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COINCIDENCE POINT THEOREMS FOR MULTI-VALUED MAPPINGS IN FUZZY METRIC SPACES

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Abstract. In this paper, by applying the countable extensions of a t-norm, we have proved a coincidence point theorem for the fuzzy Nadler type of contraction mappings. Also, a coincidence point theorem in a fuzzy metric spaces for an implicit relation is given.
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1. Introduction

It is a well known fact that a fuzzy metric space is a generalization of the concept of metric space that is based on the theory of fuzzy sets ([24]). Kramosil and Michalek (see [14]) introduced the notion of a fuzzy metric space by translating the concept of probabilistic metric space to the fuzzy environment. Further on, George and Veeramani ([2, 3]) modified the previous and obtained a Hausdorff topology for this specific kind of fuzzy metric spaces. Recently, many authors observed that various contraction mappings in metric spaces can be transferred into a fuzzy metric spaces endowed with a special t-norm. S.B. Nadler ([19]) has proved a generalization of the well known Banach contraction principle for a multi-valued mappings $f : X \to CB(X)$, where (X, d) is a classical metric space and $CB(X)$ is the family of all non-empty, closed and bounded subsets of X. There are various extensions of the Banach contraction mappings for single-valued and multi-valued mappings done in the fuzzy metric spaces' background (see [1, 4, 6, 7, 8, 9, 12, 15, 16, 17, 18]). V. Popa ([20]) introduced the idea of implicit function to prove a common fixed point theorem in metric spaces. B. Singh and S. Jain ([23]) extended the result of Popa to the fuzzy metric spaces. Many authors (see [11], [21], [23], [25]) proved a common fixed point theorem for a single-valued mappings in the fuzzy metric spaces using implicit relations. S. Sedghi et al. ([22]) proved a common fixed point theorem for a multi-valued mappings that satisfies an implicit relation on a fuzzy metric spaces for a specific t-norm.

The focus of this paper is on a coincidence point theorem for three mappings for the fuzzy Nadler type of contraction mappings. Presented result is a generalization of the Theorem 4.13 from [6]. A probabilistic version of the presented result can be found in [26]. Also, a coincidence point theorem for three mappings in a fuzzy metric space with an implicit relation is given.

A short overview of some well-known definitions and facts that are the core of the presented results follows (see [2, 4, 6, 13]).

1.1. **Preliminaries - t-norms.**

Definition 1.1 A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (a t-norm) if the following conditions are satisfied:

- $T(a,1) = a$ for all $a \in [0,1]$;
- $T(a,b) = T(b,a)$ for all $a, b \in [0,1]$;
- $a \geq b,\ c \geq d \Rightarrow\ T(a,c) \geq T(b,d)\ \ (a,b,c,d \in [0,1])$;
- $T(a,T(b,c)) = T(T(a,b),c)\ \ (a,b,c \in [0,1])$.

The following are the four basic t-norms ([13]):

$$T_M(x,y) = \min(x,y), \quad T_P(x,y) = x \cdot y, \quad T_L(x,y) = \max(x+y-1,0)$$

$$T_D(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Some important families of t-norms are given in the following example ([13]):

Example 1.2

(i) *The Dombi family of t-norms* $(T_\lambda^D)_{\lambda \in [0,\infty]}$*, which is defined by*

$$T_\lambda^D(x,\, y) = \begin{cases} T_D(x,\, y), & \lambda = 0 \\ T_M(x,\, y), & \lambda = \infty \\ \dfrac{1}{1+\left(\left(\frac{1-x}{x}\right)^\lambda + \left(\frac{1-y}{y}\right)^\lambda\right)^{1/\lambda}}, & \lambda \in (0,\, \infty). \end{cases}$$

(ii) *The Aczél-Alsina family of t-norms* $(T_\lambda^{AA})_{\lambda \in [0,\infty]}$*, which is defined by*

$$T_\lambda^{AA}(x,\, y) = \begin{cases} T_D(x,\, y), & \lambda = 0 \\ T_M(x,\, y), & \lambda = \infty \\ e^{-((-\log\, x)^\lambda + (-\log\, y)^\lambda)^{1/\lambda}}, & \lambda \in (0,\, \infty). \end{cases}$$

(iii) *Sugeno-Weber family of t-norms* $(T_\lambda^{SW})_{\lambda \in [-1,\infty]}$*, which is defined by*

$$T_\lambda^{SW}(x,\, y) = \begin{cases} T_D(x,\, y), & \lambda = -1 \\ T_P(x,\, y), & \lambda = \infty \\ \max(0,\, \frac{x+y-1+\lambda xy}{1+\lambda}), & \lambda \in (-1,\, \infty). \end{cases}$$

The following class of t-norms, that has proved itself as a highly useful tool in the fixed point theory, was introduced in [4].

