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A COINCIDENCE POINT RESULT VIA VARIATIONAL INEQUALITIES

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Abstract. In this paper, by making use of a new class of operators, we establish some existence results of the solution for an extended general variational inequality already considered in the literature. As application, we obtain a new coincidence point theorem in a Hilbert space setting.
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1. Introduction

The theory of variational inequalities has begun with the works of Stampacchia (see [15]) and Fichera (see [6]). This theory has shown to be very useful in studying problems arising in mechanics, optimization, transportation, economics equilibrium, contact problems in elasticity and other problems of practical interest.

In what follows, unless is otherwise specified, we assume that X is a real Banach space and X^* is the topological dual of X. We denote by $\langle x^*, x\rangle$ the value of the linear, continuous functional $x^* \in X^*$ in $x \in X$. Consider the set $K \subseteq X$ and let $A : K \longrightarrow X^*$ be a given operator.

Recall, that Stampacchia variational inequality, $VI_S(A, K)$, consists in finding an element $x \in K$, such that

$$\langle A(x), y - x\rangle \geq 0 \text{ for all } y \in K,$$

where the set K is convex and closed (see, for instance, [5, 6, 9, 13, 15]).

In recent years, many generalizations of this problem have been considered, studied and applied in various directions (see, for instance, [1, 5, 9, 11, 14]). General variational inequalities were introduced and studied by Noor (see [14]). It was realized that

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the general variational inequality can be used to study both the odd- and even-order free and moving boundary value problems. It has been shown that general variational inequalities provide us with a unified, simple, and natural framework to study a wide class of problems including unilateral, moving boundary, obstacle, free boundary and equilibrium problems arising in various areas of pure and applied sciences.

Recently László (see [11]), motivated by the work of Noor (see [14]), studied the so-called general variational inequality of Stampacchia type, $VI_S(A, a, K)$, which consists in finding an element $x \in K$ such that

$$\langle A(x), a(y) - a(x) \rangle \geq 0, \text{ for all } y \in K,$$

where $a : K \longrightarrow X$ is another given operator.

The problem that we shall study in this paper is the so-called extended general variational inequality, $VI(A, \psi, K)$, which was introduced in [3] and consists in finding an element $x \in K$ such that

$$\langle A(x), \psi(x, y) \rangle \geq 0, \text{ for all } y \in K,$$

where $\psi : K \times K \longrightarrow X$ is a given operator. Let us mention that this problem is a particular case of some more general problems already studied in the literature (see [7] and [10]). From another point of view, the problem studied in this paper is an extension of the general variational inequality of Stampacchia type.

In [11] some sufficient conditions that ensure the existence of the solutions for the general variational inequalities of Stampacchia type were provided. It was introduced a new class of operators, the class of operators of type ql, and it was shown that the existence results mentioned above fail outside of this class. It was also shown that the concept of operator of type ql, on the one hand, may be viewed as a generalization of the monotonicity of a real valued function of one real variable and, on the other hand, may be viewed as a generalization of a linear operator.

In the present paper we introduce a new class of operators, the class of operators of type g-ql, that contains in particular the set of operators of type ql and we extend some results already established in [11] for general variational inequalities of Stampacchia type involving operators of type ql.

The paper is organized as follows. In section 2, we introduce some new type of operators that generalize the notion of operator of type ql. An example of operator belonging to this class, that is not of type ql, is also provided. In section 3, we give some sufficient conditions that ensure the existence of the solutions for the extended general variational inequality. We also show by an example that these results fail outside of the class of operators introduced in section 2. Finally, as applications, based on the existence results of the solutions for the extended general variational inequalities established in section 3, we obtain a coincidence point result in Hilbert spaces.

2. Operators of type g-ql

In this section we generalize the concept of operator of type ql which was introduced in [11]. In what follows such an operator will be called operator of type g-ql. We also show that this concept is more general than the concept of operator of type ql, providing an example of operator of type g-ql which is not of type ql. Let us recall some definitions and results that we will need in what follows.

Let X be a real linear space. For $x, y \in X$, we denote by $[x, y] := \{(1-t)x + ty : t \in [0, 1]\}$ the closed line segment with the endpoints x and y. Let Y be another real linear space. An operator $B : D \subseteq X \longrightarrow Y$ is said to be of type *ql* (see [11]), if

$$B([x, y] \cap D) \subseteq [B(x), B(y)], \text{ for every } x, y \in D.$$

Theorem 2.1. *(Proposition 3.2,* [11]*) Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Then f is of type ql if and only if f is monotone.*

Theorem 2.2. *(Proposition 3.3,* [11]*) Let $B : X \longrightarrow Y$ be a linear operator. Then B is of type ql.*

Theorem 2.3. *(Theorem 3.2,* [11]*) Let $D \subseteq X$ be a convex set, and let $B : D \longrightarrow Y$ be an operator of type ql. Then for every $n \in \mathbb{N}$ and every $x_1, \dots, x_n \in D$, we have*

$$B(\text{co}\{x_1, \dots, x_n\}) \subseteq \text{co}\{B(x_1), \dots, B(x_n)\},$$

where $\text{co}(E)$ *denotes the convex hull of the set $E \subseteq Y$.*

In what follows we introduce the concept of an operator of type g-ql.

