Fixed Point Theory, 15(2014), No. 1, 43-58
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

EXISTENCE THEOREM FOR A FRACTIONAL MULTI-POINT BOUNDARY VALUE PROBLEM

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Abstract. This paper investigates the existence of solutions for a fractional multi-point boundary value problem at resonance by applying the coincidence degree theory. An example is given to illustrate our main result.

Key Words and Phrases: Fractional differential equations, multi-point boundary value problem, existence, coincidence degree.
2010 Mathematics Subject Classification: 26A33, 34B10, 47H11.

1. Introduction

Consider the boundary value problem (BVP for short) of the following fractional differential equation

$$\begin{cases} {}^C D^\alpha x(t) = f(t, x(t)) + e(t), \quad t \in [0,1],\ 0 < \alpha < 1, \\ \sum\limits_{i=1}^{m} A_i x(\xi_i) = 0, \end{cases} \tag{1.1}$$

where ${}^C D^\alpha$ is the Caputo fractional derivative with $0 < \alpha < 1$, $f : [0,1] \times R^n \to R^n$ and $e : [0,1] \to R^n$ are given functions satisfying some assumptions that will be specified later, A_i $(i = 1, 2, \cdots, m)$ are constant square matrices of order n, $0 \le \xi_1 < \xi_2 < \cdots < \xi_m \le 1$.

We say that BVP (1.1) is a problem at resonance, if the linear equation,

$${}^C D^\alpha x(t) = 0, \quad t \in [0,1],\ 0 < \alpha < 1,$$

This work was supported by the Natural Science Foundation of China(10971173), the National Natural Science Foundation of Hunan Province (13JJ3120), and the Construct Program of the Key Discipline in Hunan Province.

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with the boundary condition $\sum_{i=1}^{m} A_i x(\xi_i) = 0$ has nontrivial solutions. Otherwise, we call them a problem at nonresonance. In the present work, if $\sum_{i=1}^{m} A_i = 0$, then BVP (1.1) is at resonance, since equation ${}^C D^\alpha x(t)) = 0$ with boundary condition

$$\sum_{i=1}^{m} A_i x(\xi_i) = 0$$

has nontrivial solutions $x = c$, $c \in R^n$.

The theory of fractional differential equations has been extensively studied since the behavior of many physical systems can be properly described by using the fractional order system theory. There has been a great deal of interest in the solutions of fractional differential equations, see the monographs [16, 17, 21, 23], and the papers [1, 3, 4, 5, 6, 7, 11, 12, 13, 15, 18, 22, 24, 25, 26, 29, 31] and the references therein. Tools used to analyze the solvability of these fractional differential equations are mainly focused on fixed point theorems and Leray-Schauder theory. Recently, there have been few studies dealing with the existence for solutions of fractional BVP by using the coincidence degree theory [9, 10, 14, 28]. Existence results for fractional BVP with derivative order $\alpha \in (2, 3)$ are established in [9, 14], and existence results for fractional BVP with derivative order $\alpha \in (1, 2)$ are obtained in [10, 28]. However, there is no work on the existence of solutions for fractional BVP with derivative order $\alpha \in (0, 1)$. It is clear that the corresponding integral equations of fractional BVP is weakly singular if derivative order $0 < \alpha < 1$, and regular for $\alpha \geq 1$.

Motivated by [19], in which the existence to first-order multi-point BVP is proven by using the coincidence degree theory, here we investigate the existence of solutions for fractional multi-point BVP with derivative order $\alpha \in (0, 1)$. Different from the Riemann-Liouville fractional derivative used in [9, 10, 14, 28], here we consider the Caputo's one.

The outline of the remainder of this paper is as follows. In section 2, we recall some useful preliminaries. In section 3, we give the existence result of BVP (1.1) at resonance (i.e., $\sum_{i=1}^{m} A_i = 0$). In section 4, an example is given to illustrate our main results.

2. Preliminaries and Lemmas

Let us recall some notations and an abstract existences result.

Let Y, Z be real Banach spaces, $L : domL \subset Y \to Z$ be a Fredholm map of index zero and $P : Y \to Y$, $Q : Z \to Z$ be continuous projectors such that $ImP = KerL$, $KerQ = ImL$ and $Y = KerL \oplus KerP$, $Z = ImL \oplus ImQ$. It follows that $L|_{domL \cap KerP} : domL \cap KerP \to ImL$ is invertible. We denote the inverse by K_P.

If Ω is an open bounded subset of Y, and $domL \bigcap \overline{\Omega} = \emptyset$, the map $N : Y \to Z$ will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to Y$ is compact.

Theorem 2.1 [20] Let $L : domL \subset Y \to Z$ be a Fredholm operator of index zero and let $N : Y \to Z$ be $L-$compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:

(i) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [domL \setminus KerL \cap \partial\Omega] \times (0, 1)$;

(ii) $Nx \notin ImL$ for every $x \in LerL \cap \partial\Omega$;
(iii) $\deg(QN|_{KerL}, \Omega \cap KerL, 0) \neq 0$, where $Q : Z \to Z$ is a projection as above with $ImL = KerQ$.

Then the equation $Lx = Nx$ has at least one solution in $domL \bigcap \overline{\Omega}$.

We also recall the following known definitions with respect to the fractional integral and derivatives. For more details see [2, 30].

Definition 2.1 The fractional integral of order γ with the lower limit zero for a function f is defined as

$$I^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \ t > 0, \ \gamma > 0,$$

provided that the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 Riemann-Liouville derivative of order γ with the lower limit zero for a function $f : [0, \infty) \to R$ can be written as

$$D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \ t > 0, \ n-1 < \gamma < n.$$

Definition 2.3 [16] Caputo's derivative of order γ for a function $f : [0, \infty) \to R$ can be written as

$$^{C}D^{\gamma} f(t) = \ D^{\gamma} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-\gamma+1)} t^{k-\gamma}, \ t > 0, \ n-1 < \gamma < n.$$

Remark 2.1 (1) If $f(t) \in C^n([0, \infty), R)$, then

$$^{C}D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds = I^{n-\gamma} f^{(n)}(t), \ t > 0, \ n-1 < \gamma < n.$$

(2) The Caputo derivative of a constant is equal to zero, i.e., if $x(t) = c$, then $^{C}D^{\alpha} c = 0$. However, $D^{\alpha} c = \frac{ct^{-\alpha}}{\Gamma(1-\alpha)}$.

