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A FIXED POINT THEOREM FOR CORRESPONDENCES ON CONE METRIC SPACES

XUN GE

School of Mathematical Sciences, Soochow University
Suzhou 215006, P.R. China
E-mail: gexun@suda.edu.cn

Abstract. In this paper, we prove that if f is a contractive closed-valued correspondence on a cone metric space (X, d) and there is a contractive orbit $\{x_n\}$ for f at $x_0 \in X$ such that both $\{x_{n_i}\}$ and $\{x_{n_i+1}\}$ converge for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$, then f has a fixed point, which generalizes a fixed point theorem for contractive closed-valued correspondences from metric spaces to cone metric spaces.

Key Words and Phrases: Cone metric space, fixed point, contractive correspondence, closed-valued correspondence, contractive orbit.

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1. Introduction

A class of interesting cone metric spaces were firstly introduced and investigated by P. P. Zabrejko in a survey on some fixed point theorems [13]. Let $\mathbb{B}$ be an ordered linear space over $\mathbb{R}$ and $\mathbb{K}$ be a cone in $\mathbb{B}$. For a set X, P. P. Zabrejko [13] defined a function $\rho : X \times X \longrightarrow B$, which satisfies the following properties.

(a) $\rho(x, y) \geq 0$ for all $x, y \in X$.

(b) $\rho(x, y) = 0$ is equivalent to $x = y$ for all $x, y \in X$.

(c) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.

(d) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$.

Here, the function ρ is called a K-metric on X and the pair $X = (X, \rho)$ is called a K-metric space. Many interesting results in [13] are gathered in some theorems about existence and uniqueness of fixed points for operators that act in K-metric spaces, which generalized the classical Banach-Caccioppoli principle of contractive mappings.

As further investigation for K-metric spaces, another class of cone metric spaces were presented and discussed by L.G. Huang and X. Zhang in [4]. This paper refers to cone metric spaces in the sense of [4]. Recently, many interesting results, around fixed point theorems for mappings on cone metric spaces, had been obtained (see [1, 3,

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4, 5, 9, 11, 12], for example). However, can fixed point theorems for correspondence be also generalized from metric spaces to cone metric spaces? It is an interesting topic. In the classical theory of fixed points for correspondence on metric spaces, the following is a known theorem.

Theorem 1.1 ([7, 10]) *Let f be a contractive closed-valued correspondence on a metric space (X, d) and there be a contractive orbit $\{x_n\}$ for f at $x_0 \in X$ such that both $\{x_{n_i}\}$ and $\{x_{n_i+1}\}$ converge for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Then f has a fixed point.*

Thus, the following question arise naturally.

Question 1.2 *Can "metric space" in Theorem 1.1 be generalized "cone metric space"?*

In this paper, we give an affirmative answer for Question 1.2. Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all natural numbers and the set of all real numbers respectively, $\{x_n\}$ denotes the sequence $\{x_0, x_1, x_2, \cdots, x_n, \cdots\}$. A correspondence f on a space X means a set-valued mapping $f : X \longrightarrow \mathscr{P}_0(X)$, where $\mathscr{P}_0(X) = \{B \subset X : B \neq \emptyset\}$ (see [7, 8], for example). A point $x \in X$ is a fixed point for f if $x \in f(x)$.

2. Preliminaries

Definition 2.1 ([4]) Let E be a real Banach space. A subset P of E is called a cone if the following are satisfied.

(1) P is closed, $P \neq \emptyset$ and $P \neq \{0\}$.
(2) $a, b \in \mathbb{R}$, $a, b > 0$ and $x, y \in P \Longrightarrow ax + by \in P$.
(3) $x, -x \in P \Longrightarrow x = 0$.

Definition 2.2 ([4]) Let P be a cone of a real Banach space E. Some partial orderings $\leq$, $<$ and $\ll$ on E with respect to P are defined as follows, respectively. Let $x, y \in E$.

(1) $x \leq y$ if $y - x \in P$.
(2) $x < y$ if $x \leq y$ and $x \neq y$.
(3) $x \ll y$ if $y - x \in P^\circ$, where P° denotes the interior of P.