Definition 1.3 [4] Let T be a t-norm and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of t-norms given by the following:

$$T_1(x) = T(x,\, x) \quad \text{and} \quad T_{n+1}(x) = T(T_n(x),\, x).$$

A t-norm T is of the H-type if T is continuous and the sequence $\{T_n(x)\}_{n\in\mathbb{N}}$ is equicontinuous at $x = 1$.

Remark 1.4 *The family $\{T_n(x)\}_{n\in\mathbb{N}}$ of t-norms is equicontinuous at $x = 1$, if for all $\lambda \in (0, 1)$ there exists $\delta(\lambda) \in (0, 1)$ such that the following implication holds:*

$$x > 1 - \delta(\lambda) \Rightarrow T_n(x) > 1 - \lambda \quad \textit{for all} \quad n \in \mathbb{N}.$$

(see [4]). A trivial example of a t-norm of H-type is $T = T_M$. A nontrivial example can be found in [4].

An arbitrary t-norm T can be extended, due to the associativity, to an n-ary operation on $[0, 1]^n$ (see [13]):

$$T(x_1, x_2, \ldots, x_n) = \mathbf{T}_{i=1}^{n} x_i = T(\mathbf{T}_{i=1}^{n-1} x_i, x_n) \quad \text{and} \quad \mathbf{T}_{i=1}^{1} x_i = x_1.$$

Now, the four basic t-norms are extended in the following manner

$$T_L(x_1, x_2, \ldots, x_n) = \max\{\sum_{i=1}^{n} x_i - (n-1), 0\},$$
$$T_M(x_1, x_2, \ldots, x_n) = \min\{x_1, x_2, \ldots, x_n\},$$

$$T_P(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdot \cdots \cdot x_n$$

and

$$T_D(x_1, x_2, \ldots, x_n) = \begin{cases} x_i & \text{if } x_j = 1 \text{ for all } j \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

Also, a t-norm T can be extend to a countable case as follows:

$$\mathbf{T}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathbf{T}_{i=1}^{n} x_i,$$

where $(x_n)_{n\in\mathbb{N}}$ is an arbitrary sequence from $[0, 1]$. The limit on the right-hand side exists since the sequence $(\mathbf{T}_{i=1}^{n} x_i)_{n\in\mathbb{N}}$ is nonincreasing and bounded from below.

The following equivalences and proposition that will be used further on can be found in [6]:

- If $T = T_L$ or $T = T_P$, then

$$\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} x_i = 1 \Longleftrightarrow \sum_{i=1}^{\infty} (1 - x_i) < \infty.$$

- If $(T_\lambda^*)_{\lambda\in(0,\infty)}$ is the Dombi family of t-norms or the Aczél-Alsina family of t-norms and if $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements from $(0, 1]$ such that $\lim_{n\to\infty} x_n = 1$, then

$$\lim_{n\to\infty} (\mathbf{T}_\lambda^*)_{i=n}^{\infty} x_i = 1 \Longleftrightarrow \sum_{i=1}^{\infty} (1 - x_i)^\lambda < \infty. \tag{1.1}$$

- If $(T_\lambda^{SW})_{\lambda\in(-1,\infty]}$ is the Sugeno-Weber family of t-norms and $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements from $(0, 1]$ such that $\lim_{n\to\infty} x_n = 1$, then

$$\lim_{n\to\infty} (\mathbf{T}_\lambda^{SW})_{i=n}^{\infty} x_i = 1 \Longleftrightarrow \sum_{i=1}^{\infty} (1 - x_i) < \infty. \tag{1.2}$$

Proposition 1.5 [6] *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence of numbers from* $[0,\,1]$ *such that* $\lim_{n\to\infty} x_n = 1$ *and t-norm* T *is of the* H*-type. Then,*

$$\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} x_i = \lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} x_{n+i} = 1.$$

1.2. **Preliminaries - fuzzy metric spaces.**

Definition 1.6 [2] The 3-tuple $(X,\,M,\,T)$ is a fuzzy metric space if X is an arbitrary set, T is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\,\infty)$ satisfying the following conditions:

- $M(x,\,y,\,t) > 0$, for all $x,\,y \in X,\,t > 0$,
- $M(x,\,y,\,t) = 1$ for all $t > 0 \Leftrightarrow x = y$,
- $M(x,\,y,\,t) = M(y,\,x,\,t)$, for all $x,\,y \in X,\,t > 0$,
- $T(M(x,\,y,\,t),\,M(y,\,z,\,s)) \leq M(x,\,z,\,t+s)$, for all $x,\,y,\,z \in X,\,t,\,s > 0$,
- $M(x,\,y,\,\cdot) : (0,\,\infty) \to [0,\,1]$ is continuous for all $x,\,y \in X$.