Definition 2.1. *Let X and Y be two real linear spaces and let $D \subseteq X$ be a convex set. An operator $B : D \longrightarrow Y$ is said to be of type g-ql if, for every $n \in \mathbb{N}$ and every $x_1, \dots, x_n \in D$, there exist $y_1, \dots, y_n \in D$, not necessarily all different, such that for any subset $\{y_{i_1}, \dots, y_{i_k}\} \subseteq \{y_1, \dots, y_n\}$, $1 \le k \le n$, we have*

$$B(\text{co}\{y_{i_1}, \dots, y_{i_k}\}) \subseteq \text{co}\{B(x_{i_1}), \dots, B(x_{i_k})\}.$$

From Theorem 2.3 we obtain immediately, that every operator of type ql is of type g-ql. The following theorem provides us other examples.

Theorem 2.4. *Let $I \subseteq \mathbb{R}$ and let $f : I \longrightarrow \mathbb{R}$ be a function. If there exists an interval $\widetilde{I} \subseteq I$ such that $f(\widetilde{I}) = f(I)$ and the restriction of f to $\widetilde{I}$, $f_{|_{\widetilde{I}}}$, is monotone then f is of type g-ql.*

Proof. Let $n \in \mathbb{N}$ and $x_1, \dots, x_n \in I$. Since $f(\widetilde{I}) = f(I)$ then there exist $y_1, \dots, y_n \in \widetilde{I}$ such that $f(x_i) = f(y_i)$ for each $i \in \{1, \dots, n\}$. Since $f_{|_{\widetilde{I}}}$ is monotone then by Theorem 2.1 $f_{|_{\widetilde{I}}}$ is of type ql. Then by Theorem 2.3, we have

$$f(\text{co}\{y_{i_1}, \dots, y_{i_k}\}) \subseteq \text{co}\{f(y_{i_1}), \dots, f(y_{i_k})\} = \text{co}\{f(x_{i_1}), \dots, f(x_{i_k})\},$$

for any subset $\{y_{i_1}, \dots, y_{i_k}\} \subseteq \{y_1, \dots, y_n\}$, $1 \le k \le n$. Hence f is of type g-ql. □

Next we provide some examples of operators of type g-ql which are not of type ql.

Example 2.1. Let $X = \mathbb{R}$ and consider the functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$ and

$$g(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Since f and g are not monotone according to Theorem 2.1 f and g are not of the type ql. On the other hand f and g are of type g-ql (let $\widetilde{I} = [0, \infty)$ and apply Theorem 2.4).

It can be easily observed, that Theorem 2.4 can be extended for the general case as well. More precisely we have the following result.

Theorem 2.5. *Let X and Y be two real linear spaces and let $D \subseteq X$ be a convex set. Let $B : D \longrightarrow Y$ be an operator, and assume that there exists a convex subset $D_1 \subseteq D$, such that the restriction of B on D_1, $B|_{D_1} : D_1 \longrightarrow Y$ is of type ql, and $B(D_1) = B(D)$. Then B is of type g-ql.*

The proof is similar to the proof of Theorem 2.4 therefore we omit it. In what follows we extend the concept of operator of type g-ql for operators defined on subsets of a product space.

Definition 2.2. *Let X and Y be two real linear spaces, and let $D \subseteq X$ be a convex set. An operator $\psi : D \times D \longrightarrow Y$ is said to be of type g-ql if, for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in D$, there exist $y_1, \ldots, y_n \in D$, not necessarily all different, such that for any subset $\{y_{i_1}, \ldots, y_{i_k}\} \subseteq \{y_1, \ldots, y_n\}$, $1 \le k \le n$, and for every $y \in \operatorname{co}\{y_{i_1}, \ldots, y_{i_k}\}$, we have*

$$0 \in \operatorname{co}\{\psi(y, x_{i_1}), \ldots, \psi(y, x_{i_k})\}.$$

We have the following result.