Remark 2.3 *In this paper, for the sake of conveniences, we also use notations "$\geq$", "$>$" and "$\gg$'" on E with respect to P. The meanings of these notations are clear and the following hold.*

(1) $x \geq 0$ if and only if $x \in P$.
(2) $x \gg 0$ if and only if $x \in P^\circ$.
(3) If $x \geq y$ and $y \gg 0$, then $x \gg 0$ [6].
(4) If $x_1 \gg 0$ and $x_2 \gg 0$, then there is $x \gg 0$ such that $x \ll x_1$ and $x \ll x_2$ [12].

Definition 2.4 Let P is a cone of a real Banach space E.

(1) P is called strictly normal if $0 \leq x < y$ implies $\|x\| < \|y\|$ for all $x, y \in E$.
(2) P is called strongly minihedral [11] if each subset of E which is bounded from above has a supremum.

Remark 2.5 ([11]) *Let P is a cone of a real Banach space E. P is strongly minihedral if and only if each subset of E bounded below has an infimum.*

Example 2.6 *Let $E = \mathbb{R}$ and $P = \{x \in \mathbb{R} : x \geq 0\}$. Then P is a strictly normal and strongly minihedral cone of E.*

Throughout this paper, we always suppose that E is a real Banach space and P is a strictly normal and strongly minihedral cone of E. Thus, the following notations are well defined [2, 11], which will be used in our investigation.

Notation 2.7 Let (X, d) be a cone metric space.
(1) $d(x, A) = \inf\{d(x, y) : y \in A\}$, where $x \in X$ and $A \subseteq X$.
(2) $B(x, \beta) = \{y \in X : d(x, y) \ll \beta\}$, where $x \in X$ and $\beta \gg 0$.
(3) $\beta + A = \{x \in X : d(x, A) \ll \beta\}$, where $\beta \gg 0$ and $\emptyset \neq A \subseteq X$.
(4) $\rho(A, B) = \inf\{\beta \gg 0 : A \subseteq \beta + B\} \geq 0$, where $\emptyset \neq A, B \subseteq X$.
(5) $\delta(A, B) = \inf\{\beta \geq 0 : \beta \geq \rho(A, B), \beta \geq \rho(B, A)\}$, where $\emptyset \neq A, B \subseteq X$.

Definition 2.8 ([4]) Let X be a non-empty set. A mapping $d : X \times X \longrightarrow E$ is called a cone metric on X, and (X, d) is called a cone metric space if the following are satisfied.
(1) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Remark 2.9 *(1) For a cone metric space (X, d), the cone metric $d : X \times X \longrightarrow P$ from Remark 2.3(1).*

(2) If we choose E and P as Example 2.6, then the cone metric space (X, d) is a metric space.

Lemma 2.10 ([6, 12]) *Let (X, d) be a cone metric space. Put $\mathscr{B} = \{B(x, c) : x \in X, c \gg 0\}$ and $\mathscr{T} = \{U \subseteq X : \forall x \in U, \exists B \in \mathscr{B}, \text{ s.t. } x \in B \subseteq U\}$. Then $\mathscr{T}$ is a topology on X and $\mathscr{B}$ is a base for $\mathscr{T}$.*

In this paper, we always suppose that the cone metric space (X, d) is a topological space endowed the topology $\mathscr{T}$, where $\mathscr{T}$ is described as in Lemma 2.10.

Definition 2.11 ([7, 8]) Let f be a correspondence on a cone metric space (X, d).
(1) f is called closed-valued if $f(x)$ is a closed subset of X for each $x \in X$.
(2) f is called contractive if $\delta(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$.

Definition 2.12 ([7, 10]) Let f be a correspondence on a cone metric space (X, d).

(1) A sequence $\{x_n\}$ in X is called an orbit for f at x_0 if $x_n \in f(x_{n-1})$ for each $n \in \mathbb{N}$.