If $(X,\,M,\,T)$ is a fuzzy metric space, then M generates the Hausdorff topology on X with base of open sets $\{U(x,\,r,\,t) : x \in X,\,r \in (0,\,1),\,t > 0\}$, where

$$U(x,\,r,\,t) = \{y : y \in X,\,M(x,\,y,\,t) > 1 - r\}$$

(see [2]).

Since the focus is on the complete fuzzy metric spaces, the following definition is needed.

Definition 1.7 [2]

(a) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a fuzzy metric space $(X,\,M,\,T)$ is a Cauchy sequence if for all $\varepsilon \in (0,\,1)$, $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n,\,x_m,\,t) > 1 - \varepsilon$ for all $n,\,m \geq n_0$.

(b) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a fuzzy metric space $(X,\,M,\,T)$ converges to x if for all $\varepsilon \in (0,\,1)$, $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n,\,x,\,t) > 1 - \varepsilon$ for all $n \geq n_0$.

(c) A fuzzy metric space in which every Cauchy sequence is convergent is complete.

Additionally, through this paper let us assume that $\lim_{t\to\infty} M(x,y,t) = 1$.

2. Main results

Let X be an arbitrary nonempty set, $CB(X)$ the family of all non-empty, closed and bounded subsets of X, and $C(X)$ the family of all nonempty and closed subsets of X.

Let $(X,\,M,\,T)$ be a fuzzy metric space.

2.1. **Fuzzy Nadler q-contraction.** S.B. Nadler ([19]) proved on a metric space $(X,\,d)$ generalization of the Banach contraction principle for multi-valued mappings $f : X \to CB(X)$ of the form

$$D(fx,\,fy) \leq qd(x,\,y),$$

where D is the Hausdorff metric and $q \in (0,\,1)$.

A generalization of the Nadler contraction principle in a fuzzy metric space $(X,\, M, T)$ follows.

Definition 2.1 [6] Let $(X,\, M, T)$ be a fuzzy metric space, A nonempty subset of X and $F : A \to C(A)$. Mapping F is the fuzzy Nadler q-contraction, where $q \in (0,\, 1)$, if the following condition is satisfied:

for all $u,\, v \in A$, all $x \in Fu$ and all $\delta > 0$ there exists $y \in Fv$ such that for all $t > \delta$

$$M(x, y, t) \geq M(u, v, \frac{t-\delta}{q}).$$

Remark 2.2

(i) *Since the function $M(x, y, \cdot)$ is continuous, for f being a single valued mapping the fuzzy Nadler q-contraction coincides with the notion of fuzzy Banach q-contraction.*

(ii) *Connection of the probabilistic Nadler q-contraction with random operators is illustrated by Example 4.10 from* [6]. *Connection with the fuzzy metric case is analogous.*

Definition 2.3 [10] Let $(X,\, M,\, T)$ be a fuzzy metric space, $\emptyset \neq A \subset X$, $f : A \to A$ and $F : A \to C(A)$. Mapping F is a f- strongly demicompact if for every sequence $(x_n)_{n\in\mathbb{N}}$ from A, and every $\varepsilon > 0$ such that

$$\lim_{n\to\infty} M(fx_n, y_n, \varepsilon) = 1 \text{ for some sequence } (y_n)_{n\in\mathbb{N}},\ y_n \in Fx_n,\ n \in \mathbb{N},$$

there exists convergent subsequences $(fx_{n_k})_{k\in\mathbb{N}}$.

Example 2.4 *Let $X = D = \mathbb{R}$, $M(x, y, t) = M_d(x, y, t) = \frac{t}{t+d(x,y)}$ where d is the usual metric on $\mathbb{R}$ and $T = T_P$. Then (X, M, T) is a fuzzy metric space. Let $f(x) = arctg\, x$ and $F(x) = [arctg\, x - 1, arctg\, x + 1]$. Then for every sequence $(x_n)_{n\in\mathbb{N}}$, we can find a sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \in F(x_n)$, namely $y_n = arctg\, x_n + \frac{1}{n}$, and $M_d(f(x_n), y_n, t) = \frac{t}{t+d(f(x_n),y_n)} \to 1$, $n \to \infty$. Now, since the sequence $(arctg\, x_n)_{n\in\mathbb{N}}$ is bounded and it has a monotone subsequence, there exists a convergent subsequence, e.g., for $x_n = n(-1)^n$, the sequence $f(x_n) = arctg(n(-1)^n)$ has a convergent subsequence obtained by selecting even members.*

Definition 2.5 [5] A mapping $F : X \to C(X)$ is weakly commuting with $f : X \to X$ if for all $x \in X$ it holds

$$f(Fx) \subseteq F(fx).$$

Theorem 2.6 *Let $(X,\, M,\, T)$ be a complete fuzzy metric space, and A nonempty and closed subset of X. Let $f : A \to A$ be a continuous mapping and $F, G : A \to C(A)$ such that for $q \in (0,\, 1)$ the following holds:*

