathcal{F}_s]d\tau = 0$ a.s. for every $0 \le s < t \le T$, which implies that $E[N_t - N_s|\mathcal{F}_s] = 0$ a.s. for every $0 \le s < t \le T$. Finally, by the equality $x_t = x_0 + M_t + A_t$ and continuity of the process $(A_t)_{0\le t\le T}$ it follows that the process x is continuous if and only if $(M_t)_{0\le t\le T}$ is a continuous martingale. □

Remark 3.1. If the assumptions of Theorem 3.1 are satisfied and a filtratrion $\mathbb{F}$ is continuous then an $\mathbb{F}$-martingale $(M_t)_{t\ge 0}$ defined in Corollary 3.1 is continuous.

4. Viable approximation theorem

Existence of solutions of the viability problem (1.2) follows from some viable approximation theorem by applying the standard methods presented in the proofs of the existence of strong solutions for $BSDI(F,H)$ (see [2], [6]). We shall present now such type approximation theorem. Its proof is similar to the proof of viable approximation theorem presented in [1]. To begin with let us assume that $K : [0,T] \times \Omega \to Cl(\mathbb{R}^m)$ is a given set-valued process and let us define a set-valued mapping $\mathcal{K}(t)$ by setting $\mathcal{K}(t) = \{u \in \mathbb{L}(\Omega, \mathcal{F}_t, \mathbb{R}^m) : u \in K(t),\ a.s.\}$. Furthermore, assume that $F : [0,T] \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ satisfies the following conditions $(\mathcal{A})$:

(i) *F is measurable and uniformly square integrably bounded by a function $m \in L^2([0,T], \mathbb{R}_+)$,*

(ii) *$F(t,\cdot)$ is square Lipschitz continuous, i.e., there is $k \in L^2([0,T], \mathbb{R}_+)$ such that $h(F(t,x_1), F(t,x_2)) \le k(t)|x_1 - x_2|$ for a.e. $t \in [0,T]$ and $x_1, x_2 \in \mathbb{R}^m$, where h is the Hausdorff metric on $Cl(\mathbb{R}^m)$.*

Throughout this Section $\overline{D}$ denotes the Hausdorff subdistance defined on the space $\mathrm{Cl}(\mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^m))$ of all nonempty closed subsets of $\mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^m)$, whereas D denotes the Hausdorff distance defined on this space. The distance function $\mathrm{dist}(\cdot,\cdot)$ on $\mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^m) \times \mathrm{Cl}(\mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^m))$ is denoted simply by $d(\cdot,\cdot)$.

Theorem 4.1. *Assume F satisfies conditions $(\mathcal{A})$ and let $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space with a continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ such that $\mathcal{F}_T = \mathcal{F}$. Suppose $K : [0,T] \times \Omega \to Cl(\mathbb{R}^m)$ is $\mathbb{F}$-adapted set-valued process such that $\mathcal{K}(t) \ne \emptyset$ for every $0 \le t \le T$ and such that the set-valued mapping $\mathcal{K} : [0,T] \to Cl(\mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^m)$ is continuous. If*

$$\liminf_{h\to 0+} \frac{1}{h}\overline{D}\left[S\left(E\left[x + \int_{t-h}^t F(\tau,x)d\tau|\mathcal{F}_{t-h}\right]\right), \mathcal{K}(t-h)\right] = 0 \qquad (4.1)$$

is satisfied for every $(t,x) \in Graph(\mathcal{K})$, *where* $S(E[x+\int_{t-h}^{t}F(\tau,x)d\tau|\mathcal{F}_{t-h}]) = \{E[x+\int_{t-h}^{t}f_\tau d\tau|\mathcal{F}_{t-h}] : f \in S(coF\circ x)\}$, *then for every* $\varepsilon \in (0,1)$, $x_T \in \mathcal{K}(x_T)$, $a \in (0,T)$ *and a measurable process* $\phi = (\phi)_{0\le t\le T}$ *such that* $\phi_t \in \mathbb{L}(\Omega,\mathcal{F}_T,\mathbb{R}^m)$ *for* $0 \le t \le T$ *and* $\phi_T \in F(T,x_T)$ *a.s. there exist a partition* $a = t_p < t_{p-1} < ... < t_1 < t_0 = T$ *of the interval* $[a,T]$, *a step function* $\theta_\varepsilon : [a,T] \to [a,T]$, *a step stochastic process* $z^\varepsilon = (z_t^\varepsilon)_{a\le t\le T}$ *and a measurable process* $f^\varepsilon = (f_t^\varepsilon)_{a\le t\le T}$ *on* $\mathcal{P}_{\mathbb{F}}$ *such that*

(i) $t_j - t_{j+1} \le \delta$, *where* $\delta \in (0,\varepsilon)$ *is such that* $\max\{\int_t^{t+\delta}k(\tau)d\tau, \int_t^{t+\delta}m(\tau)d\tau\} \le \varepsilon^2/2^4$ *and* $D(\mathcal{K}(t+\delta),\mathcal{K}(t)) \le \varepsilon/2$ *for* $t \in [0,T]$,

(ii) $\|z_t^\varepsilon\| \le \varepsilon/2$ *for every* $a \le t \le T$, *where* $\|z_t^\varepsilon\| = E|z_t^\varepsilon|$,

(iii) $\theta_\varepsilon(t) = t_{j-1}$ *for* $t_j < t \le t_{j-1}$ *and* $\theta_\varepsilon(t_j) = t_j$ *with* $j = 1,...,p-1$ *and* $\theta_\varepsilon(t) = t_{p-1}$ *for* $a \le t \le t_{p-1}$,

(iv) $f^\varepsilon \in S(coF\circ(x^\varepsilon\circ\theta_\varepsilon))$, $|\phi_t(\omega) - f_t^\varepsilon(\omega)| = \operatorname{dist}(\phi_t, coF(t,(x^\varepsilon\circ\theta_\varepsilon)(t)))$ *for* $(t,\omega) \in [a,T]\times\Omega$, *where* $x^\varepsilon(t) = E[x_T + \int_t^T f_\tau^\varepsilon d\tau|\mathcal{F}_t] + \int_t^T z_\tau^\varepsilon d\tau$ *a.s. for* $a \le t \le T$ *and* $S(co\,F\circ(x^\varepsilon\circ\theta_\varepsilon)) = \{f \in L^2([a,T]\times\Omega,\beta_T\otimes\mathcal{F}_T,\mathbb{R}^d) : f_t \in co\,F(t,x^\varepsilon(\theta_\varepsilon(t)))$ a.s. for *a.e.* $a \le t \le T\}$,

(v) $E[\operatorname{dist}(x^\varepsilon(s), E[x^\varepsilon(t)+\int_s^t F(\tau,(x^\varepsilon\circ\theta_\varepsilon)(\tau)d\tau|\mathcal{F}_s])] \le \varepsilon$ *for* $a \le s \le t \le T$,

(vi) $d(x^\varepsilon(\theta_\varepsilon(t)),\mathcal{K}(\theta_\varepsilon(t))) = 0$ *for* $a \le t \le T$.