(2) An orbit $\{x_n\}$ for f at x_0 is called contractive if for each $n \in N$, $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$ and $d(x_{n+1}, x_{n+2}) \leq \delta(f(x_n), f(x_{n+1}))$.

Definition 2.13 ([4]) Let (X, d) be a cone metric space. A sequence $\{x_n\}$ in X is called to converge to $x \in X$ if for each $c \gg 0$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$. The sequence $\{x_n\}$ converges to x is denoted by $\lim_{n\to\infty} x_n = x$.

3. Main results

Lemma 3.1 *Let P be a cone of a real Banach space E.*

(1) If $\alpha \gg 0$, then $r\alpha \gg 0$ for each $r \in \mathbb{R}^+$.

(2) If $\beta \geq 0$ and $c \gg 0$, then $\beta + c \gg 0$.

Proof. (1) Let $\alpha \gg 0$, i.e., $\alpha \in P^\circ$. Then there is an open neighborhood B_α of α in E such that $B_\alpha \subseteq P$. If $r \in \mathbb{R}^+$, then $rB_\alpha \subseteq P$ from Definition 2.1(2). Note that $r\alpha \in rB_\alpha$ and rB_α is an open subset of E. So $r\alpha \in P^\circ$, i.e. $r\alpha \gg 0$.

(2) Let $\beta \geq 0$ and $c \gg 0$. Since $\beta + c - c = \beta \geq 0$, $\beta + c \geq c$. It follows that $\beta + c \gg 0$ from Remark 2.3(3).

Proposition 3.2 *Let (X, d) be a cone metric space.*

(1) $\rho(A, B) \leq \delta(A, B)$ for $A, B \in \mathscr{P}_0(X)$.

(2) $\rho(A, B) \leq \rho(A, C) + \rho(C, B)$ for $A, B, C \in \mathscr{P}_0(X)$.

Proof. (1) It is clear from Notation 2.7(4) and (5).

(2) Let $\rho(A, B) = \beta$, $\rho(A, C) = \beta_1$ and $\rho(C, B) = \beta_2$. Whenever $c \gg 0$, then $\beta_1 + c \gg 0$, $\beta_2 + c \gg 0$ and $\beta_1 + \beta_2 + 3c \gg 0$ from Lema 3.1. By Notation 2.7(4), $A \subseteq \beta_1 + c + C$ and $C \subseteq \beta_2 + c + B$. Let $x \in A$, then $x \in \beta_1 + c + C$, i.e., $d(x, C) \ll \beta_1 + c$. Thus, there is $y \in C$ such that $d(x, y) \leq \beta_1 + 2c$. Also, $y \in \beta_2 + c + B$, i.e., $d(y, B) \ll \beta_2 + c$. It follows that $d(x, B) \leq d(x, y) + d(y, B) \ll \beta_1 + 2c + \beta_2 + c = \beta_1 + \beta_2 + 3c$, i.e., $x \in \beta_1 + \beta_2 + 3c + B$. This proves that $A \subseteq \beta_1 + \beta_2 + 3c + B$. By Notation 2.7(4), $\beta \leq \beta_1 + \beta_2 + 3c$. By the arbitrariness of $c \gg 0$, $\beta \leq \beta_1 + \beta_2$.

Proposition 3.3 *Let f be a closed-valued correspondence on a cone metric space (X, d). Then the following are equivalent for $x, y \in X$.*

(1) $x \in f(y)$.

(2) $\rho(\{x\}, f(y)) = 0$.

Proof. (1) $\Longrightarrow$ (2): Let $x \in f(y)$. Then $0 \leq d(x, f(y)) \leq d(x, x) = 0$. So $d(x, f(y)) = 0 \ll c$ for arbitrary $c \gg 0$. It follows that $\{x\} \subseteq c + f(y)$ for arbitrary $c \gg 0$. Consequently, $\rho(\{x\}, f(y)) = \inf\{\beta \gg 0 : \{x\} \subseteq \beta + f(y)\} = 0$.