For every $u,\, v \in A$, $x \in Fu$ and $\delta > 0$, there exist $y \in Gv$ such that for all $\varepsilon > \delta$

$$M(x,\, y, \varepsilon) \geq M(fu,\, fv, \frac{\varepsilon-\delta}{q}). \tag{2.1}$$

If F and G are weakly commuting with f and the following is satisfied

(i) F or G are f-strongly demicompact

or

(ii) there exists $x_0,\, x_1 \in A$*,* $fx_1 \in Fx_0$ *and* $\mu \in (q,\, 1)$ *such that for a t-norm* T *holds*

$$\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} M(fx_0,\, fx_1,\, \frac{1}{\mu^i}) = 1.$$

Then there exist $x \in A$ *such that* $fx \in Fx \cap Gx$.

Proof. Let $x_0,\, x_1 \in A$ be such that $fx_1 \in Fx_0$. From (2.1) for $x = fx_1$, $u = x_0$, $v = x_1$ i $\delta = q$ it follows that there exist $x_2 \in A$ such that $fx_2 \in Gx_1$ and

$$M(fx_1,\, fx_2, \varepsilon) \geq M(fx_0,\, fx_1,\, \frac{\varepsilon - q}{q}).$$

Continuing in this way we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ from A such the following conditions are satisfied

(a) $fx_{2n+1} \in Fx_{2n}$ and $fx_{2n+2} \in Gx_{2n+1}$

(b) $M(fx_n,\, fx_{n+1}, \varepsilon) \geq M(fx_{n-1},\, fx_n,\, \frac{\varepsilon - q^n}{q})$.

From (b) it follows

$$M(fx_n,\, fx_{n+1}, \varepsilon) \geq M(fx_1,\, fx_0,\, \frac{\varepsilon - nq^n}{q^n}).$$

Since $\varepsilon > 0$, $\lim\limits_{n\to\infty} M(fx_1,\, fx_0,\, \frac{\varepsilon - nq^n}{q^n}) = 1$ $(\lim\limits_{n\to\infty}(\frac{\varepsilon - nq^n}{q^n}) = \infty)$, it follows

$$\lim_{n\to\infty} M(fx_n,\, fx_{n+1}, \varepsilon) = 1. \tag{2.2}$$

If we supposed that F is f-strongly demicompact, i.e., condition (i) is fulfilled, using

$$\lim_{n\to\infty} M(fx_{2n},\, fx_{2n+1}, \varepsilon) = 1 \quad \text{and} \quad fx_{2n+1} \in Fx_{2n}$$

we conclude that there exist a convergent subsequence $(fx_{2n_k})_{k\in\mathbb{N}}$ of a sequence $(fx_{2n})_{n\in\mathbb{N}}$.

It remains to be proved that the sequence $(fx_n)_{n\in\mathbb{N}}$ is convergent if T satisfies condition (ii).

Let $\sigma = \frac{q}{\mu}$. Since $\sigma \in (0,\, 1)$, $\sum\limits_{i=1}^{\infty} \sigma^i$ is convergent, there exist $m_0 \in \mathbb{N}$ such that $\sum\limits_{i=m_0}^{\infty} \sigma^i < 1$. Therefore, for all $m > m_0$ and $s \in \mathbb{N}$ it holds

$$\varepsilon > \varepsilon \sum_{i=m_0}^{\infty} \sigma^i > \varepsilon \sum_{i=m}^{m+s} \sigma^i.$$

Then,

$$\begin{aligned}
M(fx_{m+s+1}, fx_m, \varepsilon) &\geq M(fx_{m+s+1}, fx_m, \varepsilon \sum_{i=m}^{m+s} \sigma^i) \\
&\geq \underbrace{T(T(\ldots T}_{(s+1)-times}(M(fx_{m+s+1}, fx_{m+s}, \varepsilon\sigma^{m+s}) \\
&, M(fx_{m+s}, fx_{m+s-1}, \varepsilon\sigma^{m+s-1})), \ldots, M(fx_{m+1}, fx_m, \varepsilon\sigma^m)) \\
&\geq \underbrace{T(T(\ldots T}_{(s+1)-times}(M(fx_1, fx_0, \frac{\varepsilon\sigma^{m+s} - (m+s)q^{m+s}}{q^{m+s}}) \\
&, M(fx_1, fx_0, \frac{\varepsilon\sigma^{m+s-1} - (m+s-1)q^{m+s-1}}{q^{m+s-1}}), \\
&\ldots, M(fx_1, fx_0, \frac{\varepsilon\sigma^m - mq^m}{q^m})) \\
&= \underbrace{T(T(\ldots T}_{(s+1)-times}(M(fx_1, fx_0, \frac{\varepsilon}{(\frac{q}{\sigma})^{m+s}} - (m+s)) \\
&, M(fx_1, fx_0, \frac{\varepsilon}{(\frac{q}{\sigma})^{m+s-1}} - (m+s-1)), \\
&\ldots M(fx_1, fx_0, \frac{\varepsilon}{(\frac{q}{\sigma})^m} - m)) \\
&= \mathbf{T}_{i=m}^{m+s} M(fx_1, fx_0, \frac{\varepsilon}{\mu^i} - i).
\end{aligned}$$