Proof. Let $\varepsilon \in (0,1)$, $a \in (0,T)$, $x_T \in \mathcal{K}(T)$ and a measurable process $\phi = (\phi)_{0\le t\le T}$ be given. By virtue of (4.1) there exists $h_0 \in (0,\min(\delta,T))$ such that

$$\overline{D}\left[S\left(E\left[x_T + \int_{T-h_0}^{T}F(\tau,x_T)d\tau|\mathcal{F}_{T-h_0}\right]\right),\mathcal{K}(T-h_0)\right] \le \varepsilon h_0/2.$$

Let $t_1 = T - h_0$. By virtue of ([5], Th.II.3.13) there exists $f^0 \in S(coF\circ x_T)$ such that $|\phi_t(\omega) - f_t^0(\omega)| = \operatorname{dist}(\phi_t(\omega), coF(t,x_T(\omega)))$ for $(t,\omega) \in [t_1,T]\times\Omega$. Let $y_0 = E[x_T + \int_{t_1}^T f_\tau^0 d\tau|\mathcal{F}_{t_1}]$ a.s. We have $y_0 \in E[x_T + \int_{t_1}^T F(\tau,x_T)d\tau|\mathcal{F}_{t_1}]$ a.s., i.e., $y_0 \in S(E[x_T + \int_{t_1}^T F(\tau,x_T)d\tau|\mathcal{F}_{t_1}])$. Therefore, $d(y_0,\mathcal{K}(t_1)) \le \varepsilon h_0/2$. Similarly as above we can see that there exists $x_1 \in \mathcal{K}(t_1)$ such that $E|y_0 - x_1| = E[\operatorname{dist}(y_0,K(t_1))] = d(y_0,\mathcal{K}(t_1)) \le \varepsilon h_0/2$. Then $\|y_0 - x_1\| \le \varepsilon h_0/2$. Let $z_t^\varepsilon = 1/h_0(x_1 - y_0)$ a.s. for $t_1 \le t \le T$. We have $\|z_t^\varepsilon\| \le (1/h_0)\|y_0 - x_1\| \le \varepsilon/2$. Define $\theta_\varepsilon(t) = T$ for $t_1 < t \le T$ and $\theta(t_1) = t_1$. One has $f_t^0 \in \operatorname{co}F(t,x_T)$ a.s. for $t_1 < t \le T$. Let

$$x^\varepsilon(t) = E\left[x_T + \int_t^T f_\tau^0 d\tau|\mathcal{F}_t\right] + \int_t^T z_\tau^\varepsilon d\tau$$

for $t_1 < t \le T$. We have $x^\varepsilon(T) = x_T$ and $x^\varepsilon(t_1) = y_0 + h_0(1/h_0)(x_1 - y_0) = x_1$. Therefore, $d(x^\varepsilon(\theta(t)),\mathcal{K}(\theta(t))) = 0$ for $t_1 \le t \le T$ and $|\phi_t(\omega) - f_t^0(\omega)| = \operatorname{dist}(\phi_t(\omega), coF(t,x^\varepsilon(\theta_\varepsilon(t)(\omega)))$ for $(t,\omega) \in [t_1,T]\times\Omega$. Furthermore, by the definition of x^ε and properties of f^0 and x^ε one gets $E[\operatorname{dist}(x^\varepsilon(s), E[x_T + \int_s^t F(\tau,x^\varepsilon(\theta(\tau)))d\tau|\mathcal{F}_s] \le \varepsilon/2$ for $t_1 < s < t \le T$.

If $t_1 > a$ we can repeat the above procedure starting with $(t_1,x_1) \in Graph(\mathcal{K})$. Immediately from (4.1) it follows that there exists an $h_1 \in (0,\delta)$ such that

$$\overline{D}\left[S(E[x_1 + \int_{t_1-h_1}^{t_1}F(\tau,x_1)d\tau|\mathcal{F}_{t_1-h_1}]),\mathcal{K}(t_1-h_1)\right] \le \varepsilon h_1/2\,.$$

Similarly as above we can select $f^1 \in S(\text{co}F \circ x_1)$ and $x_2 \in \mathcal{K}(t_1 - h_1)$ such that $|\phi_t(\omega) - f_t^1(\omega))| = \text{dist}(\phi_t(\omega), \text{co}\,(F \circ x_1)(t,\omega)$ for $(t,\omega) \in [t_1 - h_1, t_1] \times \Omega$ and $\|y_1 - x_2\| \le \varepsilon h_1/2^2$, where $y_1 = E[x_1 + \int_{t_1-h_1}^{t_1} f_\tau^1 d\tau|\mathcal{F}_{t_1-h_1}]$ and $t_2 = t_1 - h_1$. We can extend now the step function θ_ε and the step process z^ε on the interval $[t_2, T]$ by taking $\theta_\varepsilon(t_2) = t_2$, $\theta_\varepsilon(t) = t_1$ for $t_2 < t \le t_1$ and $z_t^\varepsilon = (1/h_1)(x_2 - y_1)$ for $t_2 \le t < t_1$. We have $f_t^1 \in \text{co}\,F(t, x_1)$ a.s. for $t_2 \le t \le t_1$. We can also extend the process x^ε on the interval $(t_2, T]$ by taking

$$x^\varepsilon(t) = E\left[x_1 + \int_t^{t_1} f_\tau^1 d\tau|\mathcal{F}_t\right] + \int_t^{t_1} z_\tau^\varepsilon d\tau$$