Lemma 2.1. *Let X and Y be two real linear spaces, $D \subseteq X$ be a convex set and let $B : D \subseteq X \longrightarrow Y$ be of type g-ql. Then the operator $\psi : D \times D \longrightarrow Y$ which is defined by $\psi(x, y) = B(y) - B(x)$ is of type g-ql.*

Proof. Let $n \in \mathbb{N}$ and let $x_1, \ldots, x_n \in D$. Since $B : D \subseteq X \longrightarrow Y$ is of type g-ql , there exist $y_1, \ldots, y_n \in D$ such that for any subset $\{y_{i_1}, \ldots, y_{i_k}\} \subseteq \{y_1, \ldots, y_n\}$, $1 \le k \le n$ we have

$$B(\operatorname{co}\{y_{i_1}, \ldots, y_{i_k}\}) \subseteq \operatorname{co}\{B(x_{i_1}), \ldots, B(x_{i_k})\}.$$

Let us fix $k \in \{1, 2, \ldots, n\}$ and consider $y \in \operatorname{co}\{y_{i_1}, \ldots, y_{i_k}\}$. Then we have

$$B(y) \in \operatorname{co}\{B(x_{i_1}), \ldots, B(x_{i_k})\},$$

hence

$$0 \in \operatorname{co}\{B(x_{i_1}) - B(y), \ldots, B(x_{i_k}) - B(y)\} = \operatorname{co}\{\psi(y, x_{i_1}), \ldots, \psi(y, x_{i_k})\}. \qquad \square$$

Next we extend this concept in a Banach space context, with respect to an operator. For $x \in K$ we introduce the following notation:

$$A^+(x) = \{y \in X : \langle A(x), y\rangle \ge 0\}.$$

Definition 2.3. *Let X be a real Banach space, let X^* be its topological dual space, let $D \subseteq X$ be a convex set and let $A : X \longrightarrow X^*$ be an operator. An operator $\psi : D \times D \longrightarrow X$ is said to be of type g-ql w.r.t. A if, for every $n \in \mathbb{N}$ and every $x_1, \ldots, x_n \in D$, there exist $y_1, \ldots, y_n \in D$, not necessarily all different, such that for any subset $\{y_{i_1}, \ldots, y_{i_k}\} \subseteq \{y_1, \ldots, y_n\}$, $1 \le k \le n$, and for every $y \in \operatorname{co}\{y_{i_1}, \ldots, y_{i_k}\}$, we have*

$$\operatorname{co}\{\psi(y, x_{i_1}), \ldots, \psi(y, x_{i_k})\} \cap A^+(y) \neq \emptyset.$$

An operator $B : D \longrightarrow X$ is said to be of type g-ql w.r.t. A if the operator $\psi : D \times D \longrightarrow X$ which is defined by $\psi(x, y) = B(y) - B(x)$, for each $x, y \in D$ is of type g-ql w.r.t. A.

Remark 2.1. *Let $A : X \longrightarrow X^*$ be an arbitrary operator. Since $0 \in A^+(y)$ for each $y \in D$, we obtain that every operator of type g-ql $\psi : D \times D \longrightarrow X$ is of type g-ql w.r.t. A.*

Next we provide an example to emphasize the importance of this concept.

Example 2.2. Let $A : \mathbb{R} \longrightarrow (0, \infty)$ be a function and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an arbitrary map. Notice that for each $y \in \mathbb{R}$, $A^+(y) = [0, \infty)$. We show that f is of type g-ql with respect to A. To prove the claim consider $x_1, \dots, x_n \in \mathbb{R}$. For each $i \in \{1, \dots, n\}$, let $y_i = x_0 \in \operatorname{argmin}\{f(x) : x \in \{x_1, \dots, x_n\}\}$. Then obviously $\operatorname{co}\{y_j : j \in J\} = x_0$ for each $\emptyset \neq J \subseteq \{1, 2, \dots, n\}$. Obviously, for $y = x_0$, we have $f(x_i) - f(y) \geq 0$, for all $i \in \{1, 2, \dots, n\}$, hence

$$\operatorname{co}\{f(x_j) - f(y) : j \in J\} \cap A^+(y) \neq \emptyset,$$

for every nonempty subset $J \subseteq \{1, 2, \dots, n\}$.

3. Existence of the Solutions of Extended General Variational Inequalities

In this section, we present some existence results of the solutions for the extended general variational inequalities that we discussed in Section 1. We need the following definition of a generalized KKM mapping (see [2, 17]).

Definition 3.1. *Let E be a linear space and let $Y \subseteq E$. Let X be an arbitrary nonempty set. The mapping $G : X \rightrightarrows Y$ is called a GKKM mapping if for each finite subset $\{x_1, \dots, x_n\} \subseteq X$, there exist $y_1, \dots, y_n \in Y$ such that, for any subset $\{y_{i_1}, \dots, y_{i_k}\} \subseteq \{y_1, \dots, y_n\}$, $1 \leq k \leq n$, we have*

$$\operatorname{co}\{y_{i_1}, \dots, y_{i_k}\} \subseteq \bigcup_{j=1}^{k} G(x_{i_j})$$

The following extension of the classical KKM principle in Hausdorff topological linear spaces is due to Chang and Zhang (see [2]).