Lemma 2.1 [27] Let $n-1 < \gamma < n$, then the differential equation

$$^{C}D^{\gamma} x(t) = 0$$

has solutions $x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1}$, $c_i \in R$, $i = 0, 1, \cdots, n$.

Lemma 2.2 [27] Let $n-1 < \gamma < n$, then

$$I^{\gamma C} D^{\gamma} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1}$$

for some $c_i \in R$, $i = 0, 1, \cdots, n$.

The following basic inequalities will be used.

Lemma 2.3 [8] Let $a_1, a_2, \cdots, a_n \geq 0$, $n \in N$, then

$$\sum_{i=1}^{n} a_i^r \leq \left(\sum_{i=1}^{n} a_i \right)^r \leq n^{r-1} \left(\sum_{i=1}^{n} a_i^r \right), \quad r \geq 1,$$

and

$$n^{r-1} \left(\sum_{i=1}^{n} a_i^r \right) \leq \left(\sum_{i=1}^{n} a_i \right)^r \leq \sum_{i=1}^{n} a_i^r, \quad 0 \leq r \leq 1.$$

We denote the $n \times n$ identity matrix by E, the Banach space of all constant square matrices of order n by $M_{n\times n}$ with the norm $\|B\| = \max_{1\le i,j\le n} |b_{i,j}|$. For $\alpha = (\alpha_1, \cdots, \alpha_n)'$, define $\|\alpha\| = \max_{1\le i\le n} |\alpha_i|$. The L^p-norm in $L^p([0,1], R^n)$ is defined by $\|x\|_p = \max_{1\le i\le n} \left(\int_0^1 |x_i(t)|^p dt \right)^{1/p}$ for $1 \le p < \infty$. The $L^\infty-$norm in $C([0,1], R^n)$ is $\|x\|_\infty = \max_{1\le i\le n} \sup_{t\in[0,1]} |x_i(t)|$.

3. Main Results

In this paper, we always assume the following conditions hold.

(H$_1$) $\sum_{i=1}^m A_i = 0$ and $\det \left(\sum_{i=1}^m A_i \xi_i^\alpha \right) \ne 0$.

(H$_2$) $f : [0,1] \times R^n \to R^n$ satisfies the following conditions: $f(\cdot, x)$ is continuous for each fixed $x \in R^n$, $f(t, \cdot)$ is Lebesgue measurable for a.e. $t \in [0,1]$, and for each $r > 0$, there exists $h_r \in L^\infty([0,1], R^n)$ such that $|f_i(t,x)| \le (h_r)_i(t)$ for all $\|x\|_\infty < r$, a.e. $t \in [0,1]$, $i = 1, 2, \cdots, n$. $e \in L^1([0,1], R^n)$.

(H$_3$) There exists a constant $\alpha_1 \in (0, \alpha)$ such that $f \in L^{\frac{1}{\alpha_1}}([0,1] \times R^n, R^n)$, $e \in L^{\frac{1}{\alpha_1}}([0,1], R^n)$.

Let $Y = C([0,1], R^n)$, $Z = L^1([0,1], R^n)$. Define the linear operator $L : domL \cap Y \to Z$ with

$$domL = \left\{ x \in C([0,1], R^n) : \sum_{i=1}^m A_i x(\xi_i) = 0,\ {}^C D^\alpha x \in L^1([0,1], R^n) \right\},$$

and

$$Lx = {}^C D^\alpha x,\ x \in domL. \tag{3.1}$$

Define $N : Y \to Z$ by

$$Nx(t) = f(t, x(t)),\ t \in [0,1].$$

Then BVP (1.1) can be written as

$$Lx = Nx.$$

Lemma 3.1 Let L be defined as (3.1), then

$$KerL = \{x \in domL : x = c, c \in R^n\}, \tag{3.2}$$

and

$$ImL = \{y \in Z : \sum_{i=1}^m A_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} y(s) ds = 0\}. \tag{3.3}$$

Proof. By Lemma 2.1, ${}^C D^\alpha x(t) = 0$ has solution $x(t) = x(0)$ which implies that (3.2) holds.

For $y \in ImL$, if the equation

$$ {}^C D^\alpha x(t) = y(t) \tag{3.4}$$

has a solution $x(t)$ such that $\sum_{i=1}^{m} A_i x(\xi_i) = 0$, then from (3.4), we have

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds.$$

Applying condition $\sum_{i=1}^{m} A_i = 0$, we have

$$\begin{aligned} 0 = \sum_{i=1}^{m} A_i x(\xi_i) &= x(0) \sum_{i=1}^{m} A_i + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} A_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} y(s) ds \\ &= \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} A_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} y(s) ds, \end{aligned}$$

that is

$$\sum_{i=1}^{m} A_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} y(s) ds = 0. \tag{3.5}$$

On the other hand, if (3.5) holds, we can set

$$x(t) = d + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

where $d \in R^n$ is arbitrary. Hence, $x(t)$ is a solution of (3.4) and $\sum_{i=1}^{m} A_i x(\xi_i) = 0$ which means that $y \in ImL$. Therefore, (3.3) holds. The proof is complete. $\square$

Lemma 3.2 *$L : domL \subset Y \to Z$ is a Fredholm operator of index zero. Furthermore, the linear continuous projection operators $Q : Z \to Z$ and $P : Y \to Y$ can be defined by*

$$Qy = \left(\frac{1}{\alpha} \sum_{i=1}^{m} A_i \xi_i^{\alpha} \right)^{-1} \sum_{i=1}^{m} A_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} y(s) ds, \text{ for every } y \in Z,$$

$$Px = x(0), \text{ for every } x \in Y.$$

And the linear operator $K_P : ImL \to domL \cap KerP$ can be written by

$$K_p y = I^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds.$$

Also,

$$\|K_P y\|_\infty \le \frac{(1+\nu)^{\alpha_1 - 1}}{\Gamma(\alpha)} \|y\|_{\frac{1}{\alpha_1}}, \text{ for all } y \in ImL,$$

where $\nu = \frac{\alpha-1}{1-\alpha_1}$.

Proof. It is easy to know that $ImP = KerL$ and $P^2 x = Px$. It follows from $x = (x - Px) + Px$ that $Y = KerP + KerL$. By simple calculation, we have that $KerL \cap KerP = \{0\}$. Thus, $Y = KerP \oplus KerL$.