a.s. for $t_2 < t \le t_1$. We have $d(x^\varepsilon(\theta_\varepsilon(t)), \mathcal{K}(\theta(t))) = 0$ for $t_2 \le t \le T$ because $x^\varepsilon(t_2) = x_2$. Let $f^\varepsilon = 1\!\!1_{(t_2,t_1]} f^1 + 1\!\!1_{(t_1,T]} f^0$. We have $x^\varepsilon(t) = E[x_T + \int_t^T f_\tau^\varepsilon d\tau|\mathcal{F}_t] + \int_t^T z_\tau^\varepsilon d\tau$ a.s. for $t_2 < t \le T$. Similarly as above we can verify that $f_t^\varepsilon \in \text{co}\,F(t, x^\varepsilon(\theta_\varepsilon(t)))$ a.s. for $t_2 < t \le T$ and $|\phi_t - f_t^\varepsilon)| = \text{dist}(\phi_t, \text{co}\,F(t, x^\varepsilon(\theta(t)))$ a.s. for $t_2 < t \le T$. Furthermore, $d(x^\varepsilon(\theta_\varepsilon(t)), \mathcal{K}(\theta_\varepsilon(t))) = 0$ and $E[\text{dist}(x^\varepsilon(s), E[\int_s^t F(\tau, x^\varepsilon(\theta(\tau)))d\tau|\mathcal{F}_s] \le \varepsilon/2$ for $t_2 \le t \le T$ and $t_2 < s < t \le T$, respectively.

Suppose that for some $i \ge 1$ the inductive procedure is realized. Then there exist $t_{i-1} \in [a, T)$, such that we can extend a step function θ_ε, a step process z^ε a process x^ε and $f_t^\varepsilon \in \text{co}\,F(t, x^\varepsilon(\theta_\varepsilon(t))$ for $t_{i-1} \le t \le T$ such that $|\phi_t - f_t^\varepsilon| = \text{dist}(\phi_t, \text{co}\,F(t, x^\varepsilon(\theta_\varepsilon(t)))$, where

$$x_{i-1}^\varepsilon(t) = E[x_T + \int_t^T f_\tau^\varepsilon d\tau|\mathcal{F}_t] + \int_t^T z_\tau^\varepsilon d\tau$$

a.s. for $t_{i-1} < t \le T$. Furthermore, $d(x^\varepsilon(\theta_\varepsilon(t)), \mathcal{K}(\theta_\varepsilon(t))) = 0$ and

$$E[\text{dist}(x^\varepsilon(s), E[x^\varepsilon(t) + \int_s^t F(\tau, (x_{i-1}^\varepsilon \circ \theta_\varepsilon)(\tau))d\tau|\mathcal{F}_s])] \le \varepsilon/2$$

for $t_{i-1} < s < t \le T$. Define now a process x^ε by setting

$$x^\varepsilon(t) = E[x_T + \int_t^T f_\tau^\varepsilon d\tau|\mathcal{F}_t] + \int_t^T z_\tau^\varepsilon d\tau$$

a.s. for $t_{i-1} < t \le T$. Denote by S_i the set of all positive numbers $h \in (0, \min(\delta, t_{i-1}))$ such that

$$\overline{D}\left[S(E[x^\varepsilon(t_{i-1}) + \int_{t_{i-1}-h}^{t_{i-1}} F(\tau, x_{i-1}^\varepsilon(t_{i-1}))d\tau|\mathcal{F}_{t_{i-1}-h}]), \mathcal{K}(t_{i-1})\right] \le \varepsilon h/2.$$

By the properties of x^ε we have $(t_{i-1}, x^\varepsilon(t_{i-1})) \in Graph(\mathcal{K})$. Therefore, by virtue of (4.1), we have $S_i \neq \emptyset$ and $\sup S_i > 0$. Choose $h_{i-1} \in S_i$ such that $(1/2)\sup S_i \le h_{i-1}$. Put $t_i = t_{i-1} - h_{i-1}$. We can extend again the step function θ_ε, the step process z^ε, processes f^ε and x^ε on the interval $(t_i, T]$ such that $d(x^\varepsilon(\theta_\varepsilon(t)), \mathcal{K}(\theta(t)))) = 0$, $f_t^\varepsilon \in \text{co}F(t, x^\varepsilon(\theta_\varepsilon(t))$ and $|\phi_t - f_t^\varepsilon| = \text{dist}(\phi_t, \text{co}F(t, x^\varepsilon(\theta_\varepsilon(t))$ a.s. for $t_i < t \le T$. Furthermore

$$E[\text{dist}(x^\varepsilon(s), E[x^\varepsilon(t) + \int_s^t F(\tau, (x_{i-1}^\varepsilon \circ \theta_\varepsilon)(\tau))d\tau|\mathcal{F}_s])] \le \varepsilon/2$$

for $t_i < s < t \le T$. We can continue the above procedure up to $n \ge 1$ such that $0 < t_n \le a < t_{n-1}$. Suppose to the contrary that there does not exist such $n \ge 1$, i.e., that for every $n \ge 1$ one has $a < t_n < T$. Then we can extend the step function θ_ε, the step process z^ε and stochastic processes f^ε and x^ε on the interval $(t_n, T]$ for every $n \ge 1$ such that $x^\varepsilon(t_n) \in \mathcal{K}(t_n)$ a.s. for every $n \ge 1$ and that the above properties are satisfied on $(t_n, T]$ for every $n \ge 1$. By the boundedness of a sequence $(t_n)_{n=1}^\infty$ we can select its decreasing subsequence $(t_i)_{i=1}^\infty$ converging to $t^* \in [a, T]$. Let $(x_i)_{i=1}^\infty$ be a sequence define by $x_i = x^\varepsilon(t_i)$ a.s. for every $i \ge 0$. In particular, we have $x_i \in \mathcal{K}(t_i)$ a.s. for every $i \ge 1$. For every $j > k \ge 0$ we obtain

$$E|x_k - x_j| \le E|E[x_T|\mathcal{F}_{t_k}] - E[x_T|\mathcal{F}_{t_j}]| + \int_{t^*}^{t_k} m(t)dt + \int_{t^*}^{t_j} m(t)dt$$

$$+(t_k - t_j)E|z_t^\varepsilon| + E\left|E\left[\int_{t^*}^T f_t^\varepsilon dt|\mathcal{F}_{t_k}\right] - E\left[\int_{t^*}^T f_t^\varepsilon dt|\mathcal{F}_{t^*}\right]\right|$$

$$+E\left|E\left[\int_{t^*}^T f_t^\varepsilon dt|\mathcal{F}_{t_j}\right] - E\left[\int_{t^*}^T f_t^\varepsilon dt|\mathcal{F}_{t^*}\right]\right|.$$

