Fixed Point Theory, 15(2014), No. 1, 59-66
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

PICARD OPERATORS ON ORDERED METRIC SPACES

M. DERAFSHPOUR AND SH. REZAPOUR

Department of Mathematics
Azarbaijan Shahid Madani University
Azarshahr, Tabriz, Iran
E-mail: sh.rezapour@azaruniv.edu, rezapourshahram@yahoo.ca

Abstract. In this paper, we shall give some results about Picard operators on ordered metric spaces. In fact, we shall prove that some contractive-like mappings satisfying some conditions on ordered metric spaces are Picard operators. We shall also present an application of our results.
Key Words and Phrases: Fixed point, Picard operator, orbitally continuous.
2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

As we know, there are many papers on fixed points of contractive mappings which introduced in 1962 ([7, 18]). In 1997, Runge introduced the notion of Picard modular forms [22]. By using this notion, Weikard introduced the notion of Picard operators in 1998 [27]. Also, Rus and Muresan reviewed data dependence of the fixed points set of weakly Picard operators in 1998 [23]. Later, Rus provided some results about fiber Picard operators [24]. In 2003 by using a distinct view, Rus defined the concept of Picard operators ([25]) and we use the notion of Picard operators in the sense of Rus. For applications of the Picard operators technique see [26]. There are many works on fixed point theory in partially ordered metric spaces (for example, [1, 2, 3, 4, 5, 6, 8, 9, 12, 13, 14, 15, 16, 17, 19, 20]). Note that, not only a contractive map in ordered metric spaces is not continuous necessarily but also it is not a contraction map necessarily ([11, 21]). In this paper, we shall give some results about Picard operators on ordered metric spaces. In fact, we shall prove that some contractive-like mappings satisfying some conditions on ordered metric spaces are Picard operators.

Let $T : X \longrightarrow X$ be an operator. We denote the set of all non-empty invariant subsets by $I(T)$, that is $I(T) = \{Y \subset X | T(Y) \subseteq Y\}$. Also, we denote the fixed point set of T by $F_T = \{x \in X : x = T(x)\}$. Let $(X, \leqslant)$ be a partially ordered set, that is X is a nonempty set and $\leqslant$ is a reflexive, transitive and anti-symmetric relation on X. Denote the set of comparable elements of X by $X_{\leqslant}$. If $x, y \in X$ with $x \leqslant y$, then by $[x, y]_{\leqslant}$ we shall denote the ordered segment joining x and y. For a mapping $T : X \to X$, we denote the lower fixed point set of T by $(LF)_T := \{x \in X | x \leqslant T(x)\}$ while we denote the upper fixed point set of T by $(UF)_T := \{x \in X | x \geqslant T(x)\}$. Also, for the mappings $T : X \to X$ and $S : Y \to Y$, the cartesian product of T and S is denoted by $T \times S : X \times Y \to X \times Y$ and defined by $(T \times S)(x, y) = (T(x), S(x))$. We appeal next well-known relation in the following.

$(*)$ If $x_n \to x$, $z_n \to x$ and $x_n \leqslant y_n \leqslant z_n$ for all n, then $y_n \to x$.

In the literature, an ordered metric space is a metric space endowed with an order that, in addition, satisfy the compatibility condition $(*)$. In this paper, we use only the terminology ordered metric space and we denote it by $(X, d, \leq)$.

Here, we recall the notion of Picard operators. Let $(X, d, \leq)$ be an ordered metric space. An operator $T : X \to X$ is called a Picard operator (briefly PO) whenever $F_T = \{x^*\}$ and $(T^n(x))_{n\geq 1} \to x^*$ for all $x \in X$. Also, we say that a selfmap $T : X \to X$ is orbitally continuous whenever for each $x \in X$ and sequence $\{n(i)\}_{i\geq 1}$ with $T^{n(i)}x \to a$ for some $a \in X$, we have $T^{n(i)+1}x \to Ta$. Here, $T^{m+1} = T(T^m)$.