Since

$$\begin{aligned} Q^2y &= Q\bigg(\Big(\frac{1}{\alpha}\sum_{i=1}^m A_i\xi_i^\alpha\Big)^{-1}\sum_{i=1}^m A_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}y(s)ds\bigg) \\ &= \Big(\frac{1}{\alpha}\sum_{i=1}^m A_i\xi_i^\alpha\Big)^{-1}\sum_{i=1}^m A_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}y(s)ds \\ &= Qy. \end{aligned}$$

For $y\in Z$, set $y=(y-Qy)+Qy$. Then, $y-Qy\in KerQ=ImL$, $Qy\in ImQ$. It follows from $KerQ=ImL$ and $Q^2y=Qy$ that $ImQ\cap ImL=\{0\}$. Thus, $Z=ImL\oplus ImQ$ and

$$\dim KerL=\dim R^n=\operatorname{co}\dim ImL=n.$$

Hence, L is a Fredholm operator of index zero.

With definitions of P,K_P, it is easy to show that the generalized inverse of $L: ImL\to domL\cap KerP$ is K_P.

In fact, for $y\in ImL$, one has

$$(LK_P)y={}^CD^\alpha I^\alpha y=y,$$

and for $x\in domL\cap KerP$ which implies that $x(0)=0$, we have

$$(K_PL)x(t)=I^{\alpha C}D^\alpha x(t)=x(t).$$

This shows that $K_P=(L|_{domL\cap KerP})^{-1}$.

Let $\nu=\frac{\alpha-1}{1-\alpha_1}$, then $1+\nu>0$ since that $\alpha_1\in(0,\alpha)$, we have

$$\begin{aligned} \|K_Py\|_\infty &= \|I^\alpha y\|_\infty \\ &= \left\|\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds\right\|_\infty \\ &\le \frac{1}{\Gamma(\alpha)}\left[\int_0^t(t-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right]^{1-\alpha_1}\max_{1\le i\le n}\left\{\left(\int_0^t|y_i(s)|^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1}\right\} \\ &\le \frac{1}{\Gamma(\alpha)}\left(\frac{1}{1+\nu}t^{1+\nu}\right)^{1-\alpha_1}\|y\|_{\frac{1}{\alpha_1}} \\ &\le \frac{(1+\nu)^{\alpha_1-1}}{\Gamma(\alpha)}\|y\|_{\frac{1}{\alpha_1}}. \end{aligned}$$

The proof is complete. □

Lemma 3.3 Assume $\Omega\subset Y$ is an open bounded subset and $domL\cap\overline{\Omega}\neq\emptyset$, then N is $L-$compact on $\overline{\Omega}$.

Proof. By (H_2), we have that $QN(\overline{\Omega})$ is bounded.

Now we show that $K_P(I-Q)N:\overline{\Omega}\to Y$ is compact.

The operator $K_P(I-Q)N:Y\to Y$ is continuous in view of the continuity of f.

Since $\overline{\Omega} \subset Y$ is bounded, there exists a positive constant $M_0 > 0$ such that $\|x\|_\infty \leq M_0$ for all $x \in \overline{\Omega}$. Set

$$\begin{aligned} M_1 &= \max_{\|x\|_\infty \leq M} \bigg\| \int_0^t (t-s)^{\alpha-1}[f(s,x(s))+e(s)]ds \\ &\quad -t^\alpha \bigg(\sum_{i=1}^m A_i \xi_i^\alpha \bigg)^{-1} \sum_{i=1}^m A_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1}[f(s,x(s))+e(s)]ds \bigg\|_\infty, \end{aligned}$$

and

$$M_2 = \bigg\| \bigg(\frac{1}{\alpha} \sum_{i=1}^m A_i \xi_i^\alpha \bigg)^{-1} \sum_{i=1}^m A_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1}[f(s,x(s))+e(s)]ds \bigg\|.$$

For $x \in \overline{\Omega}$, we obtain

$$\begin{aligned} & \|(K_P(I-Q)N)x\|_\infty \\ = & \|I^\alpha\{f(t,x(t))+e(t)-Q[(f(t,x))+e(t)]\}\|_\infty \\ = & \frac{1}{\Gamma(\alpha)} \bigg\| \int_0^t (t-s)^{\alpha-1}[f(s,x(s))+e(s)]ds \\ - & \bigg(\frac{1}{\alpha} \sum_{i=1}^m A_i \xi_i^\alpha \bigg)^{-1} \sum_{i=1}^m A_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1}[f(s,x(s))+e(s)]ds \int_0^t (t-s)^{\alpha-1} ds \bigg\|_\infty \\ = & \frac{1}{\Gamma(\alpha)} \bigg\| \int_0^t (t-s)^{\alpha-1}[f(s,x(s))+e(s)]ds \\ - & t^\alpha \bigg(\sum_{i=1}^m A_i \xi_i^\alpha \bigg)^{-1} \sum_{i=1}^m A_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1}[f(s,x(s))+e(s)]ds \bigg\|_\infty \\ \leq & \frac{1}{\Gamma(\alpha)} M_1. \end{aligned}$$

Thus, $K_P(I-Q)N(\overline{\Omega}) \subset Y$ is bounded.

Let $t_1, t_2 \in [0,1]$ and $t_2 > t_1$.