By continuity of the filtration $\mathbb{F}$ it follows that $\lim_{j,k\to\infty} E|x_k - x_j| = 0$. Then $(x_i)_{i=1}^\infty$ is a Cauchy sequence of $\mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^m)$. Therefore, there is $x^* \in \mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^m$ such that $\|x_i - x^*\| \to 0$ as $i \to \infty$. We have $x_i \in \mathcal{K}(t_i))$ for every $i \ge 1$, which by continuity of $\mathcal{K}$ implies that $(t^*, x^*) \in Graph(\mathcal{K})$. Therefore, by virtue of (4.1) we can select $h^* \in (0, \min(\delta, t^*))$ such that

$$\overline{D}\left[S(E[x^* + \int_{t^*-h^*}^{t^*} F(\tau, x)d\tau|\mathcal{F}_{t^*-h^*}]), \mathcal{K}(t^* - h^*)\right] \le \varepsilon h^*/2^5.$$

Similarly as above, for every $i \ge 1$, and any $\phi_i \in S(\text{co}F \circ x_i)$ we can select $f^* \in S(\text{co}F \circ x^*)$ such that $|\phi_t^i - f_t^*)| = \text{dist}(\phi_t^i, F(t, x^*))$ a.s. for every $t^* - h^* < t \le t^*$. By continuiuty of the filtratin $\mathbb{F}$ we obtain $\|E[x^*|\mathcal{F}_{t_i-h^*}] - E[x^*|\mathcal{F}_{t^*-h^*}]\| \to 0$ and

$$E\left|E\left[\int_{t^*-h^*}^{t^*} f_\tau^* d\tau|\mathcal{F}_{t_i-h^*}\right] - E\left[\int_{t^*-h^*}^{t^*} f_\tau^* d\tau|\mathcal{F}_{t^*-h*}\right]\right| \to 0$$

as $i \to \infty$. Let $N \ge 1$ be such that for every $i \ge N$ we have $0 < t_i - t^* < \min(h^*, \delta)$, $\|x_i - x^*\| < \varepsilon h^*/(2^5 \cdot A)$, $D(\mathcal{K}(t_i - h^*), \mathcal{K}(t^* - h^*)) \le \varepsilon h^*/2^5$, $\|E[x^*|\mathcal{F}_{t_i-h^*}] - E[x^*|\mathcal{F}_{t^*-h^*}]\| \le \varepsilon h^*/2^5$, $E\int_{t_i-h^*}^{t^*-h^*} |\phi_\tau^i|d\tau \le \varepsilon h^*/2^5$, $E\int_{t^*}^{t_i} |\phi_\tau^i|dt \le \varepsilon h^*/2^5$ and $E|E[\int_{t^*-h^*}^{t^*} f_\tau^* d\tau|\mathcal{F}_{t_i-h^*}] - E[\int_{t^*-h^*}^{t^*} f_\tau^* d\tau|\mathcal{F}_{t^*-h*}]| \le \varepsilon h^*/2^5$, where $A = 1 + \int_0^T k(t)dt$. By the properties of the multifinction $F(t, \cdot)$ and the selector f^* of $F \circ x^*$ it follows that

$$\|\mathbb{1}_{[t^*-h^*,t^*]}(\phi^i - f^*)\| = E\int_{t^*-h^*}^{t^*} |\phi_\tau^i - f_\tau^*|d\tau$$

$$\le E\int_{t^*-h^*}^{t^*} h((F(t, x_i), F(t, x^*))]dt \le \|x_i - x^*\| \int_{t^*-h^*}^{t^*} k(t)dt.$$

For every $i \geq N$ one gets

$$d\left(E[x_i + \int_{t_i-h^*}^{t_i} \phi_\tau^i d\tau | \mathcal{F}_{t_i-h^*}], \mathcal{K}(t_i - h^*)\right)$$

$$\leq E\left|E[x_i + \int_{t_i-h^*}^{t_i} \phi_\tau^i d\tau | \mathcal{F}_{t_i-h^*}] - E[x^* + \int_{t^*-h^*}^{t^*} f_\tau^* d\tau | \mathcal{F}_{t^*-h*}]\right|$$

$$+d\left(E[x^* + \int_{t^*-h^*}^{t^*} f_\tau^* d\tau | \mathcal{F}_{t^*-h*}], \mathcal{K}(t^* - h^*)\right) + D(\mathcal{K}(t^* - h^*), \mathcal{K}(t_i - h^*)).$$

But for every $i \geq N$ we have

$$E\left|E\left[x_i + \int_{t_i-h^*}^{t_i} \phi_\tau^i d\tau | \mathcal{F}_{t_i-h^*}\right] - E\left[x^* + \int_{t^*-h^*}^{t^*} f_\tau^* d\tau | \mathcal{F}_{t^*-h*}\right]\right|$$

$$\leq E|E[(x_i - x^*)|\mathcal{F}_{t_i-h*}]| + E|E[x^*|\mathcal{F}_{t_i-h^*}] - E[x^*|\mathcal{F}_{t^*-h^*}]|$$

$$+E\left|E\left[\int_{t^*}^{t^*-h^*} (\phi_\tau^i - f_\tau^*) d\tau | \mathcal{F}_{t_i-h^*}\right]\right| + E\int_{t_i-h^*}^{t^*-h^*} |\phi_\tau^i| d\tau + E\int_{t^*}^{t_i} |\phi_\tau^i| dt$$

$$+E\left|E\left[\int_{t^*}^{t^*-h^*} f_\tau^* d\tau | \mathcal{F}_{t_i-h^*}\right] - E\left[\int_{t^*-h^*}^{t^*} f_\tau^* d\tau | \mathcal{F}_{t^*-h*}\right]\right| \leq 6\varepsilon h^*/2^5.$$

Therefore, for every $i \geq N$ one gets

$$d\left[E\left[x_i + \int_{t_i-h^*}^{t_i} \phi_\tau^i d\tau | \mathcal{F}_{t_i-h^*}\right], \mathcal{K}(t_i)\right] \leq 8\varepsilon h^*/2^5 = \varepsilon h^*/2^2,$$

which implies that

$$\overline{D}(S(E[x_i + \int_{t_i-h^*}^{t_i} F(\tau, x_i) d\tau | \mathcal{F}_{t_i-h^*}], \mathcal{K}(t_i)) \leq \varepsilon h^*/2^2.$$