Since $\mu \in (q, 1)$, there exist $m_1(\varepsilon) > m_0$ such that $\frac{\varepsilon}{\mu^m} - m > \frac{\varepsilon}{2\mu^m}$, for every $m > m_1(\varepsilon)$. Now, for all $s \in \mathbb{N}$ we have

$$\begin{aligned}
M(fx_{m+s+1}, fx_m, \varepsilon) &\geq \mathbf{T}_{i=m}^{m+s} M(fx_1, fx_0, \frac{\varepsilon}{2\mu^i}) \\
&\geq \mathbf{T}_{i=m}^{\infty} M(fx_1, fx_0, \frac{\varepsilon}{2\mu^i})
\end{aligned}$$

Since $\lim\limits_{m\to\infty} \mathbf{T}_{i=m}^{\infty} M(fx_1, fx_0, \frac{1}{\mu^i}) = 1$, it gives us $\lim\limits_{m\to\infty} \mathbf{T}_{i=m}^{\infty} M(fx_1, fx_0, \frac{\varepsilon}{2\mu^i}) = 1$, for all $\varepsilon > 0$. The previous implies that for all $\varepsilon > 0$, $\lambda \in (0, 1)$, there exist $m_2(\varepsilon, \lambda) > m_1(\varepsilon)$ such that $M(fx_{m+s+1}, fx_m, \varepsilon) > 1 - \lambda$, for all $m > m_2(\varepsilon, \lambda)$ and all $s \in \mathbb{N}$.

The obtained sequence $(fx_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and, since X is complete, the limit $\lim\limits_{n\to\infty} fx_n$ exists.

Therefore, in both cases (i) and (ii) there exists a subsequence $(fx_{n_k})_{k\in\mathbb{N}}$ such that

$$x = \lim_{k\to\infty} fx_{2n_k} \in A.$$

Also, from (2.2) it follows that $x = \lim\limits_{k\to\infty} fx_{2n_k+1}$.

Now, let us show that $fx \in Fx \cap Gx$. Since Fx and Gx are closed, it should be prove that $fx \in \overline{F}x \cap \overline{G}x$, i.e., that for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists

$r_1(\varepsilon,\,\lambda) \in Fx$ and $r_2(\varepsilon,\,\lambda) \in Gx$ such that

$$r_1(\varepsilon,\,\lambda) \in U(fx,\varepsilon,\,\lambda) \quad \text{and} \quad r_2(\varepsilon,\,\lambda) \in U(fx,\varepsilon,\,\lambda).$$

Since the t-norm T is continuous, we have $\sup_{x<1} T(x,\,x) = 1$ and there exists $\delta(\lambda) \in (0,\,1)$ such that

$$T(1-\delta(\lambda),\,T(1-\delta(\lambda),\,1-\delta(\lambda))) > 1-\lambda.$$

From the continuity of the mapping f and from $x = \lim_{k\to\infty} fx_{2n_k}$ follows that there exist $k_1 \in \mathbb{N}$ such that

$$M(fx, ffx_{2n_k}, \frac{\varepsilon}{3}) > 1-\delta(\lambda), \quad \text{for every } k \geq k_1.$$

Further on, condition (2.2) insures existence of $k_2 \in \mathbb{N}$ such that

$$M(ffx_{2n_k}, ffx_{2n_k+1}, \frac{\varepsilon}{3}) > 1-\delta(\lambda), \quad \text{for every } k \geq k_2.$$

Let $\delta = \frac{\varepsilon}{12}$. Since F is weakly commuting with f then

$$ffx_{2n_k+1} \in f(Fx_{2n_k}) \subseteq F(fx_{2n_k}),$$

and, due to (2.1), there exist $r_2(\varepsilon,\,\lambda) \in Gx$ such that

$$\begin{aligned} M(ffx_{2n_k+1}, r_2(\varepsilon,\,\lambda), \frac{\varepsilon}{3}) &\geq M(ffx_{2n_k}, fx, \frac{\varepsilon}{4q}) \\ &> 1-\delta(\lambda). \end{aligned}$$