Let $\mathcal{S}$ denote the class of functions $\beta : [0, \infty) \to [0, 1)$ which satisfy the condition $\beta(t_n) \to 1$ implies that $t_n \to 0$. An altering function is a non-decreasing continuous function $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(t) = 0$ if and only if $t = 0$.

2. Main results

Now, we are ready to state and prove our main results.

Theorem 2.1. *Let $(X, d, \leqslant)$ be an ordered metric space and T an operator. Suppose that*

(i) for each $x, y \in X$ with $(x, y) \notin X_\leqslant$ there exists $z \in X$ such that $(x, z) \in X_\leqslant$ and $(y, z) \in X_\leqslant$;

(ii) $X_\leqslant \in I(T \times T)$;

(iii) if $(x, y) \in X_\leqslant$ and $(y, z) \in X_\leqslant$, then $(x, z) \in X_\leqslant$;

(iv) there exists $x_0 \in X$ such that $(x_0, T(x_0)) \in X_\leqslant$;

(v) T is orbitally continuous

(vi) there exists $\beta \in \mathcal{S}$ such that $d(Tx, Ty) \leqslant \beta(d(x, y))d(x, y)$ for all $(x, y) \in X_\leqslant$;

(vii) the metric d is complete.

Then T is a **PO**.

Proof. Choose $x_0 \in X$ such that $(x_0, T(x_0)) \in X_\leqslant$. Suppose first that $x_0 \neq T(x_0)$. By using (ii), $(T^n(x_0), T^{n+1}(x_0)) \in X_\leqslant$ for all $n \geq 1$. Put $x_{n+1} = T(x_n)$. Since $\beta \in S$ and $(x_n, x_{n+1}) \in X_\leqslant$ for all $n \geq 1$, by using (vi) we get

$$d(x_{n+1}, x_n) \leqslant \beta(d(x_n, x_{n-1}))d(x_n, x_{n-1}) \leqslant d(x_n, x_{n-1}),$$

that is, for each $n \geq 1$ we have

$$d(x_{n+1}, x_n) \leqslant d(x_n, x_{n-1}). \tag{1}$$

If there exists a natural number n_0 such that $d(x_{n_0}, x_{n_0-1}) = 0$, then

$$x_{n_0} = T(x_{n_0-1}) = x_{n_0-1}$$

and so x_{n_0-1} is a fixed point of T. Suppose that $d(x_{n+1}, x_n) \neq 0$ for all $n \geq 1$. Then taking into account (1), the sequence $\{d(x_{n+1}, x_n)\}$ is decreasing and bounded below, so we can suppose that $\lim_{n\to\infty} d(x_{n+1}, x_n) = r \geqslant 0$. Assume $r > 0$. Then, we have

$$\frac{d(x_{n+1}, x_n)}{d(x_n, x_{n-1})} \leqslant \beta(d(x_n, x_{n-1})) \leqslant 1.$$

Letting $n \to \infty$ in the last inequality, we get $1 \leqslant \lim_{n\to\infty} \beta(d(x_n, x_{n-1})) \leqslant 1$ and so $\lim_{n\to\infty} \beta(d(x_n, x_{n-1})) = 1$. Since $\beta \in S$, $\lim_{n\to\infty}(d(x_{n+1}, x_n)) = 0$ which is a contradiction. Hence,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$d(x_{n(k)}, x_{m(k)}) \geqslant \varepsilon. \tag{3}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (3). Thus,

$$d(x_{n(k)-1}, x_{m(k)}) < \varepsilon. \tag{4}$$

Now, by using (3), (4) and triangular inequality, we get

$$\varepsilon \leqslant d(x_{n(k)}, x_{m(k)}) \leqslant d(x_{n(k)-1}, x_{m(k)}) + d(x_{n(k)-1}, x_{m(k)}) < d(x_{n(k)}, x_{n(k)-1}) + \varepsilon.$$