But $t^* \leq t_i$ for $i \geq 1$. Therefore, for every $i \geq N$ one has $h^* \in S_{i+1}$ and $(1/2)h^* \leq \sup S_{i+1} \leq h_i = t_i - t_{i+1}$, which contradicts to the convergence of a sequence $(t_i)_{i=1}^\infty$. Then there is a $p > 1$ such that $a = t_p < t_{p-1}, ..., t_1 < t_0 = T$. Taking $f^\varepsilon = \mathbb{1}_{[a,t_{p-1}]} f^p + \sum_{i=p-2}^{0} \mathbb{1}_{(t_{i+1},t_i]} f^i$ we obtain the desired selector of $\mathrm{co}F \circ (x^\varepsilon \circ \theta_\varepsilon)$. $\square$

Remark 4.1. The above results are also true if instead of continuity of a set-valued mapping $\mathcal{K}$ we assume that it is uniformly upper semicontinuous on $[0, T]$, i.e., that $\lim_{\delta \to 0} \sup_{0 \leq t \leq T} \overline{D}(\mathcal{K}(t + \delta), \mathcal{K}(t)) = 0$.

5. Existence of viable solutions

We shall prove now that conditions $(\mathcal{A})$ imply the existence of strong viable solutions for $BSDI(F, K)$. To begin with let us observe that immediately from the properties of the multivalued conditional expectation the following result follows.

Lemma 5.1. *If F satisfies conditions $(\mathcal{A})$, then for every $x, y \in \mathcal{S}(\mathbb{F}, \mathbb{R}^m)$ one has*

$$E\left[h\left(E\left[\int_s^t F(\tau, x_\tau) d\tau | \mathcal{F}_s\right], E\left[\int_s^t F(\tau, y_\tau) d\tau | \mathcal{F}_s\right]\right)\right] \leq \int_s^t k(\tau) E|x_\tau - y_\tau| d\tau$$

for every $0 \leq s \leq t \leq T$, where h is the Hausdorff metric on $Cl(\mathbb{R}^m)$.

We can prove now the main result of the paper.

Theorem 5.2. *Let $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space with a continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ such that $\mathcal{F}_T = \mathcal{F}$. Assume that F satisfies conditions $(\mathcal{A})$ and let $K : [0,T] \times \Omega \to Cl(\mathbb{R}^m)$ be $\mathbb{F}$-adapted set-valued process such that $\mathcal{K}(t) \neq \emptyset$ for every $0 \le t \le T$ and such that $\mathcal{K} : [0,T] \to Cl(\mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^m)$ is continuous. If $\mathcal{P}_{\mathbb{F}}$, F and K are such that (4.1) is satisfied for every $(t,x) \in Graph(\mathcal{K})$ then $BSDI(F,K)$ possesses a strong viable solution.*

Proof. Let $x_T \in \mathcal{K}(T)$ and $a \in (0,T)$ be fixed. Put $x_t^0 = x_T$ a.s. for $a \le t \le T$ and let $f^0 = (f_t^0)_{a \le t \le T}$ be a measurable process on $\mathcal{P}_{\mathbb{F}}$ such that $f_t^0 \in \mathrm{co}F(t, (x^0 \circ \theta_0)(t))$ a.s. for a.e. $a \le t \le T$, where $\theta_0(t) = T$ for $a \le t \le T$. Let $\phi_t = f_t^0$ a.s. for a.e. $a \le t \le T$. By virtue of Theorem 4.1, for $\varepsilon_1 = 1/2^{3/2}$ and the above measurable process $\phi = (\phi_t)_{a \le t \le T}$ there exist a partition $a = t_{p_1}^1 < t_{p_1-1}^1 < \dots < t_1^1 < t_0^1 = T$, a step function $\theta_1 : [a,T] \to [a,T]$, a step process $z^1 = (z_t^1)_{a \le t \le T}$ and a measurable process $f^1 = (f_t^1)_{a \le t \le T}$ on $\mathcal{P}_{\mathbb{F}}$ such that conditions (i) - (vi) of Theorem 4.1 are satisfied. In particular, $f_t^1 \in \mathrm{co}F(t, (x^1 \circ \theta_1)(t))$, $|f_t^1 - f_t^0| = \mathrm{dist}(f_t^0, \mathrm{co}F(t, (x^1 \circ \theta_1)(t)))$ a.s. for a.e. $a \le t \le T$ and $d(x^1(t), \mathcal{K}(t)) \le \varepsilon_1$ for $a \le t \le T$, because $d(x^1(t), \mathcal{K}(t)) \le |x^1(t) - x^1(\theta(t))| + d(x^1(\theta(t)), \mathcal{K}(\theta(t))) + D(\mathcal{K}(\theta(t)), \mathcal{K}(t)) \le \varepsilon_1$, where $x_t^1 = E[x_T + \int_t^T f_\tau^0 d\tau | \mathcal{F}_t] + \int_t^T z_\tau^1 d\tau$ a.s. for $a \le t \le T$. In a similar way for $\phi = (f_t^1)_{a \le t \le T}$ and $\varepsilon_2 = 1/2^3$ we can define a partition $a = t_{p_2}^2 < t_{p_2-1}^2 < \dots < t_1^2 < t_0^2 = T$, a step function $\theta_2 : [a,T] \to [a,T]$, a step process $z^2 = (z_t^2)_{a \le t \le T}$ and a measurable process $f^2 = (f_t^2)_{a \le t \le T}$ such that $f_t^2 \in \mathrm{co}F(t, (x^2 \circ \theta_2)(t))$, $|f_t^2 - f_t^1| = \mathrm{dist}(f_t^1, \mathrm{co}F(t, (x^2 \circ \theta_2)(t)))$ a.s. for a.e. $a \le t \le T$ and $d(x^2(t), \mathcal{K}(t)) \le \varepsilon_2$ for $a \le t \le T$, where $x_t^2 = E[x_T + \int_t^T f_\tau^1 d\tau | \mathcal{F}_t] + \int_t^T z_\tau^2 d\tau$ a.s. for $a \le t \le T$. Furthermore, for $i = 1,2$ we have

$$E\left[\mathrm{dist}\left(x^i(s), E\left[x^i(t) + \int_s^t F(\tau, (x^i \circ \theta_i)(\tau)) d\tau | \mathcal{F}_s\right]\right)\right] \le \varepsilon_i$$