$$\begin{aligned} & \|(K_P(I-Q)N)x(t_1) - (K_P(I-Q)N)x(t_2)\|_\infty \\ & = \|I^\alpha\{f(s,x(s))+e(s)-Q[f(s,x(s))+e(s)]\}|_{t=t_1} \\ & -I^\alpha\{f(s,x(s))+e(s)-Q[f(s,x(s))+e(s)]\}|_{t=t_2}\|_\infty \\ & = \frac{1}{\Gamma(\alpha)} \bigg\| \int_0^{t_1} (t_1-s)^{\alpha-1}[f(s,x(s))+e(s)]ds \\ & - \int_0^{t_2} (t_2-s)^{\alpha-1}[f(s,x(s))+e(s)]ds \\ & + \bigg(\frac{1}{\alpha} \sum_{i=1}^m A_i \xi_i^\alpha \bigg)^{-1} \sum_{i=1}^m A_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1}[f(s,x(s))+e(s)]ds \\ & \cdot \bigg[\int_0^{t_1} (t_1-s)^{\alpha-1} ds - \int_0^{t_2} (t_2-s)^{\alpha-1} ds \bigg] \bigg\|_\infty \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(\alpha)}\Big\| \int_0^{t_1}(t_1-s)^{\alpha-1}[f(s,x(s))+e(s)]ds \\
&\quad - \int_0^{t_2}(t_2-s)^{\alpha-1}[f(s,x(s))+e(s)]ds\Big\|_\infty \\
&+\frac{1}{\Gamma(\alpha)}\Big\|\Big(\frac{1}{\alpha}\sum_{i=1}^m A_i\xi_i^\alpha\Big)^{-1}\sum_{i=1}^m A_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}[f(s,x(s))+e(s)]ds\Big\| \\
&\quad \cdot\Big|\int_0^{t_1}(t_1-s)^{\alpha-1}ds-\int_0^{t_2}(t_2-s)^{\alpha-1}ds\Big| \\
&\le \frac{1}{\Gamma(\alpha)}\Big\|\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}][f(s,x(s))+e(s)]ds\Big\|_\infty \\
&\quad +\frac{1}{\Gamma(\alpha)}\Big\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}[f(s,x(s))+e(s)]ds\Big\|_\infty \\
&+\frac{1}{\Gamma(\alpha)}\Big\|\Big(\frac{1}{\alpha}\sum_{i=1}^m A_i\xi_i^\alpha\Big)^{-1}\sum_{i=1}^m A_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}[f(s,x(s))+e(s)]ds\Big\|\cdot\frac{t_2^\alpha-t_1^\alpha}{\alpha} \\
&\le \frac{1}{\Gamma(\alpha)}\Big\|\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]f(s,x(s))ds\Big\|_\infty \\
&\quad +\frac{1}{\Gamma(\alpha)}\Big\|\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]e(s)ds\Big\|_\infty \\
&+\frac{1}{\Gamma(\alpha)}\Big\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,x(s))ds\Big\|_\infty+\frac{1}{\Gamma(\alpha)}\Big\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}e(s)ds\Big\|_\infty \\
&+\frac{1}{\Gamma(\alpha)}\Big\|\Big(\frac{1}{\alpha}\sum_{i=1}^m A_i\xi_i^\alpha\Big)^{-1}\sum_{i=1}^m A_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}[f(s,x(s))+e(s)]ds\Big\|\cdot\frac{t_2^\alpha-t_1^\alpha}{\alpha} \\
&\le \frac{1}{\Gamma(\alpha)}\Big\{\int_0^{t_1}\Big[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\Big]^{\frac{1}{1-\alpha_1}}ds\Big\}^{1-\alpha_1}\max_{1\le i\le n}\Big\{\Big[\int_0^{t_1}|f_i(s,x(s))|^{\frac{1}{\alpha_1}}ds\Big]^{\alpha_1}\Big\} \\
&+\frac{1}{\Gamma(\alpha)}\Big\{\int_0^{t_1}\Big[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\Big]^{\frac{1}{1-\alpha_1}}ds\Big\}^{1-\alpha_1}\max_{1\le i\le n}\Big\{\Big[\int_0^{t_1}|e_i(s)|^{\frac{1}{\alpha_1}}ds\Big]^{\alpha_1}\Big\} \\
&\quad +\frac{1}{\Gamma(\alpha)}\Big[\int_{t_1}^{t_2}(t_2-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\Big]^{1-\alpha_1}\max_{1\le i\le n}\Big\{\Big[\int_{t_1}^{t_2}|f_i(s,x(s))|^{\frac{1}{\alpha_1}}ds\Big]^{\alpha_1}\Big\} \\
&+\frac{1}{\Gamma(\alpha)}\Big[\int_{t_1}^{t_2}(t_2-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\Big]^{1-\alpha_1}\max_{1\le i\le n}\Big\{\Big[\int_{t_1}^{t_2}|e_i(s)|^{\frac{1}{\alpha_1}}ds\Big]^{\alpha_1}\Big\}+\frac{M_2}{\Gamma(1+\alpha)}(t_2^\alpha-t_1^\alpha) \\
&\le \frac{1}{\Gamma(\alpha)}\Big\{\int_0^{t_1}\Big[(t_1-s)^{\frac{\alpha-1}{1-\alpha_1}}-(t_2-s)^{\frac{\alpha-1}{1-\alpha_1}}\Big]ds\Big\}^{1-\alpha_1}(\|f\|_{\frac{1}{\alpha_1}}+\|e\|_{\frac{1}{\alpha_1}}) \\
&\quad +\frac{1}{\Gamma(\alpha)}\Big[\int_{t_1}^{t_2}(t_2-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\Big]^{1-\alpha_1}(\|f\|_{\frac{1}{\alpha_1}}+\|e\|_{\frac{1}{\alpha_1}})+\frac{M_2}{\Gamma(1+\alpha)}(t_2^\alpha-t_1^\alpha) \\
&\le \frac{1}{\Gamma(\alpha)}\Big(\frac{1}{1+\nu}\Big)^{1-\alpha_1}\Big[t_1^{\frac{\alpha-1}{\alpha-\alpha_1}}+(t_2-t_1)^{\frac{\alpha-1}{\alpha-\alpha_1}}-t_2^{\frac{\alpha-1}{\alpha-\alpha_1}}\Big]^{1-\alpha_1}(\|f\|_{\frac{1}{\alpha_1}}+\|e\|_{\frac{1}{\alpha_1}})
\end{aligned}
$$

$$+\frac{1}{\Gamma(\alpha)}\left(\frac{1}{1+\nu}\right)^{1-\alpha_1}\left[(t_2-t_1)^{\frac{\alpha-1}{\alpha-\alpha_1}}\right]^{1-\alpha_1}(\|f\|_{\frac{1}{\alpha_1}}+\|e\|_{\frac{1}{\alpha_1}})+\frac{M_2}{\Gamma(1+\alpha)}(t_2^\alpha-t_1^\alpha)$$
$$\leq \frac{1}{\Gamma(\alpha)}\left(\frac{1}{1+\nu}\right)^{1-\alpha_1}(\|f\|_{\frac{1}{\alpha_1}}+\|e\|_{\frac{1}{\alpha_1}})(t_2-t_1)^{\alpha-\alpha_1}+\frac{M_2}{\Gamma(1+\alpha)}(t_2^\alpha-t_1^\alpha)$$
$$\to 0 \quad \text{as} \quad t_2\to t_1.$$

Then $K_P(I-Q)N(\overline{\Omega})$ is equicontinuous. By the Ascoli-Arzela theorem, we have that $K_P(I-Q)N:\overline{\Omega}\to Y$ is compact, then N is $L-$compact on $\overline{\Omega}$. The proof is complete. □

To obtain our main results, we also need the following conditions.