If $k \to \infty$, then by using (2) we get

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{5}$$

Again, the triangular inequality gives us

$$d(x_{n(k)}, x_{m(k)}) \leqslant d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}),$$

$$d(x_{n(k)}, x_{m(k)-1}) \leqslant d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)-1}, x_{m(k)}).$$

If $k \to \infty$, then by using (2) and (5) and above inequalities, we obtain

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon. \tag{6}$$

Since $n(k) > m(k)$ and $(x_{n(k)-1}, x_{m(k)-1}) \in X_{\leqslant}$, we have

$$\begin{aligned} d(x_{n(k)}, x_{m(k)}) &= d(T(x_{n(k)-1}), T(x_{m(k)-1})) \\ &\leqslant \beta(d(x_{n(k)-1}, x_{m(k)-1}))d(x_{n(k)-1}, x_{m(k)-1}) \leqslant d(x_{n(k)-1}, x_{m(k)-1}). \end{aligned} \tag{7}$$

If $k \to \infty$ in (7), then by using (5) and (6), we get

$$\lim_{k\to\infty} \beta(d(x_{n(k)-1}, x_{m(k)-1})) = 1.$$

Since $\beta \in S$,

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = 0.$$

The relation (6) shows that this is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

Since T is orbitally continuous, x^* is a fixed point of T. By using (vi), it is easy to see that x^* is unique. Now, let $x \in X$ be given. Then we have the following cases:

(a) If $(x, x_0) \in X_\leqslant$, then $(T^n(x), T^n(x_0)) \in X_\leqslant$ and so by using (vi) we get that

$$u_n = d(T^n(x), T^n(x_0))$$

is a non-negative decreasing sequence. Thus, there exists $u \geq 0$ such that $u_n \to u$. If $u = 0$, then $T^n(x) \to x^*$ because $T^n(x_0) = x_n \to x^*$. Let $u \neq 0$. In this case, by using (vi) for each $n \geq 1$ we obtain

$$d(T^n(x), T^n(x_0)) \leqslant \beta(d(T^{n-1}(x), x_{n-1}))d(T^{n-1}(x), x_{n-1}).$$

Therefore,

$$\liminf_{n\to\infty} \beta(d(T^{n-1}(x), x_{n-1})) = \limsup_{n\to\infty} \beta(d(T^{n-1}(x), x_{n-1})) = 1.$$

Hence,

$$\lim_{n\to\infty} d(T^{n-1}(x), x_{n-1}) = \lim_{n\to\infty} d(T^{n-1}(x), x^*) = 0$$

because $\beta \in \mathcal{S}$. Thus, $T^n(x) \to x^*$.

(b) If $(x, x_0) \notin X_\leqslant$, then by using (i) there exists $z_0 \in X_\leqslant$ such that $(x, z_0) \in X_\leqslant$ and $(z_0, x_0) \in X_\leqslant$. By using the part (a) we know that $T^n(z_0) \to x^*$. Now, put $z_{n+1} = Tz_n$ for all $n \geq 0$. Since $(x, z_0) \in X_\leqslant$, $(T^n(x), T^n(z_0)) \in X_\leqslant$ for all $n \geq 1$. Thus by using (ii) we get $w_n = d(T^n(x), T^n(z_0)) \leqslant d(T^{n-1}(x), T^{n-1}(z_0)) = w_{n-1}$ for all $n \geq 1$. Therefore, $\{w_n\}_{n\in\mathbb{N}}$ is a non-increasing and non-negative sequence. Hence, there exists $w \geq 0$ such that $w_n \to w$. If $w = 0$, then $T^n(x) \to x^*$. Let $w \neq 0$. In this case, by using (v) for each $n \geq 1$ we obtain

$$d(T^n(x), T^n(z_0)) = d(T(T^{n-1}(x)), T(z_{n-1})) \leqslant \beta(d(T^{n-1}(x), z_{n-1}))d(T^{n-1}(x), z_{n-1}).$$

Hence,

$$\lim_{n\to\infty} \beta(d(T^{n-1}(x), z_{n-1})) = 1.$$

Thus,

$$\lim_{n\to\infty} d(T^{n-1}(x), z_{n-1}) = 0$$

because $\beta \in \mathcal{S}$. Since $T^n(z_0) \to x^*$, $T^n(x) \to x^*$. □

Now by using Theorem 7 in [10], we can replace the following conditions instead the condition (vi) of Theorem 2.1. A similar cases hold for another results of this paper.