a.s. for $a \le s \le t \le T$. By the inductive procedure for $\varepsilon_k = 1/2^{3k/2}$ and $\phi^k = (f_t^k)_{a \le t \le T}$ we can select for every $k \ge 1$ a partition $a = t_{p_k}^k < t_{p_k-1}^k < \dots < t_1^k < t_0^k = T$, a step function $\theta_k : [a,T] \to [a,T]$, a step process $z^k = (z_t^k)_{a \le t \le T}$ and a measurable process $f^k = (f_t^k)_{a \le t \le T}$ such that $f_t^k \in \mathrm{co}F(t, (x^k \circ \theta_k)(t))$, $|f_t^k - f_t^{k-1}| = \mathrm{dist}(f_t^k, \mathrm{co}F(t, (x^k \circ \theta_k)(t)))$ a.s. for a.e. $a \le t \le T$ and $d(x^k(t), \mathcal{K}(t)) \le \varepsilon_k$ for $a \le t \le T$, where

$$x_t^k = E[x_T + \int_t^T f_\tau^{k-1} d\tau | \mathcal{F}_t] + \int_t^T z_\tau^k d\tau$$

a.s. for $a \le t \le T$. Furthermore,

$$E\left[\mathrm{dist}\left(x^k(s), E\left[x^k(t) + \int_s^t F(\tau, (x^k \circ \theta_k)(\tau)) d\tau | \mathcal{F}_s\right]\right)\right] \le \varepsilon_k$$

for $a \le s \le t \le T$. Of course $x^k \in \mathcal{S}(\mathbb{F}, \mathbb{R}^m)$ for $k \ge 1$. By Remark 3.1 a process x^k is continuous for every $k \ge 1$. Furthermore, by the properties of the sequence

$(f^k)_{k=1}^{\infty}$, one gets

$$|x^{k+1}(t) - x^k(t)| \leq E\left[\int_t^T |f_\tau^k - f_\tau^{k-1}|d\tau|\mathcal{F}_t\right]$$

$$+\int_t^T E|z_\tau^{k+1} - z_\tau^k|d\tau \leq E\left[\int_t^T \mathrm{dist}(f_\tau^{k-1}\text{ço}\, F(\tau,(x^k\circ\theta_k)(\tau)))d\tau|\mathcal{F}_t\right]$$

$$+\frac{9}{8}T\varepsilon_k \leq \alpha\varepsilon_k + E\left[\int_t^T k(\tau)\sup_{\tau\leq s\leq T}|x^k(s) - x^{k-1}(s)|d\tau|\mathcal{F}_t\right],$$

a.s. for $a \leq t \leq T$, where $\alpha = \frac{9}{8}T$. Therefore,

$$\sup_{t\leq u\leq T}|x^{k+1}(u) - x^k(u)|$$

$$\leq \alpha\varepsilon_k + \sup_{t\leq u\leq T} E\left[\int_u^T k(\tau)\sup_{\tau\leq s\leq T}|x^k(s) - x^{k-1}(s)|d\tau|\mathcal{F}_u\right] \leq \alpha\varepsilon_k$$

$$+\sup_{t\leq u\leq T} E\left[\int_t^T k(\tau)\sup_{\tau\leq s\leq T}|x^k(s) - x^{k-1}(s)|d\tau|\mathcal{F}_u\right]$$

a.s. for $a \leq t \leq T$ and $k = 1, 2, \ldots.$ By Doob's inequality we get

$$E\left[\sup_{t\leq u\leq T} E[\int_t^T k(\tau)\sup_{\tau\leq s\leq T}|x^k(s) - x^{k-1}(s)|d\tau|\mathcal{F}_u]\right]^2$$

$$\leq 4E\left[\int_t^T k(\tau)\sup_{\tau\leq s\leq T}|x^k(s) - x^{k-1}(s)|d\tau]\right]^2$$

for $a \leq t \leq T$. Therefore, for every $a \leq t \leq T$ and $k = 1, 2, \ldots$ we have

$$E[\sup_{t\leq u\leq T}|x^{k+1}(u) - x^k(u)|^2] \leq 2\alpha^2\varepsilon_k^2 + \beta\int_t^T k^2(\tau)E[\sup_{\tau\leq s\leq T}|x^k(s) - x^{k-1}(s)|^2]d\tau,$$

where $\beta = 8T$. By the definitions of x^1 and x^0 we obtain $E[\sup_{t\leq u\leq T}|x^1(u) - x^0(u)|^2] \leq L$, where $L = 2T(\int_0^T m^2(t)dt + T)$. Therefore,

$$E[\sup_{t\leq u\leq T}|x^2(u) - x^1(u)|^2] \leq 2\alpha^2\varepsilon_1^2 + L\beta\int_t^T k^2(\tau)d\tau$$

for $a \leq t \leq T$. Hence it follows

$$E[\sup_{t\leq u\leq T}|x^3(u) - x^2(u)|^2] \leq 2\alpha\varepsilon_2^2 + \alpha\beta\varepsilon_1^2\int_t^T k^2(\tau)d\tau$$

$$+2L\beta T\int_t^T k^2(\tau)\left(\int_\tau^T k^2(u)du\right)d\tau$$

$$\leq 2\alpha^2\varepsilon_2^2 + \alpha^2\beta\varepsilon_1^2\int_t^T k^2(\tau)d\tau + L\frac{\beta^2}{2!}\left(\int_t^T k^2(\tau)d\tau\right)^2$$

$$\le M\varepsilon_2^2\left[1+(8\beta)\int_t^T k^2(\tau)d\tau+\frac{(8\beta)^2}{2!}\left(\int_t^T k^2(\tau)d\tau\right)^2\right],$$

for $a \le t \le T$, where $M = \max(2\alpha^2, L)$. By the inductive procedure for every $k = 1, 2, ...$ and $a \le t \le T$ we obtain

$$E[\sup_{t\le u\le T}|x^{n+1}(u)-x^n(u)|^2]$$

$$\le M\varepsilon_2^2\Bigg[1+(8\beta)\int_t^T k^2(\tau)d\tau+\frac{(8\beta)^2}{2!}\bigg(\int_t^T k^2(\tau)d\tau\bigg)^2+...+\frac{(8\beta)^n}{n!}\bigg(\int_t^T k^2(\tau)d\tau\bigg)^n\Bigg]$$