Then

$$\begin{aligned} M(fx, r_2(\varepsilon,\lambda), \varepsilon) \geq\; & T(M(fx, ffx_{2n_k}, \frac{\varepsilon}{3}), T(M(ffx_{2n_k}, ffx_{2n_k+1}, \frac{\varepsilon}{3}), \\ & M(ffx_{2n_k+1}, r_2(\varepsilon,\,\lambda), \frac{\varepsilon}{3}))) > 1-\lambda. \end{aligned}$$

Now we have $r_2(\varepsilon,\,\lambda) \in U(fx,\varepsilon,\,\lambda)$.
Similarly we can prove that $r_1(\varepsilon,\,\lambda) \in Fx \cap U(fx,\varepsilon,\,\lambda)$.

Example 2.7 *Let (X, M, T) be a fuzzy metric space as in Example 2.4. Let $A = [\frac{1}{2}, 2]$ and for all $x \in A$.*

(a) Let $f(x) = 2$, $F(x) = \{1, 2\}$ and $G(x) = \{\frac{1}{x}, 1, 2\}$. Since

$$f(Fx) = \{2\} \subset F(fx) = \{1, 2\} \quad \textit{and} \quad f(Gx) = \{2\} \subset G(fx) = \{\frac{1}{2}, 1, 2\}$$

mappings F and G weakly commuting with f. Also, mappings F and G are f-strongly demicompact, for every $u, v \in A$ condition (2.1) is satisfied and the claim from the previous theorem holds.

(b) Let $f(x) = x$, $F(x) = \{1\}$ and $G(x) = \{1, x\}$. Then F and G are weakly commuting with f, F is f-strongly demicompact, for every $u, v \in A$ condition (2.1) *is satisfied and the claim from the previous theorem holds.*

Remark 2.8 *From Proposition 1.5 follows that the Theorem 2.6 holds if condition (ii) is substituted with the assumption "t-norm T is of H-type".*

Corollary 2.9 *Let (X, M, T^*) be a complete fuzzy metric space and A nonempty and closed subset of X. Let $f : A \to A$ be a continuous mapping and $F, G : A \to C(A)$ such that for $q \in (0, 1)$ the following holds:*

For every $u,\, v \in A,\, x \in Fu$ *and* $\delta > 0$, *there exist* $y \in Gv$ *such that for all* $\varepsilon > \delta$

$$M(x,\, y, \varepsilon) \geq M(fu,\, fv, \frac{\varepsilon - \delta}{q}).$$

If F *and* G *are weakly commuting with* f *and the following is satisfied*

(i) F *or* G *are* f*-strongly demicompact*

or

(ii-a) for $T^* = T^D_\lambda$ *or* $T^* = T^{AA}_\lambda$, $\lambda > 0$, *there exists* $x_0,\, x_1 \in A$, $fx_1 \in Fx_0$ *and* $\mu \in (q,\, 1)$

such that

$$\sum_{i=1}^{\infty}(1 - M(fx_0, fx_1, \frac{1}{\mu^i}))^\lambda < \infty;$$

(ii-b) for $T^* = T^{SW}_\lambda$, $\lambda > -1$, *there exists* $x_0,\, x_1 \in A$, $fx_1 \in Fx_0$ *and* $\mu \in (q,\, 1)$ *such that*

$$\sum_{i=1}^{\infty}(1 - M(fx_0, fx_1, \frac{1}{\mu^i})) < \infty.$$

Then, there exist $x \in A$ *such that* $fx \in Fx \cap Gx$.

Proof. Follows directly from equivalences (1.1) and (1.2).

2.2. **Implicit relation.** Let Φ be the set of all continuous functions $\phi : [0,1]^6 \to \mathbb{R}$ such that

(ϕ_1) $\phi(t_1, t_2, t_3, t_4, t_5, t_6)$ is non-increasing in t_6.

(ϕ_2) $\phi(u, v, v, u, 1, T(u, v)) \geq 0$ imply $u \geq v$.

(ϕ_3) $\phi(u, 1, 1, u, 1, u) \geq 0$ imply $u \geq 1$.

Example 2.10 *Let* $T = T_P$ *and* $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 t_2 t_4 - t_3 t_5 t_6$.