(a)- There exists a continuous function $\eta : [0, \infty) \to [0, \infty)$ such that $\eta^{-1}(\{0\}) = \{0\}$ and $d(Tx, Ty) \leqslant d(x, y) - \eta(d(x, y))$ holds for all $(x, y) \in X_\leqslant$.

(b)- There exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) < t$ for all $t > 0$ and $d(Tx, Ty) \leqslant \varphi(d(x, y))$ holds for all $(x, y) \in X_\leqslant$.

(c)- There exist a continuous and nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ with $\psi^{-1}(\{0\}) = \{0\}$ and a nondecreasing, right continuous function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) < t$ for all $t > 0$ and $\psi(d(Tx, Ty)) \leqslant \varphi(\psi(d(x, y)))$ holds for all $(x, y) \in X_\leqslant$.

(d)- There exist continuous and nondecreasing functions $\mu, \nu : [0, \infty) \to [0, \infty)$ with $\mu^{-1}(\{0\}) = \{0\}$, $\nu^{-1}(\{0\}) = \{0\}$ and $\mu(d(Tx, Ty)) \leqslant \mu(d(x, y)) - \nu(d(x, y))$ holds for all $(x, y) \in X_{\leqslant}$.

Remark 2.1. *A new theorem can be obtained replacing condition (vi) in Theorem 2.1 with the following condition:*
there exists a $\beta \in \mathcal{S}$ such that $\psi(d(Tx, Ty)) \leqslant \beta(d(x, y))\psi(d(x, y))$ for all $(x, y) \in X_{\leqslant}$, where ψ is an altering function.

Remark 2.2. *A new theorem can be obtained replacing condition (vi) in Theorem 2.1 with the following condition:*
there exists a $\beta \in \mathcal{S}$ such that $\psi(d(Tx, Ty)) \leqslant \beta(d(x, y))\psi(d(x, y))$ for all $(x, y) \in X_{\leqslant}$, where ψ is an altering function.

Remark 2.3. *A new theorem can be obtained replacing condition (vi) in Theorem 2.1 with the following condition:*
there exists $\beta \in \mathcal{S}$ such that

$$d(Tx, Ty) \leqslant \beta(\max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\})d(x, y),$$

for all $(x, y) \in X_{\leqslant}$.

3. An Application

In this section, we present an application of our abstract results. We will study the existence of solution for the following first-order periodic problem

$$\begin{cases} u'(t) = f(t, u(t)), & t \in [0, T] \\ u(0) = u(T), \end{cases} \tag{8}$$

where $T > 0$ and $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Consider the complete metric space $\mathcal{C}(I)$ $(I = [0, T])$ via the sup norm. The space $\mathcal{C}(I)$ can be equipped with the partial order $x \leq y$ whenever $x(t) \leq y(t)$ for all $t \in I$. It's easy to see that for each $x, y \in \mathcal{C}(I)$ there exists a lower bound $(\min\{x, y\})$ and an upper bound $(\max\{x, y\})$. Suppose that $\mathcal{A}$ denotes the class of functions $\phi : [0, \infty) \to [0, \infty)$ satisfying

(i) ϕ is nondecreasing,

(ii) $\phi(x) < x$ for $x > 0$,

(iii) $\beta(x) = \frac{\phi(x)}{x} \in \mathcal{S}$.

In fact,

$$\phi(t) = \mu \cdot t \ (0 \leq \mu < 1), \ \phi(t) = \frac{t}{1 + t}$$

and $\phi(t) = \ln(1 + t)$ are some examples of such functions. Recall now the following definition.