(H_4) There exists function a, b, $r\in L^{\frac{1}{\alpha_1}}([0,1],R)$, and constant $\theta\in[0,1)$ such that

$$\|f(t,x)\|_\infty\leq a(t)\|x\|_\infty+b(t)\|x\|_\infty^\theta+r(t), \tag{3.6}$$

for all $x\in R^n$ and $t\in[0,1]$

(H_5) There exists a constant $M>0$ such that, for $x\in domL$, if there exist some $i_0\in\{1,2,\cdots,n\}$ such that $|x_{i_0}(t)|>M$ for all $t\in[0,1]$, then

$$\sum_{i=1}^m A_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}[f(s,x(s))+e(s)]ds\neq 0. \tag{3.7}$$

(H_6) There exists a constant $M^*>0$ such that for any $c=(c_1,c_2,\cdots,c_n)'\in R^n$, if $\|c\|>M^*$, then either

$$c'\cdot\left(\frac{1}{\alpha}\sum_{i=1}^m A_i\xi_i^\alpha\right)^{-1}\sum_{i=1}^m A_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}[f(s,c)+e(s)]ds<0 \tag{3.8}$$

or

$$c'\cdot\left(\frac{1}{\alpha}\sum_{i=1}^m A_i\xi_i^\alpha\right)^{-1}\sum_{i=1}^m A_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}[f(s,c)+e(s)]ds>0. \tag{3.9}$$

Theorem 3.1 Assume (H_1)$-$(H_6) hold. Then BVP (1.1) has at least one solution $x\in C([0,1],R^n)$ provided that

$$(1+\nu)^{1-\alpha_1}\Gamma(\alpha)-2\cdot 6^{1-\alpha_1}\|a\|_{\frac{1}{\alpha_1}}>0. \tag{3.10}$$

Proof. Set

$$\Omega_1=\{x\in domL\setminus KerL: Lx=Nx \text{ for some } \lambda\in[0,1]\}.$$

Then, for $x\in\Omega_1$, $Lx=\lambda Nx$, so $\lambda\neq 0$ and $Nx\in ImL=KerQ$. Hence,

$$\sum_{i=1}^m A_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}[f(s,x(s))+e(s)]ds=0.$$

By (H_5), there exists $t_i\in[0,1]$ such that $|x_i(t_i)|\leq M$ for all $i\in\{1,2,\cdots,n\}$. Since

$$x_i(0)=x_i(t_i)-\frac{1}{\Gamma(\alpha)}\int_0^{t_i}(t_i-s)^{\alpha-1C}D^\alpha x_i(s)ds,$$

which implies that

$$\begin{aligned}|x_i(0)| &\le |x_i(t_i)| + \frac{1}{\Gamma(\alpha)}\int_0^{t_i}(t_i-s)^{\alpha-1}|^C D^\alpha x_i(t)|ds\\ &\le |x_i(t_i)| + \frac{1}{\Gamma(\alpha)}\left(\frac{1}{1+\nu}t_i^{1+\nu}\right)^{1-\alpha_1}\left(\int_0^1 |^C D^\alpha x_i(t)|^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1}\\ &\le |x_i(t_i)| + \frac{(1+\nu)^{\alpha_1-1}}{\Gamma(\alpha)}\left(\int_0^1 |^C D^\alpha x_i(t)|^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1}.\end{aligned}$$

Thus,

$$\|x(0)\|_\infty \le M + \frac{(1+\nu)^{\alpha_1-1}}{\Gamma(\alpha)}\|^C D^\alpha x\|_{\frac{1}{\alpha_1}}. \tag{3.11}$$

On the other hand,

$$\|^C D^\alpha x\|_{\frac{1}{\alpha_1}} = \|Lx\|_{\frac{1}{\alpha_1}} \le \|Nx\|_{\frac{1}{\alpha_1}}. \tag{3.12}$$

From (3.11) and (3.12), we obtain that

$$\|x(0)\|_\infty \le M + \frac{(1+\nu)^{\alpha_1-1}}{\Gamma(\alpha)}\|Nx\|_{\frac{1}{\alpha_1}}. \tag{3.13}$$

Also for $x \in \Omega_1$, $x \in domL \setminus KerL$, then $(I-P)x \in domL \cap KerP$, $LPx = 0$. Applying Lemma 3.3, we have

$$\begin{aligned}\|(I-P)x\|_\infty &= \|K_P L(I-P)x\|_\infty \le \frac{(1+\nu)^{\alpha_1-1}}{\Gamma(\alpha)}\|L(I-P)x\|_{\frac{1}{\alpha_1}}\\ &\le \frac{(1+\nu)^{\alpha_1-1}}{\Gamma(\alpha)}\|Lx\|_{\frac{1}{\alpha_1}} \le \frac{(1+\nu)^{\alpha_1-1}}{\Gamma(\alpha)}\|Nx\|_{\frac{1}{\alpha_1}}.\end{aligned} \tag{3.14}$$