$$\le M\varepsilon_n^2 exp\left[8\beta\int_t^T k^2(\tau)d\tau\right].$$

Hence, similarly as in the proof of ([12], Th.3.2.5), by Chebyschev's Inequality and Boreli-Canalli lemma it follows that a sequence $(x^k)_{k=1}^\infty$ of stochastic processes $(x^k(t))_{a\le t\le T}$ is for a.e. $\omega\in\Omega$ uniformly converging in $[a,T]$ to a continuous process $(x(t))_{a\le t\le T}$. We can verify that a sequence $(f^k)_{k=1}^\infty$ is a Cauchy sequence of $\mathbb{L}([a,T]\times\Omega,\beta_T\otimes\mathcal{F}_T,\mathbb{R}^m)$. Indeed, for every $k=0,1,2,...$ one has

$$\int_0^a E[|f_\tau^{k+1}-f_\tau^k|]d\tau$$

$$\le\int_0^a E[H(F(\tau,(x^k\circ\theta_k)(\tau))),F(\tau,(x^{k-1}\circ\theta_{k-1})(\tau))))]d\tau$$

$$\le\int_0^a k(\tau)E[\sup_{0\le u\le\tau}|x^k(u)-x^{k-1}(u)|]d\tau\,.$$

Then there is an $f\in\mathbb{L}([a,T]\times\Omega,\beta_T\otimes\mathcal{F}_T,\mathbb{R}^m)$ such that $\|f^k-f|\to 0$ as $k\to\infty$. Let $y_t=E[x_T+\int_t^T f_\tau d\tau|\mathcal{F}_t]$ a.s. for $a\le t\le T$. For every $k\ge 1$ we have

$$E[\sup_{a\le t\le T}|x(t)-y_t|]\le E[\sup_{a\le t\le T}|x(t)-x_t^k|]+E[\sup_{a\le t\le T}|x^k(t)-y_t|]$$

$$\le E[\sup_{a\le t\le T}|x(t)-x_t^k|]+E\left[\sup_{a\le t\le T}E[\int_t^T|f_\tau^k-f_\tau|d\tau|\mathcal{F}_t]\right]+\int_t^T E|z_\tau^k|d\tau$$

$$\le E[\sup_{a\le t\le T}|x(t)-x_t^k|]+E\left[E[\int_0^T|f_\tau^k-f_\tau|d\tau|\mathcal{F}_t]\right]+T\varepsilon_k^2$$

$$\le E[\sup_{a\le t\le T}|x(t)-x_t^k|]+E\int_0^T|f_\tau^k-f_\tau|d\tau+T\varepsilon_k^2,$$

which implies that $E[\sup_{a\le t\le T}|x(t)-y_t|]=0$. Then $x(t)=E[x_T+\int_t^T f_\tau d\tau|\mathcal{F}_t]$ a.s. for $a\le t\le T$. Now, for every $a\le s\le t\le T$, we get

$$E\left[\text{dist}\left(x(s),E\left[x(t)+\int_s^t F(\tau,x(\tau))d\tau|\mathcal{F}_s\right]\right)\right]$$

$$\le E\left[|x(s)-x^k(s)|\right]+E\left[\text{dist}\left(x^k(s),E\left[x^k(t)+\int_s^t F(\tau,x^k(\theta_k(\tau)))d\tau|\mathcal{F}_s\right]\right)\right]$$

$$+E\left[H\left(E\left[\int_s^t F(\tau,x^k(\theta_k(\tau)))d\tau|\mathcal{F}_s\right],E\left[\int_s^t F(\tau,x^k(\tau))d\tau|\mathcal{F}_s\right]\right)\right]$$

$$+E\left[H\left(E\left[\int_s^t F(\tau,x^k(\tau))d\tau|\mathcal{F}_s\right],E\left[\int_s^t F(\tau,x(\tau))d\tau|\mathcal{F}_s\right]\right)\right]$$

$$\leq ||x^k-x||+\varepsilon_k+E\int_a^T k(t)|x^k(\theta_k(t))-x^k(t)|dt+E\int_a^T k(t)|x^k(t)-x(t)|dt\,.$$

But

$$E[|x^k(\theta_k(t))-x^k(t)|]\leq ||x^k-x||+E[\sup_{a\leq t\leq T}|x(\theta_k(t))-x^k(t)|]$$

for every $k\geq 1$ and $a\leq t\leq T$. Then

$$E\left[\text{dist}\left(x(s),E\left[x(t)+\int_s^t F(\tau,x(\tau))d\tau|\mathcal{F}_s\right]\right)\right]$$

$$\leq\left(\int_0^T k(t)dt\right)\left\{E[\sup_{a\leq t\leq T}|x(\theta_k(t))-x^k(t)|]+E[\sup_{a\leq t\leq T}|x(t)-x_t^k|]\right\}$$

$$+\|x^k-x\|+\varepsilon_k\leq ||x^k-x||\left(1+\int_0^T k(t)dt\right)+\varepsilon_k$$

for every $k\geq 1$ and $a\leq s\leq t\leq T$. Hence it follows that

$$E\left[\text{dist}\left(x(s),E\left[x(t)+\int_s^t F(\tau,x(\tau))d\tau|\mathcal{F}_s\right]\right)\right]=0$$

for every $a\leq s\leq t\leq T$. In a similar way we also get that $d(x(t),\mathcal{K}(t))=0$ for every $a\leq t\leq T$. Then x is a strong solution of $BSDI(F,K)$ on the interval $[a,T]$.

We can extend now the above solution on the whole interval $[0,T]$. Let us denote by Λ_x the set of all extensions of the above getting viable solution x of $BSDI(F,K)$. We have $\Lambda_x\neq\emptyset$ because we can repeat the above procedure for every interval $[\alpha,T]$ with $\alpha\in(0,a]$ and get a solution x^α of $BSDI(F,K)$ on an interval $[\alpha,T]$. A process $z=1\!\!1_{[\alpha,a]}x^\alpha+1\!\!1_{(a,T]}x$ is an extension of x on the interval $[\alpha,T]$. Let us introduce in Λ_x the partial order relation $\preceq$ by setting $x\preceq z$ if and only if $a_z\leq a_x$ and $x=z|_{[a_x,T]}$, where $a_x,a_z\in(0,a)$ are such that x and z are strong viable solutions for $BSDI(F,K)$ on $[a_x,T]$ and $[a_z,T]$, respectively and $z|_{[a_x,T]}$ denotes the restriction of the solution z to the interval $[a_x,T]$. Let $\psi:[\alpha,T]\to\mathbb{R}^d$ be an extension of x on $[\alpha,T]$ with $\alpha\in(0,a]$ and denote by $P_x^\psi\subset\Lambda_x$ the set containing ψ and all its restrictions $\psi|_{[\beta,T]}$ for every $\beta\in(\alpha,a)$. It is clear that each completely ordered subset of Λ_x is of the form P_x^ψ determined by some extension ψ of x. Then by Kuratowski and Zorn's Lemma there exists the maximal element γ of Λ_x . It has to be $a_\gamma=0$, where $a_\gamma\in[0,T)$ is such that γ is a strong viable solution of $BSDI(F,K)$ on the interval $[a_\gamma,T]$. Indeed, if it would be $a_\gamma>0$ then we could repeat the above procedure and extend γ, as a viable strong solution $\xi\in\Lambda_x$ of $BSFI(F,K)$, to the interval $[b,T]$ with $0\leq b<a_\gamma$. It would be imply that $\gamma\preceq\xi$. A contradiction to the assumption that γ is a maximal element of Λ_x. Then x can be extended on the whole interval $[0,T]$. □