Definition 3.1. *A lower solution for (8) is a function $\alpha \in \mathcal{C}^1(I)$ such that*

$$\begin{cases} \alpha'(t) \leq f(t, \alpha(t)), & (t \in I) \\ \alpha(0) \leq \alpha(T). \end{cases}$$

Similarly $\alpha \in \mathcal{C}^1(I)$ is an upper solution for (8) whenever

$$\begin{cases} \alpha'(t) \geq f(t, \alpha(t)), & (t \in I) \\ \alpha(0) \geq \alpha(T). \end{cases}$$

Now, we present the following theorem about the existence of a solution for the problem (8) in presence of a lower or upper solution. The existence of a solution has been proved only for lower solution phase ([5]).

Theorem 3.1. *Consider the problem (8) with a continuous function $f : I \times \mathbb{R} \to \mathbb{R}$. Suppose that there exist numbers λ, α such that $\alpha \leq (\frac{2\lambda(e^{\lambda t}-1)}{T(e^{\lambda t}+1)})^{\frac{1}{2}}$ and for each $x, y \in \mathbb{R}$ we have $f(t, y) + \lambda y - [f(t, x) + \lambda x] \leq \alpha\sqrt{|y - x|\phi(y - x)}$, where $\phi \in \mathcal{A}$. Then the existence of a lower or upper solution for (8) provides the existence of a unique solution for (8).*

Proof. The problem (8) can be rewrite as

$$\begin{cases} u'(t) + \lambda u(t) = f(t, u(t)) + \lambda u(t), & (t \in [0, T]) \\ u(0) = u(T). \end{cases}$$

This problem is equivalent to the integral equation

$$u(t) = \int_0^T G(t, s)[f(s, u(s)) + \lambda u(s)]ds,$$

where $G(t, s)$ is a green function given by

$$G(t, s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{(e^{\lambda T}-1)}, & 0 \leq s < t \leq T \\ \frac{e^{\lambda(s-t)}}{(e^{\lambda T}-1)}. & 0 \leq t < s \leq T \end{cases}$$

Define $F : \mathcal{C}(I) \to \mathcal{C}(I)$ by

$$F(u)(t) = \int_0^T G(t, s)[f(s, u(s)) + \lambda u(s)]ds.$$

If $u \in \mathcal{C}(I)$ is a fixed point of F, then $u \in \mathcal{C}^1(I)$ is a solution for (8). We check that F satisfies the conditions of Proposition 2.1. It has been proved that for $(u, v) \in \mathcal{C}(I)_{\leq}$ we have ([5])

$$d(Fu, Fv)^2 \leq \frac{\phi(d(u, v))}{d(u, v)} \cdot d(u, v)^2.$$

Define

$$\psi(x) = x^2 \text{ and } \beta = \frac{\phi(x)}{x}.$$

Since $\phi \in \mathcal{A}$, $\beta \in \mathcal{S}$. Also, note that ψ is an altering function. Thus,

$$\psi(d(Fu, Fv)) \leq \beta(d(u, v))\dot{\psi}(d(u, v))$$

for all $(u, v) \in \mathcal{C}(I)_{\leq}$. It is easy to see that $\mathcal{C}(I)_{\leq} \in I(F \times F)$. Also, there exists $x_0 \in \mathcal{C}(I)$ such that $(x_0, F(x_0)) \in \mathcal{C}(I)_{\leq}$. In fact if $\alpha(t)$ be a lower solution for (8), from [4] we know that $\alpha(t) \leq (F\alpha)(t)$ for all $t \in I$. Similarly, If $\alpha(t)$ is an upper solution for (8), then we have $\alpha(t) \geq (F\alpha)(t)$, for all $t \in I$. Therefore, F satisfies the conditions of Proposition 2.1. Thus, F is a Picard operator and so the problem (8) has a unique solution. □

Acknowledgment. The authors express their gratitude to the referees for their helpful suggestions which improved final version of this paper.

References

[1] R.P. Agarwal, M.A. El-Gebiely, D. O'Regan, *Generalized contractions in partially ordered metric spaces*, Appl. Analysis, **87**(2008), 109-116.