From condition (H_4) and Lemma 2.3, for $i = 1, 2, \cdots, n$ we have

$$\begin{aligned}&\left[\int_0^1 |f_i(t,x(t)) + e_i(t)|^{\frac{1}{\alpha_1}}dt\right]^{\alpha_1} \le \left[\int_0^1 (|f_i(t,x(t))| + |e_i(t)|)^{\frac{1}{\alpha_1}}dt\right]^{\alpha_1}\\ &\le \left[2^{\frac{1}{\alpha_1}-1}\int_0^1 (|f_i(t,x(t))|^{\frac{1}{\alpha_1}} + |e_i(t)|^{\frac{1}{\alpha_1}})dt\right]^{\alpha_1}\\ &\le 2^{1-\alpha_1}\left\{\left[\int_0^1 |f_i(t,x(t))|^{\frac{1}{\alpha_1}}dt\right]^{\alpha_1} + \left[\int_0^1 |e_i(t)|^{\frac{1}{\alpha_1}}dt\right]^{\alpha_1}\right\}\\ &\le 2^{1-\alpha_1}\left\{\int_0^1\left[a(t)\|x\|_\infty + b(t)\|x\|_\infty^\theta + r(t)\right]^{\frac{1}{\alpha_1}}dt\right\}^{\alpha_1} + 2^{1-\alpha_1}\|e\|_{\frac{1}{\alpha_1}}\\ &\le 6^{1-\alpha_1}\left\{\int_0^1\left[\left(|a(t)|\|x\|_\infty\right)^{\frac{1}{\alpha_1}} + \left(|b(t)|\|x\|_\infty^\theta\right)^{\frac{1}{\alpha_1}} + |r(t)|^{\frac{1}{\alpha_1}}\right]dt\right\}^{\alpha_1} + 2^{1-\alpha_1}\|e\|_{\frac{1}{\alpha_1}}\\ &\le 6^{1-\alpha_1}\left[\left(\int_0^1 |a(t)|^{\frac{1}{\alpha_1}}dt\right)^{\alpha_1}\|x\|_\infty + \left(\int_0^1 |b(t)|^{\frac{1}{\alpha_1}}dt\right)^{\alpha_1}\|x\|_\infty^\theta\right.\\ &\left.+\left(\int_0^1 |r(t)|^{\frac{1}{\alpha_1}}dt\right)^{\alpha_1}\right] + 2^{1-\alpha_1}\|e\|_{\frac{1}{\alpha_1}}\end{aligned}$$

$$= 6^{1-\alpha_1}\left(\|a\|_{\frac{1}{\alpha_1}}\|x\|_\infty + \|b\|_{\frac{1}{\alpha_1}}\|x\|_\infty^\theta + \|r\|_{\frac{1}{\alpha_1}}\right) + 2^{1-\alpha_1}\|e\|_{\frac{1}{\alpha_1}},$$

which yields that

$$\|f+e\|_{\frac{1}{\alpha_1}} \le 6^{1-\alpha_1}\left(\|a\|_{\frac{1}{\alpha_1}}\|x\|_\infty + \|b\|_{\frac{1}{\alpha_1}}\|x\|_\infty^\theta + \|r\|_{\frac{1}{\alpha_1}}\right) + 2^{1-\alpha_1}\|e\|_{\frac{1}{\alpha_1}}. \quad (3.15)$$

Combining (3.13), (3.14) and (3.15), we have

$$\begin{aligned}
\|x\|_\infty &\le \|Px\|_\infty + \|(I-P)x\|_\infty \\
&= \|x(0)\|_\infty + \|(I-P)x\|_\infty \\
&\le \frac{2(1+\nu)^{\alpha_1-1}}{\Gamma(\alpha)}\|Nx\|_{\frac{1}{\alpha_1}} + M \\
&\le \frac{2(1+\nu)^{\alpha_1-1}}{\Gamma(\alpha)}\|f+e\|_{\frac{1}{\alpha_1}} + M \\
&\le \frac{2(1+\nu)^{\alpha_1-1}}{\Gamma(\alpha)}\Big(6^{1-\alpha_1}\|a\|_{\frac{1}{\alpha_1}}\|x\|_\infty + 6^{1-\alpha_1}\|b\|_{\frac{1}{\alpha_1}}\|x\|_\infty^\theta \\
&\quad +6^{1-\alpha_1}\|r\|_{\frac{1}{\alpha_1}} + 2^{1-\alpha_1}\|e\|_{\frac{1}{\alpha_1}}\Big) + M. \qquad (3.16)
\end{aligned}$$

Thus, from (3.10) and (3.16) we have

$$\begin{aligned}
\|x\|_\infty \le\ & \frac{2\cdot 6^{1-\alpha_1}\|b\|_{\frac{1}{\alpha_1}}}{(1+\nu)^{1-\alpha_1}\Gamma(\alpha) - 2\cdot 6^{1-\alpha_1}\|a\|_{\frac{1}{\alpha_1}}}\|x\|_\infty^\theta \\
&+\frac{2\cdot 6^{1-\alpha_1}\|r\|_{\frac{1}{\alpha_1}} + 2\cdot 2^{1-\alpha_1}\|e\|_{\frac{1}{\alpha_1}} + (1+\nu)^{1-\alpha_1}\Gamma(\alpha)M}{(1+\nu)^{1-\alpha_1}\Gamma(\alpha) - 2\cdot 6^{1-\alpha_1}\|a\|_{\frac{1}{\alpha_1}}}.
\end{aligned}$$

Since $0 \in [0,1)$, from above the inequality, there exists $M_3 > 0$ such that

$$\|x\|_\infty \le M_3,$$

which implies that Ω_1 is bounded.

Let

$$\Omega_2 = \{x \in KerL : Nx \in ImL\}.$$

For $x \in \Omega_2$, $x \in KerL = \{x \in domL : x = c, c \in R^n\}$, and $QNx = 0$, we can get

$$\sum_{i=1}^m A_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1}[f(s,c) + e(s)]ds = 0,$$

and $\|c\| \le M^*$. Otherwise, if $\|c\| > M^*$, from condition (H_6), we have

$$\sum_{i=1}^m A_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1}[f(s,c) + e(s)]ds \ne 0,$$

which is a contradiction. Thus, Ω_2 is bounded.

Next, we define the isomorphism $J : ImQ \to KerL$ by

$$Jc = c,\ \forall c \in R^n.$$

According to condition (H_6), for any $c \in R^n$, if $\|c\| > M^*$, then either (3.8) or (3.9) holds.

If (3.8) holds, set

$$\Omega_3 = \{x \in KerL : -\lambda Jx + (1-\lambda)QNx = 0, \lambda \in [0,1]\}.$$

For any $x = c_0 \in \Omega_3$, we have

$$\lambda c_0 = (1-\lambda)\cdot\left(\frac{1}{\alpha}\sum_{i=1}^{m} A_i\xi_i^{\alpha}\right)^{-1}\sum_{i=1}^{m} A_i\int_0^{\xi_i}(\xi_i - s)^{\alpha-1}[f(s,c_0)+e(s)]ds.$$

If $\lambda = 1$, then $c_0 = 0$. If $\|c_0\| > M^*$, from (3.8) we have

$${c_0}'\cdot\left(\frac{1}{\alpha}\sum_{i=1}^{m} A_i\xi_i^{\alpha}\right)^{-1}\sum_{i=1}^{m} A_i\int_0^{\xi_i}(\xi_i - s)^{\alpha-1}[f(s,c_0)+e(s)]ds < 0.$$

Thus, $\lambda {c_0}' c_0 < 0$ which contradicts $\lambda {c_0}' c_0 \geq 0$. Therefore, Ω_3 is bounded.