Theorem 3.1. *Let X be a nonempty subset of a Hausdorff topological linear space E and $G : X \rightrightarrows E$ be a GKKM mapping with nonempty, closed values. Then, the family $\{G(x) : x \in X\}$ has the finite intersection property, i.e.,*

$$\bigcap_{x \in S} G(x) \neq \emptyset,$$

for every finite subset S of X. Moreover, if there exists $x_0 \in X$ such that $G(x_0)$ is compact, then

$$\bigcap_{x \in X} G(x) \neq \emptyset.$$

Let us mention, that this theorem is actually a reformulation of Theorem 3.1 from [8]. In [12] Lin, Chuang and Yu proved the following more general result.

Theorem 3.2. *Let X be an arbitrary nonempty set and let Y be a closed nonempty subset of a topological linear space E. Let $G : X \rightrightarrows Y$ be a GKKM mapping with nonempty, closed values. Then, the family $\{G(x) : x \in X\}$ has the finite intersection property, i.e.,*

$$\bigcap_{x\in S} G(x) \neq \emptyset,$$

for every finite subset S of X. Moreover, if there exists a finite subset S of X such that $\bigcap_{x\in S} G(x)$ is compact, then

$$\bigcap_{x\in X} G(x) \neq \emptyset.$$

In what follows, let X and Y be two real Banach spaces and let X^* the topological dual of X. Recall that an operator $T : X \longrightarrow Y$ is called weak to $\|\cdot\|$-sequentially continuous at $x \in X$, if for every sequence $\{x_k\} \subseteq X$ that converges weakly to x, we have that $\{T(x_k)\} \subseteq Y$ converges to $T(x)$ in the topology of the norm of Y. An operator $T : X \longrightarrow Y$ is called weak to weak-sequentially continuous at $x \in X$, if for every sequence $\{x_k\} \subseteq X$ that converges weakly to x, we have that $\{T(x_k)\} \subseteq Y$ converges weakly to $T(x)$.

The next results will be very useful in the proof of our main existence result bellow.

Lemma 3.1. *If $P \subset Q \subset X$, where Q is weakly compact and P is weakly sequentially closed then P is weakly compact.*

Proof. Indeed, by Eberlein-Šmulian theorem, Q is weakly sequentially compact. Let $\{x_k\} \subseteq P$, hence $\{x_k\} \subseteq Q$, which is weakly sequentially compact. Hence, there exists $\{x_{k_n}\} \subseteq \{x_k\}$, weakly convergent to a point $x \in Q$. But obviously $\{x_{k_n}\} \subseteq P$, which is weakly sequentially closed, hence $x \in P$. Thus P is weakly sequentially compact, and according to Eberlein-Šmulian theorem (see, for instance, [4]), P is weakly compact. □

Lemma 3.2. *Consider a bounded net $\{(x_i, x_i^*)\}_{i\in I} \subset X \times X^*$, and assume that one of the following conditions is fulfilled:*

- a) *$\{x_i\}_{i\in I}$ converges to x in the weak topology of X and $\{x_i^*\}_{i\in I}$ converges to x^* in the topology of norm of X^*.*
- b) *$\{x_i\}_{i\in I}$ converges to x in the topology of norm of X and $\{x_i^*\}_{i\in I}$ converges to x^* in the weak* topology of X^*.*

Then $\langle x_i^*, x_i\rangle \longrightarrow \langle x^*, x\rangle$.

Proof. We prove $a)$ the proof in case $b)$ is similar. Assume that $a)$ is satisfied. We have $|\langle x_i^* - x^*, x_i\rangle| \leq \|x_i^* - x^*\|\|x_i\|$, hence $\langle x_i^*, x_i\rangle - \langle x^*, x_i\rangle \longrightarrow 0$ which shows that $\langle x_i^*, x_i\rangle \longrightarrow \langle x^*, x\rangle$. □

Consider the set $K \subseteq X$ and let $A : K \longrightarrow X^*$, $a : K \longrightarrow X$ and $\psi : K \times K \longrightarrow X$ be given operators. Now, we are ready to state our first main existence result.

Theorem 3.3. *Let K be a weakly compact and convex set and let ψ be of type g-ql with respect to A. Assume that one of the following conditions is fulfilled.*

a) *A is weak to $\|\cdot\|$-sequentially continuous and $\psi(\cdot, y)$ is weak to weak-sequentially continuous for each $y \in K$.*
b) *A is weak to weak-sequentially continuous and $\psi(\cdot, y)$ is weak to $\|\cdot\|$-sequentially continuous for each $y \in K$.*

Then $VI(A, \psi, K)$ admits solutions.