Theorem 2.11 *Let* (X, M, T) *be a complete fuzzy metric space and* A *nonempty and closed subset of* X. *Let* $f : A \to A$ *a continuous function and let* $F, G : X \to C(A)$ *be a multi-valued functions such that for some* $\phi \in \Phi$ *there exists a constant* $k \in (0, 1)$ *such that for all* $u, v \in A$ *and every* $t > 0$, $x \in Fu$ *there exists* $y \in Gv$ *such that*

$$\phi\Big(M(x, y, kt), M(fu, fv, t), M(x, fu, t),$$
$$M(y, fv, kt), M(x, fv, t), M(y, fu, (k+1)t)\Big) \geq 0 \qquad (2.3)$$

If F *and* G *are weakly commuting with* f *and the following is satisfied*

(i) F *or* G *are* f*-strongly demicompact*

or

(ii) there exists $x_0, x_1 \in A$ *such that for* $fx_1 \in Fx_0$ *and* $\mu \in (k, 1)$ *the following holds:*

$$\lim_{n\to\infty} \mathbf{T}^{\infty}_{i=n} M(fx_0, fx_1, \frac{1}{\mu^i}) = 1.$$

Then there exists $x \in A$ *such that* $fx \in Fx \cap Gx$.

Proof. Let x_0 and $x_1 \in A$ such that $fx_1 \in Fx_0$. From (2.3) for $x = fx_1$, $u = x_0$, $v = x_1$ it follows that there exist $x_2 \in A$ such that $fx_2 \in Gx_1$ and

$$\phi(M(fx_1, fx_2, kt), M(fx_0, fx_1, t), M(fx_1, fx_0, t),$$
$$M(fx_2, fx_1, kt), M(fx_1, fx_1, t), M(fx_2, fx_0, (k+1)t)) \geq 0.$$

Since $M(fx_2, fx_0, (k+1)t) \geq T(M(fx_2, fx_1, kt), M(fx_1, fx_0, t))$, by applying (ϕ_1) and (ϕ_2), it is obtained

$$\begin{aligned}&\phi(M(fx_1, fx_2, kt), M(fx_0, fx_1, t), M(fx_1, fx_0, t),\\ &M(fx_2, fx_1, kt), M(fx_1, fx_1, t), T(M(fx_2, fx_1, kt), M(fx_1, fx_0, t))) \geq 0\end{aligned}$$

and $M(fx_1, fx_2, kt) \geq M(fx_0, fx_1, t)$, i.e., $M(fx_1, fx_2, t) \geq M(fx_0, fx_1, \frac{t}{k})$. Continuing with the previous procedure, we can obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ from A such that

(a) $fx_{2n+1} \in Fx_{2n}$ and $fx_{2n+2} \in Gx_{2n+1}$
(b) $M(fx_n, fx_{n+1}, t) \geq M(fx_{n-1}, fx_n, \frac{t}{k}) \geq \cdots \geq M(fx_1, fx_0, \frac{t}{k^n})$, i.e.,

$$\lim_{n\to\infty} M(fx_n, fx_{n+1}, t) = 1.$$

If F is f-strongly demicompact, from $\lim\limits_{n\to\infty} M(fx_{2n}, fx_{2n+1}, t) = 1$ and $fx_{2n+1} \in Fx_{2n}$ follows existence of a convergent subsequence $(fx_{2n_p})_{p\in\mathbb{N}}$ of a sequence $(fx_{2n})_{n\in\mathbb{N}}$.

It remains to be proved that the sequence $\{fx_n\}$ is a convergent if the t-norm T satisfies condition (ii). Let $\sigma = \frac{k}{\mu}$. Since $0 < \sigma < 1$ the series $\sum_{i=1}^{\infty} \sigma^i$ is convergent and there exists $n_0 \in \mathbb{N}$ such that $\sum_{i=n_0}^{\infty} \sigma^i < 1$. Hence, for $n > n_0$ and all $m \in \mathbb{N}$, it holds

$$t > t\sum_{i=n_0}^{\infty} \sigma^i > t\sum_{i=n}^{n+m-1} \sigma^i.$$

Then we have,

$$\begin{aligned}
&M(fx_{n+m}, fx_n, t) \geq M(fx_{n+m}, fx_n, t\sum_{i=n}^{n+m-1} \sigma^i) \geq \\
\geq\ & \underbrace{T(T(\dots T}_{m-times}(M(fx_{n+m}, fx_{n+m-1}, t\sigma^{n+m-1}), \dots, M(fx_{n+1}, fx_n, t\sigma^n))) \\
\geq\ & \underbrace{T(T(\dots T}_{m-times}(M(fx_0, fx_1\frac{t\sigma^{n+m-1}}{k^{n+m-1}}), \dots, M(fx_0, fx_1, \frac{t\sigma^n}{k^n}))) \\
=\ & \underbrace{T(T(\dots T}_{m-times}(M(fx_0, fx_1\frac{t}{\mu^{n+m-1}}), \dots, M(fx_0, fx_1, \frac{t}{\mu^n}))) \\
=\ & \mathbf{T}_{i=n}^{n+m-1} M(fx_0, fx_1, \frac{t}{\mu^i}) \geq \mathbf{T}_{i=n}^{\infty} M(fx_0, fx_1, \frac{t}{\mu^i}).
\end{aligned}$$