If (3.9) holds, set

$$\Omega_3' = \{x \in KerL : \lambda Jx + (1-\lambda)QNx = 0, \lambda \in [0,1]\}.$$

Similar to the above argument, we also have that Ω_3' is bounded.

Let $\Omega \supset \cup_{i=1}^{3}\overline{\Omega_i} \cup \{0\}$ (or $\Omega \supset \cup_{i=1}^{2}\overline{\Omega_i} \cup \overline{\Omega_3'} \cup \{0\}$) be a bounded open subset of Y. It follows from Lemma 3.3 that N is $L-$compact on $\overline{\Omega}$. Then by the above argument, we have that conditions (i) and (ii) of Theorem 2.1 are satisfied, and we need only prove condition (iii) of Theorem 2.1 hold.

Take

$$H(x,\lambda) = \pm\lambda Jx + (1-\lambda)QNx.$$

In view of the argument to the sets Ω_3 and Ω_3', we have that $H(x,\lambda) \neq 0$ for all $\partial\Omega \cap KerL$. By the homotopy of degree, we get that

$$\begin{aligned}
\deg(QN|_{KerL}, \Omega\cap KerL, 0) &= \deg(H(\cdot,0), \Omega\cap KerL, 0)\\
&= \deg(H(\cdot,1), \Omega\cap KerL, 0)\\
&= \deg(\pm J, \Omega\cap KerL, 0)\\
&\neq 0,
\end{aligned}$$

which means that condition (iii) of Theorem 2.1 is satisfied. By Theorem 2.1, $Lx = Nx$ has at least one solution in $domL \cap \overline{\Omega}$, then BVP (1.1) has at least one solution in $C([0,1], R^n)$. This completes of the proof. □

4. An Example

As the application of our main result, we consider the following example.

Example 4.1 Consider BVP

$$\begin{cases} {}^CD^{\frac{1}{2}}x_1 = \frac{1}{24}x_1(1+\cos^2 x_2) + 3\sin(x_1)^{\frac{1}{3}} + \cos^2 t + 1, \\ {}^CD^{\frac{1}{2}}x_2 = \frac{1}{24}x_2(1+e^{-\sin^2 x_1}) + 3\sin(x_2)^{\frac{1}{3}} + \sin^2 t + 1, \end{cases} \tag{4.1}$$

with the boundary condition

$$\begin{cases} -\frac{3}{2}x_1(0) + x_1(\frac{1}{4}) + \frac{1}{2}x_1(1) = 0, \\ -3x_1(0) - 2x_2(0) + 2x_1(\frac{1}{4}) + x_2(\frac{1}{4}) + x_1(1) + x_2(1) = 0. \end{cases} \tag{4.2}$$

From (4.1), we have that $f(t,x) = (f_1(t,x), f_2(t,x))'$, $e(t) = (e_1(t), e_2(t))'$, where $f_1(t,x) = \frac{1}{24}x_1(1+\cos^2 x_2) + 3\sin(x_1)^{\frac{1}{3}}$, $f_2(t,x) = \frac{1}{24}x_2(1+e^{-\sin^2 x_1}) + 3\sin(x_2)^{\frac{1}{3}}$, $e_1(t) = \cos^2 t + 1$, $e_2(t) = \sin^2 t + 1$. For any $x \in C([0,1], R^2)$ and $t \in [0,1]$, f, e satisfies condition (H_2). Taking $\alpha_1 = \frac{1}{4}$, then $\alpha_1 < \frac{1}{2} = \alpha$, $f \in L^{\frac{1}{4}}([0,1] \times R^2, R^2)$ and $e \in L^{\frac{1}{4}}([0,1], R^2)$. Thus condition ($\mathrm{H}_3$) is satisfied.

Let $\xi_1 = 0, \xi_2 = \frac{1}{4}, \xi_3 = 1$, (4.2) can be written

$$A_1(x_1(\xi_1), x_2(\xi_1))' + A_2(x_1(\xi_2), x_2(\xi_2))' + A_3(x_1(\xi_3), x_2(\xi_3))' = 0,$$

where

$$A_1 = \begin{pmatrix} -\frac{3}{2} & 0 \\ -3 & -2 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},\ A_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 1 \end{pmatrix},$$

we have

$$A_1 + A_2 + A_3 = 0, \quad \text{and } \det\left(A_1\xi_1^{\frac{1}{2}} + A_2\xi_2^{\frac{1}{2}} + A_3\xi_3^{\frac{1}{2}}\right) = \frac{3}{2} \neq 0,$$

then condition (H_1) holds.

Taking $a(t) = \frac{1}{12}, b(t) = 3$, then $\|a\|_{\frac{1}{\alpha_1}} = \frac{1}{12}$ and

$$\|f(t,x)\|_\infty \leq a(t)\|x\|_\infty + b(t)\|x\|_\infty^{\frac{1}{3}},$$

which implies that condition (H_4) holds.

Moreover,

$$\begin{aligned}
&\sum_{i=1}^{3} A_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} f(s, x(s))ds \\
= &\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{-\frac{1}{2}} [f_1(s,x(s)) + e_1(s)]ds \\ \int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{-\frac{1}{2}} [f_2(s,x(s)) + e_2(s)]ds \end{pmatrix} \\
&+ \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \int_0^1 (1 - s)^{-\frac{1}{2}} [f_1(s,x(s)) + e_1(s)]ds \\ \int_0^1 (1 - s)^{-\frac{1}{2}} [f_2(s,x(s)) + e_2(s)]ds \end{pmatrix} \\
= &\begin{pmatrix} F_1(x_1, x_2, e_1, e_2) \\ F_2(x_1, x_2, e_1, e_2) \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
&F_1(x_1, x_2, e_1, e_2) \\
= &\int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{-\frac{1}{2}} [f_1(s,x(s)) + e_1(s)]ds + \frac{1}{2}\int_0^1 (1-s)^{-\frac{1}{2}} [f_1(s,x(s)) + e_1(s)]ds, \\
&F_2(x_1, x_2, e_1, e_2) \\
= &2\int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{-\frac{1}{2}} [f_1(s,x(s)) + e_1(s)]ds + \int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{-\frac{1}{2}} [f_2(s,x(s)) + e_2(s)]ds \\
&+ \int_0^1 (1-s)^{-\frac{1}{2}} [f_1(s,x(s)) + e_1(s)]ds + \int_0^1 (1-s)^{-\frac{1}{2}} [f_2(s,x(s)) + e_2(s)]ds.
\end{aligned}$$

Take $M = 121$ and assume $|x_1(t)| > M$ for any $t \in [0,1]$, since x_1 is continuous, then either $x_1(t) > M$ or $x_1(t) < -M$ hold for any $t \in [0,1]$.