(2) $\Longrightarrow$ (1): Let $\rho(\{x\}, f(y)) = 0$. If $x \notin f(y)$, then there is $c \gg 0$ such that $B(x, c) \bigcap f(y) = \emptyset$ because $f(y)$ is closed in (X, d). It follows that $d(x, f(y)) \geq c \gg 0$. Thus, $\{x\} \not\subseteq c/2 + f(y)$. Consequently, $\rho(\{x\}, f(y)) = \inf\{\beta \gg 0 : \{x\} \subseteq \beta + f(y)\} > c/3 \gg 0$. this contradicts that $\rho(\{x\}, f(y)) = 0$.

Now we give the main theorem of this paper.

Theorem 3.4 *Let (X, d) be a cone b-metric space (X, d) with coefficient $s \geq 1$ and f be a contractive mapping on (X, d). If there is a contractive orbit $\{x_n\}$ for f at some $x_0 \in X$ such that $\{x_{n_i}\}$ converges for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$, then f has a fixed point.*

Proof. Let $\{x_n\}$ be a contractive orbit for f at some $x_0 \in X$ such that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges to some $a \in X$. Then, for arbitrary $\varepsilon \gg \theta$, there is $t \in \mathbb{N}$ such that $d(x_{n_i}, a) \ll \varepsilon$ for all $i \geq t$. Furthermore, we write $b = f(a)$, then $d(x_{n_i+1}, b) = d(f(x_{n_i}), f(a)) \leq d(x_{n_i}, a) \ll \varepsilon$ for all $i \geq t$. So $\{x_{n_i+1}\}$ converges to b. We only need to prove that $a = b$.

Put $\Delta = \{(x, x) : x \in X\}$, i.e., Δ is the diagonal of $X \times X$. Let

$$g : (X \times X) - \Delta \longrightarrow \mathbb{R},$$

where $\mathbb{R}$ is the set of all real numbers and g is defined as follows.

Whenever $(x, y) \in (X \times X) - \Delta$, then $x \neq y$, so $d(x, y) \neq 0$, and hence $\|d(x, y)\| \neq 0$. Put

$$g(x, y) = \frac{\|d(f(x), f(y))\|}{\|d(x, y)\|}.$$

Then g is continuous because g is a quotient of two continuous functions $\|d(f(x), f(y))\|$ and $\|d(x, y)\|$. For each $(x, y) \in (X \times X) - \Delta$, since f is contractive, $\theta \leq d(f(x), f(y)) < d(x, y)$. By the strictly normality of the cone P, $\|d(f(x), f(y))\| < \|d(x, y)\|$. It follows that $g(x, y) < 1$ for all $(x, y) \in (X \times X) - \Delta$.

If $a \neq b$, then $(a, b) \in (X \times X) - \Delta$, and hence $g(a, b) < 1$. Thus, there are disjoint neighborhoods U and V of a and b respectively, such that $g(x, y) \leq \lambda$ for all $(x, y) \in U \times V$ and for some $\lambda < 1$. There are $\beta_a \gg 0$ and $\beta_b \gg 0$ such that $B(a, \beta_a) \subseteq U$ and $B(b, \beta_b) \subseteq V$. By Remark 2.3(4), there is $\beta \gg 0$ such that $\beta \ll \beta_a$ and $\beta \ll \beta_b$. Without loss of generality, we can choose $\beta \gg 0$ such that $2\beta < d(a, b)$. Then

$$d(a, b) - 2\beta > 0, \ B(a, \beta) \subset U, \ B(b, \beta) \subset V.$$

Since

$$a = \lim_{i \to \infty} x_{n_i}, \ b = \lim_{i \to \infty} x_{n_i+1},$$

there is $m \in \mathbb{N}$ such that $x_{n_i} \in B(a, \beta)$ and $x_{n_i+1} \in B(b, \beta)$ for all $i \geq m$. So, if $i \geq m$, then

$$d(a, b) \leq d(a, x_{n_i}) + d(x_{n_i}, x_{n_i+1}) + d(x_{n_i+1}, b) \leq 2\beta + d(x_{n_i}, x_{n_i+1}),$$