Proof. Define the set-valued mapping $G : K \rightrightarrows K$ by

$$G(y) = \{x \in K : \langle A(x), \psi(x, y)\rangle \geq 0\}, \text{ for all } y \in K.$$

It is easy to see that the existence of the solution of $VI(A, \psi, K)$ is equivalent to

$$\bigcap_{y \in X} G(y) \neq \emptyset.$$

We show that G is a $GKKM$ mapping with weakly compact values hence, the assumptions of Theorem 3.1 are satisfied. Since ψ is of type g-ql with respect to A, according to Definition 2.3, $G(y) \neq \emptyset$. Now, we show that for each $y \in K$, $G(y)$ is weakly compact, thus it is weakly closed as well. To show this, for $y \in K$ consider a sequence $\{x_k\} \subseteq G(y)$ that converges weakly to $x \in K$. We show that $x \in G(y)$.

If $a)$ is satisfied, then according to Lemma 3.2 $a)$ we have

$$\lim_{k \longrightarrow \infty} \langle A(x_k), \psi(x_k, y)\rangle = \langle A(x), \psi(x, y)\rangle,$$

and from $\langle A(x_k), \psi(x_k, y)\rangle \geq 0$ we get that $\langle A(x), \psi(x, y)\rangle \geq 0$.

If $b)$ is satisfied, then according to Lemma 3.2 $b)$ we have

$$\lim_{k \longrightarrow \infty} \langle A(x_k), \psi(x_k, y)\rangle = \langle A(x), \psi(x, y)\rangle,$$

and from $\langle A(x_k), \psi(x_k, y)\rangle \geq 0$ we get that $\langle A(x), \psi(x, y)\rangle \geq 0$.

Hence $x \in G(y)$, which means that $G(y)$ is weakly sequentially closed for all $y \in K$. Since K is weakly compact and $G(y) \subseteq K$, by Lemma 3.1 we obtain that $G(y)$ is weakly compact.

Now we prove that G is a GKKM mapping. Let $y_1, \dots, y_n \in K$.

Since $\psi : K \times K \longrightarrow X$ is of type g-ql with respect to A, for every finite subset $\{y_1, \dots, y_n\}$ of K, there exist $z_1, \dots, z_n \in K$ such that for every subset $\{z_{i_1}, \dots, z_{i_k}\} \subseteq \{z_1, \dots, z_n\}$, $1 \leq k \leq n$, and for every $z \in \operatorname{co}\{z_{i_1}, \dots, z_{i_k}\}$, we have

$$\operatorname{co}\{\psi(z, y_{i_1}), \dots, \psi(z, y_{i_k})\} \cap A^+(z) \neq \emptyset.$$

We show that

$$\operatorname{co}\{z_{i_1}, \dots, z_{i_k}\} \subseteq \bigcup_{j=1}^{k} G(y_{i_j}),$$

consequently G is a GKKM mapping. Suppose the contrary, that is, there exists $z \in \operatorname{co}\{z_{i_1}, \dots, z_{i_k}\}$ such that $z \notin G(y_{i_j})$, for every $j \in \{1, 2, \dots, k\}$. Hence, $\langle A(z), \psi(z, y_{i_j})\rangle < 0$, for every $j \in \{1, 2, \dots, k\}$.

Since $\mathrm{co}\{\psi(z, y_{i_j}) : j \in \{1, 2, \ldots, k\}\} \cap A^+(z) \neq \emptyset$, there exist $\lambda_{i_j} \geq 0, j \in \{1, 2, \ldots, k\}$ with $\sum_{j=1}^{k} \lambda_{i_j} = 1$ such that

$$\sum_{j=1}^{k} \lambda_{i_j} \psi(z, y_{i_j}) \in A^+(z),$$

or equivalently

$$\left\langle A(z), \sum_{j=1}^{k} \lambda_{i_j} \psi(z, y_{i_j}) \right\rangle \geq 0.$$

On the other hand, by multiplying the inequalities $\langle A(z), \psi(z, y_{i_j})\rangle < 0$ one by one with λ_{i_j}, $j \in \{1, 2, \ldots, k\}$ and summing up them from 1 to k, we obtain that

$$\left\langle A(z), \sum_{j=1}^{k} \lambda_{i_j} \psi(z, y_{i_j}) \right\rangle < 0,$$

a contradiction.

According to Theorem 3.1 $\bigcap_{y \in X} G(y) \neq \emptyset$, i.e., there exists $x \in K$, such that $\langle A(x), \psi(x, y)\rangle \geq 0$ for all $y \in K$. □

As an immediate consequence we obtain the following result, a generalization of Theorem 4.1 from [11].