[2] A. Amini-Harandi, H. Emami, *A fixed point theorem for contractions in partially ordered metric spaces and application to ordinary differantial equations*, Nonlinear Anal., **72**(2010), 2238–2242.

[3] T.G. Bhashkar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65**(2006), 1379–1393.

[4] D. Burgec, S. Kalabusic, M.R.S. Kulanovic, *Global attractivity results for mixed monotone mappings in partially ordered copmlete metric spaces*, Fixed Point Theory and Appl., 2009, Article ID 762478.

[5] J. Caballero, J. Harjani, K. Sadarngani, *Contractive-like mapping principles in ordered metric spaces and application to ordinary differential equations*, Fixed Point Theory and Appl., 2010, Article ID 916064.

[6] L. Ciric, N. Cakid, M. Rjovi, J.S. Ume, *Monotone generalized nonlinear contractions in partially ordered metric spaces*, Fixed Point Theory and Appl., 2008, Article ID 131294.

[7] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc., **37**(1962), 74–79.

[8] J. Harjani, K. Sadarangani, *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Anal., **71**(2009), 3403–3410.

[9] J. Harjani, K. Sadarangani, *Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations*, Nonlinear Anal., **72**(2010), 1188–1197.

[10] J. Jachymski, *Equivalent conditions for generalized contractions on (ordered) metric spaces*, Nonlinear Analysis, **74**(2011), 768–774.

[11] Z. Kadelburg, M. Pavlovic, S. Radenovic, *Common fixed point theorems for ordered contractions and quasi-contractions in ordered cone metric spaces*, Comput. Math. Appl., **59**(2010), No. 9, 3148–3159.

[12] V. Lakshmikantham, L. Ciric, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., **70**(2009), 4341–4349.

[13] J.J. Nieto, R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22**(2005), 223–239.

[14] J.J. Nieto, R.L. Pous, R. Rodriguez-Lopez, *Fixed point theorems in ordered abstract spaces*, Proc. Amer. Math. Soc., **135**(2007), 2505–2517.

[15] J.J. Nieto, R. Rodriguez-Lopez, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sinica, **23**(2007), 2205–2212.

[16] D. O'Regan, A. Petrusel, *Fixed point theorems for generalized contractions on ordered metric spaces*, J. Math. Anal. Appl., **341**(2008), 1241–1252.

[17] A. Petrusel, I.A. Rus, *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc., **134**(2006), 411–418.

[18] E. Rakotch, *A note on contractive mappings*, Proc. Amer. Math. Soc., **13**(1962), 459–465.

[19] A.C.M. Ran, M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132**(2004), 1435–1443.

[20] Sh. Rezapour, P. Amiri, *Fixed point of multivalued operators on ordered generalized metric spaces*, Fixed Point Theory, **13**(2012), 173–179.

[21] Sh. Rezapour, M. Derafshpour, N. Shahzad, *Best proximity points of cyclic φ-contractions in ordered metric spaces*, Topol. Methods Nonlinear Analysis, **37**(2011), 193–202.

[22] B. Runge, *On Picard Modular Forms*, Math. Nachr., **184**(1997), 259–273.

[23] I.A. Rus, S. Muresan, *Data dependence of the fixed points set of weakly Picard operators*, Studia Univ. Babes-Bolyai Math., **43**(1998), No. 1, 79–83.

[24] I.A. Rus, *Fiber Picard operators theorem and applications*, Studia Univ. Babes-Bolyai Math., **44**(1999), No. 3, 89–97.

[25] I.A. Rus, *Picard operators and applications*, Sci. Math. Jpn., **58**(2003), 191–219.

[26] I.A. Rus, *Some nonlinear functional differential and integral equations via weakly Picard operator theory: A survey*, Carpathian J. Math., **26**(2010), 230–258.

[27] R. Weikard, *Picard operators*, Math. Nachr., **195**(1998), 251–266.

Received: May 3, 2012; Accepted: June 21, 2012.