If $x_1(t) > M$ holds for any $t \in [0,1]$, we have

$$\begin{aligned}
& F_1(x_1, x_2, e_1, e_2) \\
= \ & \int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{-\frac{1}{2}} \left[\frac{1}{24} x_1(1 + \cos^2 x_2) + 3\sin(x_1)^{\frac{1}{3}} + \cos^2 t + 1\right] ds \\
& + \frac{1}{2} \int_0^1 (1 - s)^{-\frac{1}{2}} \left[\frac{1}{24} x_1(1 + \cos^2 x_2) + 3\sin(x_1)^{\frac{1}{3}} + \cos^2 t + 1\right] ds \\
\geq \ & \int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{-\frac{1}{2}} \left[\frac{1}{24} x_1(t) - 2\right] ds + \frac{1}{2} \int_0^1 (1 - s)^{-\frac{1}{2}} \left[\frac{1}{24} x_1(t) - 2\right] ds \\
> \ & \left[\frac{1}{24} M - 2\right] \int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{-\frac{1}{2}} ds + \frac{1}{2} \left[\frac{1}{24} M - 2\right] \int_0^1 (1 - s)^{-\frac{1}{2}} ds \\
= \ & \frac{1}{24} M - 2 > 0.
\end{aligned}$$

If $x_1(t) < -M$ holds for any $t \in [0,1]$, we have

$$\begin{aligned}
& F_1(x_1, x_2, e_1, e_2) \\
\leq \ & \int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{-\frac{1}{2}} \left[\frac{1}{24} x_1(t) + 5\right] ds + \frac{1}{2} \int_0^1 (1 - s)^{-\frac{1}{2}} \left[\frac{1}{24} x_1(t) + 5\right] ds \\
< \ & \left[-\frac{1}{24} M + 5\right] \int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{-\frac{1}{2}} ds + \frac{1}{2} \left[-\frac{1}{24} M + 5\right] \int_0^1 (1 - s)^{-\frac{1}{2}} ds \\
= \ & -\frac{1}{24} M + 5 < 0.
\end{aligned}$$

Hence, $\sum_{i=1}^{3} A_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} f(s, x(s)) ds \neq 0$, then condition (H$_5$) holds.

Taking $M^* = 241$, for any $c \in R^2$, when $\|c\| > M^*$, then either $\|c\| = |c_1| > M^*$ or $\|c\| = |c_2| > M^*$.

If $\|c\| = |c_1| > M^*$, then $|c_1| \geq |c_2|$. We have

$$\begin{aligned}
& c' \cdot \left(\frac{1}{\alpha} \sum_{i=1}^{3} A_i \xi_i^{\alpha}\right)^{-1} \sum_{i=1}^{3} A_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} f(s, c) ds \\
& = (c_1, c_2) \left(2 \begin{pmatrix} 1 & 0 \\ 2 & \frac{3}{2} \end{pmatrix}\right)^{-1} \begin{pmatrix} F_1(c_1, c_2, e_1, e_2) \\ F_2(c_1, c_2, e_1, e_2) \end{pmatrix} \\
& = \frac{1}{2}(c_1, c_2) \begin{pmatrix} 1 & 0 \\ -\frac{4}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} F_1(c_1, c_2, e_1, e_2) \\ F_2(c_1, c_2, e_1, e_2) \end{pmatrix} \\
& = \frac{1}{2}\left[(c_1 - \frac{4}{3} c_2) F_1(c_1, c_2, e_1, e_2) + \frac{2}{3} c_2 F_2(c_1, c_2, e_1, e_2)\right] \\
& = \frac{1}{2}\left[\frac{1}{12} c_1^2 (1 + \cos^2 c_2) + 6 c_1 \sin(c_1)^{\frac{1}{3}} + 2c_1 + c_1 \int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{-\frac{1}{2}} \cos^2 s ds\right.
\end{aligned}$$

$$+\frac{c_1}{2}\int_0^1 (1-s)^{-\frac{1}{2}}\cos^2 sds + \frac{1}{12}c_2^2(1+e^{-\sin^2 c_1}) + 6c_2\sin(c_2)^{\frac{1}{3}} + 2c_2$$
$$+\frac{2}{3}c_2\int_0^{\frac{1}{4}}(\frac{1}{4}-s)^{-\frac{1}{2}}\sin^2 sds + \frac{2}{3}c_2\int_0^1(1-s)^{-\frac{1}{2}}\sin^2 sds\Big]$$
$$\geq \frac{1}{24}c_1^2 - 10|c_1| > 0.$$

Similarly, if $\|c\| = |c_2| > M^*$, then $|c_2| \geq |c_1|$. We have

$$c' \cdot \left(\frac{1}{\alpha}\sum_{i=1}^{3} A_i\xi_i^\alpha\right)^{-1}\sum_{i=1}^{3} A_i\int_0^{\xi_i}(\xi_i - s)^{\alpha-1}f(s,c)ds \geq \frac{1}{24}c_2^2 - 10|c_2| > 0.$$

Thus, condition (H_6) is satisfied.

On the other hand,

$$\begin{aligned}(1+\nu)^{1-\alpha_1}\Gamma(\alpha) - 2\cdot 6^{1-\alpha_1}\|a\|_{\frac{1}{\alpha_1}} &= \left(1+\frac{\frac{1}{2}-1}{1-\frac{1}{4}}\right)^{1-\frac{1}{4}}\Gamma(\frac{1}{2}) - 2\times 6^{1-\frac{1}{4}}\times\frac{1}{12}\\ &\approx 0.1386 > 0,\end{aligned}$$

then (3.10) is satisfied.

Thus, BVP (4.1) has at least one solution $x \in C([0,1], R^2)$ by using Theorem 3.1.

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Received: February 28, 2012; Accepted: May 31, 2012.