Thus

$$d(x_{n_i}, x_{n_i+1}) \geq d(a, b) - 2\beta > 0.$$

On the other hand, for each $i \geq m$, $(x_{n_i}, x_{n_i+1}) \in B(a, \beta) \times B(b, \beta) \subseteq U \times V$, so $g(f(x_{n_i}), f(x_{n_i+1})) < \lambda$, i.e., $\|d(f(x_{n_i}), f(x_{n_i+1}))\| \leq \lambda \|d(x_{n_i}, x_{n_i+1})\|$. Since $\{x_n\}$ is a contractive orbit for f at x_0, $d(x_{n_i+1}, x_{n_i+2}) \leq \delta(f(x_{n_i}), f(x_{n_i+1}))$, and hence, by the strong normality of the cone P,

$$\|d(x_{n_i+1}, x_{n_i+2})\| \leq \|\delta(f(x_{n_i}), f(x_{n_i+1}))\| \leq \lambda \|d(x_{n_i}, x_{n_i+1})\|.$$

Furthermore, we have

$$\|d(x_{n_{i+1}}, x_{n_{i+1}+1})\| \leq \|d(x_{n_{i+1}-1}, x_{n_{i+1}})\| \leq \cdots$$
$$\cdots \leq \|d(x_{n_i+1}, x_{n_i+2})\| \leq \lambda \|d(x_{n_i}, x_{n_i+1})\|.$$

Iterating this inequality, we have $\|d(x_{n_k}, x_{n_k+1})\| \leq \lambda^{k-i} \|d(x_{n_i}, x_{n_i+1})\|$ for all $k > i \geq m$. In particular, $\|d(x_{n_k}, x_{n_k+1})\| \leq \lambda^{k-m} \|d(x_{n_m}, x_{n_m+1})\|$ for all $k > m$. It follows that

$$\lim_{k \to \infty} \|d(x_{n_k}, x_{n_k+1})\| = 0.$$

This contradicts that $\|d(x_{n_i}, x_{n_i+1})\| \geq \|d(a, b)\| - 2\|\beta\| > 0$ for all $i \geq m$.

Thus, we have proved that $a = b$.

Theorem 3.4 improves Theorem 1.1 by relaxing "metric space" in Theorem 1.1 to cone metric spaces. which gives an affirmative answer for Question 1.2. As an application of Theorem 3.4, we also have the following corollary, which improves a fixed point theorem of contractive mapping on cone metric spaces (see [4]).

Corollary 3.5 *Let (X, d) be a sequentially compact cone metric space. If the mapping $T : X \longrightarrow X$ is a contractive mapping, i.e., $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, then T has a fixed point, i.e., $Tx = x$ for some $x \in X$.*

Proof. Let $T : X \longrightarrow X$ be a contractive mapping. Put $f : X \longrightarrow \mathscr{P}_0(X)$ by $f(x) = \{Tx\}$ for each $x \in X$. Then T has a fixed point if and only if f has a fixed point.

(1) Since each single-point set is closed in X, f is a closed-valued correspondence on X.

(2) It is clear that $\delta(f(x), f(y)) = d(Tx, Ty)$ for all $x, y \in X$. It follows that $\delta(f(x), f(y)) = d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. So f is a contractive correspondence.

(3) Choose $x_0 \in X$ and for each $n \in \mathbb{N}$, put $x_n = Tx_{n-1}$, i.e., $\{x_n\} = f(x_{n-1})$. Then the sequence $\{x_n\}$ is an orbit for f at x_0. Since $d(x_{n+1}, x_{n+2}) = \delta(\{x_{n+1}\}, \{x_{n+2}\}) = \delta(f(x_n), f(x_{n+1})) < d(x_n, x_{n+1})$, the sequence $\{x_n\}$ is a contractive orbit for f at x_0.

(4) Since X is sequentially compact, there is a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$. It is clear that the mapping T is continuous, $\{Tx_{n_i}\}$ converges, i.e., $\{x_{n_{i+1}}\}$ converges.