Theorem 3.4. *Let $K \subseteq X$ be a weakly compact and convex set and let a be of type g-ql. Assume that one of the following conditions is fulfilled.*

a) *A is weak to $\|\cdot\|$-sequentially continuous and a is weak to weak-sequentially continuous.*
b) *A is weak to weak-sequentially continuous and a is weak to $\|\cdot\|$-sequentially continuous.*

Then $VI_S(A, a, K)$ admits solutions.

Proof. Let $\psi(x, y) = a(y) - a(x)$ for each $x, y \in K$. Since a is of type g-ql according to Lemma 2.1 ψ is of type g-ql. Hence according to Remark 2.1, ψ is of type g-ql w.r.t. A. The conclusion follows immediately from Theorem 3.3. □

Remark 3.1. *An anonymous referee emphasized the following extension of problem $VI(A, \psi, K)$. Find an element $x \in K$ such that*

$$\langle A(x), \psi(x, y)\rangle \geq 0, \text{ for all } y \in L,$$

where L is an arbitrary nonempty set and $\psi : K \times L \longrightarrow X$ is a given operator. Let us introduce the map $G : L \rightrightarrows K$, $G(y) = \{x \in K : \langle A(x), \psi(x, y)\rangle \geq 0\}$, for all $y \in L$.

An existence result of solution for this variational inequality, similar to Theorem 3.3, can be established using as argument Theorem 3.2 instead of Theorem 3.1.

The following result provides sufficient conditions for the existence of solutions of $VI(A, \psi, K)$, without any continuity assumptions imposed on the operators A and a.

Theorem 3.5. *Let $K \subseteq X$ be a weakly compact and convex set. Suppose that the following conditions are satisfied:*

(a) *ψ is of type g-ql with respect to A;*
(b) *If $\{x_k\} \subseteq K$ converges weakly to $x \in K$, then*

$$\liminf_{k \longrightarrow \infty} \langle A(x_k), \psi(x_k, y) \rangle \leq \langle A(x), \psi(x, y) \rangle, \text{ for all } y \in K.$$

Then $VI(A, \psi, K)$ admits solutions.

Proof. Let us define the mapping $G : K \rightrightarrows K$ as in the proof of Theorem 3.3. We show that $G(y)$ is weakly sequentially closed for all $y \in K$, the rest of the proof is similar to the proof of Theorem 3.3 and will be omitted.
Since ψ is of type g-ql with respect to A, according to Definition 2.3, $G(y) \neq \emptyset$. For $y \in K$, consider a sequence $\{x_k\} \subseteq G(y)$ that converges weakly to $x \in K$. We show that $x \in G(y)$. Indeed, we have $\langle A(x_k), \psi(x_k, y) \rangle \geq 0$ for all $k \in \mathbb{N}$. Then according to (b) we have

$$0 \leq \liminf_{k \longrightarrow \infty} \langle A(x_k), \psi(x_k, y) \rangle \leq \langle A(x), \psi(x, y) \rangle.$$

Hence $0 \leq \langle A(x), \psi(x, y) \rangle$, which shows that $x \in G(y)$. Consequently $G(y)$ is weakly sequentially closed for all $y \in K$. □

The next result extends Theorem 3.2 from [16] and Theorem 4.3 from [11].

Theorem 3.6. *Let $K \subseteq X$ be a weakly compact and convex set. Suppose that the following conditions are satisfied:*

(a) *a is of type g-ql ;*
(b) *If $\{x_k\} \subseteq K$ converges weakly to $x \in K$, then*

$$\liminf_{k \longrightarrow \infty} \langle A(x_k), y) \rangle \leq \langle A(x), y) \rangle$$

for all $y \in K$;
(c) *The function $x \rightarrowtail \langle A(x), a(x) \rangle$, mapping K into $\mathbb{R}$ is sequentially weakly lower semicontinuous.*

Then $VI_S(A, a, K)$ admits solutions.

Proof. It suffices to show that the condition (b) of Theorem 3.5 is satisfied by $\psi(x, y) = a(y) - a(x)$. To show this let $\{x_k\} \subseteq K$ be a sequence that converges weakly to $x \in K$. Then from (b) and (c), we have

$$\liminf_{k \longrightarrow \infty} \langle A(x_k), a(y) - a(x_k) \rangle \leq \liminf_{k \longrightarrow \infty} \langle A(x_k), a(y) \rangle - \liminf_{k \longrightarrow \infty} \langle A(x_k), a(x_k) \rangle \leq$$
$$\leq \langle A(x), a(y) \rangle - \langle A(x), a(x) \rangle = \langle A(x), a(y) - a(x) \rangle.$$ □

The next example shows, that the condition that the operator ψ is of type g-ql with respect to A is essential in the hypotheses of the previous theorems.