The limit $\lim\limits_{n\to\infty} \mathbf{T}_{i=n}^{\infty} M(fx_0, fx_1, \frac{1}{\mu^i}) = 1$ implies $\lim\limits_{n\to\infty} \mathbf{T}_{i=n}^{\infty} M(fx_0, fx_1, \frac{t}{\mu^i}) = 1$ for all $t > 0$. Now, for every $t > 0$ and every $\lambda \in (0,1)$ there exist $n_1(t,\lambda)$ such that $M(fx_{n+m}, fx_n, t) > 1 - \lambda$ for every $n \geq n_1(t,\lambda)$ and every $m \in \mathbb{N}$. Therefore, the sequence $\{fx_n\}$ is a Cauchy sequence and the completeness of X insures that there exists $x \in A$ such that $\lim\limits_{n\to\infty} fx_n = x$. Now, for both cases there exists a subsequence $(fx_{n_p})_{p\in\mathbb{N}}$ such that $x = \lim\limits_{p\to\infty} fx_{2n_p} \in A$.

The remaining step is to prove that $fx \in Fx \cap Gx$. Since Fx and Gx are closed, it should be proved that$fx \in \overline{F}x \cap \overline{G}x$, i.e., that for all $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exist $r_1(\varepsilon, \lambda) \in Fx$ and $r_2(\varepsilon, \lambda) \in Gx$ such that

$$r_1(\varepsilon, \lambda) \in U(fx, \varepsilon, \lambda) \text{ and } r_2(\varepsilon, \lambda) \in U(fx, \varepsilon, \lambda).$$

From the continuity of the mapping f and $x = \lim_{p\to\infty} fx_{2n_p}$ it follows that

$$fx = \lim_{p\to\infty} ffx_{2n_p} = \lim_{p\to\infty} ffx_{2n_{p+1}}.$$

Since F is weakly commuting with f we have $ffx_{2n_{p+1}} \in f(Fx_{2n_p}) \subseteq F(fx_{2n_p})$, and by applying (2.3) on $x = ffx_{2n_{p+1}}$, $u = fx_{2n_p}$, existence of $r_2(\varepsilon, \lambda) \in Gx$ such that

$$\begin{aligned}&\phi(M(ffx_{2n_{p+1}}, r_2(\varepsilon,\lambda), kt), M(ffx_{2n_p}, fx, t), M(ffx_{2n_{p+1}}, ffx_{2n_p}, t),\\ &M(r_2(\varepsilon,\lambda), fx, kt), M(ffx_{2n_{p+1}}, fx, t), \mathcal{M}(r_2(\varepsilon,\lambda), ffx_{2n_p}, (k+1)t))) \geq 0\end{aligned}$$

is obtained. Now, (ϕ_1) insures

$$\begin{aligned}&\phi\Big(M(ffx_{2n_{p+1}}, r_2(\varepsilon,\lambda), kt), M(ffx_{2n_p}, fx, t),\\ &M(ffx_{2n_{p+1}}, ffx_{2n_p}, t), M(r_2(\varepsilon,\lambda), fx, kt), M(ffx_{2n_{p+1}}, fx, t),\\ &T(M(r_2(\varepsilon,\lambda), fx, kt), M(fx, ffx_{2n_p}, t)))\Big) \geq 0.\end{aligned}$$

Letting $p \to \infty$ the following is obtained

$$\begin{aligned}&\phi\Big(M(fx, r_2(\varepsilon,\lambda), kt), M(fx, fx, t), M(fx, fx, t),\\ &M(r_2(\varepsilon,\lambda), fx, kt), M(fx, fx, t), T(M(r_2(\varepsilon,\lambda), fx, kt), M(fx, fx, t)))\Big) \geq 0,\end{aligned}$$

i.e., $\varphi(M(fx, r(\varepsilon,\lambda), kt), 1, 1, M(r_2(\varepsilon,\lambda), fx, kt), 1, M(r_2(\varepsilon,\lambda), fx, kt)) \geq 0$, and from (ϕ_3) follows $M(fx, r_2(\varepsilon,\lambda), kt) \geq 1$. Therefore $r_2(\varepsilon, \lambda) \in U(fx, \varepsilon, \lambda)$. Similarly it can prove that $r_1(\varepsilon, \lambda) \in Fx \cap U(fx, \varepsilon, \lambda)$.

Example 2.12 *Let X, M, T, A, f, F and G be as in Example 2.7 and let $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - t_2$. Then, all conditions of the previous theorem are satisfied and the claim holds.*

Remark 2.13 *Again the second condition can be substituted with assumption "t-norm T is of H-type". Also, analogously to the Corollary 2.9, the previous theorem can be reformulated for a t-norm from Dombi, Aczél-Alsina or Sugeno-Weber class.*

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