Remark 5.1. The above existence theorem is also true if $\mathcal{K}(t) = \{u \in \mathbb{L}(\Omega, \mathcal{F}_0, \mathbb{R}^d) : u \in K(t)\}$. In such a case instead of (4.1) we can assume that $\liminf_{h\to 0+} \overline{D}[S((x + \int_{t-h}^{t} F(\tau, x)d\tau), \mathcal{K}(t)] = 0$ for every $(t, x) \in Graph(\mathcal{K})$.

Proof. For every $(t, x) \in Graph(\mathcal{K})$, $f \in S(\text{co}F \circ x)$ and $u \in \mathcal{K}(t)$ we have

$$E\left(\left|E[x + \int_{t-h}^{t} f_\tau d\tau|\mathcal{F}_{t-h}] - u\right|\right)$$
$$= E\left(\left|E[x + \int_{t-h}^{t} f_\tau d\tau|\mathcal{F}_{t-h}] - E[u|\mathcal{F}_{t-h}]\right|\right)$$
$$\leq E\left(E\left[\left|x + \int_{t-h}^{t} f_\tau d\tau - u\right| \Big|\mathcal{F}_{t-h}\right]\right) = E\left|x + \int_{t-h}^{t} f_\tau d\tau - u\right|.$$

Therefore, $d(E[x + \int_{t-h}^{t} f_\tau d\tau|\mathcal{F}_{t-h}], \mathcal{K}(t)) \leq d(x + \int_{t-h}^{t} f_\tau d\tau, \mathcal{K}(t))$ for every $f \in S(\text{co}F \circ x)$. Then

$$\overline{D}\left[S(E[x + \int_{t-h}^{t} F(\tau, x)d\tau|\mathcal{F}_{t-h}]), \mathcal{K}(t-h)\right]$$
$$\leq \overline{D}\left[x + \int_{t-h}^{t} F(\tau, x)d\tau, \mathcal{K}(t-h)\right]$$

for every $(t, x) \in Graph(\mathcal{K})$. Then $\liminf_{h\to 0+} \overline{D}[S(x+\int_{t-h}^{t} F(\tau, x)d\tau), \mathcal{K}(t-h)] = 0$ implies that (4.1) is satisfied. □

It can be verified that the requirement $P(\{X_t \in K(t)\}) = 1$ for $0 \leq t \leq T$ in some above viability problems is too strong to be satisfied. For example the stochastic differential equation $dX_t = f(X_t) + dB_t$ with Lipschitz continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$ does not have any solution $X = (X_t)_{0\leq t\leq T}$ with X_t belonging to a compact set $K \subset \mathbb{R}$ a.s. for every $0 \leq t \leq T$. It is a consequence (see [10]) of the following theorem.

Theorem 5.4. *Let $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space and $B = (B_t)_{t\geq 0}$ a real valued $\mathbb{F}$-Brownian motion on $\mathcal{P}_{\mathbb{F}}$. Assume that $\xi = (\xi_t)_{0\leq t\leq T}$ is an Itô diffusion such that $d\xi_t = \alpha_t(\xi)dt + dB_t$, $\xi_0 = 0$ for $0 \leq t \leq T$. Then $P(\{\int_0^T \alpha_t^2(\xi)dt < \infty\}) = 1$ and $P(\{\int_0^T \alpha_t^2(B)dt < \infty\}) = 1$ if and only if ξ and B have the same distributions as C_T- random variables on $\mathcal{P}_{\mathbb{F}}$, where $C_T = C([0, T], \mathbb{R})$.*

Example 5.1. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and Lipschitz continuous. Let $\mathcal{P}_{\mathbb{F}}$ and B be such as in Theorem 5.4. Put $\alpha_t(x) = f(e_t(x))$ for $x \in C_T$, where $C_T = C([0, T], \mathbb{R})$ and e_t is the evaluation mapping on C_T, i.e., $e_t(x) = x(t)$ for $x \in C_T$ and $0 \leq t \leq T$. Assume that K is a nonempty compact subset of $\mathbb{R}$ such that $0 \in K$ and consider the viable problem

$$\begin{cases} dX_t = f(X_t)dt + dB_t \quad a.s.\ for\ \ 0 \leq t \leq T \\ X_t \in K \quad a.s.\ for\ \ t \in [0, T] \end{cases} \tag{5.1}$$

Suppose there is a solution X, an Itô diffusion, of (5.1) such that $X_0 = 0$. By the properties of f we have $\int_0^T f^2(X_t)dt < \infty$ and $\int_0^T f^2(B_t)dt < \infty$ a.s. Therefore, by virtue of Theorem 5.4, for every $A \in \beta(C_T)$ with $PX^{-1}(A) = 1$ one has $PX^{-1}(A) =$

$PB^{-1}(A)$. By the properties of the process X one has $P(\{X_t \in K\}) = 1$. But $P(\{X_t \in K\}) = P(\{e_t(X) \in K\}) = PX^{-1}(e_t^{-1}(K))$. Hence it follows that $1 = PX^{-1}(e_t^{-1}(K)) = PB^{-1}(e_t^{-1}(K)) = P(\{B_t \in K\}) < 1$. A contradiction. Then the problem (5.1) does not have any K-viable strong solution.

Remark 5.3. It is possible to consider viability problems with weaker viable requirements of the form $P(\{X_t \in K(t)\}) \in (\varepsilon, 1)$ for $0 \leq t \leq T$ and a given sufficiently large $\varepsilon \in (0,1)$.

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