Example 3.1. Let us consider the operators $A : K \longrightarrow \mathbb{R}^2$, $A(x, y) = (1, x)$, and $\psi : K \times K \longrightarrow \mathbb{R}^2$, $\psi((x, y), (u, v)) = (uv - xy, y - v)$, where $K = [(-1, 1), (1, -1)] \subseteq \mathbb{R}^2$. Then all the assumptions in the hypothesis of Theorem 3.3 are satisfied excepting the one, that ψ is of type g-ql with respect to A. We show that the conclusion of Theorem 3.3 fails, that is, $VI(A, \psi, K)$ has no solutions.

Obviously A is continuous, K is compact and convex, and $\psi((\cdot,\cdot),(u,v))$ is continuous for every $(u,v)\in K$. It is also obvious that $\psi((x,y),(x,y))=(0,0)$ for all $(x,y)\in K$. This shows in particular, that the mapping G introduced in the proof of Theorem 3.3 has nonempty values.

We show that ψ is not of type g-ql with respect to A. Suppose the contrary and consider $(x,y)=(-1,1)\in K$ and $(u,v)=(1,-1)\in K$. Then there exist $(p_1,q_1),(p_2,q_2)\in K$ such that $\psi((p_1,q_1),(-1,1))\cap A^+(p_1,q_1)\neq\emptyset$, $\psi((p_2,q_2),(1,-1))\cap A^+(p_2,q_2)\neq\emptyset$ and for all $(p,q)\in[(p_1,q_1),(p_2,q_2)]$ we have

$$[\psi((p,q),(-1,1)),\psi((p,q),(1,-1))]\cap A^+(p,q)\neq\emptyset.$$

From $\psi((p_1,q_1),(-1,1))\cap A^+(p_1,q_1)\neq\emptyset$ we obtain $\langle(-1-p_1q_1,q_1-1),(1,p_1)\rangle\geq 0$, hence $p_1\leq -1$, and taking into account that $(p_1,q_1)\in K$, we get $(p_1,q_1)=(-1,1)$.

From $\psi((p_2,q_2),(1,-1))\cap A^+(p_2,q_2)\neq\emptyset$ we obtain $\langle(-1-p_2q_2,q_2+1),(1,p_2)\rangle\geq 0$, hence $p_2\geq 1$, and taking into account that $(p_2,q_2)\in K$, we get $(p_2,q_2)=(1,-1)$.

Hence, for all $(p,q)\in[(p_1,q_1),(p_2,q_2)]=K$ we have

$$[\psi((p,q),(-1,1)),\psi((p,q),(1,-1))]\cap A^+(p,q)\neq\emptyset.$$

Let $(p,q)=(0,0)\in K$. Then $[\psi((p,q),(-1,1)),\psi((p,q),(1,-1))]=[(-1,-1)(-1,1)]$ and $A^+(p,q)=(1,0)$, hence

$$[\psi((p,q),(-1,1)),\psi((p,q),(1,-1))]\cap A^+(p,q)=\emptyset,\ \text{contradiction.}$$

We showed that ψ is not of type g-ql with respect to A, and it remained to show that $VI(A,\psi,K)$ has no solution. Let us suppose that there exists $(x,y)\in K$, a solution of the problem $VI(A,\psi,K)$. Then for every $(u,v)\in K$ we have $\langle A(x,y),\psi((x,y),(u,v))\rangle\geq 0$ or equivalently $v(u-x)\geq 0$. For $u=1$, $v=-1$ we get $1-x\leq 0$, but $x\in[-1,1]$, hence $x=1$.

For $u=-1$, $v=1$ we get $1+x\leq 0$, but $x\in[-1,1]$, hence $x=-1$. Contradiction.

4. A coincidence point result

In this section we apply an existence result obtained in the previous section, to establish a coincidence point result involving operators of type g-ql. As particular case we obtain Brouwer's fixed point theorem. Let X and Y be two arbitrary sets and $f,g\colon X\longrightarrow Y$ be two given mappings. Recall that a point $x\in X$ is a coincidence point of f and g if $f(x)=g(x)$. Our coincidence point result is stated as follows.

Theorem 4.1. *Let H be a real Hilbert space identified with its dual. Let $K\subseteq H$ be a convex and weakly compact set, let $f:K\longrightarrow H$ be weak to weak sequentially continuous, $g:K\longrightarrow H$ be weak to norm sequentially continuous and of type g-ql. Assume that $f(K)\subseteq g(K)$. Then there exists $x_0\in K$ such that $f(x_0)=g(x_0)$.*

Proof. Consider the operator $A:K\longrightarrow H$, $A(x)=g(x)-f(x)$. It can be easily verified that the operator A is weak to weak sequentially continuous.