By the above (1),(2),(3) and (4), the orbit $\{x_n\}$ and the correspondence f satisfy the conditions in Theorem 3.4, so f has a fixed point. It follows that T has a fixed point.

4. Some Example

In this section, we give some examples to illuminate the importance of the conditions in Theorem 3.4.

Example 4.1 *"f is contractive" in Theorem 3.4 can not be omitted.*

Proof. Let (M, d) be a cone metric space having a infinite dense proper subset D. Choose $x_0 \in M - D$ and $\varepsilon \gg 0$, then $B(x_0, \varepsilon) \bigcap D \neq \emptyset$ and choose $x_1 \in B(x_0, \varepsilon) \bigcap D$. If $x_1, x_2, \cdots, x_n \in M$ have been chosen, then we choose $x_{n+1} \in B(x_0, d(x_0, x_n)/10) \bigcap D$. By induction, we construct a sequence $\{x_n\}$ in (M, d) such that $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ and $n \neq m$, and $x_n \neq x_0$ for all $n \in \mathbb{N}$. Put $X = \{x_n : n \in \mathbb{N} \bigcup \{0\}\}$ and put $f(x_{n-1}) = \{x_n\}$ for each $n \in \mathbb{N}$. Then f is a correspondence on the cone metric space (X, d), where the restriction of d on X is still denoted by d. It is not difficult to check that the following (1)–(5) are true.

(1) f is closed-valued.

(2) $\{x_n\}$ is a contractive orbit for f at x_0.

(3) $\{x_n\}$ converges, hence both $\{x_{n_i}\}$ and $\{x_{n_i+1}\}$ converge for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

(4) f is not contractive.

(5) f has not any fixed point.
So "f is contractive" in Theorem 3.4 can not be omitted.

Example 4.2 *"f is closed-valued" in Theorem 3.4 can not be omitted.*
Proof. Let (X, d) be the cone metric space described in the proof of Example 4.1. For each $n \in \mathbb{N}$, put $f(x_{n-1}) = \{x_m : m \geq n\}$. Then f is a correspondence on the cone metric space (X, d). It is not difficult to check that the following (1)–(5) are true.
(1) f is contractive.
(2) $\{x_n\}$ is a contractive orbit for f at x_0.
(3) $\{x_n\}$ converges, hence both $\{x_{n_i}\}$ and $\{x_{n_i+1}\}$ converge for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$.
(4) f is not closed-valued.
(5) f has not any fixed point.
So "f is closed-valued" in Theorem 3.4 can not be omitted.

Example 4.3 *"Both $\{x_{n_i}\}$ and $\{x_{n_i+1}\}$ converge for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$" in Theorem 3.4 can not be omitted.*
Proof. Let (M, d) be a cone metric space having a infinite dense proper subset D. Choose $x \in M - D$ and $\varepsilon \gg 0$, then $B(x, \varepsilon) \bigcap D \neq \emptyset$ and choose $x_0 \in B(x, \varepsilon) \bigcap D$. If $x_0, x_1, \cdots, x_n \in M$ have been chosen, then we choose $x_{n+1} \in B(x, d(x, x_n)/10) \bigcap D$. By induction, we construct a sequence $\{x_n\}$ in (M, d) such that $x_n \neq x_m$ for all $n, m \in \mathbb{N} \bigcup \{0\}$ and $n \neq m$. Put $X = \{x_n : n \in \mathbb{N} \bigcup \{0\}\}$ and put $f(x_{n-1}) = \{x_n\}$ for each $n \in \mathbb{N}$. Then f is a correspondence on the cone metric space (X, d), where the restriction of d on X is still denoted by d. It is not difficult to check that the following (1)–(4) are true.
(1) f is contractive closed-valued.
(2) $\{x_n\}$ is a contractive orbit for f at x_0.
(3) $\{x_n\}$ has not any convergent subsequence.
(4) f has not any fixed point.
So "both $\{x_{n_i}\}$ and $\{x_{n_i+1}\}$ converge for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$" in Theorem 3.4 can not be omitted.

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