Hence, the condition of Theorem 3.4 *b)* is satisfied (with $a=g$), thus there exists $x_0\in K$ such that

$$\langle A(x_0),g(y)-g(x_0)\rangle\geq 0$$

for all $y\in K$, or, equivalently $\langle g(x_0)-f(x_0),g(y)-g(x_0)\rangle\geq 0$ for all $y\in K$.

Since $f(K) \subseteq g(K)$ there exists $y \in K$ such that $g(y) = f(x_0)$. Then we have $\langle g(x_0) - f(x_0), f(x_0) - g(x_0)\rangle \geq 0$, or, equivalently $-\|f(x_0) - g(x_0)\|^2 \geq 0$ which shows that $f(x_0) = g(x_0)$. □

It is well known that in finite dimensional spaces the weak and strong topologies coincide. As a corollary we obtain the following coincidence point result, a generalization of Brouwer fixed point theorem.

Corollary 4.1. *Let $K \subseteq \mathbb{R}^n$ be a convex and compact set and let $f, g : K \longrightarrow \mathbb{R}^n$ be continuous. Assume that $f(K) \subseteq g(K)$ and that g is of type g-ql. Then there exists $x_0 \in K$ such that $f(x_0) = g(x_0)$.*

As an immediate consequence we obtain Brouwer's fixed point theorem. Indeed, if we take $g : K \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $g(x) = x$, then g obviously is of type g-ql and $g(K) = K$. We have the following corollary.

Corollary 4.2. *Let $K \subseteq \mathbb{R}^n$ be a convex and compact set and let $f : K \longrightarrow \mathbb{R}^n$ be continuous. Assume that $f(K) \subseteq K$. Then there exists $x_0 \in K$ such that $f(x_0) = x_0$.*

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References

[1] A. Bensoussan, J.L. Lions, *Applications des Inequations Variationelles en Control et Stochastiques*, Dunod, Paris, 1978.

[2] S.S. Chang, Y. Zhang, *Generalized KKM theorem and variational inequalities*, J. Math. Anal. Appl., **159**(1991), 208-223.

[3] P.K. Das, *An Iterative Method for (AGDDVIP) in Hilbert Space and the Homology Theory to Study the (GDDCP$_n$) in Riemannian n-manifolds in the Presence of Fixed Point Inclusion*, European Journal of Pure and Applied Mathematics, **4**(2011), no. 4, 340-360.

[4] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, V. Zizler, *Functional Analysis and Infinite-Dimensional Geometry*, Springer-Verlag, New York, 2001.

[5] R. Ferrentino, *Variational inequalities and optimization problems*, Appl. Math. Sci., **1**(2007), 2327-2343.

[6] G. Fichera, *Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno*, Atti Accad. Naz. Lincei, Mem. Cl. Sci. Fis. Mat. Natur. Sez., **7**(8)(1963-1964), 91-140.

[7] S.S. Irfan, R. Ahmad, *Generalized multivalued vector variationallike inequalities*, J. Global Optim., **46**(2010), 25-30.

[8] G. Kassay, I. Kolumbán, *On the Knaster-Kuratowski-Mazurkiewicz and Ky Fan's theorem*, Seminar on Mathematical Analysis, Babes-Bolyai Univ. Cluj-Napoca, **90-7**(1990), 87-100.

[9] D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and their Applications*, Academic Press, New York, 1980.

[10] B.T. Kien, N.C. Wong, J.C. Yao, *Generalized vector variational inequalities with star-pseudomonotone and discontinuous operators*, Nonlinear Anal., **68**(2008), 2859-2871.

[11] S. László, *Some existence results of solutions for general variational inequalities*, J. Optim. Theory Appl., **150**(2011), 425-443.

[12] L.J. Lin, C.S. Chuang, Z.T. Yu, *Generalized KKM theorems and common fixed point theorems*, Nonlinear Anal., **74**(2011), no. 16, 5591-5599.

[13] A. Maugeri, F. Raciti, *On existence theorems for monotone and nonmonotone variational inequalities*, Journal of Convex Anal., **16**(2009), 899-911.

[14] M.A. Noor, *General variational inequalities*, Appl. Math. Lett., **1**(1988), 119-121.

[15] G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convexes*, C.R. Acad. Sci. Paris, Ser I. Math., **258**(1964), 4413-4416.

[16] J.C. Yao, *General variational inequalities in Banach spaces*, Appl. Math. Lett., **5**(1992), 51-54.

[17] L.C. Zeng, S.Y. Wu, J.C. Yao, *Generalized KKM theorem with applications to generaized minimax inequalities and generalized equilibrium problems*, Taiwanese J. Math., **10**(2006), no. 6, 1497-1514.

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